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If Al is taller than Bill and Bill is taller than Charlie, we may conclude that Al is taller than Charlie. This fact is abstracted mathematically by the statement that the relation "is taller than" is a transitive relation. Many other relations are also transitive: e.g., "greater than", "less than", "is isomorphic to", and "equals".

Certainly, if all relations were transitive, it would not be an interesting property to study. The relation "does not divide" (written $\nmid$) is not transitive, for from the facts $3 \nmid 5$ and $5 \nmid 12$, it does not follow that $3 \nmid 12$.

Intuitively one feels that relations having to do with dominance like "is better than" or "wins at chess from" or "is wiser than" should be transitive. But they are not, and this is surprising. Stories abound of chess masters who can beat everybody but a certain nemesis. This nemesis may be a rather second-rate player and be beaten regularly by many of the players that the chess master beats. Thus we have a case where A (the master) beats B and B beats C (the nemesis), but C beats A.

This example is to a certain extent unsatisfying because the reasons for it are unclear, or at least the problem may not be mathematical. Perhaps the fault lies in some imprecision in the definitions. So let us consider a better defined situation, one involving objects and events that can be described in detail, and one which involves some interesting mathematics.

We will consider a game between two players involving three dice, colored red, white, and blue for purposes of identification. Each player chooses a die and rolls it, and the one who rolls the higher number wins. The dice have been specially made for the game, and each face has an integer between one and nine on it; opposite faces of each die are identical; and the dice are fair in the sense that each side is equally likely. Surprisingly, this game, which sounds perfectly fair, can be rigged in such a way that the player who chooses the first die will lose an average of five out of nine games. Let us see how this is possible. The three dice have the following distinct numbers on their faces (each repeated twice, recall).

red:	1	5	9
white:	3	4	8
blue:	2	6	7

If the first player chooses blue, the second chooses white; if the first chooses white, the second chooses red; and if the first chooses red, the second chooses blue. Look at the possible outcomes of rolling two dice in each case (asterisks mark those for which the second player wins):

blue	white	white	red	red	blue
2	3*	3	1	1	2*
2	4*	3	5*	1	6*
2	8*	3	9*	1	7*
6	3	4	1	5	2
6	4	4	5*	5	6*
6	8*	4	9*	5	7*
7	3	8	1	9	2
7	4	8	5	9	6
7	8*	8	9*	9	7
5 wins for white		5 wins for red		5 wins for blue	

The "dominant" die wins five out of the nine possible rolls, and thus whichever die the first player chooses, there remains a "better" die for the second player to choose.

In two recent columns [1, 2] Martin Gardner has presented a number of other non-transitive games and situations. In particular there is what is known as the voters' paradox in which the voters when presented with pair-wise choices prefer A to B, B to C, and C to A. This is sometimes called the cyclic majority problem. This problem has been studied by several people [3, 4], and Usiskin shows that as the number of candidates increases without bound there is a limit to how many of the voters prefer A to B, B to C, etc. (assuming that all such preferences are equal). This limit is 3/4 of the voters preferring the stronger candidate and 1/4 the weaker. For small numbers N of candidates, Usiskin derives values for P the maximum preference: for N = 3, P = .618; for N = 4, P = 2/3; for N = 10, P = .732.

In his column [1] Gardner notes that Efron was the first to point out the symmetry between the cyclic majority paradox and the non-transitive dominance relation among dice, with dice playing the role of candidates and the odds in favor of the dominant die being equivalent to the preference for the stronger candidate of each pair.

In this paper we will do two things: first we will show that as the number of sides per die increases there is also a bound on the odds in favor of the dominant die and that this bound tends toward 3/4 as the number of sides goes to infinity. Second, we will present an algorithm for finding sets of dice with d dice of s sides each.

The Odds as a Function of the Number of Sides

Standard dice are made of cubes with 6 sides. Other regular solids have 4, 8, 12 and 20 sides. Given N, an integer greater than 2, one might be able to construct a solid with N sides each equally likely to "come up". Rather simpler is the resort to an old child's toy called a dreidel. Historically a dreidel consists of a top with four flat faces each bearing one of four letters of the Hebrew alphabet. This top is spun and when it settles, one of the four faces is uppermost. It is easy to conceive of an N-sided dreidel. This requires that for N odd one either reads the face touching the floor, or better, replaces the N-agon by a 2N-agon and duplicates each face, retaining thus N different faces, each appearing exactly twice.

To shorten up the following discussion we present a few definitions and conventions:

1. The dreidels are numbered 1, 2, ..., d, and when considered as pairs i dominates i + 1 where arithmetic is done cyclicly so that d - 1 dominates d and d dominates 1, etc.
2. Each of the d·s faces will contain a unique integer drawn from the set {1, ..., d·s}.
3. The number on the j^{th} face of the i^{th} dreidel will be symbolized by $A_{i,j}$. We agree to order the faces on each dreidel in monotonically increasing order so that $A_{i,j} < A_{i,j+1}$ for all i and all j. Sometimes we will refer to the i^{th} dreidel as A_i.
4. x_i is defined to be the number of different rolls of dreidels i and i + 1 such that i wins from i + 1 (i.e., x_i is the number of pairs $\langle j, k \rangle$ such that $A_{i,j} > A_{i+1,k}$). Since there are s^2 possible rolls of two s-sided dreidels, the probability that i wins is x_i/s^2.
5. Since the first player (otherwise known as the shill) is presumed to be intelligent and is free to pick any dreidel he wishes, we would like to make the smallest of the x_i's as large as possible. Usiskin shows, and a moments thought will convince the reader, that this can still occur when all the x_i's are equal.
6. $\alpha_{i,j}$ is defined as the number of faces of dreidel i + 1 that the j^{th} face of dreidel i is larger than. It is the number of ways that i can still win from i + 1 given that the i^{th} dreidel came up $A_{i,j}$. We have immediately that

$$x_i = \sum_{j=1}^{s} \alpha_{i,j} = \alpha_{i,1} + \alpha_{i,2} + \dots + \alpha_{i,s}$$

Further we know that if side j of dreidel i is larger than α_{ij} faces of dreidel i + 1 then side j + 1 of dreidel i must be larger than at least α_{ij} faces of dreidel i + 1, ($\alpha_{i,j+1} \geq \alpha_{1,j}$) since $A_{i,j+1} > A_{i,j}$ by assumption 3.

Now we are ready to find the bounds on x_i as a function of s.

Consider any column j in the table A_{ij}. For some adjacent pair (i, i + 1) it must be the case that $A_{i,j} < A_{i+1,j}$ (because < is transitive and each of $\{A_{1,j}, \dots, A_{d,j}\}$ is distinct). Let us refer to dreidels A_i and A_{i+1} as B and C to simplify subscripts. For this pair, the maximum possible advantage of B over C would be achieved if all the sides $B_{j+1}, \dots, B_s$ were greater than all the sides of C. This would allow B to win over C in s(s - j) of the possible spins. Furthermore, to maximize the chance of B winning over C, $B_1, \dots, B_j$ should be chosen large enough to win against $C_1, \dots, C_{j-1}$ even though they must loose to $C_j, \dots, C_s$; in short, they can account for j(j - 1) winning spins. Adding these together, we see that the advantage x of B over C is bounded by

$$x \leq s(s - j) + j(j - 1) = s^2 - s_j + j^2 - j$$

and this bound must hold for each column j. We now ask for which column j does this analysis give the smallest x? Simple calculus and some integer arithmetic shows that the value $j = \left\lceil \frac{s}{2} \right\rceil$ is the most constraining (where $\lceil z \rceil$ is the smallest integer $\geq$ z). This gives a bound on x of $x \leq \left\lfloor \frac{3s^2 - 2s}{4} \right\rfloor$ (where $\lfloor z \rfloor$ is the largest integer $\leq$ z). (The reader is warned that there are two cases to consider: one for even s and another for odd s; the formula holds for both cases.)

Table 1 displays the probability that dreidel i will win from dreidel i + 1. This probability, p, is found by dividing x by s^2; it approaches 3/4 as s approaches infinity, which is the same limit found by Usiskin as d approaches infinity. We will show by construction that this bound on x can be reached when d is large enough so this is a tight bound. For the general set of d dreidels with s sides the smaller of the two bounds (ours and Usiskin's) applies.

Table 1

Maximum probability of winning (p) for various numbers of sides (s)

s	p	
2	1/2	.500
3	5/9	.555
4	5/8	.625
5	16/25	.640
6	2/3	.666
10	7/10	.700
∞	3/4	.750

Construction of Dreidels

One may now ask how to construct all sets of d dreidels with s sides for which dreidel i will win x out of the s^2 possible rolls with dreidel i + 1. The construction consists of three phases: first, generate all appropriate partitions of x; second, build advantage tables from the partitions; and third, attempt to construct sets of dreidels corresponding to advantage tables.

The easiest way to understand the construction is to work backwards from a set of dreidels. Consider the set of three four-sided dreidels with faces

$$A_1 = 1 \quad 7 \quad 8 \quad 10$$
$$A_2 = 4 \quad 5 \quad 6 \quad 11$$
$$A_3 = 2 \quad 3 \quad 9 \quad 12$$

For this set of dreidels, A_i wins 9 of the 16 equally likely rolls with A_{i+1}. For example, face one of A_1 dominates no face of A_2, but the other faces of A_1 each dominate three faces of A_2. Similarly, the first three faces of A_2 dominate only two faces of A_3, and the fourth face of A_2 dominates three faces of A_3. This information can be succinctly summarized in what we call an advantage table. For this example, the advantage table is:

0	3	3	3
2	2	2	3
1	1	3	4

Since A_i was constructed to win 9 of the 16 equally likely rolls with A_{i+1}, each row must add to 9. Since each entry in the table represents the number of faces of a dreidel dominated by a certain face, no number in the table may be larger than 4, the number of sides on a dreidel in this example. Also, since we list the sides of dreidels in increasing order, the entries in the rows of an advantage table must be in non-decreasing order (see assumption 6 above). Finally, note that in column j the smallest entry must be less than j. To see that this is true, consider the numbers on the faces of the set of dreidels (the $A_{i,j}$'s) as a matrix. The smallest number in column j of this matrix cannot dominate more than j - 1 faces of its successor dreidel, for if it did, then the number for face j of the successor dreidel would have to be smaller than this number which was assumed the smallest in column j; a contradiction.

Note that the choice of which dreidel to label A_1 is arbitrary; but once that choice is made, the other names (A_2 and A_3) are given by the dominance relations that obtain.

For the case of d dreidels with s sides and an advantage of x out of s^2 between adjacent dreidels, the rows of an advantage table consist of partitions of x into s integral parts arranged from left-to-right in non-decreasing order, where no part is less than zero or greater than s. The partitions of 9 into 4 parts subject to these constraints are:

$P_1 = 0\ \ 1\ \ 4\ \ 4$

$P_2 = 0\ \ 2\ \ 3\ \ 4$

$P_3 = 0\ \ 3\ \ 3\ \ 3$

$P_4 = 1\ \ 1\ \ 3\ \ 4$

$P_5 = 1\ \ 2\ \ 2\ \ 4$

$P_6 = 1\ \ 2\ \ 3\ \ 3$

$P_7 = 2\ \ 2\ \ 2\ \ 3$

(An interesting side problem, not explored here, is how many such partitions of x exist.)

One can now easily list all matrices that could be advantage tables for d dreidels selecting d (not necessarily distinct) partitions of x as rows, subject to the constraint that column j of the matrix have an entry which is less than j. In listing all such matrices, we are not interested in those whose rows are cyclic permutations of one another. Thus, if the matrix with rows corresponding to partitions P_1, P_3, P_5 were listed, the matrices P_3, P_5, P_1 and P_5, P_1, P_3 should be omitted. When this is done for the example we have been considering, twenty-two potential advantage tables are found.

Now that we see how to generate possible advantage tables, we wish a procedure that will, if possible, assign integers to the sides of the dreidels, and if this is not possible, will tell us so as quickly as may be. We present an algorithm that accomplishes this with reasonable dispatch.

With each row i of an advantage table we associate a counter C_i and a pointer L_i which tells us at which element of the row the counter is pointing. All counters are initialized to 0 and all pointers to 1, so that every counter points at the left most element of its row. A counter is said to be "satisfied" if the number it holds (the contents of C_i) is equal to the element of the advantage table at which it points.

Let N be an integer initialized to 1. The algorithm has 6 steps:

1. Find a counter C_i such that C_i is satisfied and C_{i-1} is not satisfied. If no such counter exists, no set of dreidels exists for this advantage table, and the algorithm halts.
2. Assign the integer N to the side corresponding to the element at which this counter is pointing. (That is, if $L_i = j$, assign N to side $A_{i,j}$.)
3. Increment N by 1.

4. Increment L_i by 1. (Move counter i one position to the right.)

5. Increment C_{i-1} by 1 (increment the unsatisfied counter above C_i).

6. Repeat steps 1 - 5 until either all counters move off the right end of their rows ("moved off" counters are considered to be not satisfied) or the algorithm halts in step 1.

Note that L_i is always exactly one greater than C_{i-1} for all i, but the algorithm seems easier to understand if we introduce the pointers explicitly.

Each pass through the algorithm assigns a number to one side of one of the dreidels so in d·s passes we are guaranteed to exit from step 6 unless we fail earlier.

Because in step 5 we increment the counter of the row preceding the row to which we assign the integer N, each counter tallies the number of sides on the succeeding dreidel that have been "taken care of" (by having integers assigned to them). Since we are assigning integers in increasing order, we cannot assign a value to face j of dreidel i until α_{ij} sides of its successor have been assigned smaller integers, where (α_{ij}) is the advantage table. Thus it is that we wish to work with satisfied counters. But now consider two successive satisfied counters. If the lower one is processed first, it will mean that another side of the second of the pair of dreidels will be assigned. But since the upper counter is satisfied, we know that we have already assigned just exactly enough sides to the second dreidel. Consequently, we look for a satisfied counter C_i such that C_{i-1} is not satisfied. Figure 1 shows the application of this algorithm to the advantage table shown. Stars are used to mark satisfied counters.

We have programmed this algorithm in APL (see Figure 2). The inputs are simply ADVANTAGETABLE, the matrix (α_{ij}); D, the number of dreidels; and SIDES, the number of sides. The output consists of a matrix FACES, containing in the i,jth position the number of the jth face of the ith dreidel, and a logical flag SUCCESS which is set to 1 if a set of dreidels is constructed and to 0 if none is. This flag is used by the output routine PRINT which displays the dreidels.

Advantage Table

P_1 =	0	1	4	4
P_5 =	1	2	2	4
P_6 =	1	2	3	3

Faces of Dreidels

D_1 =	0	3		
D_2 =	2	5	6	
D_3 =	1	4		

Counters and their Pointers at the beginning of each Pass

C_1	0*	0	0	1*	1	1	2	3
L_1	1	2	2	2	3	3	3	3
C_2	0	0	1*	1	1	2*	2*	2
L_2	1	1	1	2	2	2	3	4
C_3	0	1*	1	1	2*	2	2	2
L_3	1	1	2	2	2	3	3	3
Pass	1	2	3	4	5	6	7	8

Figure 1. A time history of the assignment algorithm. In step 8 no satisfied counter is found so the algorithm fails.

```
     ∇ DREIDEL ADVANTAGETABLE;AT;FACES;CANDIDATE;POINTER;COUNTER;VALUE;SUCCESS
[1]    AT←ADVANTAGETABLE,Dρ¯1
[2]    VALUE←0
[3]    COUNTER←Dρ0
[4]    POINTER←Dρ1
[5]    FACES←(D,SIDES)ρ¯1
[6]   NEXT:FIND COUNTER
[7]    →OUTPUT IF~SUCCESS
[8]    FACES[CANDIDATE;POINTER[CANDIDATE]]←VALUE←VALUE+1
[9]    POINTER[CANDIDATE]←POINTER[CANDIDATE]+1
[10]   COUNTER[ABOVE CANDIDATE]←COUNTER[ABOVE CANDIDATE]+1
[11]   →NEXT IF(∨/POINTER≤SIDES)
[12]  OUTPUT:PRINT
     ∇


     ∇ FIND COUNTER
[1]    CANDIDATE←(((POINTER-1)⌽AT)[;1]=COUNTER)/⍳D
[2]    →FAILURE IF 0=ρCANDIDATE
[3]    SUCCESS←1
[4]    CANDIDATE←CANDIDATE[1]BACKUP 1↓CANDIDATE
[5]    →0
[6]   FAILURE:SUCCESS←0
     ∇


     ∇ B←C BACKUP L
[1]    B←C
[2]    →0 IF~(ABOVE C)∊L
[3]    B←(ABOVE C)BACKUP(L≠ABOVE C)/L
     ∇


     ∇ S←ABOVE R
[1]    S←1+D|R-2
     ∇


     ∇ L←A IF B
[1]    L←B/A
     ∇
```

Figure 2

We used this routine, together with several others, to generate advantage tables, etc., to try to construct sets of dreidels with up to seven sides. Some of the results are tabulated in Figure 3.

Compression

The set of four six-sided dreidels (dice) presented in Figure 3 is claimed to be the only set with an advantage of 2/3 and yet they look almost nothing like Efron's set presented by Gardner with the same advantage. The reason for this is that we have chosen to make each side unique. To convert our set of four six-sided dreidels to Efron's dice we employ a technique called compression. If on one particular dreidel two successive integers (N and N + 1) appear, replace N + 1 by N, N + 2 by N + 1 and so on for all integers (on all the dreidels) larger than N. Repeat until no successive integers appear on the same dreidel. This converts the set given in Figure 3 to:

1	1	5	5	5	5
4	4	4	4	4	4
3	3	3	3	7	7
2	2	2	6	6	6

which is the same as Efron's, except that we start with 1, and he starts with 0.

Number of Sides	Number of Dreidels	Advantage	Number of Sets of Dreidels	First Advantage Table	Faces of First Set of Dreidels
3	3	5/9	2	0 2 3 1 1 3 1 2 2	1 5 9 3 4 8 2 6 7
4	3	10/16	0	-----	-----
4	3	9/16	6	0 1 4 4 0 3 3 3 1 2 2 4	1 4 10 11 2 7 8 9 3 5 6 12
4	4	10/16	5	0 2 4 4 1 3 3 3 2 2 2 4 1 1 4 4	1 9 12 13 5 8 10 11 4 6 7 16 2 3 14 15
5	4	16/25	4	0 1 5 5 5 0 4 4 4 4 2 3 3 3 5 2 2 2 5 5	1 3 15 16 17 2 11 12 13 14 6 8 9 10 20 4 5 7 18 19
5	3	15/25	2	0 0 5 5 5 3 3 3 3 3 1 2 2 5 5	1 3 11 12 13 6 7 8 9 10 2 4 5 14 15
5	3	16/25	0	-----	-----
6	4	24/36	1	0 0 6 6 6 6 4 4 4 4 4 4 3 3 3 3 6 6 2 2 2 6 6 6	1 2 16 17 18 19 10 11 12 13 14 15 6 7 8 9 23 24 3 4 5 20 21 22
6	3	23/36	0	-----	-----
7	4	33/49	0	-----	-----
7	5	33/49	>16	0 0 5 7 7 7 7 3 5 5 5 5 5 5 3 4 4 4 4 7 7 3 3 3 3 7 7 7 1 2 2 7 7 7 7	1 3 20 23 24 25 26 13 16 17 18 19 21 22 9 11 12 14 15 34 35 6 7 8 10 31 32 33 2 4 5 27 28 29 30

Figure 3

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