DECOMPOSABLE MACHINES AND SIMPLE RECURSION[1]

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ABSTRACT

We introduce the class of decomposable machines, present a uniform realization theory for this class, and note that it yields not only the well-known theory for linear machines, but also the recent theory of group machines. In particular, we give a derivation of Kalman's module-theoretic approach to linear systems <u>in which linearity plays no role</u>.

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1. Summary of the Input-Process Approach.

Here we give a brief summary of the general framework given in [2a] for the study of machines in a category. In the next section we shall present those notions of category theory not contained in [2a] which are required for our introduction of decomposable machines in Section 3:

DEFINITION: A <u>process</u> in an arbitrary category $\mathcal{K}$ is a functor $X: \mathcal{K} \longrightarrow \mathcal{K}$. $\mathrm{Dyn}(X)$ denotes the <u>category of X-dynamics</u> whose objects are pairs (Q,δ) with Q an object of $\mathcal{K}$ and $QX \xrightarrow{\delta} Q$ a morphism in $\mathcal{K}$, and whose morphisms $(Q,\delta) \xrightarrow{f} (Q',\delta')$-the <u>X-dynamorphisms-</u> are $\mathcal{K}$-morphisms $Q \xrightarrow{f} Q'$ which render

$$\begin{array}{ccc} QX & \xrightarrow{\delta} & Q \\ {\scriptstyle fX}\downarrow & & \downarrow{\scriptstyle f} \\ Q'X & \xrightarrow{\delta'} & Q' \end{array}$$

commutative. Composition and identities are defined as in $\mathcal{K}$ so that $\mathrm{Dyn}(X)$ is a category.

We say that X is an <u>input process</u> if the forgetful functor $\mathrm{Dyn}\,(X) \longrightarrow \mathcal{K}: \ (Q,\delta) \mapsto Q$ has a left adjoint -that is, just in case for each K in $\mathcal{K}$ there exists a "free machine" $(KX^{@}, K\mu_0)$ with "inclusion of the generators" $K\eta : K \longrightarrow KX^{@}$, so that for any (Q,δ) in $\mathrm{Dyn}(X)$ and any $K \xrightarrow{f} Q$ in $\mathcal{K}$, there is exactly one dynamorphism ψ extending f; i.e. there exists a unique ψ which renders the following diagrams commutative:

$$\begin{array}{ccc} K & \xrightarrow{K\eta} & KX^{@} \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \psi} \\ & & Q \end{array} \qquad \begin{array}{ccc} (KX^{@})X & \xrightarrow{K\mu_0} & KX^{@} \\ {\scriptstyle \psi X}\downarrow & & \downarrow{\scriptstyle \psi} \\ QX & \xrightarrow{\delta} & Q \end{array}$$

Having defined an input process, we could then make the following general definition:

DEFINITION: A *machine* in a category $\mathcal{K}$ is a 7-tuple

$$M = (X, Q, \delta, I, \tau, Y, \beta)$$

where $X: \mathcal{K} \longrightarrow \mathcal{K}$ is an input process

$(Q,\delta) \in \mathrm{Dyn}(X)$ -we call Q the *state object*

I is an object of $\mathcal{K}$, the *initial state object*

$I \xrightarrow{\tau} Q$ is a $\mathcal{K}$-morphism called the *initial state*

Y is an object of $\mathcal{K}$, the *output object*

$Q \xrightarrow{\beta} Y$ is the *output map*.

We call $IX^@$ the *object of inputs*, and then use our freeness axiom to define the *reachability map* r as the unique dynamorphic extension of τ:

$$\begin{array}{ccc} I & \xrightarrow{I\eta} & IX^@ \\ & \searrow^{\tau} & \downarrow r \\ & & Q \end{array} \qquad \begin{array}{c} (IX^@, I\mu_0) \\ \downarrow r \\ (Q,\delta) \end{array}$$

Again, by freeness, we define the *run map* $\delta^@ : QX^@ \longrightarrow Q$ as the unique dynamorphic extension of the identity map

$$\begin{array}{ccc} Q & \xrightarrow{Q\eta} & QX^@ \\ & \searrow^{id_Q} & \downarrow \delta^@ \\ & & Q \end{array} \qquad \begin{array}{c} (QX^@, Q\mu_0) \\ \downarrow \delta^@ \\ (Q,\delta) \end{array}$$

Finally, by the *response* (or *system behaviour*) of M, we mean

$$IX^@ \xrightarrow{r} Q \xrightarrow{\beta} Y$$

For fixed I, Y and X we call any $\mathcal{K}$-morphism

$$IX^{@} \xrightarrow{f} Y$$

a response morphism. We say M realizes (or is a realization of) f just in case f is the system behavior of M. Our main aim in this paper is to present very general conditions under which decomposable machines - which include both linear machines and group machines - exist which are minimal realizations of an appropriate class of response morphisms f.

2. THE IDENTITY PROCESS

Given a set Q, a q_0 in Q and a map $Q \xrightarrow{\delta} Q$, we can inductively define a sequence $\underline{N} \xrightarrow{q} Q$ as the orbit of f starting at q_0:

$$q_0 = q_0$$
$$q_{n+1} = (q_n)\delta = (q_0)\delta^{n+1}$$

We say $\underline{N} \xrightarrow{q} Q$ is defined by simple recursion on q_0 and f. Note that if $q: \underline{N} \longrightarrow Q$ has $q_1 = q_0$ and is defined by simple recursion, then $q_n = q_0$ for all $n \in \underline{N}$.

We may think of an orbit $\underline{N} \xrightarrow{q} Q$ as the state-trajectory of a one-input sequential machine:

Example 1: In $\mathcal{K}=\mathcal{S}$ we may identify the identity functor X with $- \times X_0$ where X_0 is a one-element set $\{x_0\}$. Then

$$IX^{@} = I \times X_0^{*} \cong I \times \underline{N} .$$

In particular, if I has one element, and $I \xrightarrow{\tau} Q$ yields initial state $q_0 = (1)\tau$, we have $r(1,n) = (q_0)\delta^n$ for a dynamics $Q \xrightarrow{\delta} Q$.

We now re-express the above notion arrow-theoretically prior to formulating the general categorical definition:

Consider $(\underline{\underline{N}}, I \xrightarrow{z} \underline{\underline{N}}, \underline{\underline{N}} \xrightarrow{s} \underline{\underline{N}})$ where I is a one-element set, $I \xrightarrow{z} \underline{\underline{N}}$ is the zero map with image 0, and $\underline{\underline{N}} \xrightarrow{s} \underline{\underline{N}}$ is the successor map $n \mapsto n + 1$. Then $I \xrightarrow{z} \underline{\underline{N}} \xrightarrow{s} \underline{\underline{N}}$ has the universal property that for all $I \xrightarrow{\tau} Q \xrightarrow{\delta} Q$ there exists exactly one $\underline{\underline{N}} \xrightarrow{q} Q$ such that

$$\begin{array}{ccccc} I & \xrightarrow{z} & \underline{\underline{N}} & \xrightarrow{s} & \underline{\underline{N}} \\ & \searrow^{\tau} & \downarrow q & & \downarrow q \\ & & Q & \xrightarrow[\delta]{} & Q \end{array}$$

commutes.

This clearly generalizes to the following:

DEFINITION: If I is an object in the category $\mathcal{K}$, a <u>simple recursive object with basis</u> I is an

$$(\hat{I}, I \xrightarrow{z} \hat{I}, \hat{I} \xrightarrow{s} \hat{I})$$

such that for all $(Q, I \xrightarrow{\tau} Q, Q \xrightarrow{\delta} Q)$ there exists exactly one $\hat{I} \xrightarrow{q} Q$ such that

$$\begin{array}{ccc} I & \xrightarrow{z} & \hat{I} \\ & \searrow^{q_0} & \downarrow q \\ & & Q \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \hat{I} & \xrightarrow{s} & \hat{I} \\ q \downarrow & & \downarrow q \\ Q & \xrightarrow{f} & Q \end{array}$$

commute. We say that q is defined by <u>simple recursion</u> (on q_0 and f).

This definition was first formulated by Lawvere [4, p. 1507, Axiom 3; 5, p. 292]. We shall see in Section 3 that $\hat{I}$ can be constructed in most categories, but now let us see the implications of its existence in any category $\mathcal{K}$:

Let, then, $X: \mathcal{K} \longrightarrow \mathcal{K}$ be the identity functor, i.e. $KX = K$ and $(K \xrightarrow{f} L)X = K \xrightarrow{f} L$. An X-dynamics is then just a map $Q \xrightarrow{\delta} Q$.

It is clear that $(\hat{I}, \hat{I} \xrightarrow{s} \hat{I}, I \xrightarrow{z} \hat{I})$ is a simple recursive object with basis I if and only if (via $\mu_0 = s$ and $\eta = z$) it is just the free dynamics of the identity functor on I - note that this ties in with $\mathcal{S}$:

Example 2: Returning to Example 1, we see that in the category $\mathcal{S}$,

$$\hat{I} = I \times \underline{\underline{N}} \quad \text{with}$$

$$I \xrightarrow{z} I \times \underline{\underline{N}}: i \mapsto (i,0); \quad \text{and}$$

$$I \times \underline{\underline{N}} \xrightarrow{s} I \times \underline{\underline{N}}: (i,n) \mapsto (i,n+1) .$$

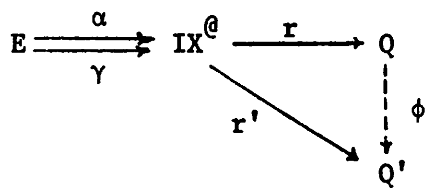
DEFINITION: We say that $\mathcal{K}$ is a <u>simple-recursive</u> category if $\hat{I}$ exists for all $I \in \mathcal{K}$.

DEFINITION: A <u>decomposable machine</u> $M = (Q,F,X_0,G,Y,H)$ in the simple recursive category $\mathcal{K}$ is a machine of the identity process: $Q \xrightarrow{F} Q$ is the dynamics, $X_0 \xrightarrow{G} Q$ is the initial state morphism, and $Q \xrightarrow{H} Y$ is the output morphism. The reachability map $\hat{X}_0 \xrightarrow{r} Q$ is then the morphic extension of G, while $rH: \hat{X}_0 \longrightarrow Y$ is the system behavior.

We shall see in Section 3 that this does indeed yield linear machines if $\mathcal{K}$ is the category of R-modules, and yields group machines [1] if $\mathcal{K}$ is the category of groups. But here let us develop the realization theory for morphisms $\hat{X}_0 \longrightarrow Y$ in a simple recursive category $\mathcal{K}$ by showing that they have a minimal decomposable machine realization.

We start by recalling that in [2a] we called an arbitrary machine $M = (X,Q,\delta,I,z,Y,\beta)$ in a category $\mathcal{K}$ <u>coequalizer-reachable</u> just in case the reachability map $IX^@ \xrightarrow{r} Q$ is a coequalizer, i.e. just in case there exists a pair of maps $E \overset{\alpha}{\underset{\gamma}{\rightrightarrows}} IX^@$ such that $\alpha r = \gamma r$, and such that for every map r' for which $\alpha r' = \gamma r'$, there is a unique map ϕ

such that the following diagram commutes:



We may thus state the following problem (which we shall see in Section 3 does indeed reduce to the usual problem for linear machines):

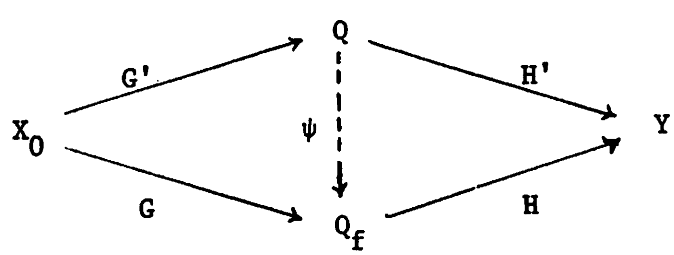
REALIZATION PROBLEM FOR DECOMPOSABLE MACHINES:

Given a response morphism $f: \hat{X}_0 \longrightarrow Y$ in the simple recursive category $\mathcal{K}$, find a decomposable machine $M_f = (Q_f, F, X_0, G, Y, H)$ which is the minimal coequalizer-reachable realization of f. In other words:

(i) M_f realizes f, i.e. $f = r_f \cdot H$

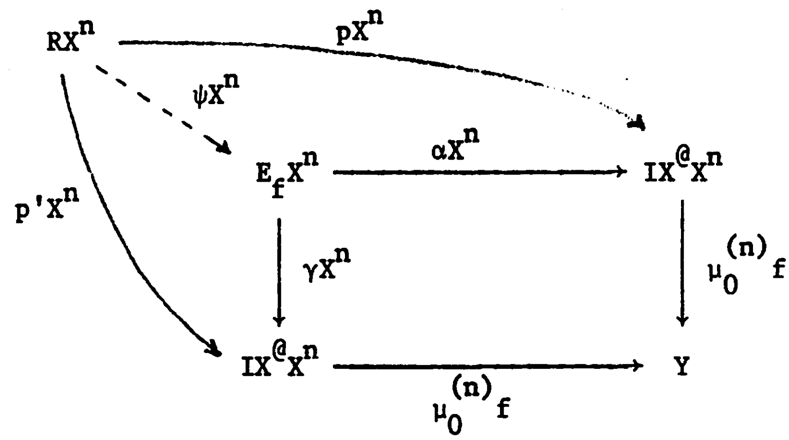
(ii) The reachability map $\hat{X}_0 \xrightarrow{r_f} Q$ of M_f is a coequalizer

(iii) M_f is minimal in that for all $M = (Q, F', X_0, G', Y, H')$ satisfying (i) and (ii) there exists a unique dynamorphism ψ for which we have the commutativity of



We now approach our main theorem which not only includes the realization theory for linear machines [3, Chapter 10; 7, Chapter 8] but also the more recent realization theory for group machines [1].

Recall from [2a] the general defining diagram for the Nerode equivalence $E_f \overset{\alpha}{\underset{\gamma}{\rightrightarrows}} IX^@$ (when it exists) of an arbitrary response morphism $f\colon\ IX^@ \longrightarrow Y$:

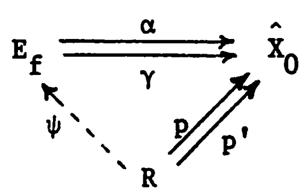


In the case of present interest $I = X_0$, $X = id_{\mathcal{K}}$ and $\mu_0^{(n)} = s^n\colon \hat{X}_0 \longrightarrow \hat{X}_0$. Thus our general definition reduces to the following for $X = id_{\mathcal{K}}$:

DEFINITION: Let $f\colon \hat{X}_0 \longrightarrow Y$ be a response morphism for the identity process of a simple recursive category $\mathcal{K}$. The <u>Nerode equivalence</u> of f is defined to be a pair of morphisms of the form

$$E_f \overset{\alpha}{\underset{\gamma}{\rightrightarrows}} \hat{X}_0$$

such that $\alpha s^n f = \gamma s^n f$ for all $n = 0, 1, \ldots$ and universal with that property, that is whenever p, p' is another



pair of maps with $ps^n f = p's^n f$, then there exists unique ψ with $\psi\alpha = p$ and $\psi\gamma = p'$.

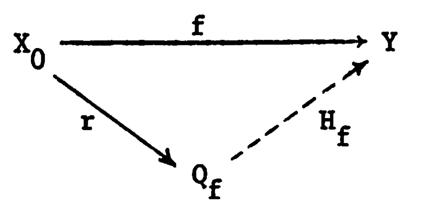
This is an example of what categorists call a <u>limit construction</u> [6] and as such E_f exists in most categories. For example, in <u>R-Mod</u>, groups or topological spaces, E_f is the set of all (x,y) in $\hat{X}_0 \times \hat{X}_0$ such that $xs^n f = ys^n f$ for all n, and α, γ are the restrictions of the coordinate projections.

NERODE REALIZATION THEOREM FOR DECOMPOSABLE MACHINES

Let $f: \hat{X}_0 \longrightarrow Y$ be a response morphism for the identity process in the simple-recursive category $\mathcal{K}$. Assume that the Nerode equivalence $E_f \underset{\gamma}{\overset{\alpha}{\rightrightarrows}} \hat{X}_0$ of f exists and that there exists a coequalizer $\hat{X}_0 \xrightarrow{r} Q_f$ of α and γ. Then

(i) There exists a unique dynamics $Q_f \xrightarrow{F_f} Q_f$ and initial state morphism $X_0 \xrightarrow{G_f} Q_f$ whose reachability morphism is r.

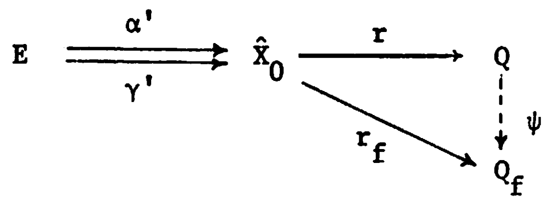
(ii) There exists a unique $\mathcal{K}$-morphism H_f:



Moreover, $M_f = (Q_f, F_f, X_0, G_f, Y, H_f)$ is the minimal coequalizer-reachable realization of f.

Proof: This follows from the Nerode Realization theorem of [2a], since Postulates 1 and 2 of [2a] were assumed, while Postulates 3 and 4 are guaranteed by the Lemma of [2a] following that theorem since it is trivial that the identity process preserves coequalizer diagrams. □

Of course, in the present case, a direct verification is also straightforward: We can find the desired ψ from

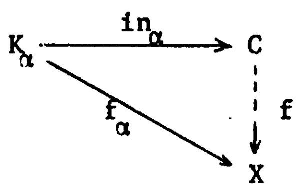


once we verify that $\alpha' r_f = \gamma' r_f$. Now $\alpha' s^n f = \alpha' r F^n H = \gamma' r F^n H = \gamma' s^n f$, and so by the definition of Nerode equivalence, there exists a unique ψ with $\psi\alpha = \alpha'$ and $\psi\gamma = \gamma'$. But this yields $\alpha' r_f = \psi\alpha r_f = \psi\gamma r_f = \gamma' r_f$. □

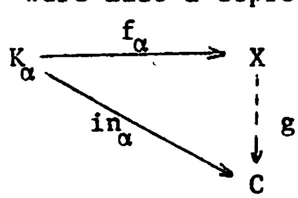
3. COPRODUCTS

We now show that any category $\mathcal{K}$ with suitable coproducts is simple recursive:

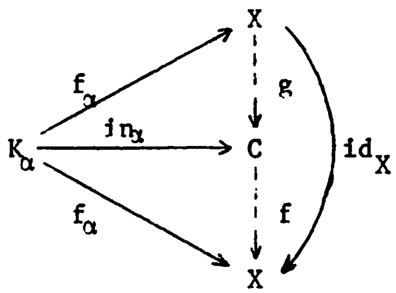
DEFINITION: Let $\{K_\alpha \mid \alpha \in I\}$ be a family of objects in $\mathcal{K}$. A <u>coproduct of</u> $\{K_\alpha\}$ is an object C together with a family of $\mathcal{K}$-morphisms $\{K_\alpha \xrightarrow{in_\alpha} C\}$ with the universal property that for any family $\{K_\alpha \xrightarrow{f_\alpha} X\}$ there exists exactly one f for which the following diagram commutes:

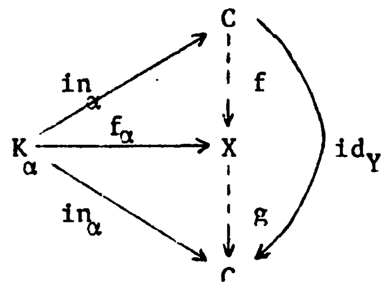


Note that the above definition implies that C is unique up to isomorphism, for if $K_\alpha \xrightarrow{f_\alpha} X$ were also a coproduct with



we infer from the diagrams





and the uniqueness condition in the definition of coproduct that

$$gf = id_X \quad \text{and} \quad fg = id_Y$$

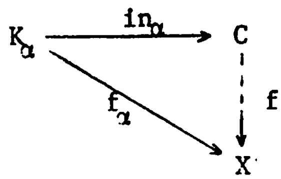
so that f and g are isomorphisms, and $f_\alpha = in_\alpha \cdot f$ while $in_\alpha = f_\alpha \cdot g$.

Most categories we care about have coproducts:

Example: In the category $\mathcal{S}$ of sets, the coproduct of a family $\{K_\alpha | \alpha \in I\}$ of sets is the disjoint union

$$C = \coprod_{\alpha \in I} K_\alpha = \{(x,\alpha) \mid x \in K_\alpha, \alpha \in I\}$$

with $K_\alpha \xrightarrow{in_\alpha} C : x \mapsto (\alpha, x)$. For, given any family of maps $K_\alpha \xrightarrow{f_\alpha} X$ we may define $C \xrightarrow{f} X$ uniquely by $(\alpha, x) \mapsto f_\alpha(x)$ to ensure that

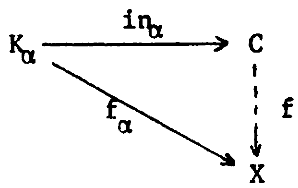


commutes.

Example: In the category R-Mod of modules over the ring R, the coproduct of a family $\{K_\alpha \mid \alpha \in I\}$ of R-modules is the direct sum

$$C = \coprod_{\alpha \in I} K_\alpha = \bigoplus_{\alpha \in I} K_\alpha = \{(x_\alpha) \mid x_\alpha \neq 0 \text{ for only finitely many } \alpha\}$$

with $K_\beta \xrightarrow{in_\beta} C: \; x_\beta \mapsto (\delta_{\alpha\beta} x_\beta)$. For, given any family of R-linear maps $K_\alpha \xrightarrow{f_\alpha} X$ we may define $C \xrightarrow{f} X$ uniquely by $(x_\alpha) \mapsto \sum_{\substack{\alpha \in I \\ x_\alpha \neq 0}} f_\alpha(x_\alpha)$ to ensure that



commutes.

Example: In the category $\underline{\underline{Gp}}$ of groups, the coproduct of a family $\{K_\alpha \mid \alpha \in I\}$ of groups is the "free product"

$$C = \coprod K_\alpha$$

defined by the following rather elaborate construction:

1. The elements of C consist of all finite sequences (including the empty one, Λ) of elements of the form

   $$(k,\alpha) \quad \text{for which} \quad k \in K_\alpha \quad \text{and} \quad \alpha \in I$$

   subject to the conditions:

   (i) No (k,α) in the string has k = identity e_α of K_α

   (ii) No string $(k_1,\alpha_1)(k_2,\alpha_2)\ldots(k_n,\alpha_n)$ has $\alpha_j = \alpha_{j+1}$ for any j, $1 \leq j < n$.

2. Multiplication in C is simply concatenation of sequences, save that we must apply the operations

   (iii) replace consecutive elements of the form $(k,\alpha)(k',\alpha)$ – α – by the single element (kk',α) – using multiplication in K_α.

   (iv) Delete elements of the form (e_α,α) until obtaining a sequence (probably empty) which satisfies (i) and (ii).

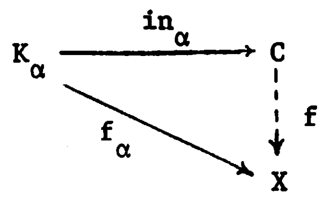
It is clear that Λ is the identity, and that $(k_1,\alpha_1)\ldots(k_n,\alpha_n)$ has inverse $(k_n^{-1},\alpha_n)\ldots(k_1^{-1},\alpha_1)$. One simply checks associativity to confirm that $\coprod K_\alpha$ is indeed a group.

Then, given any family of homomorphisms $K_\alpha \xrightarrow{f_\alpha} X$ we may define

$C \xrightarrow{f} X$ uniquely by

$$[k_1,\alpha_1)\dots(k_n,\alpha_n)]f = (k_1)f_{\alpha_1}\dots(k_n)f_{\alpha_n}$$

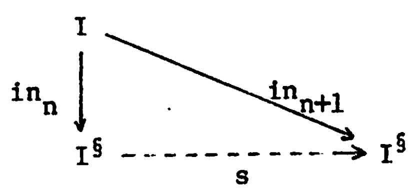
to ensure that



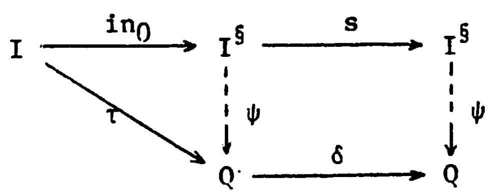
commutes.

We may now appreciate that it is easy to find examples of simple-recursive categories because any category which has countable copowers is simple-recursive. More precisely, let us write $(I \xrightarrow{in_n} I^{\S}: n=0,1,\dots)$ for the countably infinite copower of copies of I. The following theorem says that "mathematical induction is stronger than simple recursion".

THEOREM: $(I^{\S}, I \xrightarrow{in_0} I^{\S}, I^{\S} \xrightarrow{s} I^{\S})$ is a simple-recursive object with basis I, where s is defined by the "induction hypothesis"



Proof: We have to check that given $I \xrightarrow{\tau} Q \xrightarrow{\delta} Q$ there exists exactly one ψ such that



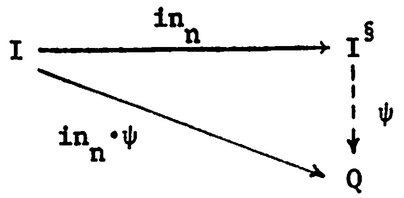
But from the left-hand triangle we read

$$in_0 \cdot \psi = \tau$$

while from the right-hand square, and the fact that $in_n \cdot s = in_{n+1}$, we read

$$in_{n+1} \cdot \psi = (in_n \cdot \psi) \cdot \delta \qquad \text{for each } n \cdot \underline{\underline{N}}$$

This defines $in_n \cdot \psi$ for each $n \in \underline{\underline{N}}$, and so defines ψ uniquely by applying the coproduct property to



□

With this background we may see how neatly the general theory of Section 2 embraces the theory of linear machines. (The reader may supply a corresponding link to the group machines of [2a]):

Lemma: A <u>linear</u> machine (X_0, Q, F, G, Y, H) is coequalizer-reachable just in case every state is of the form $\sum_{j=0}^{n} x_j GF^j$, i.e. just in case (F,G) is <u>reachable</u> in the conventional system-theoretic sense of the term:

Proof: We note that in R-Mod $X_0^{\S} = \coprod_{\alpha \in \underline{\underline{N}}} X_0 = \{(x_n, \ldots, x_1, x_0) \quad n \in \underline{\underline{N}}$ with

$$\text{each } x_j \in X_0, \quad n \geqslant j \geqslant 0$$

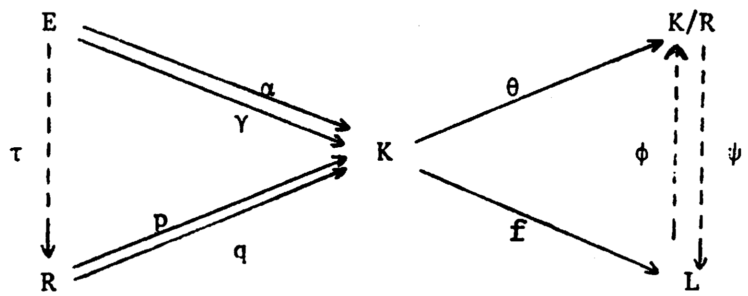
so that $r\colon X_0^{\S} \longrightarrow Q$ is precisely the map

$$(x_n, \ldots, x_1, x_0) \mapsto \sum_{j=0}^{n} x_j GF^j .$$

Thus, our assertion follows immediately from the following lemma which asserts in particular that every coequalizer in <u>R-Mod</u> is onto. □

Lemma: Let $\mathcal{K}$ be a category of universal algebras and let $K \xrightarrow{f} L$ be a coequalizer. Then f is onto.

Proof: Consider the diagram



where f is the coequalizer of α and γ, R is the congruence $\{(x,y)\colon\ xf = yf\}$ of f (p, q being the restrictions of the coordinate projections) and θ is the canonical projection to the quotient algebra. Because $\alpha f = \gamma f$ there exists τ with $\tau p = \alpha$, $\tau q = \gamma$. Thus, $\alpha\theta = \gamma\theta$, inducing unique ϕ with $f\phi = \theta$. As $pf = qf$ there exists unique ψ with $\theta\psi = f$. As $f\phi\psi = f$ and f is a coequalizer, $\phi\psi = id$. Thus ψ is onto and $f = \theta\psi$ is onto. $\square$

It is now apparent that our general Nerode realization theorem of Section 2 yields the Kalman realization theory of [3, Chapter 10] as a special case: If $X_0 = K^m$ for some field K, then $X_0^{\S} \cong K[s]^m$, the ring of m-tuples of polynomials in the indeterminate s (we eschew Kalman's use of z here for obvious reasons) with coefficients in K. Then the Nerode equivalence

$$\{(x,y) \mid (x,y) \in X_0^{\S} \times X_0^{\S} \quad \text{and} \quad xs^n f = ys^n f\}$$

is the equivalence of the $K[s]$-linear map

$$\tilde{f} : K[s]^m \longrightarrow Y^{\underline{N}} : x \mapsto (xf, xsf, xs^2f, \ldots, xs^nf, \ldots)$$

Just as a $K[s]$-module is any K-module equipped with a linear map, and Kalman shows us how to go from $\tilde{f}$ (or the corresponding $f : K[s]^m \longrightarrow Y$) to the minimal $K[s]$-module $Q_f \xrightarrow{F} Q_f$ for which there exist G and H such that f is the behavior of (K^m, G, Q_f, F, Y, H) so do we show how, in <u>any</u>

simple recursive category $\mathcal{K}$ to go from a $\mathcal{K}$-morphism $\hat{X}_0 \xrightarrow{f} Y$ to the minimal (G,Q_f,X_0,F,Y,H) which realizes it. The crucial insight of our approach is that linearity plays no role.

We state here that if both K and its dual K^{op} are simple-recursive then the identity process is state-behavior in the sense of [2b] and the usual observability and duality theory of linear systems can be formulated. See [2b] for details.

4. AN ALTERNATIVE FORMULATION

To conclude, it will prove insightful to give a uniform format for linear and group machines which yields a second general notion of decomposable machine within our framework of machines in a category. We first note some further properties of the coproduct:

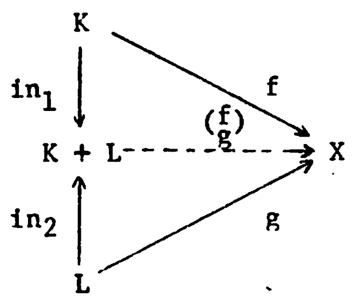
If we consider $\{K_\alpha \mid \alpha \in I\}$ for I the empty set $\emptyset$, the condition for C to be their coproduct is simply that for any X in $\mathcal{K}$, there is a unique map $C \xrightarrow{f} X$. Such a C is called initial. For $\mathcal{K} = \mathcal{S}$, we have $C = \emptyset$; for $\mathcal{K}$ = R-Mod, it is the 1-element module $\{0\}$; and for $\mathcal{K}$ = Gp, it is the 1-element group $\{e\}$.

Where no ambiguity can result, we may write $\emptyset$ for the initial object in any category ($\emptyset$ is thus unique up to isomorphism if it exists). We shall often write K + L for the coproduct of two objects K and L, It can easily be checked that

1. $K + \emptyset \cong K$
2. $K + L \cong L + K$
3. $(K + L) + M \cong K + (L + M) \cong K + L + M$

We now show how to combine maps using the coproduct construction:

DEFINITION: (i) Given $K \xrightarrow{f} X$ and $L \xrightarrow{g} X$, and some specific choice of $K + L$, we define the map $\binom{f}{g}: K + L \longrightarrow X$ by the diagram

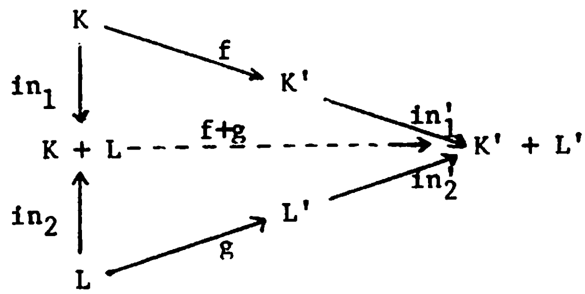


(ii) Given $K \xrightarrow{f} K'$ and $L \xrightarrow{g} L'$ we define

$$K + L \xrightarrow{f+g} K' + L'$$

by $f + g = \begin{pmatrix} f\ in_1' \\ g\ in_2' \end{pmatrix}$

i.e. we have



Example: For R-modules,

$$\binom{f}{g}(k,\ell) = f(k) + g(\ell) \qquad \text{in } X$$

$$(f + g)(k,\ell) = (f(k),g(\ell)) \quad \text{in } K \oplus L$$

For groups

$$\binom{f}{g}(k_1\ell_1k_2\ell_2\ldots) = f(k_1)g(\ell_1)f(k_2)g(\ell_2)\ldots \qquad \text{in } X$$

$$(f+g)(k_1\ell_1k_2\ell_2\ldots) = f(k_1)g(\ell_1)f(k_2)g(\ell_2)\ldots \quad \text{in } K + L.$$

As an interesting aside, we note that if $\hat{I}$ and $I + \hat{I}$ exist, then $\binom{z}{s} : I + \hat{I} \longrightarrow \hat{I}$ is an isomorphism, which may be interpreted by saying that $\hat{I}$ is an "infinite" object (think of $\mathcal{S}$, and I a non-empty set).

With these definitions we may succinctly state our alternative general format:

A <u>linear machine</u> is an R-linear map $\binom{F}{G} : Q + X_0 \longrightarrow Q$ together with an R-linear map $H: Q \longrightarrow Y$.

A <u>group machine</u> is a homomorphism $\binom{F}{G} : Q + X_0 \longrightarrow Q$ together with an R-linear map $H: Q \longrightarrow Y$.

We suddenly understand a source of great confusion in general system theory. For R-modules, the product $Q \times X_0$ and the coproduct $Q + X_0$ are isomorphic, and thus we could always treat linear systems as if the next-state map were from $Q \times X_0$ to Q. However, it is clear that $Q \times X_0$ and $Q + X_0$ are <u>not</u> isomorphic for groups (<u>unless they are abelian</u> - can you see why?), and it was the attempt to provide a realization theory for group machines that forced us [1] to see that coproducts provided the appropriate algebraic setting.

With this motivation, we are led to consider the following process in <u>any</u> category $\mathcal{K}$: Pick an object $X_0 \in \mathcal{K}$, and let X be the process $X = - + X_0 : \mathcal{K} \longrightarrow \mathcal{K} : Q \mapsto Q + X_0$. We assume each $Q + X_0$ exists; but then, the action on morphisms is given by

$$\begin{array}{ccc} Q & \xrightarrow{\quad in_1 \quad} & Q + X_0 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f+id_{X_0}} \\ Q' & \xrightarrow[\quad in'_1 \quad]{} & Q' + X_0 \end{array}$$

It is clear that X is indeed a functor. It is in fact an input process (i.e. admits free machines):

THEOREM: Let X_0 be any object of a simple recursive category $\mathcal{K}$ with finite products. Then the functor $X = - + X_0$ is an input process on $\mathcal{K}$. Moreover, the free dynamics on I is determined as follows:

$$IX^@ = \hat{I} + \hat{X}_0$$

Denoting the zero and successor maps, respectively, by

$$I \xrightarrow{\bar{z}} \hat{I} \quad \text{and} \quad \hat{I} \xrightarrow{\bar{s}} \hat{I} \quad \text{for } I\text{; and}$$

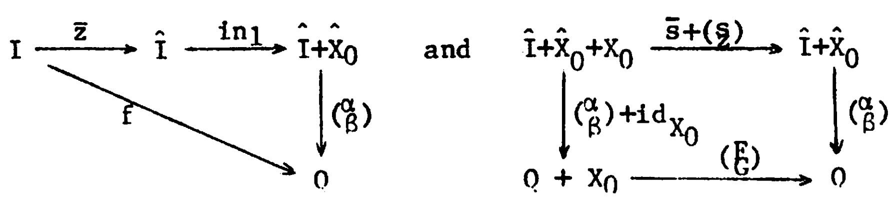
$$X_0 \xrightarrow{z} \hat{X}_0 \quad \text{and} \quad \hat{X}_0 \xrightarrow{s} \hat{X}_0 \quad \text{for } X_0\text{;}$$

we have

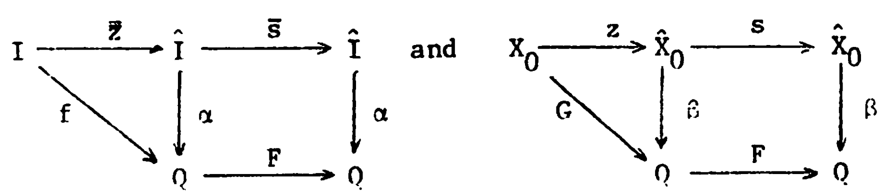
$$(IX^@)X \xrightarrow{I\mu_0} IX^@ \quad \text{is} \quad \hat{I} + \hat{X}_0 + X_0 \xrightarrow{\bar{s}+\binom{s}{z}} \hat{I} + \hat{X}_0$$

$$I \xrightarrow{I\eta} IX^@ \quad \text{is} \quad I \xrightarrow{\bar{z}} \hat{I} \xrightarrow{in_1} \hat{I} + \hat{X}_0$$

Proof: Given any $Q + X_0 \xrightarrow{\binom{F}{G}} Q$ and $I \xrightarrow{f} Q$ we must check that there exists a unique $\hat{I} + \hat{X}_0 \xrightarrow{\binom{\alpha}{\beta}} Q$ such that the following diagrams commute:



Reading off the 'top' of the two diagrams spliced together, and off the 'middle' and 'bottom' of the right-hand diagram we obtain



and these are clearly simple recursions, and so define α and β uniquely. □

Having assured ourselves that $- + X_0$ is an input process, we can then make our alternative definition:

DEFINITION: A decomposable machine (Mark II) in the simple recursive category with finite coproducts is a 6-tuple

$$(X_0,Q,F,G,Y,H)$$

where $Q + X_0 \xrightarrow{\binom{F}{G}} Q$ is an $X = - + X_0$ dynamics

$Q \xrightarrow{H} Y$ is the output map

and it is understood that we take the initial state to be the unique map $\emptyset \xrightarrow{\tau} Q$. We denote the machine by (F,G,H) for short.

This indeed fits in with our convention of taking 0 as the initial state of a linear machine and e as the initial state of a group machine.

Now it is easy to check that $\hat{\emptyset} = \emptyset$ so that

$$\emptyset X^{@} = \hat{\emptyset} + \hat{X}_0 \cong \emptyset + \hat{X}_0 = \hat{X}_0.$$

Thus the reachability map of (F,G,H) is simply the map

$$r: \hat{X}_0 \longrightarrow Q$$

which is the "β" for $f = \tau$ in the last proof. If $\hat{I} = I^{\S}gr$ satisfies

$$in_n r = GF^n \qquad \text{for all } n \in \underline{\underline{N}}.$$

The system behavior f equals rH, and is the unique $f : X_0^{\S} \longrightarrow Y$ satisfying

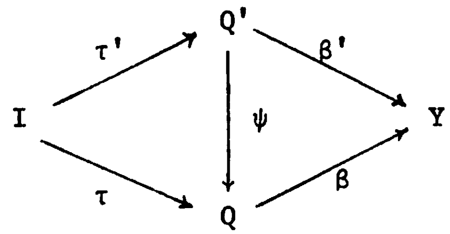
$$in_n\, f = GF^nH \qquad \text{for all } n \in \underline{\underline{N}},$$

a formula very familiar from linear system theory (which is a special case).

To conclude, then, let us check that our two notions of decomposable machine mesh. Given a fixed input process X, initial state object I and output object Y, we may define a category $\mathcal{M}_{X,I,Y}$ whose objects are the machines

$$(X,Q,\delta,I,\tau,Y,\beta)$$

for arbitrary Q, δ, τ and β, and whose morphisms $M' \xrightarrow{\psi} M$ are the X-dynamorphisms $\psi: Q' \longrightarrow Q$ which satisfy



Noting that the identity process in a category $\mathcal{K}$ with finite products equals $- + \emptyset$, we have our equivalence result in the following form:

THEOREM: Let $\mathcal{K}$ be a simple recursive category with finite coproducts, and let X_0 and Y be fixed objects of $\mathcal{K}$.

Set $\mathcal{M}_1 = \mathcal{M}_{-+\emptyset,X_0,Y}$ and $\mathcal{M}_2 = \mathcal{M}_{-+X_0,\emptyset,Y}$

Then the maps

$$(- + \emptyset,Q,F,X_0,G,Y,H) \mapsto (- + X_0,Q,\binom{F}{G},\emptyset,\tau,Y,H)$$

$$\psi: Q' \longrightarrow Q \mapsto \psi: Q' \longrightarrow Q$$

define a behavior and reachability preserving isomorphism Φ from $\mathcal{M}_1$ to $\mathcal{M}_2$.

Proof: The crucial point is that

$$X_0(- + \emptyset)^@ = \hat{X}_0 + \hat{\emptyset} \cong \hat{X}_0 \cong \hat{\emptyset} + \hat{X}_0 = \emptyset(- + X_0)^@$$

The rest follows by routine calculation. □

In particular, it follows that a machine M is a minimal coequalizer-reachable realization of $f\colon X_0^{\S} \longrightarrow Y$ in $\mathcal{M}_1$ iff $\Phi(M)$ is a minimal coequalizer-reachable realization of f in $\mathcal{M}_2$.

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