NATURAL STATE TRANSFORMATIONS[1]

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ABSTRACT

The concept of generalized[3] sequential machines in arbitrary categories is developed in the paper. The change in viewpoint from the previous studies comes from the appropriate choice of a monoidal category. Thus a monad, rather than a monoid in the category of sets, becomes the crucial notion of this development. By re-expressing the old notion of a generalized sequential machine, a concise framework is developed that easily yields results on, for example, bottom-up and top-down tree transformations. Transformations, i.e., maps that change the underlying structure, rather than sequential machines, are emphasized and natural state transformations are defined as certain morphisms of monads. On this basis, a duality theory for direct and inverse state transformations is developed, which lays bare the relationship between the two models of finite state (tree) transformations mentioned above.

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Introduction

The objective of the research reported in this paper was to generalize the study of finite state mappings on trees to categories other than the category of sets, to provide the appropriate notion of a monoid that will play the role of an ordinary monoid in the category of sets and to investigate in this general framework the nature of the duality for which a good example is top-down and bottom-up finite state transformations on trees.

The general state mappings introduced here are natural transformations called state transformations. Engelfriet's deterministic bottom-up tree transformations and Thatcher's generalized[2] sequential machine maps become realizations of our two general dual concepts, direct and inverse state transformations. The nature of this duality is explicated and proved through the categorical notion of adjoints. The material presented grew to a great extent from the Arbib-Manes categorical automata theory, but the interest here is shifted to morphisms which need not preserve the underlying structure (e.g., in mapping strings, they need not be length-preserving--cf. generalized sequential machines) and of which sequential machines are a particular case.

We propose the notion of a monad, i.e., a monoid in the endofunctor category $\mathcal{K}^{\mathcal{K}}$, as a natural structure for investigating the properties of these models. The use of categorical language in the paper led to quite different proof techniques in which certain universal properties are used rather than a classical scheme of mathematical induction. We believe that this is of interest to automata theorists in its own right and that it tells us a lot about the nature of induction in a conventional theory.

The paper is organized in three sections. In the first, expository, one an attempt has been made to provide a choice of relevant material from category theory and the Arbib-Manes automata theory; and a number of examples important to the theory of machines and transformations. The second and the third sections deal with direct, resp. inverse, state transformations, and contain a number of (sometimes surprising) theorems and examples about the two models and the relationship between them.

1. Machines, Monads and Algebras

Let $\mathcal{K}$ be a category† and consider a category $\mathcal{K}^{\mathcal{K}}$ with objects functors, $R,S,T,\ldots : \mathcal{K}\to\mathcal{K}$ and morphisms natural transformations $\tau : R \dot{\to} S,\ldots$ with the operation of vertical (i.e., ordinary) composition. $\mathcal{K}^{\mathcal{K}}$ is equipped with a bifunctor

$$\mathcal{K}^{\mathcal{K}} \times \mathcal{K}^{\mathcal{K}} \to \mathcal{K}^{\mathcal{K}}$$

defined by

$$\begin{array}{c} T \\ \alpha\big\downarrow \\ \bar{T} \end{array} \quad \begin{array}{c} K \\ \big\downarrow\beta \\ \bar{K} \end{array} \;\mapsto\; \begin{array}{ccccc} & & TK & & \\ & \alpha K \swarrow & \big\downarrow \alpha\beta & \searrow T\beta & \\ \bar{T}K & & & & T\bar{K} \\ & \bar{T}\beta \searrow & & \swarrow \alpha\bar{K} & \\ & & \bar{T}\bar{K} & & \end{array} \qquad (1.1)$$

where T,K are objects of $\mathcal{K}^{\mathcal{K}}$ and the diagram on the right (which commutes by α a natural transformation) defines a horizontal composition $\alpha\beta$ of natural transformations α and β.

A monoid in this (strictly) monoidal category (Mac Lane [8] p. 157) is called a monad (where we replace a map $\mu : T\times T\to T$ by a natural transformation $\mu : TT \dot{\to} T$ but still require associativity, etc.):

(1.2) Definition

(i) A monad $\langle T,\eta,\mu\rangle$ in a category $\mathcal{K}$ consists of a functor $T : \mathcal{K}\to\mathcal{K}$ and two natural transformations

$$\eta : I \dot{\to} T, \quad \mu : TT \dot{\to} T$$

(where I is the identity functor) which make the following diagrams commute

$$(1.3)\quad \begin{array}{ccc} TTT & \xrightarrow{\mu T} & TT \\ T\mu\big\downarrow & & \big\downarrow\mu \\ TT & \xrightarrow[\mu]{} & T \end{array} \qquad\qquad \begin{array}{ccccc} IT & \xrightarrow{\eta T} & TT & \xleftarrow{T\eta} & TI \\ & 1 \searrow & \big\downarrow\mu & \swarrow 1 & \\ & & T & & \end{array} \quad (1.4)$$

†We assume the reader familiar with the category theory presented in Arbib-Manes [1]; or with the first 4 chapters of Mac Lane [8].

(1.3) is called the associativity axiom and (1.4) the unitary axiom.

(ii) Let $\langle T,\eta,\mu\rangle$ and $\langle \bar{T},\bar{\eta},\bar{\mu}\rangle$ be monads in the category $\mathcal{K}$. By a <u>morphism</u> $\langle T,\eta,\mu\rangle \to \langle \bar{T},\bar{\eta},\bar{\mu}\rangle$ <u>of monads</u> we mean a natural transformation $\bar{\tau} : T \dot{\to} \bar{T}$ such that

$$\begin{array}{c} T \\ \downarrow \bar{\tau} \\ \bar{T} \end{array} \qquad \begin{array}{ccc} TT & \xrightarrow{\mu} & T \\ \downarrow \bar{\tau}\bar{\tau} & & \downarrow \bar{\tau} \\ \bar{T}\bar{T} & \xrightarrow{\bar{\mu}} & \bar{T} \end{array} \qquad \begin{array}{ccc} T & \xrightarrow{\eta} & I \\ \downarrow \bar{\tau} & & \downarrow 1 \\ \bar{T} & \xleftarrow{\bar{\eta}} & I \end{array} \qquad (1.5)$$

There is a monad in the category <u>Set</u> of sets associated with any monoid. In particular, if X_0 is a finite alphabet, the usual monad associated with $X_0{*}$ is defined by

$$T = -\mathrm{x}X_0{*}$$

$$(1.6) \qquad \mu : -\mathrm{x}X_0{*}\mathrm{x}X_0{*} \dot{\to} -\mathrm{x}X_0{*} \qquad \eta : I \dot{\to} -\mathrm{x}X_0{*}$$

$$\bullet,w,w' \mapsto \bullet,ww' \qquad \bullet \to \bullet,\Lambda$$

Clearly, the above definitions satisfy the axioms (1.3) and (1.4).

Let Σ be a ranked alphabet, i.e., a set Σ together with a set-valued function $r : \Sigma \to \underline{\underline{N}}^{\wp}$ where $\underline{\underline{N}}$ is the set of nonnegative integers and $\underline{\underline{N}}^{\wp}$ is, for the time being, the set of finite subsets of $\underline{\underline{N}}$. Denote $\Sigma_n = \{\sigma | \langle\sigma,n\rangle r\}$ and let Z be a countable set of variables. The set $T_{\Sigma,Z}$ of finite Σ-trees on Z generators is defined inductively as follows:

$$Z \cup \Sigma_0 \subset T_{\Sigma,Z}$$

$$(1.7) \qquad \sigma \in \Sigma_n \text{ and } t_1,\ldots,t_n \in T_{\Sigma,Z} \Rightarrow$$

$$t_1 \ldots t_n \sigma \in T_{\Sigma,Z}$$

Define a functor T : <u>Set</u> $\to$ <u>Set</u> to have the object function $ZT = T_{\Sigma,Z}$ (i.e., T sends a set Z to the set of finite Σ-trees on Z generators) and the mapping function

$$\begin{array}{ccc} Z & & ZT = T_{\Sigma,Z} \\ \Big\downarrow f & \mapsto & \Big\downarrow fT \\ U & & UT = T_{\Sigma,U} \end{array}$$

where fT is defined inductively by:

$$\langle z\rangle fT = \langle zf\rangle$$

(1.8)

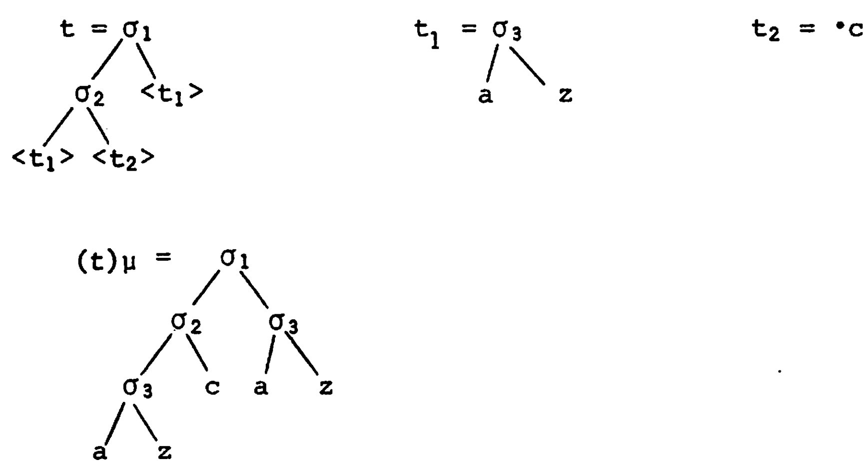
$$(t_1\ldots t_n\sigma)fT = (t_1)fT\ldots(t_n)fT\sigma$$

i.e., fT simply relabels each initial node labelled $z\varepsilon Z$ with $f(z)\varepsilon U$. Here we sometimes write $\langle z\rangle$ to distinguish the one node tree from the variable z.

The tree monad over Σ consists of the above functor T and the natural transformations η and μ where η is the inclusion of generators

$$\eta : z \mapsto \langle z\rangle$$

and μ applied to a tree on trees as generators gives a tree on variables (removes the brackets) as in the example:



(1.9) <u>Definition</u>

(i) If $T = \langle T,\eta,\mu\rangle$ is a monad in $\mathcal{K}$, a <u>T-algebra</u> $\langle A,h\rangle$ is a pair consisting of an object $A\varepsilon\mathcal{K}$ (the underlying object of the algebra) and a morphism

$h : AT \to A$ of $\mathcal{K}$ (the structure map of the algebra) such that

$$\begin{array}{ccc} ATT & \xrightarrow{hT} & AT \\ \downarrow{\scriptstyle A\mu} & & \downarrow{\scriptstyle h} \\ AT & \xrightarrow{h} & A \end{array} \qquad\qquad \begin{array}{ccc} A & \xrightarrow{A\eta} & AT \\ & {\scriptstyle 1}\searrow & \downarrow{\scriptstyle h} \\ & & A \end{array} \tag{1.10}$$

The first diagram is called the associativity axiom and the second the unitary axiom.

(ii) A <u>morphism</u> $f : \langle A,h\rangle \to \langle A',h'\rangle$ of T-algebras is a morphism $f : A \to A'$ of $\mathcal{K}$ which renders commutative the diagram

$$\begin{array}{ccc} AT & \xrightarrow{h} & A \\ \downarrow{\scriptstyle fT} & & \downarrow{\scriptstyle f} \\ A'T & \xrightarrow{h'} & A' \end{array} \tag{1.11}$$

For the monad associated with a monoid, T-algebras are precisely monoid actions. In particular, for $T = -xX_0{}^*$ we get the following: Let $\langle X_0,Q,\delta,Y_0,\lambda\rangle$ be a sequential machine, where $Y_0 \xleftarrow{\lambda} QxX_0 \xrightarrow{\delta} Q$. δ can be extended to $\delta^* : QxX_0{}^* \to Q$ inductively by $\langle q,\Lambda\rangle\delta^* = \Lambda$ and $\langle q,wx\rangle\delta^* = \langle\langle q,w\rangle\delta^*,x\rangle\delta$. Then a T-algebra is simply $\langle Q,\delta^*\rangle$ since

$$\begin{array}{ccc} Q & \xrightarrow{\eta} & QxX_0{}^* \\ & {\scriptstyle 1}\searrow & \downarrow{\scriptstyle \delta^*} \\ & & Q \end{array} \qquad\qquad \begin{array}{ccc} QxX_0{}^*xX_0{}^* & \xrightarrow{\delta^*x1} & QxX_0{}^* \\ \downarrow{\scriptstyle 1x\mu} & & \downarrow{\scriptstyle \delta^*} \\ QxX_0{}^* & \xrightarrow{\delta^*} & Q \end{array} \tag{1.12}$$

$$\langle q,\Lambda\rangle\delta^* = q \qquad\qquad \langle q,ww'\rangle\delta^* = \langle\langle q,w\rangle\delta^*,w'\rangle\delta^*$$

A Σ-algebra is a pair $\langle A,\delta\rangle$ where A is a set called the carrier of the algebra and δ assigns to each $\sigma \in \Sigma$, a function $\delta_\sigma : A_\sigma \to A$ where $A_\sigma = A^n \coprod_{\langle\sigma,n\rangle r}$. For $\sigma \in \Sigma_0$ we write δ_σ for $(\lambda)\delta_\sigma$. Then a T-algebra with

T the tree functor is a pair $<A,h>$ where $h : T_{\Sigma,A} \to A$ is defined by

$$(1.13) \qquad (t_0)h = \begin{cases} \delta_\sigma & \text{if } t_0 = \sigma \in \Sigma_0 \\ t_0 & \text{if } t_0 = a \in A \end{cases}$$

$$(t_1 \ldots t_n \sigma)h = ((t_1)h \ldots (t_n)h)\delta_\sigma$$

The reader can easily check that the unitary and associativity axioms for T-algebras are satisfied.

We now introduce a category that will play an important role in our development:

(1.14) <u>Definition</u>

(Arbib-Manes [1]) Let $X : \mathcal{K} \to \mathcal{K}$. The category Dyn(X) is defined to have as objects pairs $<Q,\delta>$ where $Q \in \mathcal{K}$ and $\delta : QX \to Q$. A pair $<Q,\delta>$ is called an <u>X-dynamics</u> (dynamics, for short). A morphism $f : <Q,\delta> \to <Q',\delta'>$ of dynamics (<u>dynamorphism</u>) is a morphism $f : Q \to Q'$ of $\mathcal{K}$ such that

$$(1.15) \qquad \begin{array}{ccc} QX & \xrightarrow{\delta} & Q \\ \downarrow{\scriptstyle fX} & & \downarrow{\scriptstyle f} \\ Q'X & \xrightarrow{\delta'} & Q' \end{array}$$

For sequential machines $Y_0 \xleftarrow{\lambda} QxX_0 \xrightarrow{\delta} Q$ we would have $X = -xX_0$ and the category of dynamics would simply have as objects $<Q,\delta>$ where $\delta : QxX_0 \to Q$. In a more general case, define a functor Σ by $A\Sigma = A^n \coprod_{<\sigma,n>r}$ and

$$(1.16) \qquad \begin{array}{ccc} A & & A^n \coprod_{<\sigma,n>r} \\ \downarrow{\scriptstyle f} & \mapsto & \downarrow{\scriptstyle \coprod_{<\sigma,n>r} f^n} \\ A' & & A'^n \coprod_{<\sigma,n>r} \end{array}$$

Then the category $\mathrm{Dyn}(\Sigma)$ is just the category of Σ-algebras, for a homomorphism of Σ-algebras $\langle A,\delta\rangle \to \langle A',\delta'\rangle$ is just a function $f : A \to A'$ such that

(1.17)
$$\begin{array}{ccc} A^n & \xrightarrow{\ \delta_\sigma\ } & A \\ \Big\downarrow{\scriptstyle f^n} & & \Big\downarrow{\scriptstyle f} \\ A'^n & \xrightarrow{\ \delta'_\sigma\ } & A' \end{array}$$

for every n,σ such that $\langle\sigma,n\rangle r$.

(1.18) <u>Definition</u>

An <u>adjunction</u> $\langle F,U,\eta,\varepsilon\rangle : \mathcal{K} \to \mathcal{A}$ consists of a pair of functors F, U where $\mathcal{K} \underset{U}{\overset{F}{\rightleftarrows}} \mathcal{A}$ and a pair of natural transformations

$$\eta : I \xrightarrow{\ \bullet\ } FU \qquad \varepsilon : UF \xrightarrow{\ \bullet\ } I$$

such that the (triangular) identities

(1.19)
$$\begin{array}{ccc} U & \xrightarrow{\ U\eta\ } & UFU \\ & {\scriptstyle 1}\searrow & \Big\downarrow{\scriptstyle U} \\ & & U \end{array} \qquad \begin{array}{ccc} FUF & \xleftarrow{\ \eta F\ } & F \\ \Big\downarrow{\scriptstyle F} & \swarrow{\scriptstyle 1} & \\ F & & \end{array}$$

hold.

It is well known (MacLane [8] pp. 78-81) that the above statement is equivalent to the statement that for every $A \in \mathcal{K}$ there is a pair $\langle AF, A\eta\rangle$ where $AF \in \mathcal{A}$ and $A\eta : A \to AFU$ is a morphism of $\mathcal{K}$ with the following universal property: For any other such pair $\langle B,g\rangle$ where $B \in \mathcal{A}$ and $g : A \to BU$ a morphism of $\mathcal{K}$, there exists a unique morphism $f : AF \longrightarrow B$ of $\mathcal{A}$ such that $A\eta \circ fU = g$ as in the diagrams:

$$\begin{array}{ccc} A & \xrightarrow{A\eta} & AFU \\ & {\scriptstyle g}\searrow & \Big\downarrow {\scriptstyle fU} \\ & & BU \end{array} \qquad\qquad \begin{array}{c} AF \\ \Big\downarrow {\scriptstyle f} \\ B \end{array} \tag{1.20}$$

Such a pair $\langle AF, A\eta\rangle$ is said to be <u>free</u> over A with respect to U. F is said to be a <u>left adjoint</u> to U.

Observe that the above statement establishes the bijection ϕ which sends every $AF \xrightarrow{f} B$ to $A \xrightarrow{g} BU$. It follows from the Yoneda proposition ([8] pp. 59 & 78-81) that ϕ is, in fact, a natural equivalence and one can show that it has the form

$$\begin{aligned} (f)\phi &= A\eta \circ FU \\ (g)\phi^{-1} &= gF \circ B\varepsilon \end{aligned} \tag{1.21}$$

Following the Arbib-Manes approach to categorical automata theory we introduce the following fundamental postulate:

(1.22) <u>Postulate</u>

The forgetful functor $U : Dyn(X) \to \mathcal{K}$ has a left adjoint, i.e., there exists a functor $F : \mathcal{K} \to Dyn(X)$ such that there is an adjunction

$$\langle F, U, \eta, \varepsilon\rangle : \mathcal{K} \rightharpoonup Dyn(X)$$

When the above postulate holds we call $X : \mathcal{K} \to \mathcal{K}$ an <u>input process</u>. Denote $FU = X^{@}$ and we have (according to 1.20) that each $f : Q \to Q'$ extends to a morphism ψ of dynamics

$$\begin{array}{ccc} Q & \xrightarrow{Q\eta} & QX^{@} \\ & {\scriptstyle f}\searrow & \Big\downarrow {\scriptstyle \psi} \\ & & Q' \end{array} \qquad\qquad \begin{array}{ccc} QX^{@}X & \xrightarrow{Q\mu_0} & QX^{@} \\ \Big\downarrow {\scriptstyle \psi X} & & \Big\downarrow {\scriptstyle \psi} \\ Q'X & \xrightarrow{\delta'} & Q' \end{array} \tag{1.23}$$

$\langle QX^{@}, Q\mu_0\rangle$ is the <u>free dynamics</u> on Q.

The following theorem relates monads and adjoints.

(1.24) <u>Theorem</u>

An adjunction $\langle F,U,\eta,\varepsilon\rangle : \mathcal{X} \to \mathcal{A}$ determines a monad $\langle FU,\eta,F\varepsilon U\rangle$ in the category $\mathcal{X}$.

Proof: This is a standard argument. From the triangular identities (1.19) putting F in front in the first diagram and U behind in the second one we get the unitary axioms (1.4) for the monad $\langle FU,\eta,F\varepsilon U\rangle$. The diagram

$$\begin{array}{ccc} UFUF & \xrightarrow{\varepsilon UF} & UF \\ \Big\downarrow{\scriptstyle UF\varepsilon} & & \Big\downarrow{\scriptstyle \varepsilon} \\ UF & \xrightarrow{\varepsilon} & I \end{array}$$

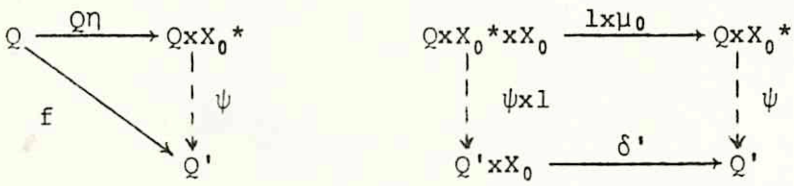
is just the definition of horizontal composition (1.1) and commutes by ε a natural transformation. Putting F in front and U behind in the above diagram we get

$$\begin{array}{ccc} (FU)(FU)(FU) & \xrightarrow{(F\varepsilon U)FU} & (FU)(FU) \\ \Big\downarrow{\scriptstyle FU(F\varepsilon U)} & & \Big\downarrow{\scriptstyle F\varepsilon U} \\ (FU)(FU) & \xrightarrow{F\varepsilon U} & FU \end{array}$$

the associativity axiom (1.3) of $\langle FU,\eta,F\varepsilon U\rangle$

The above theorem constructs a monad in $\mathcal{X}$ from the fundamental postulate (1.22). We'll investigate more closely the meaning of the postulate and the theorem 1.24 in two examples familiar to automata theorists:

For ordinary sequential machines we know that $X = -xX_0$ and $X^{@} = -xX_0{}^*$. In other words, the monad constructed from the theorem 1.24 for $X = -xX_0$ is exactly the familiar monad associated with the free monoid $X_0{}^*$. The diagrams that prove the existence of a free dynamics $\langle Q \times X_0^*, Q\mu_0 \rangle$ for any $Q \in$ Set are:



(1.25)

Here $\mu_0 : -\times X_0^* \times X_0 \xrightarrow{\bullet} -\times X_0^*$

$$\bullet, w, x \longrightarrow \bullet, wx$$

Chosing the diagrams in elements we get

$$\langle q,\Lambda\rangle\psi = qf \qquad \langle q,wx\rangle\psi = \langle\langle q,w\rangle\psi,x\rangle\delta'$$

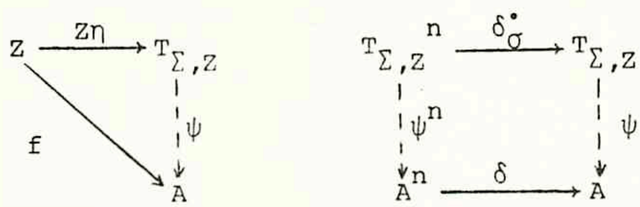
The above equations determine the dynamorphism ψ inductively and uniquely as $\langle q,w\rangle\psi = \langle qf,w\rangle(\delta')^*$.

The totally free Σ-algebra on Z generators is a pair $\langle T_{\Sigma,Z},\delta^\bullet\rangle$ where

$$\delta^\bullet_\sigma = \sigma \quad \text{for } \sigma \in \Sigma_0$$

$$(t_1,\ldots,t_n)\delta^\bullet_\sigma = t_1\ldots t_n\sigma$$

Our postulate is affirmed for the tree functor T by the existence of the above Σ-algebra for any set Z of generators. $\langle T_{\Sigma,Z},\delta^\bullet\rangle$ has the following universal property: (observe T as defined before is equal to FU)



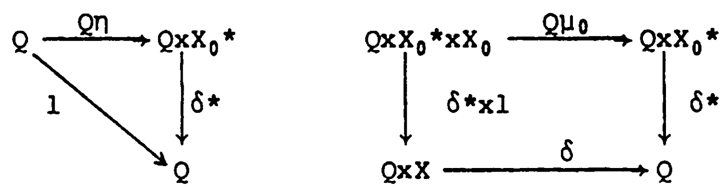
(1.26)

Any assignment $f : Z \to A$ of values in A to the variables in Z determines a unique Σ-algebra homomorphism ψ from $\langle T_{\Sigma,Z},\delta^\bullet\rangle$ into $\langle A,\delta\rangle$. This becomes clear from chosing the above diagrams in elements.

$$z\psi = zf \qquad (t_1 \ldots t_n\sigma)\psi = ((t_1)\psi \ldots (t_n)\psi)\delta_\sigma$$

We see that this is essentially the same inductive definition as the one in (1.25). So what we have is the adjunction $\underline{\text{Set}} \underset{U}{\overset{F}{\rightleftarrows}} \Sigma\text{-}\underline{\text{alg}}$ where $\Sigma\text{-}\underline{\text{alg}}$ is the category of Σ-algebras. This adjunction constructs the tree monad discussed previously.

For ordinary sequential machines we know that to $\langle Q,\delta\rangle$ there corresponds a unique $X^@$-algebra, namely $\langle Q,\delta^*\rangle$. The diagrams that define δ^* inductively have the form



(1.27)

and they say

$$\langle q,\Lambda\rangle\delta^* = q \qquad \langle q,wx\rangle\delta^* = \langle\langle q,w\rangle\delta^*,x\rangle\delta$$

This in fact establishes a bijection $\text{Dyn}(-xX_0) \simeq \underline{\text{Set}}^{X^@}$ between the category of $-xX_0$-dynamics, and the category of $X^@$-algebras. A much more general form of the above passage is proved in the next construction:

Theorem 1.24 states that the adjunction $\langle F,U,\eta,\varepsilon\rangle : \mathcal{K} \to \text{Dyn}(X)$ constructs a monad in $\mathcal{K}$. For this monad, denote it $\langle X^@,\eta,\mu\rangle$, we can construct the category $\mathcal{K}^{X^@}$ of $X^@$-algebras and the adjunction $\langle \bar{F},\bar{U},\bar{\eta},\bar{\varepsilon}\rangle : \mathcal{K} \to \mathcal{K}^{X^@}$ where $\bar{U}$ is the standard forgetful functor $\bar{U} : \mathcal{K}^{X^@} \to \mathcal{K}$. The existence of the letter adjunction follows from the fact that for each $A \in \mathcal{K}$ the pair $\langle AX^@, A\mu : AX^@X^@ \to AX^@\rangle$ is an $X^@$-algebra (the free $X^@$-algebra on A generators) in view of the associativity and (left) unity axioms for the monad $\langle X^@,\eta,\mu\rangle$. More about this see [8] p. 136. The following theorem which relates the two adjunctions plays an important role in general automata theory:

(1.28) Theorem (Comparison)

There exists a unique functor $K : \mathrm{Dyn}(X) \to X^{@}$ such that

$$\begin{array}{ccc} \mathrm{Dyn}(X) & \xrightarrow{\ K\ } & X^{@} \\ {\scriptstyle F}\nwarrow\searrow{\scriptstyle U} & & {\scriptstyle \bar{F}}\nearrow\swarrow{\scriptstyle \bar{U}} \\ & \mathcal{K} & \end{array} \qquad (1.29)$$

Moreover, K is an isomorphism of categories.

Proof: By (1.20)--with $g=1$ and $B=(Q,\delta)$--we may define the object function of K by

$$K : \langle Q,\delta\rangle \to \langle Q,Q\delta^{@}\rangle$$

where $Q\delta^{@} = \langle Q,\delta\rangle\varepsilon U$ is the unique dynomorphic extension (from the free dynamics) of the identity. We call it the run morphism of (Q,δ). Thus

$$\begin{array}{ccc} Q & \xrightarrow{\ Q\eta\ } & QX^{@} \\ & {\scriptstyle 1}\searrow & \downarrow{\scriptstyle Q\delta^{@}} \\ & & Q \end{array} \qquad (1.30)$$

which gives us the unitary axiom of $\langle Q,Q\delta^{@}\rangle$.

The diagram

$$\begin{array}{ccc} \langle Q,\delta\rangle UFUFU & \xrightarrow{\ \langle Q,\delta\rangle\varepsilon UFU\ } & \langle Q,\delta\rangle UFU \\ \downarrow{\scriptstyle \langle Q,\delta\rangle UF\varepsilon U} & & \downarrow{\scriptstyle \langle Q,\delta\rangle\varepsilon U} \\ \langle Q,\delta\rangle UFU & \xrightarrow{\ \langle Q,\delta\rangle\varepsilon U\ } & \langle Q,\delta\rangle U \end{array}$$

is just the definition of horizontal composition (1.1) for $\varepsilon\varepsilon U$ and therefore commutes. With $Q\mu = \langle Q,\delta\rangle UF\varepsilon U$ (theorem 1.24) and $\langle Q,\delta\rangle\varepsilon UFU = Q\delta^{@}X^{@}$ observe that the above diagram is just the associativity axiom for $\langle Q,Q\delta^{@}\rangle$. The mapping function of K is well defined since

$$\begin{array}{ccc} QX & \xrightarrow{\delta} & Q \\ \Big\downarrow{\scriptstyle fX} & & \Big\downarrow{\scriptstyle f} \\ Q'X & \xrightarrow{\delta'} & Q' \end{array} \quad \text{implies} \quad \begin{array}{ccc} QX^{@} & \xrightarrow{Q\delta^{@}} & Q \\ \Big\downarrow{\scriptstyle fX^{@}} & & \Big\downarrow{\scriptstyle f} \\ Q'X^{@} & \xrightarrow{Q'\delta^{@}} & Q' \end{array}$$

The second diagram commutes by $\delta^{@}$ a natural transformation. Thus K does indeed map $\mathrm{Dyn}(X)$ to $\mathcal{K}^{X^{@}}$. The reader can check that $FK = \bar{F}$, $K\bar{U} = U$ and that K is uniquely defined (see [8] p. 139).

The above proof is really a standard argument. As in many other cases, K turns out to be an isomorphism of categories, which is not difficult to show and the interested reader may consult [2] or [4].

Following this procedure for Σ-algebras we get the formulas (1.13) for ordinary tree machines.

We conclude this section by noting that because of some applications to the theory of treetransformations we will sometimes use a notational variant for the functor Σ defined in (1.16) and write $Z\Sigma = \{\langle z_1\rangle \ldots \langle z_n\rangle\sigma \mid \sigma\varepsilon\Sigma_n\}$. It should be clear that with this interpretation Σ becomes a functor under the substitution of variables as defined for the tree functor T in (1.8).

2. Direct State Transformations

In this section we introduce, using the language developed so far, some quite general models that give as particular cases generalized sequential machines, deterministic bottom-up finite state (tree) transformations and true automata. To provide enough intuition, we first work out in detail the particular case of generalized sequential machines:

Let $\langle X_0, Q, \delta, Y_0, \lambda\rangle$ where $Y_0{}^* \xleftarrow{\lambda} QxX_0 \xrightarrow{\delta} Q$ be a generalized sequential machine (g.s.m.). [The following theory thus applies to sequential machines, for which $\lambda : QxX_0 \to Y_0$, as a special case.] δ and λ can be extended to $\delta^* : QxX_0{}^* \to Q$ and $\lambda^* : QxX_0{}^* \to Y_0{}^*$ inductively as follows

(2.1)
$$\langle q,\Lambda\rangle\delta^* = q \qquad\qquad \langle q,\Lambda\rangle\lambda^* = \Lambda$$
$$\langle q,wx\rangle\delta^* = \langle\langle q,w\rangle\delta^*,x\rangle\delta \qquad\qquad \langle q,wx\rangle\lambda^* = \langle q,w\rangle\lambda^*\langle\langle q,w\rangle\delta^*,x\rangle\lambda$$

Now we make a passage from <u>Set</u> to $\underline{\text{Set}}^{\underline{\text{Set}}}$ via the full and faithful functor

(2.2)
$$\begin{array}{ccc} Q & & -xQ \\ \downarrow f & \mapsto & \downarrow -xf \\ Q' & & -xQ' \end{array}$$

This passage says that "sets are functors." In this spirit, we redefine a g.s.m. by saying that a g.s.m. mapping is a natural transformation

(2.3)
$$\tau : -xQxX_0 \xrightarrow{\bullet} -xY_0{}^*xQ$$
$$\bullet,q,x \longmapsto \bullet,\langle q,x\rangle\lambda,\langle q,x\rangle\delta$$

Just as in (2.1), we may extend τ to a natural transformation

(2.4)
$$\bar{\tau} : -xQxX_0{}^* \xrightarrow{\bullet} -xY_0{}^*xQ$$
$$\bullet,q,w \longmapsto \bullet,\langle q,w\rangle\lambda^*,\langle q,w\rangle\delta^*$$

inductively by the formulae

$$\langle q,\Lambda\rangle\bar{\tau} = \langle \Lambda,q\rangle = \langle q,\Lambda\rangle\lambda,\langle q,\Lambda\rangle\delta \qquad \langle q,wx\rangle\bar{\tau} = \langle q,w\rangle\lambda^{*}\langle\langle q,w\rangle\delta^{*},x\rangle\lambda$$
$$\langle\langle q,w\rangle\delta^{*},x\rangle\delta$$

However, for our general theory, it is more instructive to note that this definition is captured in the commutative diagrams

$$(2.5) \qquad \begin{array}{ccc} -\times Q & \xrightarrow{1\times\eta} & -\times Q\times X_0{}^{*} \\ & \searrow^{\bar{\eta}\times 1} \quad \text{I} & \downarrow \bar{\tau} \\ & & -\times Y_0{}^{*}\times Q \end{array} \qquad \begin{array}{ccccc} -\times Q\times X_0{}^{*}\times X_0 & & \xrightarrow{1\times\mu_0} & & -\times Q\times X_0{}^{*} \\ \downarrow \bar{\tau}\times 1 & & \text{II} & & \downarrow \bar{\tau} \\ -\times Y_0{}^{*}\times Q\times X_0 & \xrightarrow{1\times\tau} & -\times Y_0{}^{*}\times Y_0{}^{*}\times Q & \xrightarrow{\bar{\mu}\times 1} & -\times Y_0{}^{*}\times Q \end{array}$$

From (2.5), we may easily deduce the commutativity of

$$(2.6) \qquad \begin{array}{ccccc} -\times Q\times X_0{}^{*}\times X_0{}^{*} & & \xrightarrow{1\times\mu} & & -\times Q\times X_0{}^{*} \\ \downarrow \bar{\tau}\times 1 & & & & \downarrow \bar{\tau} \\ -\times Y_0{}^{*}\times Q\times X_0{}^{*} & \xrightarrow{1\times\bar{\tau}} & -\times Y_0{}^{*}\times Y_0{}^{*}\times Q & \xrightarrow{\bar{\mu}\times 1} & -\times Y_0{}^{*}\times Q \end{array}$$

for in elements we get

$$\langle q,ww'\rangle\bar{\tau} = \langle q,w\rangle\lambda^{*}\langle\langle q,w\rangle\delta^{*},w'\rangle\lambda^{*},\langle\langle q,w\rangle\delta^{*},w'\rangle\delta^{*}$$

and since

$$\langle q,ww'\rangle\bar{\tau} = \langle q,ww'\rangle\lambda^{*},\langle q,ww'\rangle\delta^{*}$$

this reduces to the familiar formulas

$$\langle q,ww'\rangle\lambda^{*} = \langle q,w\rangle\lambda^{*}\langle\langle q,w\rangle\delta^{*},w'\rangle\lambda^{*}$$
$$\langle q,ww'\rangle\delta^{*} = \langle\langle q,w\rangle\delta^{*},w'\rangle\delta^{*}.$$

Even if we replace the free monoid Y^{*} by an arbitrary monoid M', (2.5) still makes sense. Moreover, in the combined diagram formed from the last diagram and part I of (2.5), we may also replace $X_0{}^{*}$ by an arbitrary monoid M to obtain

$$(2.6)\qquad \begin{array}{ccc} -xQ & \xrightarrow{1x\eta} & -xQxM \\ & \searrow^{\bar{\eta}x1} & \downarrow \bar{\tau} \\ & & -xM'xQ \end{array} \qquad \begin{array}{ccc} -xQxMxM & \xrightarrow{1x\mu} & -QxM \\ \downarrow \bar{\tau}x1 & & \downarrow \bar{\tau} \\ -xM'xQxM & \xrightarrow{(1x\bar{\tau})(\bar{\mu}x1)} & -xM'xQ \end{array}$$

We call any $\bar{\tau}$ which satisfies these diagrams a direct state transformation. In the case $M = X_0{*}$, it is clear that $\bar{\tau}$ can be obtained as the extension of some τ by (2.5) (with M' replacing $Y_0{*}$). In this case $\bar{\tau}$ will be called a direct state transformation on a free monad. With this change of viewpoint, generalized sequential machines can be placed in a much more general framework to which we now turn.

First, observe that a functor $X : \mathcal{K} \to \mathcal{K}$ determines a functor $.X : \mathcal{K}^{\mathcal{K}} \to \mathcal{K}^{\mathcal{K}}$ defined to be just the composition with X. From this remark follows immediately the following:

(2.7) <u>Theorem</u>

If X is an input process in $\mathcal{K}$ then $.X$ is an input process in $\mathcal{K}^{\mathcal{K}}$.

<u>Proof</u>: If follows from (1.23) that given any functor F in $\mathcal{K}^{\mathcal{K}}$, natural transformation $F'X \xrightarrow{\Delta'} F'$, and natural transformation $F \xrightarrow{f} F'$, there exists, <u>for any Q in $\mathcal{K}$</u>, a unique morphism of $\mathcal{K}$ $QFX^{@} \xrightarrow{Q\psi} QF'$ such that

$$\begin{array}{ccc} QF & \xrightarrow{QF\eta} & QFX^{@} \\ & \searrow^{Qf} & \downarrow Q\psi \\ & & QF' \end{array} \qquad \begin{array}{ccc} QFX^{@}X & \xrightarrow{QF\mu_0} & QFX^{@} \\ \downarrow Q\psi X & & \downarrow Q\psi \\ QF'X & \xrightarrow{Q\Delta'} & QF' \end{array} \qquad (2.8)$$

To prove that $<.X^{@},\mu_0>$ is a free dynamics in $Dyn(.X)$ we need only check that ψ is a natural transformation. The outer diagram

$$\begin{array}{ccc} Q \\ \downarrow g \\ Q' \end{array} \qquad \begin{array}{ccccc} & & \overset{Qf}{\longrightarrow} & & \\ QF & \xrightarrow{QF\eta} & QFX^{@} & \xrightarrow{Q\psi} & QF' \\ \downarrow gF & \text{II.} & \downarrow gFX^{@} & \text{I.} & \downarrow gF' \\ Q'F & \xrightarrow{Q'F\eta} & Q'FX^{@} & \xrightarrow{Q'\psi} & Q'F' \\ & & \underset{Q'f}{\longrightarrow} & & \end{array} \qquad (2.9)$$

commutes by f a natural transformation and I. by η natural. Therefore II. commutes preceded by the inclusion of generators. Now observe that from μ_0 and Δ' natural transformations we have

$$\begin{array}{ccc} QFX^{@}X & \xrightarrow{QF\mu_0} & QFX^{@} \\ \downarrow gFX^{@}X & & \downarrow gFX^{@} \\ Q'FX^{@}X & \xrightarrow[Q'F\mu_0]{} & Q'FX^{@} \end{array} \qquad \begin{array}{ccc} QF'X & \xrightarrow{Q\Delta'} & QF' \\ \downarrow gF'X & & \downarrow gF' \\ Q'F'X & \xrightarrow[Q'\Delta']{} & Q'F' \end{array} \qquad (2.10)$$

This means that all morphisms in II. are dynamorphisms, and so are the two composites $<QFX^{@},QF\mu_0> \underset{gfX^{@}\circ Q'}{\overset{Q\psi\circ gF'}{\rightrightarrows}} <Q'F',Q'\Delta'>$. All told, we have the commutativity of

$$\begin{array}{ccc} QF & \xrightarrow{QF\eta} & QFX^{@} \\ & {\scriptstyle gF\circ Q'f} \searrow & \downarrow Q\psi\circ gF' \;\; \downarrow gFX^{@}\circ Q'\psi \\ & & Q'F' \end{array} \qquad (2.11)$$

Since $<QFX^{@},QF\mu_0>$ is a free dynamics from (1.23) we have that $Q\psi\circ gF = gFX^{@}\circ Q'\psi$ which means that II. commutes and therefore ψ is natural.

(2.12) <u>Definition</u>

A <u>direct state transformation on a free monad</u> consists of the following

(i) X - an input process in $\mathcal{K}$. The corresponding monad $<X^{@},\eta,\mu>$ will be called the <u>input monad</u>

(ii) $<\bar{T},\bar{\eta},\bar{\mu}>$ - a monad in $\mathcal{K}$ (the <u>output monad</u>)

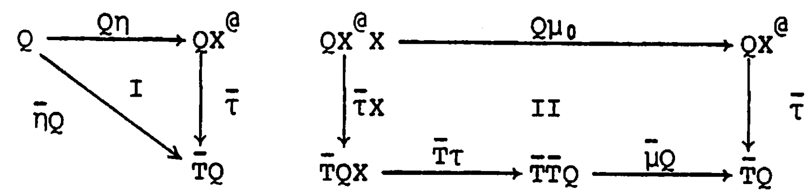
(iii) Q - an object of $\mathcal{K}^{\mathcal{K}}$ (the <u>state functor</u>)

(iv) $\tau : QX \xrightarrow{\bullet} \bar{T}Q$ - a natural transformation (the <u>output morphism</u>)

Bearing in mind the motivating diagram (2.5) we may extend τ to $\bar{\tau}$ as follows:

(2.13) <u>Proposition</u>

The diagrams:

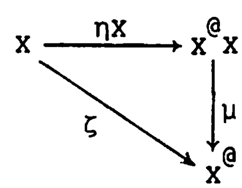


define a unique extension of $\tau : QX \xrightarrow{\bullet} \bar{T}Q$ to a natural transformation (the <u>state transformation</u>) $\bar{\tau} : QX^{@} \xrightarrow{\bullet} \bar{T}Q$

Proof: I and II define $\bar{\tau}$ as the unique dynamorphic extension $\langle QX^{@}, Q\mu_0\rangle \to \langle \bar{T}Q, \bar{T}\tau \circ \bar{\mu}Q\rangle$ of $\bar{\eta}Q$ (by the previous theorem).

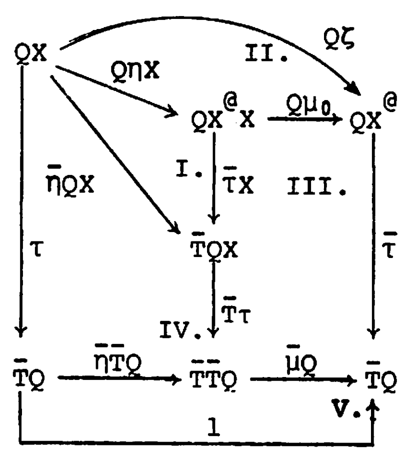
If we define the inclusion (of generators, to be explained later) $\zeta : X \xrightarrow{\bullet} X^{@}$ as

(2.14)



we get

(2.15)



In this diagram, I. and III. commute by the definition of $\bar{\tau}$ (2.13), IV. by τ (and $\bar{\eta}$) natural, II. by (2.14) and V. is the unity law for $\langle \bar{T},\bar{\eta},\bar{\mu}\rangle$. Therefore the whole diagram commutes and we have

(2.16)
$$\begin{array}{ccc} QX & \xrightarrow{Q\zeta} & QX^{@} \\ & \searrow^{\tau} & \downarrow \bar{\tau} \\ & & \bar{T}Q \end{array}$$

This diagram provides the natural expression of the first step of the inductive extension (2.13) of τ to $\bar{\tau}$.

As a particular case ($\mathcal{K}$ = <u>Set</u>) of our general model we get deterministic bottom-up finite state (tree) transformations (Engelfriet [5]). A bottom-up finite state transformation is a 5-tuple $\langle \Sigma,Q,\tau,\Omega,Q_0\rangle$ where Σ and Ω are ranked alphabets, Q a set of states, Q_0 a set of final states and τ is an output function specified by a set of rules of the form

(2.17) $$\langle z_1,q_1\rangle\langle z_2,q_2\rangle \cdots \langle z_K,q_K\rangle\sigma \to \langle t_1 q_K\rangle \in T_{\Omega,Z_K}xQ$$

where $Z_K = \{z_1,\ldots,z_K\}$ are free variables and $Z_0 = \phi$ so that for that case $(K=0)$ we have a rule of the form

$$\sigma \to \langle t_1 q\rangle \in T_{\Omega}xQ$$

The condition (2.17) is just the condition for τ to be a natural transformation in the following sense. With the notation introduced in (1.7) and (1.8) let $ZT = T_{\Sigma,Z}$ and $Z\bar{T} = T_{\Omega,Z}$ and we have

(2.18) $$Z\tau : (ZxQ)\Sigma \xrightarrow{\bullet} Z\bar{T}xQ$$

which can be extended in a unique fashion to

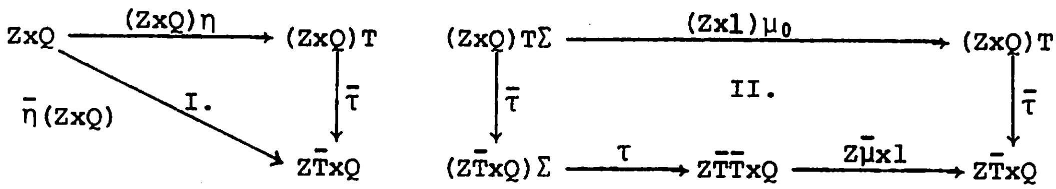
(2.19) $$Z\bar{\tau} : (ZxQ)T \xrightarrow{\bullet} Z\bar{T}xQ$$

according to (2.13). A particular component of τ and $\bar{\tau}$ is of most interest to the theory of treetransformations, namely the one for $Z = \phi$. In this case we get

(2.20)
$$\phi\tau : \Sigma \to T_\Omega x Q$$
$$\phi\bar{\tau} : T_\Sigma \to T_\Omega x Q$$

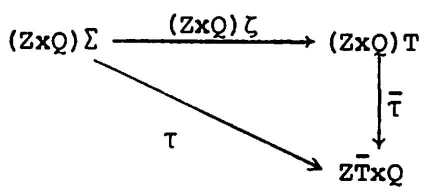
(2.13) gives the following diagrams

(2.21)



I. says $\langle z,q\rangle\bar{\tau} = \langle z,q\rangle\tau$ and II. (hard to chose in elements with elegance) is the inductive extension whose first step is defined by (2.16)

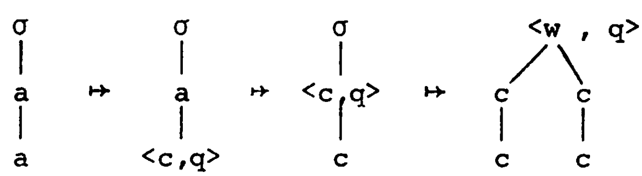
(2.22)



which says $(t)\bar{\tau} = (t)\tau$ if $t = \langle z_1,q_1\rangle \cdots \langle z_K,q_K\rangle\sigma$ For example, given a bottom-up finite state transformation

$a \to \langle c,q\rangle$	$\Sigma_0 = \{a\}$ $\Sigma_1 = \{a,\sigma\}$
$\langle x,q\rangle a \to \langle xc,q\rangle$	$\Omega_0 = \{c\}$ $\Omega_1 = \{c\}$ $\Omega_2 = \{w\}$
$\langle x,q\rangle\sigma \to \langle xxw,q\rangle$	and $Q = \{q\}$

and a tree $aa\sigma$ we map it into a pair $\langle ccccw,q\rangle$ using (2.21) as follows



Tree automata form a particular class of bottom-up finite state transformations. In this case

(2.23) $$\bar{\tau} : T_\Sigma \to T_\Sigma x Q$$

and the natural transformation τ is specified by rules of the form

$$\langle z_1, q_1\rangle \cdots \langle z_K, q_K\rangle\sigma \to \langle z_1 \ldots z_K\sigma, q\rangle$$

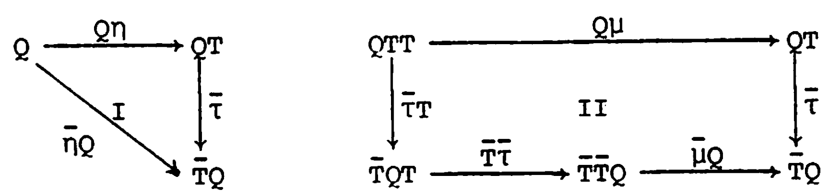
which means that it is an identity transformation on the input tree. If $(t)\bar{\tau} = \langle t,q\rangle$ and $q \in Q_0$ (some set of final states) we say that a tree t is accepted. This unconventional but natural definition can be found in a different framework in [5] and generalized according to (2.13) in a straightforward way. We now turn to the general concept of a direct state transformation (not necessarily on a free monad) motivated by diagram (2.6):

(2.24) Definition

Let $\langle T,\eta,\mu\rangle$ and $\langle \bar{T},\bar{\eta},\bar{\mu}\rangle$ be monads in a category $\mathcal{K}$, and Q a functor $\mathcal{K} \to \mathcal{K}$. By a direct state transformation of the above monads we mean a natural transformation

$$\bar{\tau} : QT \xrightarrow{\cdot} \bar{T}Q$$

such that the following diagrams commute:



(2.25)

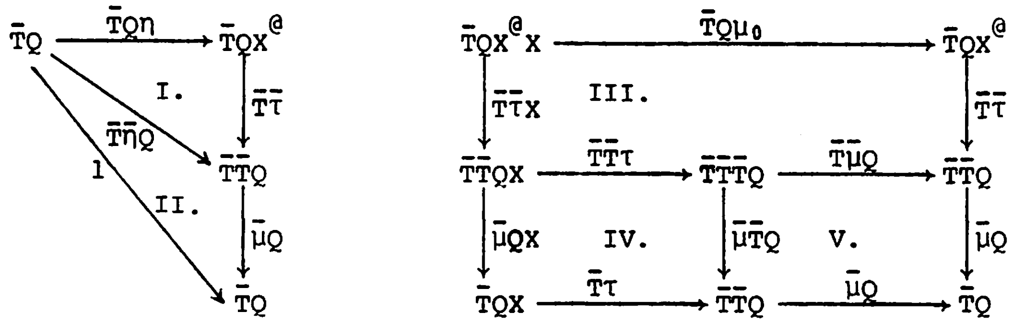
We must now verify that our intuition from (2.5) and (2.6) is in fact valid, and that our terminology is thus justified:

(2.26) Theorem

The direct state transformation on a free monad $\bar{\tau} : QX^@ \xrightarrow{\cdot} \bar{T}Q$ is a direct state transformation: $\langle X^@, \eta, \mu\rangle \to \langle \bar{T}, \bar{\eta}, \bar{\mu}\rangle$.

Proof: Diagram II in (2.13) defines $\bar{\tau}$ as a morphism $\langle QX^@, Q\mu_0\rangle \to \langle \bar{T}Q, \bar{T}\tau\circ\bar{\mu}Q\rangle$ in Dyn(.X). Using the comparison theorem 1.28 we'll find the corresponding diagram of $X^@$-algebra homomorphism. Firstly observe that in

(2.26)



I. and III. commute by the definition (2.13).

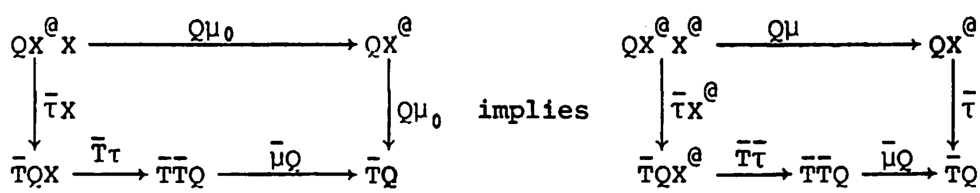
II. is a unitary law for $\langle \bar{T}, \bar{\eta}, \bar{\mu}\rangle$ and V. the associativity law of the same monad.

IV. commutes by τ (or $\bar{\mu}$) a natural transformation.

The two outer diagrams together say (see theorem 1.28) that the run morphism of $\bar{T}\tau\circ\bar{\mu}Q$ is $\bar{T}\bar{\tau}\circ\bar{\mu}Q$. Since $\mu = (\mu_0)^@$ we have that in fact the comparison functor K does the following

$$\langle QX^@, Q\mu_0\rangle \to \langle QX^@, Q\mu\rangle \quad ; \quad \langle \bar{T}Q, \bar{T}\tau\circ\bar{\mu}Q\rangle \to \langle \bar{T}Q, \bar{T}\bar{\tau}\circ\bar{\mu}Q\rangle$$

Therefore, the comparison theorem gives us that



(2.27)

which completes the proof.

In case of bottom-up finite state transformations we get the result dual to Thatcher's ([13] p. 353) but expressed properly in terms of monads and not the monoids in Set which Thatcher calls pretheories. The diagram that corresponds to (2.27) (right) in this case is:

(2.28)
$$\begin{array}{ccccc} (ZxQ)TT & & \xrightarrow{(Zx1)\mu} & & (ZxQ)T \\ \downarrow{\scriptstyle \bar{\tau}T} & & & & \downarrow{\scriptstyle \bar{\tau}} \\ (Z\bar{T}xQ)T & \xrightarrow{\bar{T}\bar{\tau}} & Z\bar{T}\bar{T}xQ & \xrightarrow{Z\bar{\mu}x1} & Z\bar{T}xQ \end{array}$$

(2.29) Corollary

$\langle X^{@},\eta,\mu\rangle$ is a free monad on X generators.

Proof: Take $Q = I$ (identity functor) in (2.16) and (2.26). According to the proposition (2.13) we have that given any other monad $\langle\bar{T},\bar{\eta},\bar{\mu}\rangle$ and a natural transformation $\tau : X \xrightarrow{\bullet} \bar{T}$ there exists a unique morphism $\bar{\tau} : \langle X^{@},\eta,\mu\rangle \to \langle\bar{T},\bar{\eta},\bar{\mu}\rangle$ of monads such that $\zeta\circ\bar{\tau} = \tau$ as in the diagrams:

(2.30)
$$\begin{array}{ccc} X & \xrightarrow{\zeta} & X^{@} \\ & \searrow^{\tau} & \downarrow{\scriptstyle \bar{\tau}} \\ & & \bar{T} \end{array} \qquad \begin{array}{ccc} X^{@}X^{@} & \xrightarrow{\mu} & X^{@} \\ \downarrow{\scriptstyle \bar{\tau}\bar{\tau}} & & \downarrow{\scriptstyle \bar{\tau}} \\ \bar{T}\bar{T} & \xrightarrow{\bar{\mu}} & \bar{T} \end{array} \qquad \begin{array}{ccc} X^{@} & \xleftarrow{\eta} & I \\ \downarrow{\scriptstyle \bar{\tau}} & & \downarrow{\scriptstyle 1} \\ \bar{T} & \xleftarrow{\bar{\eta}} & I \end{array}$$

We remark that our construction for the case $Q = I$ reduces to the one of Barr [4].

(2.31) Definition

A state transformation $\bar{\tau} : QT \xrightarrow{\bullet} \bar{T}Q$ is called pure when Q is the identity functor.

(2.32) Corollary

A pure state transformation $\bar{\tau} : T \xrightarrow{\bullet} \bar{T}$ is a morphism $\langle T,\eta,\mu\rangle \to \langle\bar{T},\bar{\eta},\bar{\mu}\rangle$ of monads.

(2.33) <u>Definition</u>

Let T,K,L be functors and

$$\bar{\tau} : QT \xrightarrow{\bullet} KQ$$

$$\bar{\rho} : SK \longrightarrow LS$$

be direct state transformations. Define the <u>composite</u> transformation $\bar{\tau}\circ\bar{\rho}$ by

(2.34)
$$\begin{array}{ccc} SQT & \xrightarrow{S\bar{\tau}} & SKQ \\ & \searrow^{\bar{\tau}\circ\bar{\rho}} & \downarrow{\bar{\rho}Q} \\ & & LSQ \end{array}$$

Since the above composition is just the vertical composition of natural transformations, it satisfies the associativity and the unity axiom for categories. We can therefore define a category $\mathcal{D}$ whose objects are monads $\langle T,\eta,\mu\rangle$, $\langle K,\bar{\eta},\bar{\mu}\rangle,\ldots$ and whose morphisms $\bar{\tau} : \langle T,\eta,\mu\rangle \to \langle K,\bar{\eta},\bar{\mu}\rangle$ are direct state transformations.

Observe that the corollary (2.29) says that both the monad of $-xX_0{}^*$ and the tree monad are free, the first one on $-xX_0$ generators and the second one on Σ generators.

The following construction, interesting for category and homotopy theorists (see Manes [9] and Meyer [10]) is the immediate consequence of the results that we have proved so far. Just as in theorem (2.7) we showed that the input process $X : \mathcal{K} \to \mathcal{K}$ can be lifted to the functor category $\mathcal{K}^{\mathcal{K}}$, so it turns out that the direct state transformation $\bar{\tau} : QT \xrightarrow{\bullet} \bar{T}Q$ allows us to define a lifting of the state functor Q to the category of T-algebras in the sense of the following definition:

(2.35) <u>Definition</u>

Let $\langle T,\eta,\mu\rangle$ and $\langle\bar{T},\bar{\eta},\bar{\mu}\rangle$ be monads in a category $\mathcal{K}$, Q a functor

$\mathcal{K} \to \mathcal{K}$, and $\mathcal{K}^{\bar{T}}$ and $\mathcal{K}^{T}$ the categories of $\bar{T}$ and T-algebras. By a <u>lifting</u> of Q we mean a functor $\bar{Q} : \mathcal{K}^{\bar{T}} \to \mathcal{K}^{T}$ such that

$$\begin{array}{ccc} \mathcal{K}^{\bar{T}} & \xrightarrow{\bar{Q}} & \mathcal{K}^{T} \\ \downarrow{\scriptstyle \bar{U}} & & \downarrow{\scriptstyle U} \\ \mathcal{K} & \xrightarrow{Q} & \mathcal{K} \end{array}$$

where $\bar{U}$ and U are standard forgetful functors.

(2.36) <u>Theorem</u>

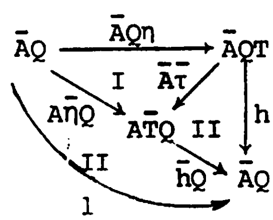
Given monads $\langle T,\eta,\mu\rangle$ and $\langle \bar{T},\bar{\eta},\bar{\mu}\rangle$ in the category $\mathcal{K}$, there is a 1-1 correspondence between the direct state transformations $\bar{\tau} : QT \xrightarrow{\bullet} \bar{T}Q$ and the liftings $\bar{Q} : \mathcal{K}^{\bar{T}} \to \mathcal{K}^{T}$ of Q to the category of algebras.

Proof: Let $\bar{\tau} : QT \xrightarrow{\bullet} \bar{T}Q$ be given, then $\bar{Q} : \langle \bar{A},\bar{h}\rangle \to \langle A,h\rangle$ is defined by $A = \bar{A}Q$ and

$$\begin{array}{ccc} \bar{A}QT & \xrightarrow{\bar{A}\bar{\tau}} & \bar{A}\bar{T}Q \\ & \searrow{\scriptstyle h} & \downarrow{\scriptstyle \bar{h}Q} \\ & & \bar{A}Q \end{array} \tag{2.37}$$

Here $\langle \bar{A},\bar{h}\rangle$ is an object in $\mathcal{K}^{\bar{T}}$ and we have to show that $\langle A,h\rangle$ is an object in $\mathcal{K}^{T}$. We get the unity law for $\langle A,h\rangle$ from

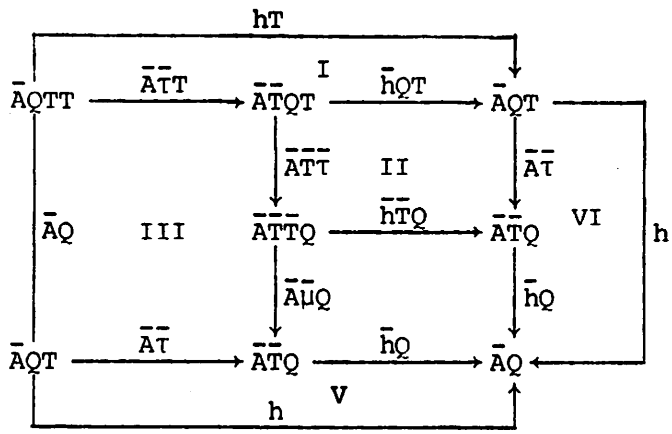
(2.38)



observing that I commutes by $\bar{\tau}$ a direct state transformation, II by the definition of $\bar{Q}$ and III is just the unity law for $\langle \bar{A},\bar{h}\rangle$.

We get the associativity axiom for $\langle A,h\rangle$ from the diagram

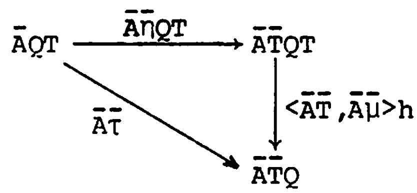
(2.39)



observing that I, V and VI commute by the definition of $\bar{Q}$, III by $\bar{\tau}$ a direct state transformation, II by $\bar{\tau}$ natural, while IV is the associativity law for $\langle\bar{A},\bar{h}\rangle$.

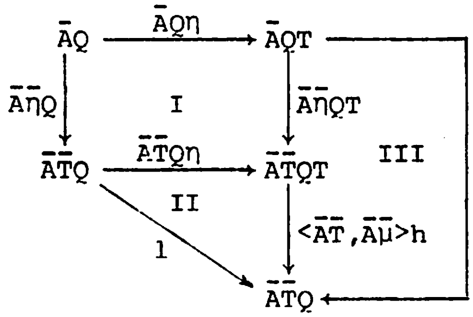
Conversely, given a lifting $\bar{Q}$ of Q we have $\langle\bar{A},\bar{h}\rangle\bar{Q} = \langle\bar{A}Q,\langle\bar{A},\bar{h}\rangle h\rangle$ where $\langle\bar{A},\bar{h}\rangle h$ is clearly a natural transformation. Define now $\bar{\tau} : QT \xrightarrow{\bullet} \bar{T}Q$ by

(2.40)



Then we can show that in fact $\bar{\tau}$ is a direct state transformation. Firstly, in the diagram

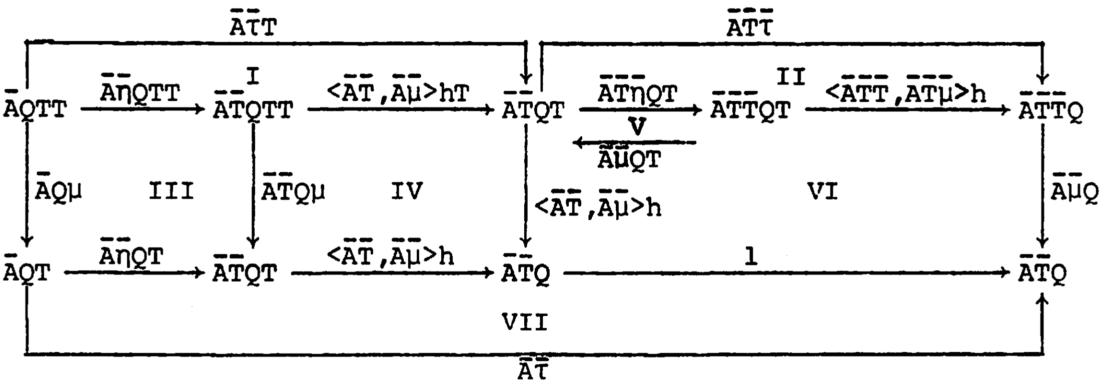
(2.41)



I commutes by η a natural transformation, III by the definition of $\bar{\tau}$ and II is the unity law for $\langle\bar{A}\bar{T}Q,\langle\bar{A}\bar{T},\bar{A}\bar{\mu}\rangle h\rangle$. Therefore, the whole diagram commutes and this gives us axiom I from the definition (2.24).

Furthermore we observe that in

(2.42)



I, II, VII commute by the definition of $\bar{\tau}$, III by $\bar{\eta}$ a natural transformation, IV is the associativity law for $\langle \bar{A}\bar{T}Q, \langle \bar{A}\bar{T}, \bar{A}\bar{\mu}\rangle h\rangle$ and V follows from the unity law for $\langle \bar{T}, \bar{\eta}, \bar{\mu}\rangle$. Finally, VI commutes by $\bar{\mu}$ a natural transformation and therefore the whole diagram shows that the second axiom for a direct state transformation is satisfied for $\bar{\tau}$ as defined in (2.40).

A straightforward computation shows that the two passages constructed above are inverse to each other, which completes the proof.

3. Inverse State Transformations

In this section we develop a model which is dual to direct state transformations. A natural way to get an intuitive feel for the nature of this duality seems to be by studying a particular case--Thatcher's generalized[2] sequential machine maps, which we will call top-down finite state transformations (f.s.t.). One would intuitively think that top-down f.s.t. are dual in some sense to bottom-up f.s.t. studied in the previous section. We investigate this more carefully, and show that the nature of this duality is adjointness. Moreover, there is no reason for restricting ourselves to the category Set of sets.

We first give the definitions for top-down f.s.t. and invite the reader to go back to the example of bottom-up f.s.t. for comparison. The proofs and justifications will be given in a general framework following the brief exposition of this particular case:

A top-down f.s.t. is a 5-tuple $<\Sigma,Q,\tau,\Omega,q_0>$, where Σ and Ω are ranked alphabets, Q is the set of states, $q \in Q$ is the initial state and τ is the output function specified by the set of rules of the form

(3.1) $$<z_1 \ldots z_K \sigma, q> \to t \qquad \text{where } t \in T_{\Omega, Z_K x Q_K}$$

One checks easily that this is a condition for τ to be a natural transformation under the substitution of variables. So dually to (2.18) and (2.19) we have

(3.2) $$Z\tau : Z\Sigma xQ \xrightarrow{\bullet} (ZxQ)\bar{T}$$

Observe that $-xQ$ has a right adjoint via $\underline{\text{Set}} \underset{(-)^Q}{\overset{-xQ}{\rightleftarrows}} \underline{\text{Set}}$. $Z\tau$ can be extended to $Z\bar{\tau} : ZTxQ \xrightarrow{\bullet} (ZxQ)\bar{T}$. The diagrams that define this extension are: (compare Thatcher [13] p. 352)

(3.3)

$$\begin{array}{ccc} Z\times Q & \xrightarrow{Z\eta\times 1} & ZT\times Q \\ & \searrow^{(Z\times Q)\bar{\eta}} & \downarrow{\scriptstyle Z\bar{\tau}} \\ & & (Z\times Q)\bar{T} \end{array} \qquad \begin{array}{ccccc} ZT\Sigma\times Q & & \xrightarrow{Z\mu_0\times 1} & & ZT\times Q \\ \downarrow{\scriptstyle ZT\tau} & & & & \downarrow{\scriptstyle Z\bar{\tau}} \\ (ZT\times Q)\bar{T} & \xrightarrow{Z\bar{\tau}\bar{T}} & (Z\times Q)\bar{T}\bar{T} & \xrightarrow{(Z\times 1)\bar{\mu}} & (Z\times Q)\bar{T} \end{array}$$

The first diagram says $\langle z,q\rangle\bar{\tau} = \langle z,q\rangle$ and the second how to proceed.

The diagram dual to (2.22) in case of top-down f.s.t. has the form

(3.4)

$$\begin{array}{ccc} Z\Sigma\times Q & \xrightarrow{Z\eta\times 1} & ZT\times Q \\ & \searrow^{Z\tau} & \downarrow{\scriptstyle Z\bar{\tau}} \\ & & (Z\times Q)\bar{T} \end{array}$$

The above diagram defines the initial step of inductive definition of τ to $\bar{\tau}$ and says

$$\langle t,q\rangle\bar{\tau} = \langle t,q\rangle\tau \qquad \text{if} \quad t = z_1\ldots z_K\sigma$$

A particular component of τ and $\bar{\tau}$ is of most interest and that's the one for $Z = \phi$. In this case we get

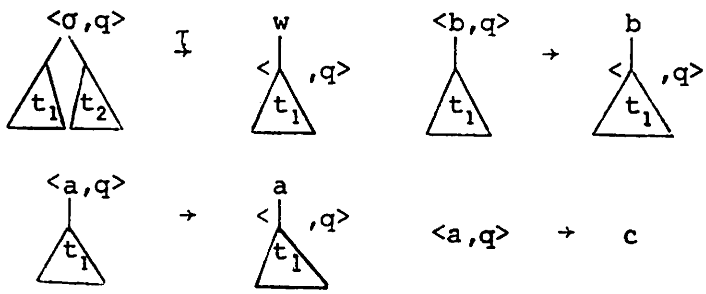
(3.5)

$$\phi\tau \;:\; \Sigma\times Q \to T_\Omega$$
$$\phi\bar{\tau} \;:\; T_\Sigma\times Q \to T$$

We conclude this introduction by giving an example of a top-down finite state transformation.

$$\langle z_1 z_2\sigma,q\rangle \to \langle z_1,q\rangle w \qquad \langle z_1 b,q\rangle \to \langle z_1,q\rangle b$$
$$\langle z_1 a,q\rangle \to \langle z_1,q\rangle a \qquad \langle a,q\rangle \to c$$
$$\langle b,q\rangle \to a$$

This definition of τ has the following pictorial representation:

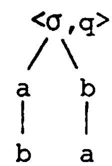
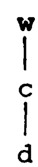


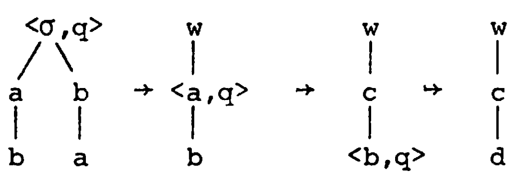
$\langle a,q\rangle \to c$

$\langle b,q\rangle \to d$

Here $\Sigma_0 = \{a,b\}$ $\Sigma_1 = \{a,b\}$ $\Sigma_2 = \{\sigma\}$ $\Omega_0 = \{c,d\}$ $\Omega_1 = \{c,d\}$

$\Omega_2 = \{w\}$ $Q = \{q\}$

Given a pair  we map it into a tree  in a sequential manner following (3.3) and (3.4) as follows



With this intuition in mind, we can now turn to general considerations.

(3.8) Definition

An inverse state transformation on a free monad consists of the following:

X - an input process in $\mathcal{K}$. The corresponding monad $\langle X^{@},\eta,\mu\rangle$ is called the input monad.

$\langle\bar{T},\bar{\eta},\bar{\mu}\rangle$ - the output monad.

$Q : \mathcal{K} \to \mathcal{K}$ - the state functor, with the property that Q has a right adjoint $Q^{\cdot}$

$\tau : XQ \xrightarrow{\cdot} Q\bar{T}$ - a natural transformation (the output morphism).

Just as in Section 2, we may extend τ to $\bar{\tau}$ and verify the morphic properties of $\bar{\tau}$:

(3.9) <u>Theorem</u>

The diagrams:

$$\begin{array}{ccc} Q & \xrightarrow{\eta Q} & X^{@}Q \\ & \searrow^{Q\bar{\eta}} \quad \text{I} & \downarrow \bar{\tau} \\ & & Q\bar{T} \end{array} \qquad \begin{array}{ccccc} X^{@}XQ & & \xrightarrow{\mu_0 Q} & & X^{@}Q \\ \downarrow X^{@}\tau & & \text{II} & & \downarrow \bar{\tau} \\ X^{@}Q\bar{T} & \xrightarrow{\bar{\tau}\bar{T}} & Q\bar{T}\bar{T} & \xrightarrow{Q\bar{\mu}} & Q\bar{T} \end{array}$$

define a unique extension of the output morphism $\tau : XQ \xrightarrow{\bullet} Q\bar{T}$ to a natural transformation $\bar{\tau} : X^{@}Q \xrightarrow{\bullet} Q\bar{T}$ called the <u>state transformation</u>.

Proof: We first give the outline of this quite involved proof.

The reader not interested in details may wish to read only this outline.

From the properties of adjoints (1.21) we know that there exists a natural equivalence ϕ such that to every $f : AQ \to B$ corresponds $(f)\phi = A\eta' \circ fQ^{\cdot} : A \to BQ^{\cdot}$ and to every $g : A \to BQ^{\cdot}$ $(g)\phi^{-1} = gQ \circ B\varepsilon^{\cdot} : AQ \to B$ where $\langle Q, Q^{\cdot}, \eta^{\cdot}, \varepsilon^{\cdot}\rangle : \mathcal{K} \to \mathcal{K}$

Moreover, the nature of this correspondence is such that a natural transformation is mapped to a natural transformation. So, $\tau : XQ \xrightarrow{\bullet} Q\bar{T}$ is mapped to $(\tau)\phi = \rho : X \xrightarrow{\circ} Q\bar{T}Q^{\cdot} = L$ where $L : \mathcal{K} \to \mathcal{K}$ and $\bar{\tau} : X^{@}Q \xrightarrow{\bullet} Q\bar{T}$ to $(\bar{\tau})\phi = \bar{\rho} : X^{@} \xrightarrow{\bullet} Q\bar{T}Q^{\cdot} = L$.

Careful checking shows that $\langle L, \eta', \mu'\rangle$ is a monad in the category $\mathcal{K}$ and that to the diagrams I and II in (3.9) correspond (via ϕ) the diagrams [<u>heuristically</u>, multiply (3.9) by $Q^{\cdot}$ on the right, and interchange $QQ^{\cdot}$ and I as convenient]:

(3.10)
$$\begin{array}{ccc} I & \xrightarrow{\eta} & X^{@} \\ & \searrow^{\eta'} \quad \text{I} & \downarrow \bar{\rho} \\ & & L \end{array} \qquad \begin{array}{ccccc} X^{@}X & & \xrightarrow{\mu_0} & & X^{@} \\ \downarrow X^{@}\rho & & \text{II} & & \downarrow \bar{\rho} \\ X^{@}L & \xrightarrow{\bar{\rho}L} & LL & \xrightarrow{\mu'} & L \end{array}$$

But now, it follows from the definition (1.1) of horizontal composition that

$$\begin{array}{ccc} X^{@}X & \xrightarrow{\bar{\rho}X} & LX \\ \downarrow{\scriptstyle X^{@}\rho} & & \downarrow{\scriptstyle L\rho} \\ X^{@}L & \xrightarrow{\bar{\rho}L} & LL \end{array}$$

so that after all the diagram on the right in (3.10) has the form:

(3.11)
$$\begin{array}{ccccc} X^{@}X & & \xrightarrow{\quad\mu_0\quad} & & X^{@} \\ \downarrow{\scriptstyle \bar{\rho}X} & & & & \downarrow{\scriptstyle \bar{\rho}} \\ LX & \xrightarrow{L\rho} & LL & \xrightarrow{\mu'} & L \end{array}$$

All told, we proved that to every inverse state transformation on a free monad there corresponds a pure direct state transformation on a free monad and therefore the theorem follows from (2.13), (2.31) and (2.32). Simple checking similar to (2.15) shows that the diagram

(3.12)
$$\begin{array}{ccc} XQ & \xrightarrow{\zeta Q} & X^{@}Q \\ & \searrow{\scriptstyle \tau} & \downarrow{\scriptstyle \bar{\tau}} \\ & & Q\bar{T} \end{array}$$

dual to (2.16) commutes. The above diagram defines the initial step of the extension of τ to $\bar{\tau}$.

Now we turn to careful checking that the above construction really works: We first observe that the monad $\langle\bar{T},\bar{\eta},\bar{\mu}\rangle$ is in fact determined by the adjunction $\langle\bar{F},\bar{U},\bar{\eta},\bar{\mu}\rangle : \mathcal{K} \to \mathcal{K}^{\bar{T}}$ ([8] p. 136).

The two given adjunctions

$$\langle Q,Q^{\cdot},\eta^{\cdot},\varepsilon^{\cdot}\rangle : \mathcal{K} - \mathcal{K} \qquad\qquad \langle\bar{F},\bar{U},\bar{\eta},\bar{\mu}\rangle : \mathcal{K} \to \mathcal{K}^{\bar{T}}$$

$$\mathcal{K} \underset{Q^{\cdot}}{\overset{Q}{\rightleftarrows}} \mathcal{K} \underset{\bar{U}}{\overset{\bar{F}}{\rightleftarrows}} \mathcal{K}^{\bar{T}}$$

give the composite adjunction ([8] p. 101)

$$\langle Q\bar{F}, \bar{U}Q^{\bullet}, \eta^{\bullet}\circ Q\bar{\eta}Q^{\bullet}, \bar{U}\varepsilon^{\bullet}\bar{F}\circ\bar{\varepsilon}\rangle : \mathcal{K}\to\mathcal{K}^{\bar{T}}$$

According to the theorem (1.24) this adjunction determines a monad $\langle L,\eta',\mu'\rangle$ where

$$L = Q\bar{F}\bar{U}Q^{\bullet} = Q\bar{T}Q^{\bullet} \qquad \eta' = \eta^{\bullet}\circ Q\bar{\eta}Q^{\bullet}$$
$$\mu' = Q\bar{F}\varepsilon'\bar{U}Q^{\bullet} \qquad \varepsilon' = \bar{U}\varepsilon^{\bullet}\bar{F}\circ\bar{\varepsilon} \tag{3.13}$$

According to (1.21)

$$\begin{array}{c} AX^{@}Q \\ \downarrow A\bar{\tau} \\ AQ\bar{T} \end{array} \qquad \begin{array}{ccc} AX^{@} & \xrightarrow{AX^{@}\eta^{\bullet}} & AX^{@}QQ^{\bullet} \\ & {\scriptstyle (A\bar{\tau})\phi = A\bar{\rho}}\searrow & \downarrow A\bar{\tau}Q^{\bullet} \\ & & AQ\bar{T}Q^{\bullet} = AL \end{array} \tag{3.14}$$

Using the same argument and the formula for $\eta^{\bullet}$ we see that

$$\begin{array}{c} AQ \\ \downarrow AQ\bar{\eta} \\ AQ\bar{T} \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{A\eta^{\bullet}} & AQQ^{\bullet} \\ & {\scriptstyle (AQ\bar{\eta})\phi = A\eta'}\searrow & \downarrow AQ\bar{\eta}Q^{\bullet} \\ & & AQ\bar{T}Q^{\bullet} \end{array} \tag{3.15}$$

Furthermore, ϕ is not simply a bijection, but a natural equivalence ([8] p. 78) which in particular means

$$\begin{array}{ccc} (AX^{@}Q \xrightarrow{A\bar{\tau}} AQ\bar{T}) & \xrightarrow{\phi} & (AX^{@} \xrightarrow{A\rho} AQ\bar{T}Q^{\bullet}) \\ \downarrow (A\eta Q)^{*} & & \downarrow (A\eta)^{*} \\ (AQ \xrightarrow{A\eta Q} AX^{@}Q \xrightarrow{A\bar{\tau}} AQ\bar{T}) & \xrightarrow{\phi} & (A \xrightarrow{A\eta} AX^{@} \xrightarrow{A\bar{\rho}} AQ\bar{T}Q^{\bullet}) \end{array} \tag{3.16}$$

where $(A\eta)^{*}$ and $(A\eta Q)^{*}$ denote the composition with $A\eta$ resp. $A\eta Q$.
All told, to the diagram I in (3.9) corresponds via ϕ the diagram I in (3.10).
Following the similar procedure we can show that $(\tau)\phi = \rho : X \xrightarrow{\bullet} L$ where $\rho = X\eta^{\bullet}\circ\tau Q^{\bullet}$ and

$$(X^{@}XQ \xrightarrow{\mu_0 Q} X^{@}Q \xrightarrow{\bar{\tau}} Q\bar{T}) \xrightarrow{\phi} (X^{@}X \xrightarrow{\mu_0} T \xrightarrow{\bar{\rho}} L) \qquad (3.17)$$

Finally, using the formula (1.21) and the fact that ϕ is natural not only in A but also in B ([8] p. 78) we get the formula dual to (3.16) which says:

$$\begin{array}{c} (X^{@}XQ \xrightarrow{X^{@}\tau} X^{@}Q\bar{T} \xrightarrow{\bar{\tau}\bar{T}} Q\bar{T}\bar{T} \xrightarrow{Q\bar{\mu}} A\bar{T}) \\ \downarrow \phi \\ (X^{@}X \xrightarrow{X^{@}\rho} X^{@}Q\bar{T}Q^{\cdot} \xrightarrow{\bar{\tau}\bar{T}Q^{\cdot}} Q\bar{T}\bar{T}Q^{\cdot} \xrightarrow{Q\bar{\mu}Q^{\cdot}} Q\bar{T}Q^{\cdot}) \end{array} \qquad (3.18)$$

But then we observe that in

$$\begin{array}{ccccc} X^{@}Q\bar{T}Q^{\cdot} & & \xrightarrow{\rho Q\bar{T}Q^{\cdot}=\rho L} & & Q\bar{T}Q^{\cdot}Q\bar{T}Q^{\cdot} \\ & {\scriptstyle \bar{\tau}\bar{T}Q^{\cdot}} \searrow & \text{I} & \swarrow {\scriptstyle Q\bar{T}\varepsilon^{\cdot}\bar{T}Q^{\cdot}} & \downarrow {\scriptstyle \mu'} \\ & & Q\bar{T}\bar{T}Q^{\cdot} & \text{II} & \\ & & & {\scriptstyle Q\bar{\mu}Q^{\cdot}} \searrow & \\ & & & & Q\bar{T}Q^{\cdot}=L \end{array} \qquad (3.19)$$

I commutes by the second formula in (1.21) and II by the formula for μ' in (3.13)

$\mu' = Q\bar{F}\varepsilon'\bar{U}Q^{\cdot} = Q\bar{F}(\bar{U}\varepsilon^{\cdot}\bar{F}\circ\bar{\varepsilon})\bar{U}Q^{\cdot} = Q\bar{F}\bar{U}\varepsilon'\bar{F}\bar{U}Q^{\cdot}\circ Q\bar{F}\bar{\varepsilon}\bar{U}Q^{\cdot} = Q\bar{T}\varepsilon^{\cdot}\bar{T}Q^{\cdot}\circ Q\bar{\mu}Q^{\cdot}$ exactly as in II above since $\bar{\mu} = \bar{F}\bar{\varepsilon}\bar{U}$ according to (1.24).

By (3.17), (3.18) and (3.19) we showed that every path of the diagram II in (3.9) is mapped into the corresponding path of the diagram II in (3.10). Observe that from II in (3.10) we get (3.11) as explained before. Using the fact that ϕ is natural in both A and B the reader can easily show that it maps a natural transformation to a natural transformation.

The theorem that we just proved tells us, in particular, that (3.3) does define a unique extension $\bar{\tau}$ of τ. Also, it says that to every top-down f.s.t. corresponds a pure transformation, which is not a treetransformation any more, in the sense that its domain is not the set of trees. The assumption that Q has a right adjoint allowed us to extend uniquely $\tau : XQ \xrightarrow{\cdot} Q\bar{T}$ to $\bar{\tau} : X^{@}Q \xrightarrow{\cdot} Q\bar{T}$. But if we have $\bar{\tau} : TQ \xrightarrow{\cdot} Q\bar{T}$ available, we do not need this assumption, neither we have to require the freeness of the input monad.

Hence the definition:

(3.20) <u>Definition</u>

For $\langle T,\eta,\mu\rangle$ and $\langle \bar{T},\bar{\eta},\bar{\mu}\rangle$ monads in $\mathcal{K}$ and $Q : \mathcal{K} \to \mathcal{K}$, an <u>inverse state transformation</u> of the above monads is a natural transformation

$$\bar{\tau} : TQ \xrightarrow{\cdot} Q\bar{T}$$

such that

$$\begin{array}{ccc} Q & \xrightarrow{\eta Q} & TQ \\ & \searrow^{Q\bar{\eta}} & \downarrow{\bar{\tau}} \\ & & Q\bar{T} \end{array} \qquad \begin{array}{ccccc} TTQ & & \xrightarrow{\mu Q} & & TQ \\ \downarrow{T\bar{\tau}} & & & & \downarrow{\bar{\tau}} \\ TQ\bar{T} & \xrightarrow{\bar{\tau}\bar{T}} & Q\bar{T}\bar{T} & \xrightarrow{Q\bar{\mu}} & Q\bar{T} \end{array}$$

Just as in the case of direct state transformations, we have:

(3.21) <u>Corollary</u>

The state transformation on the free monad $\bar{\tau} : X^{@}Q \xrightarrow{\cdot} Q\bar{T}$ is an inverse state transformation: $\langle X^{@},\eta,\mu\rangle \to \langle \bar{T},\bar{\eta},\bar{\mu}\rangle$

Proof: Following the same strategy as in theorem (3.9) prove that ϕ maps the diagram

$$\begin{array}{ccccc} X^{@}X^{@}Q & & \xrightarrow{\mu Q} & & TQ \\ \downarrow{X^{@}\bar{\tau}} & & & & \downarrow{\bar{\tau}} \\ X^{@}Q\bar{T} & \xrightarrow{\bar{\tau}\bar{T}} & Q\bar{T}\bar{T} & \xrightarrow{Q\bar{\mu}} & Q\bar{T} \end{array}$$

into the diagram

$$\begin{array}{ccc} X^{@}X^{@} & \xrightarrow{\mu} & X^{@} \\ \downarrow{\bar{\rho}\bar{\rho}} & & \downarrow{\bar{\rho}} \\ LL & \xrightarrow{\mu'} & L \end{array}$$

Indeed, from (3.14) using the same argument as in (3.18) and (3.19) we prove

$$\begin{array}{c}(X^@X^@Q \xrightarrow{X^@\bar{\tau}} X^@Q\bar{T} \xrightarrow{\bar{\tau}\bar{T}} Q\bar{T}\bar{T} \xrightarrow{Q\bar{\mu}} Q\bar{T}) \\ \updownarrow \phi \\ (X^@X^@ \xrightarrow{X^@\bar{\rho}} X^@L \xrightarrow{\bar{\rho}L} LL \xrightarrow{\mu'} L)\end{array}$$

and also

$$(X^@X^@Q \xrightarrow{\mu Q} X^@Q \xrightarrow{\bar{\tau}} Q\bar{T}) \xrightarrow{\phi} (X^@X^@ \xrightarrow{\mu} T \xrightarrow{\bar{\rho}} L)$$

But then the proof of the corollary follows from (2.26) and (2.32).

In the case of top-down f.s.t. we get the result that corresponds to Thatcher [13] p. 353 lemma 6.7. The diagram for that case is

$$\begin{array}{ccccc} ZTTxQ & & \xrightarrow{Z\mu x1} & & ZTxQ \\ \downarrow{\scriptstyle ZT\bar{\tau}} & & & & \downarrow{\scriptstyle Z\bar{\tau}} \\ (ZTxQ)\bar{T} & \xrightarrow{Z\bar{\tau}\bar{T}} & (ZxQ)\bar{T}\bar{T} & \xrightarrow{(Zx1)\bar{\mu}} & (ZxQ)\bar{T} \end{array}$$

(3.22) <u>Definition</u>

Let $<T,\eta,\mu>$, $<K,\bar{\eta},\bar{\mu}>$, $<L,\eta',\mu'>$ be monads in $\mathcal{K}$ and let

$$\bar{\tau} : TQ \xrightarrow{\bullet} QK$$

$$\bar{\rho} : KS \xrightarrow{\bullet} SL$$

be inverse state transformations. Define the <u>composite</u> transformation $\bar{\tau}\circ\bar{\rho}$ by

$$\begin{array}{ccc} TSQ & \xrightarrow{\bar{\tau}Q} & SKQ \\ & \searrow^{\bar{\tau}\circ\bar{\rho}} & \downarrow{\scriptstyle S\bar{\rho}} \\ & & SQL \end{array}$$

As in the case of direct transformations, this definition gives rise to a category with objects $<T,\eta,\mu>$, $<K,\bar{\eta},\bar{\mu}>$, $\cdots$ and morphisms

$\bar{\tau} : \langle T,\eta,\mu\rangle \to \langle K,\bar{\eta},\bar{\mu}\rangle$ inverse state transformations. This definition gives, as a particular case, the standard definition of composition of finite state transformations ([13] p. 354). As in (2.32) inverse pure state transformations become morphisms of monads and in particular, pure f.s.t. morphisms of tree monads.

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