DUALITY THEORY

FOR

DISCRETE-TIME LINEAR SYSTEMS

by

Stephen J. Hegner†

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UNIVERSITY OF MASSACHUSETTS AT AMHERST
AMHERST, MASSACHUSETTS 01003
U.S.A.

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†The author is currently with Lawrence Livermore Laboratory, University of California, P.O. Box 808, L-156, Livermore, California 94550.

ABSTRACT

Previous duality theories for discrete-time linear systems over a field K have been restricted to those cases in which the input, state, and output spaces are finite-dimensional. Direct attempts to extend such a theory to infinite-dimensional systems fail, because the category K-LS of linear spaces over the field K is not self-dual and hence does not, by itself, provide an adequate framework for a general duality theory of discrete linear systems. Instead, it is necessary to consider categories of linearly-topologized spaces over K, and to use topological rather than algebraic duals. Using this approach, the dimensionality of the system is of no consequence, and so finite- and infinite-dimensional systems are handled with equal ease.

A general categorical duality of discrete-time linear systems is first developed within the framework of the self-dual category K-DP of dual pairs over K, so that the essential character of the theory is algebraic rather than topological. K-DP is equivalent to sK-LTS, the category of weak linearly-topoligized spaces, and also to kK-LTS, the category of Mackey linearly-topologized spaces. This provides a linearly-topologized-space framework for discrete-time linear systems, with topological dualization the underlying duality functor.

Using the duality of maximal and minimal dual pairs, the category cK-LTS of linearly-compact linearly-topologized spaces is the proper framework for studying the duals of machines in K-LS. Again, topological dualization is the underlying duality functor.

DUALITY THEORY FOR DISCRETE-TIME LINEAR SYSTEMS

INTRODUCTION

Duality theory for finite-dimensional discrete linear systems has been around since at least 1960[7]. Since then, the theory of discrete-time linear systems has evolved to an elegant categorical approach over an arbitrary ring R, where the concept of dimensionality is not even meaningful[1]. Along with a categorical theory of systems comes a categorical theory of duality; $(R\text{-MOD})^{op}$ provides the mathematically-ideal framework in which to base the dual M^{op} of a discrete-time linear system M modelled in R-mod, the category of left R modules. However, from a structural point of view, little can be said about $(R\text{-MOD})^{op}$, in general, other than that it is the opposite of R-MOD. What is desired in a system-theoretic duality theory is an approach in which a linear system in a category K has a dual M' which is also a linear system in K, and which is equivalent (in a categorical sense) to M^{op}. Unfortunately, R-MOD is not in general a self-dual category; if R is a field, R-MOD is never self dual. For a field K, let K-LS be the category of K-linear spaces; K-LS = K-MOD. The full subcategory of K-LS consisting of the finite-dimensional linear spaces is self-dual, and this explains, at least in part, why concrete duality theories for linear systems have been, until now, restricted to those cases in which the input, state, and output spaces are finite-dimensional.

It is the purpose of this paper to extend the theory of duality of linear systems over a field $\mathbb{K}$ to those cases in which the input, state, and output spaces are not necessarily finite-dimensional. To do this, linearly-topologized spaces over $\mathbb{K}$ are used, rather than untopologized linear spaces. However, all of the topologies used are essentially algebraic in character, in that they are completely described by dual pairs. To emphasize this fact, the entire duality theory is first constructed within the category $\mathbb{K}$-$\mathbb{DP}$ of dual pairs over $\mathbb{K}$, and then converted, via equivalences of categories, to s$\mathbb{K}$-$\mathbb{LTS}$, the category of weak linearly-topologized spaces, and k$\mathbb{K}$-$\mathbb{LTS}$, the category of Mackey linearly-topologized spaces. $\mathbb{K}$-$\mathbb{DP}$ is self-dual, and so the dual system will lie in the same category as the original system. However, $\mathbb{K}$-$\mathbb{DP}$ is not balanced, and has many image-factorization systems. Consequently, there are several concepts of reachability and observability in systems so modelled. Three are discussed in this paper.

An alternative theory of duality in which the system is modelled directly in $\mathbb{K}$-$\mathbb{LS}$ is also presented. The dual system in this case is modelled not in $\mathbb{K}$-$\mathbb{LS}$, but in c$\mathbb{K}$-$\mathbb{LTS}$, the category of linearly-compact linearly-topologized spaces. This approach has the advantage that there is only one concept of reachability and observability.

The paper is divided into three parts. The first, §1-§3, deals with the general theory of duality of decomposable systems. The second, §4-§7, deals with particular models of duality. The third, §8, contains examples illustrating the theory. Some remarks on systems modelled in the category of Hilbert spaces, some remarks on the literature, and two appendices are also included.

While it is assumed that the reader is familiar with elementary linear algebra, category theory and topology, the treatment of linearly-topologized spaces is self-contained, with the reader referred to other sources only for motivation and proofs.

§0 Notation

The notation in this paper is drawn from several sources. The notation of [1] is adhered to whenever concepts from that paper are used. For categorical concepts not covered in [1], standard categorical notation, as can be found in [5], [13], and [15], is used. For concepts concerning topology, linear algebra, and linearly-topologized spaces, the text [11] is the reference. The following is a guide to the otherwise unexplained notation of this paper.

The symbol 1 usually means an identity (morphism, functor, etc.). When more specific notation is needed, a subscript is used. For example 1_A means the identity (morphism) on A; 1_{K} means the identity functor on the category K.

The symbol $\cong$ is shorthand for "is isomorphic to". The frame of reference is usually clear. In particular, when the symbol is used between functors it means that there is a natural isomorphism from one functor to the other.

The exponent op means opposite (dual) in categorical terms, and is generally used only when confusion would otherwise result. Thus, for example, K^{op} is the opposite category of K, and if F: $K \to H$ is a functor,

$F^{op}: K^{op} \to H^{op}$ is the opposite functor (which is identical to F as a function). The convention that K^{op} has the *same* objects and morphisms as K, as in [5] and [15], will be used.

If K is a category $Obj(K)$ denotes the class of all K objects, and $Mor(K)$ denotes the class of all K morphisms.

The symbol $\aleph_0$ denotes the first infinite cardinal and ω the first infinite ordinal.

Throughout this paper, K will denote an arbitrary (but fixed) field with $0 \neq 1$, unless specifically otherwise noted.

If $f\colon A \to B$ is a linear map of linear spaces, $\mathit{ker} f = \{a \in A \mid f(a) = 0\}$ (denoted N[A] in [11]). $\mathit{dim}(A)$ denotes the dimension of the linear space A (denoted d(A) in [11]).

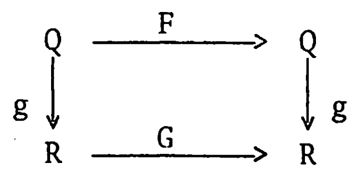
Within each section, formal facts are numbered consecutively, starting with 1. Within its section, reference to a fact is made by giving its number, e.g., (3) means fact 3 of the currect section. When a fact of another section is referenced, both the section number and the fact number are given, e.g., 2.(3) means the third fact of section 2.

§1 Review of Decomposable Systems

The system-theoretic framework of this paper is based upon the theory of decomposable systems of Arbib and Manes, as presented in [1]. It is presumed that the reader is familiar with the terminology and results presented there. However, in the interest of providing a unified presentation, a terse review of the concepts in [1] which are necessary to the theory presented here is given.

Throughout this section, fix a category K.

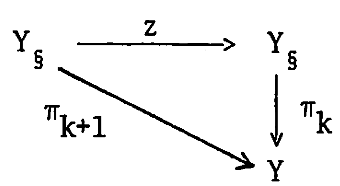
A <u>system dynamics</u> in K is an ordered pair (Q,F) where Q is a K object and $F: Q \to Q$ is a K-morphism. A K-morphism $g: Q \to R$ is called a dynamorphism for the system dynamics (Q,F), (R,G) provided that the diagram



commutes. $g: (Q,F) \to (R,G)$ is sometimes written.

A <u>decomposable system</u> in K is a 6-tuple $M = (Q,F,I,G,Y,H)$ such that (Q,F) is a system dynamics in K, I and Y are K objects such that I has a countable copower and Y has a countable power, and $G: I \to Q$ and $H: Q \to Y$ are K morphisms. Q is called the <u>state space</u>, F the <u>state-transition map</u>, I the <u>input space</u>, Y the <u>output space</u>, G the <u>input map</u>, and H the <u>output map</u> of M.

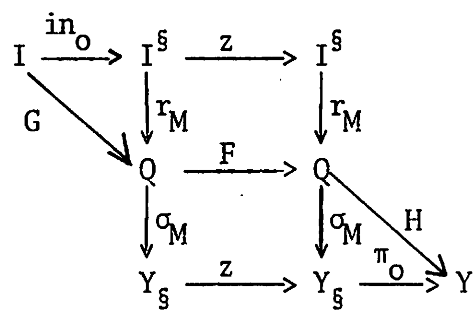
Let Y be a K object with a countable power $(Y_{\S}\{\pi_k | k\in\omega\})$. The <u>power-shift morphism</u> $z: Y_{\S} \to Y_{\S}$ is the unique morphism making the diagrams



$(k\in\omega)$

commute. For a K object I with a countable copower $(I^{\S},\{in_k | k\in\omega\})$, the <u>copower-shift morphism</u> is defined dually, and is also denoted z.

(1) *For every decomposable system* M = (Q,F,I,G,Y,H) *in K with* $(I^{\S},\{in_k|k\epsilon\omega\})$ *a countable copower for* I *and* $(Y_{\S},\{\pi_k|k\epsilon\omega\})$ *a countable power for* Y, *there are unique K morphisms* $r_M: I^{\S} \to Q$ *and* $\sigma_M: Q \to Y_{\S}$ *such that the diagram*



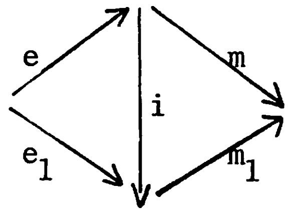
commutes.

Proof. Define r_M by $r_M \circ in_o = G$, $r_M \circ in_{k+1} = F^{k+1} \circ G$. Define σ_M dually. ∎

r_M is called the reachability map of M and σ_M is called the observability map of M. The total response of M is $\sigma_M \circ r_M$ and is denoted by $f_M^{\blacktriangle}$. r_M, σ_M, and $f_M^{\blacktriangle}$ are unique only up to a choice of countable copower for I and countable power for Y.

To deal with the problems of reachability and observability, the concept of image-factorization system is used.

An image-factorization system for K is an ordered pair $(\mathbb{E},\mathbb{M})$ such that $\mathbb{E}$ is a class of epimorphisms and $\mathbb{M}$ is a class of monomorphisms, each closed under composition and each containing all isomorphisms, such that each K morphism f has a factorization $f = m \circ e$ with $e \in \mathbb{E}$ and $m \in \mathbb{M}$ which is unique up to isomorphism in the sense that if $f = m_1 \circ e_1$ is another such factorization, then there is an isomorphism i such that the diagram



commutes. (e,m) is called an $(\mathcal{E},\mathcal{M})$-factorization of f.

Let M be a decomposable system in K, and let $(\mathcal{E},\mathcal{M})$ be an image-factorization system for K. M is $\mathcal{E}$-reachable if $r_M \in \mathcal{E}$, and M is $\mathcal{M}$-observable if $\sigma_M \in \mathcal{M}$.

Given K objects I and Y such that I has a countable copower and Y has a countable power, and a dynamorphism g: $(I^{\S},z) \to (Y_{\S},z)$, a realization of g is a decomposable system M in K such that the total response $f_M^{\blacktriangle} = g$. Given an image-factorization $(\mathcal{E},\mathcal{M})$, a realization of g which is both reachable and observable is called a $(\mathcal{E},\mathcal{M})$-canonical realization of g. Reachability, observability, and canonicity only depend upon the choice of $(\mathcal{E},\mathcal{M})$, and not on the choice of countable powers and countable copowers.

Let M = (Q,F,I,G,Y,H) be a decomposable system in K. M^{op} = (Q,F,Y,H,I,G) is called the dual system of M. The following are simple consequences of categorical duality.

(2) *Let* M *be a decomposable system in* K.

(a) M^{op} *is a decomposable system in* K^{op}.

(b) *The reachability map of* M *is the observability map of* M^{op}, *and vice-versa. The total response of* M *is the total response of* M^{op}.

(c) *If* $(\mathcal{E},\mathcal{M})$ *is an image-factorization system for* K, *then* $(\mathcal{E},\mathcal{M})$ *is an image-factorization system for* K^{op} *and* M *is* $\mathcal{E}$*-reachable if and only if* M^{op} *is* $\mathcal{E}$*-observable, and* M *is* $\mathcal{M}$*-observable if and only if* M^{op} *is* $\mathcal{M}$*-reachable.* ∎

In (b), the convention that the canonical copower of I in K and its power in K^{op} are the *same* (not just isomorphic) is followed; a similar convention is followed for Y.

§2 Theory of Equivalent Systems

Recall that a functor F: $K \to H$ is an equivalence of categories provided that there is a functor G: $H \to K$ such that $G \circ F \cong 1_K$ and $F \circ G \cong 1_H$. Note that the definition is symmetric, so that G is also an equivalence. Equivalences preserve all essential categorical properties. In fact, a property is called categorical provided that it is preserved by equivalences.

(1) *The following properties are categorical: epimorphism, monomorphism, countable power, countable copower, commutative diagram.*
Proof: Consult [5], 12.2, 12.8, 12.10, and 24.11. ∎

There is a useful interpretation of preservation of image-factorization systems under equivalence. Let K be a category, and let F be a class of K morphisms. The smallest class of morphisms containing F as well as all isomorphisms and which is closed under composition is called the closure of F and is denoted $\overline{\mathsf{F}}$.

(2) *Let* E: $K \to H$ *be an equivalence of categories.*

(a) *If* E *is the class of all* K *epimorphisms, then* $\overline{E(\mathsf{E})}$ *is the class of all* H *epimorphisms.*

(b) *If* M *is the class of all* K *monomorphisms, then* $\overline{E(\mathsf{M})}$ *is the class of all* H *monomorphisms.*
Proof: Consult Appendix 2. ∎

(3) *Let* F: $K \to H$ *and* G: $H \to K$ *be equivalences of categories with* $G \circ F \cong 1_K$ *and* $F \circ G \cong 1_H$.

(a) $(\overline{F(\mathcal{E})},\overline{F(\mathcal{M})})$ *is an image-factorization system for* H.

(b) $(G(\overline{F(\mathcal{E})}),G(\overline{F(\mathcal{M})})) = (\mathcal{E},\mathcal{M})$.

Proof: Consult Appendix 2. ∎

It is now possible to discuss the concept of equivalence of decomposable systems. In doing so, it is important to point out the need for specifying a particular choice for countable copower and countable power. For example, in the category of linear spaces over the field $\mathbb{K}$, for any object I, $I^{\S} \cong \{(i_0,i_1,i_2,\ldots) \mid i_k \in \mathbb{K}$ and only finitely many i_k nonzero$\}$ if I is finite-dimensional (nonzero), and $I^{\S} \cong I$ if I is infinite dimensional. The specific input space which is desired is $\{(i_0,i_1,i_2,\ldots) \mid i_k \in I$ and only finitely many nonzero$\}$, and neither of the two mentioned above (unless, by chance, they happen to coincide).

Let $\mathcal{Sys}(K)$ denote the <u>category of decomposable systems</u> in K. A morphism from $M_1 = (Q_1,F_1,I_1,G_1,Y_1,H_1)$ to $M_2 = (Q_2,F_2,I_2,G_2,Y_2,H_2)$ is an ordered triple of K morphisms ($a: I_1 \to I_2$, $b: Q_1 \to Q_2$, $c: Y_1 \to Y_2$) such that

$$\begin{array}{ccccccc} I_1 & \xrightarrow{G_1} & Q_1 & \xrightarrow{F_1} & Q_1 & \xrightarrow{H_1} & Y_1 \\ {\scriptstyle a}\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle c}\downarrow \\ I_2 & \xrightarrow{G_2} & Q_2 & \xrightarrow{F_2} & Q_2 & \xrightarrow{H_2} & Y_2 \end{array}$$

commutes. Note that the two systems are isomorphic if and only if a, b, and c are each K isomorphisms.

Let H be another category and let P: $K \to H$ be a functor which preserves countable powers and countable copowers. The commutativity of the above diagram implies the commutativity of

$$\begin{array}{ccccccc}
P(I_1) & \xrightarrow{P(G_1)} & P(Q_1) & \xrightarrow{P(F_1)} & P(Q_1) & \xrightarrow{P(H_1)} & P(Y_1) \\
{\scriptstyle P(a)}\downarrow & & {\scriptstyle P(b)}\downarrow & & {\scriptstyle P(b)}\downarrow & & {\scriptstyle P(c)}\downarrow \\
P(I_2) & \xrightarrow{P(G_2)} & P(Q_2) & \xrightarrow{P(F_2)} & P(Q_2) & \xrightarrow{P(H_2)} & P(Y_2)
\end{array} ,$$

$P(I_1)$ and $P(I_2)$ have countable copowers, and $P(Y_1)$ and $P(Y_2)$ have countable powers, so that P induces a functor, denoted $\check{P}$: $\mathit{Sys}(\mathcal{K}) \to \mathit{Sys}(\mathcal{H})$, given by $(Q,F,I,G,Y,H) \mapsto (P(Q),P(F),P(I),P(G),P(Y),P(H))$ on objects and $(a,b,c) \mapsto (P(a),P(b),P(c))$ on morphisms.

Let R: $\mathcal{K} \to \mathcal{H}$ be another functor which preserves countable powers and countable copowers and let τ: P $\to$ R be a natural transformation. Define $\check{\tau}$: $\mathit{Obj}(\mathit{Sys}(\mathcal{K})) \to \mathit{Mor}(\mathit{Sys}(\mathcal{H}))$ by $(Q,F,I,G,Y,H) \mapsto (\tau(I),\tau(Q),\tau(Y))$. The verification of the following is routine.

(4) *Let* P *and* R *be functors which preserve countable powers and countable copowers and let* τ: P $\to$ R *be a natural transformation.*

(a) $\check{\tau}$ *is a natural transformation from* P *to* R.

(b) *If* τ *is a natural isomorphism, so is* $\check{\tau}$. ∎

The above result is significant because it leads to the following.

(5) *Let* P *be a functor. If* P *is an equivalence* $\check{P}$ *exists and is also an equivalence.*

Proof: It suffices to note that by (1), equivalences preserve countable powers and countable copowers, and then to apply (4). ∎

If $M = (Q,F,I,G,Y,H)$ is a decomposable system in K and $E\colon K \to H$ is an equivalence, then in particular $E(M)$ is a decomposable system in H and the diagram

$$
\begin{array}{ccccccc}
E(I) & \xrightarrow{E(in_o)} & E(I^{\S}) & \xrightarrow{E(z)} & E(I^{\S}) & & \\
 & \searrow^{E(G)} & \downarrow{\scriptstyle E(r_M)} & & \downarrow{\scriptstyle E(r_M)} & & \\
 & & E(Q) & \xrightarrow{E(F)} & E(Q) & \searrow^{E(H)} & \\
 & & \downarrow{\scriptstyle E(\sigma_M)} & & \downarrow{\scriptstyle E(\sigma_M)} & & \\
 & & E(Y_{\S}) & \xrightarrow{E(z)} & E(Y_{\S}) & \xrightarrow{E(\pi_o)} & E(Y)
\end{array}
$$

commutes. Thus, if $(E(I^{\S}), \{E(in_k) | k\epsilon\omega\})$ is regarded as the *canonical* countable copower of $E(I)$ in H and $(E(Y_{\S}), \{E(\pi_k) | k\epsilon\omega\})$ is regarded as the *canonical* countable power of $E(Y)$ in H, then the reachability map of $E(M)$ *is* $E(r_M)$ and the observability map of $E(M)$ *is* $E(\sigma_M)$. Note that in general the reachability (resp. observability) map of $E(M)$ is unique only up to isomorphism, and truly unique only after choice of a canonical countable copower for $E(I)$ (resp. countable power for $E(Y)$). In any case, the following is always true.

(6) *Let* $E\colon K \to H$ *be an equivalence, and let* M *be a decomposable system in* K.

(a) $\breve{E}(M)$ *is* $\overline{E(\mathcal{E})}$*-reachable if and only if* M *is* $\mathcal{E}$*-reachable.*

(b) $\breve{E}(M)$ *is* $\overline{E(\mathcal{M})}$*-observable if and only if* M *is* $\mathcal{M}$*-observable.*

(c) $\breve{E}(M)$ *is* $(\overline{E(\mathcal{E})},\overline{E(\mathcal{M})})$*-canonical if and only if* M *is* $(\mathcal{E},\mathcal{M})$*-canonical.*

Proof: This follows immediately from (1), (3), and (5). ∎

This shows that all essential properties of decomposable systems are invariant under transformation by equivalence. Equivalence provides the essential machinery for transformation of a system to one which is categorically the same. A most important case of this is the following.

Let K and H be categories. A dual equivalence of K and H is a pair (F,G) where F: $K^{op} \to H$ and G: $H^{op} \to K$ are equivalences such that $G \circ F^{op} \cong 1_K$ and $F \circ G^{op} \cong 1_H$. If $F \circ G^{op} = 1_K$ and $G \circ F^{op} = 1_H$, (F,G) is called a dual isomorphism of K and H.

The dual equivalence is used to remove from abstraction the idea of the dual machine. Let M be a decomposable system in K, and let (F,G) be a dual equivalence of K and H. The abstract dual M^{op} of M is converted to the equivalent system $\check{F}(M)$ in H.

The concept of dual equivalence is completely symmetric, since a functor F: $K^{op} \to H$ is also a functor F: $K \to H^{op}$, etc. If N is a decomposable system in H, then its abstract dual N^{op} is equivalent to G(N).

§3 Duality in Linear Systems

As noted above, the concept of dual equivalence permits the replacement of the abstract dual machine M^{op} in the category K^{op} by an equivalent machine in another category H. The particular case of linear spaces is the subject of this section. $\mathcal{K}$-$\mathcal{LS}$ denotes the category of all $\mathcal{K}$-linear spaces (hereafter called just linear spaces) over the field $\mathcal{K}$, with $\mathcal{K}$-linear maps (hereafter just linear maps) as morphisms. It will now be shown that $\mathcal{K}$-$\mathcal{LS}$ is not self-dual, i.e., there is no dual equivalence of $\mathcal{K}$-$\mathcal{LS}$ with itself.

Familiarity with the structure of powers and copowers in $\mathbb{K}$-LS is presumed. Briefly, the d^{th} power of A, A^d (d a cardinal), is just the cartesian product of d copies of A with componentwise addition and scalar multiplication. The d^{th} copower of A, ${}^{d}A$, is the subspace of A^d consisting of precisely those vectors for which all but finitely many of the projections are 0.

(1) *Let* d *be an arbitrary cardinal.*

(a) *The dimension of* ${}^{d}\mathbb{K}$ *is* d.

(b) *If* d *is infinite, the dimension of* $\mathbb{K}^d$ *is* k^d*, where* k *is the cardinality of* $\mathbb{K}$.

Proof: (a) is obvious. For (b), consult [11], §9.5(3). ∎

(2)(a) *Every* $\mathbb{K}$*-linear space is isomorphic to a direct sum (copower) of copies of* $\mathbb{K}$.

(b) *There exist* $\mathbb{K}$*-linear spaces which are not isomorphic to a product (power) of copies of* $\mathbb{K}$.

Proof: (a) is immediate from (1a), since linear spaces with the same dimension are isomorphic.

(b) follows from (1b), since the cardinality k of $\mathbb{K}$ is always at least 2, and so $k^d > \aleph_0$ for infinite d. Since the dimension of $\mathbb{K}^d$ for finite d is clearly d, it follows that no power of $\mathbb{K}$ can have dimension $\aleph_0$, and so spaces of such dimension are not isomorphic to any power of $\mathbb{K}$. ∎

(3) $\mathbb{K}$-LS *is not self dual.*

Proof: Power and copower are clearly dual concepts; they are also categorical ([5],24.11). By (2a), every $\mathbb{K}$-linear space is isomorphic to a power of copies of $\mathbb{K}$. However, there is no linear space A such that every linear space is isomorphic to a product of A. This is because, by 2.(6b), $\mathbb{K}$ will not work, yet $\mathbb{K}$ (or an isomorphic copy) is the only possibility, for $\mathbb{K}$ is clearly not isomorphic to any power of any space of dimension greater than 1. Hence power and copower do not have dual properties, and so $\mathbb{K}$-$\mathbb{LS}$ is not self dual. ∎

Hence, some other framework must be used for the modelling of the duals of linear systems. Recall that a concrete category is a pair (H,U) where H is a category and $U: H \to \mathbf{SET}$ is a faithful functor ($\mathbf{SET}$ is the category of sets, with functions as morphisms). The interpretation of a concrete category (H,U) is that H is a category of sets with additional structure; the functor U forgets this structure. This idea may clearly be generalized. Let K be a category. A K-concrete category is an ordered pair (H,U) where H is a category and $U: H \to K$ is a faithful functor.

The following special case will prove to be central to the rest of this paper. Let $K = \mathbb{K}$-$\mathbb{LS}$ and let H be a category whose objects are $\mathbb{K}$-linear spaces which also have a topology, and whose morphisms are continuous linear maps. The functor $U: H \to \mathbb{K}$-$\mathbb{LS}$ forgets the topology, assigning to each linear space-topology pair (E,T) the underlying space E. The morphisms are mapped identically, so as to preserve the underlying function.

A categorical duality for linear systems consists of the following.

1. Two $\mathbb{K}$-$\mathbb{LS}$ concrete categories (H,U) and (J,V).
2. A dual equivalence of H and J $(F:H^{op} \to J, G:J^{op} \to H)$.

3. A construction for countable powers and copowers in each of the categories $\mathcal{H}$ and $\mathcal{J}$.

4. An exhibition of (possibly several) image-factorization systems for $\mathcal{H}$ and $\mathcal{J}$, together with rules for computing their transformations under the functors F and G, in a sense which will become apparent as the theory is developed.

§4 Duality with Dual Pairs

The theory of dual pairs has seen wide application in the theory of locally-convex spaces, and also in the theory of linearly-topologized spaces (consult [11]). The purpose of the present treatment is to develop the essential categorical properties of dual pairs using entirely algebraic machinery, without any reference to topologized vector spaces. The category of dual pairs will not be used as a framework for discrete linear systems; equivalent categories of linearly-topologized spaces will be used. The purpose of using dual pairs in the initial treatment is (a) to simplify the overall presentation, and (b) to show that topology is auxillary (although convenient) to the theory.

Let E and F be linear spaces. A <u>dual pair</u> of E and F is a bilinear map $\langle\ ,\ \rangle: E\times F \to \mathbb{K}$ which satisfies the following two laws.

(D1) $(x\in E \text{ and } (\forall y\in F)(\langle x,y\rangle=0)) \implies x = 0.$

(D2) $(y\in F \text{ and } (\forall x\in E)(\langle x,y\rangle=0)) \implies y = 0.$

The dual pair is denoted $\langle E,F\rangle$.

In a dual pair $\langle E,F\rangle$, each element $f \in F$ can be identified with an element $\tilde{f}$ of the algebraic dual E^* of E via the rule $\tilde{f}(e) = \langle e,f\rangle$. Denote by $\tilde{F}$ the subspace of E^* consisting of these elements. $\tilde{F}$ is a <u>total</u> subspace of E^*, in the precise sense that for each $e \in E$, there is an $x \in \tilde{F}$ with $x(e) \neq 0$. Conversely, there is a canonical pairing between E and any total subspace F of E^*. Thus, the concept of the dual pair is a generalization of the concept of a linear space and its algebraic dual.

Let $\langle E_1,F_1\rangle$ and $\langle E_2,F_2\rangle$ be dual pairs, and let $g\colon E_1 \to E_2$ be a linear map. g is <u>compatible</u> (for $\langle E_1,F_1\rangle$ and $\langle E_2,F_2\rangle$) provided that there is a linear map $g'\colon F_2 \to F_1$ such that $(\forall x \in E_1)(\forall y \in F_2)(\langle g(x),y\rangle = \langle x,g'(y)\rangle)$. If g is compatible, the notation $g\colon \langle E_1,F_1\rangle \to \langle E_2,F_2\rangle$ will be used. It is easily seen that g' is unique (if it exists) and $g'\colon \langle F_2,E_2\rangle \to \langle F_1,E_1\rangle$. g' is called the <u>adjoint</u> of g (for the pairs $\langle E_1,F_1\rangle$ and $\langle E_2,F_2\rangle$). Clearly $g'' = g$.

The category of dual pairs, denoted $\mathbb{K}$-$\mathbf{DP}$, has as objects the dual pairs over $\mathbb{K}$ and as morphisms the compatible linear maps. A $\mathbb{K}$-$\mathbf{DP}$ morphism g is an isomorphism if and only if both g and g' are bijections.

The functor $\mathfrak{U}_{DP}\colon \mathbb{K}$-$\mathbf{DP} \to \mathbb{K}$-$\mathbf{LS}$ defined by $\langle E,F\rangle \mapsto E$ on objects and $(g\colon \langle E_1,F_1\rangle \to \langle E_2,F_2\rangle) \mapsto (g\colon E_1 \to E_2)$ on morphisms is clearly faithful.

(1) $(\mathbb{K}$-$\mathbf{DP},\mathfrak{U}_{DP})$ *is a* $\mathbb{K}$-$\mathbf{LS}$ *concrete category.* ∎

The association which sends the dual pair $\langle E,F\rangle$ to the pair $\langle F,E\rangle$ and the compatible linear map g to its adjoint g' is easily seen to be a functor from $(\mathbb{K}$-$\mathbf{DP})^{op}$ to $\mathbb{K}$-$\mathbf{DP}$. Denote this functor by $\mathfrak{P}$.

(2) K-DP *is isomorphic to its dual.* $(\mathfrak{P},\mathfrak{P})$ *is a dual equivalence of* K-DP *with itself.*

Proof: $\mathfrak{P}$ is its own inverse, hence bijective and an isomorphism. ∎

The category K-DP has countable powers and copowers, the construction of which is based upon those of linear spaces. Let E be a linear space. In K-LS, a countable power of E is given by an ω-indexed cartesian product of copies of E, with componentwise operations, and is denoted $E_{\S}$. The projections π_k $(k \in \omega)$ are just the canonical maps $(e_o,e_1,e_2,\ldots,e_n,\ldots) \mapsto e_k$. A countable copower of E is given by the subspace of $E_{\S}$ consisting of those elements which have only finitely many nonzero projections, and is denoted $E^{\S}$. The injections in_k $(k \in \omega)$ are just the injections $e \mapsto (0,\ldots,0,e,0,\ldots)$ (e in k^{th} place).

(3) K-DP *has countable powers and countable copowers. Let* <E,F> *be a dual pair.*

(a) *A countable power of* <E,F> *is given by* $\langle E_{\S},F^{\S}\rangle$ *with the rule* $\langle (e_o,e_1,e_2,\ldots),(f_o,f_1,f_2,\ldots)\rangle \mapsto \sum_{i=0}^{\infty} \langle e_i,f_i\rangle$. *The projections are just the linear-space projections, i.e.* $\pi_i = (e_o,e_1,e_2,\ldots) \mapsto e_i$.

(b) *A countable copower of* <E,F> *is given by* $\langle E^{\S},F_{\S}\rangle$ *with the same rule. The injections are just the linear-space injections, i.e.* $in_i = e \mapsto (0,\ldots,0,e,0,\ldots)$, *with the* e *in the* i^{th} *place.*

Proof: (a) To prove this part, it is necessary to prove that $\langle E_{\S},F^{\S}\rangle$ is a dual pair, and that π_i is compatible from $\langle E_{\S},F^{\S}\rangle$ to <E,F>, for each $i \in \omega$. The map on $E^{\S} \times F_{\S}$ given by $((e_o,e_1,e_2,\ldots),(f_o,f_1,f_2,\ldots)) \mapsto \sum_{i=0}^{\infty} \langle e_i,f_i\rangle$ is well-defined, since $f_i \neq 0$ for only finitely-many i. It is clearly bilinear into K. Conditions (D1) and (D2) are clearly satisfied, so that $\langle E_§, F^§\rangle$ is a dual pair. For $(e_o, e_1, e_2, \ldots) \in E_§$, $f_j \in F_j$, $\langle \pi_i(e_o, e_1, e_2, \ldots), f_j\rangle = \langle e_j, f_j\rangle = \sum_{i=0}^{\infty} \langle e_i, f_i\rangle$ where $f_i = 0$ if $i \neq j$. However, $\sum_{i=0}^{\infty} \langle e_i, f_i\rangle = \langle (e_o, e_1, e_2, \ldots), in_j(f_j)\rangle$ in this case, so that π_j is compatible with adjoint in_j. The universality of this pair is verified as follows. Suppose $\{g_i: \langle G,H\rangle \to \langle E,F\rangle | i\epsilon\omega\}$ is a set of compatible maps. Define $g: G \to E^§$ by $g(x) = (g_o(x), g_1(x), g_2(x), \ldots)$. $g: \langle G,H\rangle \to \langle E_§, F^§\rangle$, for g' is given by $g\ ((f_o, f_1, f_2, \ldots)) = \sum_{i=0}^{\infty} g_i(f_i)$, i.e., g' is the coproduct map induced by $g_o', g_1', g_2', \ldots$. The known universality of g in K-LS completes the proof.

(b) Use (2) to dualize (a). ∎

From a system-theoretic point of view, the countable copower (resp. countable power) construction in (3) is the correct one, in the sense of a proper representation of input (resp. output) signals (see [1]).

The concept of orthogonality is central to the theory of dual pairs. Let $\langle E,F\rangle$ be a dual pair, and let $S \subset E$. The orthogonal of S, denoted $S^\perp$, is given by $S^\perp = \{f\epsilon F | (\forall s\epsilon S)(\langle s,f\rangle = 0)\}$. The following properties are routinely verified.

(4) *Let* $\langle E,F\rangle$ *be a dual pair with* $S \subset E$.

(a) $S^\perp$ *is a linear subspace of* F.

(b) $S \subset S^{\perp\perp}$.

(c) $S^\perp = S^{\perp\perp\perp}$. ∎

If $S = S^{\perp\perp}$, S is said to be orthogonally-closed. A compatible linear map $g: \langle E_1, F_1\rangle \to \langle E_2, F_2\rangle$ is dense if $g(E_1)^\perp = 0$.

(5) *Suppose* $g: \langle E_1, F_1\rangle \to \langle E_2, F_2\rangle$. $g(E_1)^\perp = ker\, g'$. *In particular,* g *is injective if and only if* g' *is dense.*

Proof: Let $f_2 \in F_2$. $g'(f_2) = 0 \iff (\forall e_1 \in E_1)(\langle e_1, g'(f_2)\rangle = 0) \iff (\forall e_1 \in E_1)(\langle g(e_1), f_2\rangle = 0) \iff f_2 \in g(E_1)^\perp$. ∎

(6) *A* $\mathbb{K}$-$\mathbb{DP}$ *morphism* $g: \langle E_1,F_1\rangle \to \langle E_2,F_2\rangle$ *is a monomorphism if and only if it is injective, and an epimorphism if and only if it is dense.*

Proof: Clearly every injective morphism is monomorphic. Conversely, suppose g is not injective. Pick $x \in \mathit{ker}\, g \backslash \{0\}$, and let $\rho: \mathbb{K} \to E_1$ be defined by $k \mapsto kx$. Let $\langle \mathbb{K}, \mathbb{K}\rangle$ denote the canonical pairing $(x,y) \mapsto xy$. $\rho: \langle \mathbb{K}, \mathbb{K}\rangle \mapsto \langle E_1,F_1\rangle$, i.e., ρ is compatible, since it is easily verified that $\rho': f_1 \mapsto \langle \rho(1), f_1\rangle$. The identically zero map $0: \mathbb{K} \to E_1$ is clearly compatible for these same pairs. Since $g\circ 0 = g\circ\rho$, it follows that g is not monomorphic. Combine (2) and (5) with the preceding for the epimorphism characterization. ∎

(7) *Let* $\langle E,F\rangle$ *be a dual pair, and* G *a linear subspace of* E.

(a) $\langle G, F/G^\perp\rangle$ *is a dual pair with* $\langle g,[f]\rangle = \langle g,f\rangle$.

(b) *The canonical injection* $i: G \to E$ *is compatible, and* $i': F \to F/G^\perp$ *is the canonical surjection.*

Proof: Obvious. ∎

A compatible linear map $g: \langle E_1,F_1\rangle \to \langle E_2,F_2\rangle$ is an <u>embedding</u> provided that there is an isomorphism $i: \langle g(E_1), F_2/g(E_1)^\perp\rangle \to \langle E_1,F_1\rangle$ such that $g\circ i$ is the canonical injection; $g\circ i: \langle g(E_1), F_2/g(E_1)^\perp\rangle \to \langle E_2,F_2\rangle$.

(8) *Suppose* $g: \langle E_1,F_1\rangle \to \langle E_2,F_2\rangle$. g *is an embedding if and only if* g' *is a surjection.*

Proof: Suppose g is an embedding. Combining (7b) with the definition of embedding shows that g' is surjective. Conversely, suppose g' is surjective.

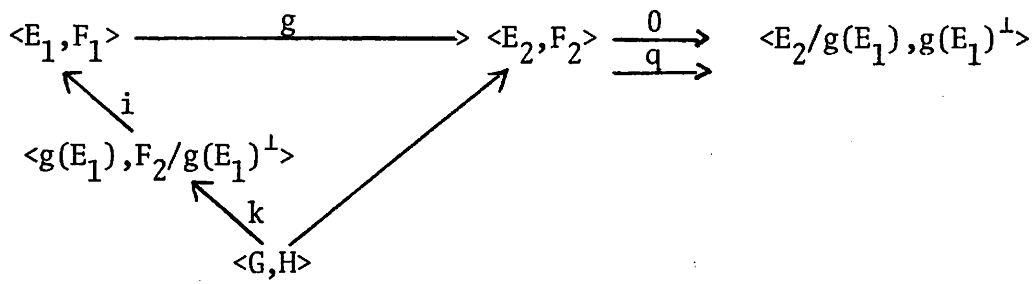
Select an isomorphism j: $F_1 \to F_2/\mathit{ker}\,g'$ such that j g' is the canonical surjection. By (5), $\mathit{ker}\,g' = g(E_1)^\perp$. Hence, by (7a), $j \circ g'$: $\langle F_2, E_2\rangle \to \langle F_2/g(E_1)^\perp, g(E_1)\rangle$, so that $g \circ j'$ is the canonical injection. Hence g is an embedding. ∎

An embedding g: $\langle E_1, F_1\rangle \to \langle E_2, F_2\rangle$ is <u>closed</u> if $g(E_1)$ is orthogonally-closed in E_2. A surjection g is <u>open</u> if g' is a closed embedding.

A standard result in category theory states that if every morphism f in a category *K* has a factorization $f = m \circ e$ with e an epimorphism and m an equalizer, then (epimorphisms, equalizers) is an image-factorization system for *K*. See [15], 18.4.7 dual, for example. This result will be used in developing image-factorization systems for K-DP. The following lemma is the crucial step.

(9) *Let* g *be a* K-DP *morphism. If* g *is a closed embedding, then it is an equalizer.*

Proof: Suppose g: $\langle E_1, F_1\rangle \to \langle E_2, F_2\rangle$ is a closed embedding. There is an isomorphism i: $\langle g(E_1), F_2/g(E_1)^\perp\rangle \to \langle E_1, F_1\rangle$ such that $g \circ i$ is the canonical injection, and $g(E_1) = g(E_1)^{\perp\perp}$, by definition. Let q: $\langle E_2, F_2\rangle \to \langle E_2/g(E_1), g(E_1)^\perp\rangle$ be the canonical surjection. Clearly $q \circ g \circ i = 0 \circ g \circ i$, where 0: $\langle E_2, F_2\rangle \to \langle E_2/g(E_1), g(E_1)\rangle$ is the identically-zero map. Let h: $\langle G, H\rangle \to \langle E_2, F_2\rangle$ be any K-DP morphism such that $q \circ h = 0 \circ h$. Define k: $G \to g(E_1)$ by $x \mapsto h(x)$. k is clearly into $g(E_1)$, because $0 = q \circ h(x)$, which implies $h(x) \in \mathit{ker}(q) = g(E_1)$. Furthermore, k: $\langle G, H\rangle \to \langle g(E_1), F_2/g(E_1)^\perp\rangle$, because k': $F_2/g(E_1)^\perp \to H$: $[y] \mapsto h'(y)$ is well-defined, since $[y_1] = [y_2] \implies y_1 - y_2 \in g(E_1)$. Hence the diagram



commutes. $i\circ k$ is clearly unique, since g is a monomorphism, by (6). Hence g is the equalizer of 0 and q. ∎

It is now possible to exhibit some of the important image-factorization systems which $\mathbb{K}$-$\mathbb{DP}$ possesses.

(10) *The pair* $(\mathbb{E},\mathbb{M})$ *where* $\mathbb{E}$ *is the class of all dense maps and* $\mathbb{M}$ *is the class of all closed embeddings is an image-factorization system for* $\mathbb{K}$-$\mathbb{DP}$.

Proof: In view of (9) and the remarks preceding (9), it suffices to show that each $\mathbb{K}$-$\mathbb{DP}$ morphism $g: \langle E_1,F_1\rangle \to \langle E_2,F_2\rangle$ has a factorization $g = m\circ e$ with e a dense map and m a closed embedding. However,

$$\langle E_1,F_1\rangle \xrightarrow{\ e\ } \langle g(E_1)^{\perp\perp},F_2/g(E_1)^{\perp}\rangle \xrightarrow{\ m\ } \langle E_2,F_2\rangle$$

with e defined by $x \mapsto g(x)$ and m the canonical injection is clearly such a factorization. Hence, (dense maps, closed embeddings) is an image-factorization system for $\mathbb{K}$-$\mathbb{DP}$. ∎

By duality, the following image-factorization system follows at once.

(11) *The pair* (E,M), *where* E *is the class of all open surjections and* M *is the class of all injections is an image-factorization system for* K-DP. *The image* $(\mathfrak{M}(\mathsf{M}),\mathfrak{E}(\mathsf{E}))$ *of this image-factorization system is the system* (dense maps, closed embeddings).

Proof: Dualize (10), using (5). ∎

One more image-factorization system is evident.

(12) *The pair* (E,M) *where* E *is the class of all surjections and* M *is the class of all embeddings is an image-factorization system for* K-DP. *The image* $(\mathfrak{M}(\mathsf{M}),\mathfrak{E}(\mathsf{E}))$ *of this image-factorization system is the system itself.*

Proof: Let $g: \langle E_1,F_1\rangle \to \langle E_2,F_2\rangle$ be a K-DP morphism. Write g as

$$\langle E_1,F_1\rangle \xrightarrow{h} \langle g(E_1),F_2/g(E_1)^{\perp}\rangle \xrightarrow{k} \langle E_2,F_2\rangle.$$

h is defined by $e \mapsto g(e)$ while k is the canonical injection. Embeddings are closed under composition by (8), since surjections are closed under composition. The rest of the image-factorization properties are clear. The duality is an immediate consequence of (8). ∎

§5 Linear Weak Duality

Let E be a linear space over the field K. A separated topology on E is called <u>linear</u> provided that it is translation-invariant and has a neighborhood filter at 0 with a basis of linear subspaces. It is easy to verify that addition and scalar multiplication are continuous, when K is discretely topologized. A pair (E,T), where E is a linear space and T is a linear topology is called a <u>linearly-topologized space</u>, abbreviated l.t.s. (E,T) is usually denoted $E[T]$, or just E if the topology is otherwise known or unimportant.

Given a dual pair $\langle E,F\rangle$, there is a natural linear topology on E. Let $\mathcal{F}$ be the set of all finite-dimensional linear subspaces of F, and let $U = \{G \in E \mid (\exists h \in \mathcal{F})(G = h^{\perp})\}$. U forms the basis at 0 for a linear topology on E, called the <u>linear weak topology.</u>
This topology is denoted $\mathfrak{T}_{ls}(F)$.

A converse relationship may also be established. Let $E[T]$ be a l.t.s. The set of all $f \in E^*$ which are continuous for T ($\mathbb{K}$ discrete) is called the <u>dual</u> of $E[T]$ and denoted $E[T]'$ or just E'. E' is clearly a linear subspace of E^*.

(1) *Let* $\langle E,F\rangle$ *be a dual pair.* $E[\mathfrak{T}_{ls}(F)]' = \tilde{F}$. *In particular,* $\langle E[\mathfrak{T}_{ls}(F)], E[\mathfrak{T}_{ls}(F)]'\rangle$ *is a dual pair and* $\mathfrak{T}_{ls}(F) = \mathfrak{T}_{ls}(E')$. *Furthermore, if* $\langle G,H\rangle$ *is another dual pair, a linear map* $g\colon E \to G$ *is compatible if and only if it is continuous for the topologies* $\mathfrak{T}_{ls}(F)$ *on* E *and* $\mathfrak{T}_{ls}(H)$ *on G.*

Proof: Consult [11], §10,12.(1). ∎

Thus, the dual pair $\langle E,F\rangle$ may be recovered, up to isomorphism, from just E and the topology $\mathfrak{T}_{ls}(F)$. This is now developed formally.

A *l.t.s.* $E[T]$ is <u>weak</u> provided that there is a dual pair $\langle E,F\rangle$ such that $T = \mathfrak{T}_{ls}(F)$. The category s$\mathbb{K}$-LTS has as objects the weak l.t.s.'s and as morphisms the continuous linear maps. The functor $\mathfrak{S}_1$: $\mathbb{K}$-DP $\to$ s$\mathbb{K}$-LTS is defined by $\langle E,F\rangle \mapsto E[\mathfrak{T}_{ls}(F)]$ on objects and the corresponding identity on morphisms. The functor $\mathfrak{S}_2$: s$\mathbb{K}$-LTS $\to$ $\mathbb{K}$-DP is defined by $E \mapsto \langle E,E'\rangle$ on objects and the corresponding identity on morphisms. In view of (1), the following result is immediate.

(2) $\mathfrak{S}_1$ *and* $\mathfrak{S}_2$ *are equivalences of categories, and* $\mathfrak{S}_1 \circ \mathfrak{S}_2 = 1_{\mathrm{sK\text{-}LTS}}$, $\mathfrak{S}_2 \circ \mathfrak{S}_1 \cong 1_{\mathrm{K\text{-}DP}}$. ∎

Using the above along with 4.(2), the next result is a routine verification. Denote by $\mathfrak{D}_s$ the functor $\mathfrak{S}_1 \circ \mathfrak{P} \circ \mathfrak{S}_2^{op}$: $(\mathrm{sK\text{-}LTS})^{op} \to \mathrm{sK\text{-}LTS}$.

(3) $(\mathfrak{D}_s, \mathfrak{D}_s)$ *is a dual equivalence of* sK-LTS *with itself.* ∎

Furthermore, setting $\mathfrak{U}_s = \mathfrak{U}_{DP} \circ \mathfrak{S}_2$,

(4) $(\mathrm{sK\text{-}LTS}, \mathfrak{U}_s)$ *is a* K-LS *concrete category.* ∎

The equivalence of (2) allows the immediate transfer of categorical properties of K-DP to sK-LTS.

(5) sK-LTS *has countable powers and copowers. Let* E *be a weak* LTS.

(a) *A countable power of* E *is given by* $E_{\S}[\mathfrak{T}_{ls}((E')^{\S})]$.

(b) *A countable copower of* E *is given by* $E^{\S}[\mathfrak{T}_{ls}((E')_{\S}]$. ∎

In order to transfer the image-factorization systems of K-DP to sK-LTS , it is necessary to list some of the properties of weak l.t.s.'s. These properties will not be proved here, as proofs can be found in [11].

(6) *Let* E[T] *be a weak l.t.s. and let* F *be a linear subspace of* E.

(a) *Under the induced topology,* F *is a weak l.t.s. The induced topology the same as the topology* $\mathfrak{T}_{ls}(E'/F^{\perp})$, *computed for the pair* $\langle F, E'/F^{\perp}\rangle$.

(b) F *is T-closed if and only if* F *is orthogonally-closed for the pair* <E,E'>. *Assuming* F *is closed,* E/F *is a l.t.s. under the quotient topology. This quotient topology is the same as the topology* $\mathfrak{T}_{ls}(F^{\perp})$ *computed for the pair* $\langle E/F, F^{\perp}\rangle$ $(F^{\perp} \subset E')$. ∎

Using these facts, the following associations may be given.

(7) *The functors* $\mathfrak{S}_1$ *and* $\mathfrak{S}_2$ *each preserve and reflect the following properties:* (a) *dense map,* (b) *open surjection,* (c) *embedding,* (d) *closed embedding. Furthermore a* K-LTS *morphism is an isomorphism if and only if it is a homeomorphism.* ∎

The three image-factorization systems of K-DP transfer easily to sK-LTS.

(8) *Each of the following* $(\mathbb{E}_i, \mathbb{M}_i)$ *is an image-factorization system for* sK-LTS.

(a) $\mathbb{E}_1$ = *open surjections,* $\mathbb{M}_1$ = *injections;*

(b) $\mathbb{E}_2$ = *surjections,* $\mathbb{M}_2$ = *embeddings;*

(c) $\mathbb{E}_3$ = *dense maps,* $\mathbb{M}_3$ = *closed embeddings.*

Their behavior under the duality functor $\mathfrak{D}_s$ *is*

(d) $(\overline{\mathfrak{D}_s(\mathbb{M}_1)}, \overline{\mathfrak{D}_s(\mathbb{E}_1)}) = (\mathbb{E}_3, \mathbb{M}_3)$;

(e) $(\overline{\mathfrak{D}_s(\mathbb{M}_2)}, \overline{\mathfrak{D}_s(\mathbb{E}_2)}) = (\mathbb{E}_2, \mathbb{M}_2)$;

(f) $(\overline{\mathfrak{D}_s(\mathbb{M}_3)}, \overline{\mathfrak{D}_s(\mathbb{E}_3)}) = (\mathbb{E}_1, \mathbb{M}_1)$.

Proof: ((a)-(c)). The proof of these parts is based upon the following two observations:

(i) The functor $\mathfrak{S}_1$ is surjective (on objects and morphisms).

(ii) Each of the classes E_i and M_i for $1 \le i \le 3$ is closed under composition and contains all isomorphisms.

Thus, applying $\mathfrak{S}_1$ to each of the image-factorization systems of K-ΠP listed in 4.(10), 4.(11), and 4.(12), the result follows.

((d)-(f)). On one hand, it is clear from the definition of $\mathfrak{D}_s$, along with 4.(7) and 4.(8), that $\overline{\mathfrak{D}_s(E_i)} \subset M_{4-i}$ and $\overline{\mathfrak{D}_s(M_i)} \subset E_{4-i}$ for $1 \le i \le 3$. On the other hand, since E and M determine each other in an image-factorization system (E,M) (Consult [5], 33.6), it follows from (a)-(c) that each of these inclusions is an equality. ∎

§6 Linear Mackey Duality

This section presents a theory of linearly-topologized spaces which is entirely parallel to that of §5. A l.t.s. E is linearly-compact if every filter on E which has a base consiting of linear submanifolds of E has an adherent point in E.

Given a dual pair <E,F>, a natural linear topology can be defined on E as follows. Let C be the set of all $\mathfrak{T}_{ls}$(E)-linearly-compact subspaces of F, and set $U = \{G \subset E \mid (\exists c \in C)(G = c^{\perp})\}$. It can be shown (consult [11]), that U is the base at 0 for a linear topology on E, called the linear Mackey topology. This topology is denoted by $\mathfrak{T}_{lk}$(F). A l.t.s. E[T] is called Mackey provided that there is a dual pair <E,F> such that $T = \mathfrak{T}_{lk}$(F). The following result corresponds to 5.(1).

(1) *Let* $\langle E,F\rangle$ *be a dual pair.* $E[\mathfrak{T}_{lk}(F)]' = \tilde{F}$. *In particular,* $\langle E[\mathfrak{T}_{lk}(F)], E[\mathfrak{T}_{lk}(F)]'\rangle$ *is a dual pair and* $\mathfrak{T}_{lk}(F) = \mathfrak{T}_{lk}(E')$. *If* $\langle G,H\rangle$ *is another dual pair, a linear map* $g\colon E \to G$ *is compatible if and only if it is continuous for the topologies* $\mathfrak{T}_{lk}(F)$ *on* E *and* $\mathfrak{T}_{lk}(H)$ *on* G.

Proof: Consult [11], §10,12.(1). ∎

The category k$\mathcal{K}$-LTS has as objects the Mackey l.t.s.'s and as morphisms the continuous linear maps. The crucial observation is the following. Define a functor $\mathfrak{I}$: s$\mathcal{K}$-LTS → k$\mathcal{K}$-LTS by $E[\mathfrak{T}_{ls}(E')] \to E[\mathfrak{T}_{lk}(E')]$ on objects and the corresponding identity on morphisms.

(2) $\mathfrak{I}$ *is bijective and hence an isomorphism of the categories* s$\mathcal{K}$-LTS *and* k$\mathcal{K}$-LTS.

Proof: Combine the results of (1) and 5.(1). ∎

This isomorphism cuts greatly the amount of work which must be done in developing the properties of k$\mathcal{K}$-LTS. Let $\mathfrak{D}_k = \mathfrak{I}\circ\mathfrak{D}_s(\mathfrak{I}^{-1})^{op}$.

(3) $(\mathfrak{D}_k,\mathfrak{D}_k)$ *is a dual equivalence of* k$\mathcal{K}$-LTS *with itself.* ∎

Define the functor $\mathfrak{U}_k$: k$\mathcal{K}$-LTS → $\mathcal{K}$-LS by $\mathfrak{U}_k = \mathfrak{U}_s\circ\mathfrak{I}^{-1}$

(4) (k$\mathcal{K}$-LTS, $\mathfrak{U}_k$) *is a* $\mathcal{K}$-LS *concrete category.* ∎

(5) k$\mathcal{K}$-LTS *has countable powers and copowers. Let* E *be a Mackey* LTS.

(a) *A countable power of* E *is given by* $E_{\S}[\mathfrak{T}_{lk}((E')^{\S})]$.

(b) *A countable copower of* E *is given by* $E^{\S}[\mathfrak{T}_{lk}((E')_{\S})]$. ∎

In order to transfer image-factorization systems, some non-categorical results are necessary. The proofs are not given here, as they can be found in [11].

(6) *Let* $E[T]$ *be a Mackey l.t.s., and let* F *be a linear subspace on* E.

(a) F *is not necessarily a Mackey l.t.s. under the induced topology.*

(b) F *is* T*-closed if and only if* F *is orthogonally-closed for the pair* $\langle E,E'\rangle$.

Assuming F *is closed,* E/F *is a Mackey l.t.s. under the quotient topology. The quotient topology is the same as the topology* $\mathfrak{T}_{lk}(F^{\perp})$, *computed for the pair* $\langle E/F,F^{\perp}\rangle$ $(F^{\perp}\subset E')$. ∎

Call an injection $g\colon E[T_E] \to F[T_F]$ of Mackey l.t.s.'s a <u>Mackey injection</u> if the canonical map $i\colon E[T_E] \to g(E)[\mathfrak{T}_{lk}(E'/g(E)^{\perp})]$ is an isomorphism. The following is an easily-verified chart of transformation under $\mathfrak{Y}$.

(7) *The following chart gives the transition of selected classes of morphisms under transition of the isomorphism* $\mathfrak{Y}$.

Concept in sK-LTS		*Concept in* kK-LTS.
dense map	$\longleftrightarrow$	*dense map*
open surjection	$\longleftrightarrow$	*open surjection*
embedding	$\longleftrightarrow$	*Mackey injection*
closed embedding	$\longleftrightarrow$	*closed Mackey injection.* ∎

(8) *The following pairs* $(\mathsf{E}_i,\mathsf{M}_i)$ *are image-factorization systems for* kK-LTS.

(a) E_1 = *open surjections,* M_1 = *dense maps;*

(b) E_2 = *surjections,* M_2 = *Mackey injections;*

(c) E_3 = *dense maps,* M_3 = *closed embeddings.*

The behavior under the duality functor $\mathfrak{D}_k$ *is*

(d) $(\overline{\mathfrak{D}_k(\mathsf{M}_1)}, \overline{\mathfrak{D}_k(\mathsf{E}_1)}) = (\mathsf{E}_3,\mathsf{M}_3)$;

(e) $(\overline{\mathfrak{D}_k(\mathsf{M}_2)}, \overline{\mathfrak{D}_k(\mathsf{E}_2)}) = (\mathsf{E}_2,\mathsf{M}_2)$;

(f) $(\overline{\mathfrak{D}_k(\mathsf{M}_3)}, \overline{\mathfrak{D}_k(\mathsf{E}_3)}) = (\mathsf{E}_1,\mathsf{M}_1)$. ∎

§7 Discrete and Linearly-Compact Duality

Sections 4-6 presented two duality theories in which the category in which the systems were modelled was dually-equivalent to itself. However, in neither case could such a category be equivalent to K-LS (see 3.(3)). In this section, a dual equivalence involving K-LS is developed. As in the previous development, the theory is first developed in the framework of dual pairs without any mention of topology, and then equivalences to categories of l.t.s.'s are exhibited.

A dual pair <E,F> is <u>maximal</u> if $\tilde{F} = E^*$ (or equivalently, $\langle E,F\rangle \cong \langle E,E^*\rangle$). The category K-MD is the full subcategory of K-DP whose objects are precisely the maximal dual pairs. Conversely, a dual pair <E,F> is <u>minimal</u> if $\tilde{E} = F^*$ (or equivalently $\langle E,F\rangle = \langle E^*,E\rangle$). The category K-DM is the full subcategory of K-DP whose objects are precisely the minimal dual pairs.

Note that the functor $\mathfrak{P}$ maps K-MD^{op} into K-DM and K-DM^{op} into K-MD. Define the functors $\mathfrak{P}_{MD}$: $\text{K-MD}^{op} \to \text{K-DM}$ and $\mathfrak{P}_{DM}$: $\text{K-DM}^{op} \to \text{K-MD}$ to be the restrictions of the functor $\mathfrak{P}$. The following result is immediate.

(1) *The categories* K-MD *and* K-DM *are dually-isomorphic. The pair* $(\mathfrak{P}_{MD},\mathfrak{P}_{DM})$ *is a dual equivalence (in fact a dual isomorphism) of* K-MD *and* K-DM. ∎

The following illustrates the importance of this duality.

(2) *Let* $\langle E_1,F_1\rangle$ *and* $\langle E_2,F_2\rangle$ *be dual pairs, and* $g\colon E_1 \to E_2$ *a linear map. If* $\langle E_1,F_1\rangle$ *is maximal, then* g *is compatible. In particular, in* K-MD *every linear map is compatible.*

Proof: Assume $\langle E_1,F_1\rangle$ is maximal. The map $h\colon F_2 \to E_1^*$ defined by $f_2 \mapsto \langle g(\cdot),f_2\rangle$ is clearly linear. Let $i\colon \tilde{F}_1 \to E_1^*$ be the canonical isomorphism $f_1 \mapsto \langle \cdot,f_1\rangle$. The map $i^{-1}\circ h$ is the transpose of g, so g is compatible. ∎

Define the functor $\mathcal{L}_1$: K-MD $\to$ K-LS by $\langle E,F\rangle \mapsto E$ on objects and the corresponding identity on morphisms. The functor $\mathcal{L}_2$: K-LS $\to$ K-MD is defined by $E \mapsto \langle E,E^*\rangle$ on objects and the corresponding identity on morphisms.

(3) $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *are equivalences of categories, and* $\mathcal{L}_1\circ\mathcal{L}_2 = 1_{\text{K-LS}}$, $\mathcal{L}_2\circ\mathcal{L}_1 \cong 1_{\text{K-MD}}$.

Proof: The proof is an immediate consequence of the definitions. ∎

The dual of K-LS is characterized by the following. The proof is not given, as it can be found in [11].

(4) *A l.t.s.* E *is linearly-compact if and only if it is isomorphic to* $F^*[\mathfrak{T}_{ls}(F)]$ *for some linear space* F. ∎

Let cK-LTS denote the full subcategory of sK-LTS consisting of precisely the linearly-compact *l.t.s.'s*. Note that by (4), the functor $\mathfrak{S}_1$ maps K-DM into cK-LTS and $\mathfrak{S}_2$ maps cK-LTS into K-DM. Define $\mathfrak{E}_1$: K-DM → cK-LTS and $\mathfrak{E}_2$: cK-LTS → K-DM to be the restrictions of these functors. In view of (4), the following holds.

(5) $\mathfrak{E}_1$ *and* $\mathfrak{E}_2$ *are equivalences of categories and* $\mathfrak{E}_1 \circ \mathfrak{E}_2 = 1_{cK\text{-}LTS}$, $\mathfrak{E}_2 \circ \mathfrak{E}_1 \cong 1_{K\text{-}DM}$. ∎

The dual equivalence of K-LS and cK-LTS now emerges. Define the functors $\mathfrak{D}_d$: K-LS^{op} → cK-LTS by $\mathfrak{D}_d = \mathfrak{E}_1 \circ \mathfrak{P}_{MD} \circ \mathfrak{L}_2{}^{op}$ and $\mathfrak{D}_c$: cK-LTS^{op} → K-LS by $\mathfrak{D}_c = \mathfrak{L}_1 \circ \mathfrak{P}_{DM} \circ \mathfrak{E}_2{}^{op}$. The proof of the following is a routine verification.

(6) $(\mathfrak{D}_d, \mathfrak{D}_c)$ is a dual equivalence of K-LS and cK-LTS. ∎

Familiarity with the construction of countable powers and copowers in K-LS has already been assumed. To translate these results to cK-LTS, the path K-LS → K-MD → K-DM → cK-LTS will be used.

(7) K-MD *has countable powers and copowers. Let* <E,F> *be a maximal dual pair.*

(a) *A countable power of* <E,F> *is given by* $\langle E_{\S}, (E_{\S})^* \rangle$. *The projections are the linear-space projections.*

(b) *A countable copower of* $\langle E,F\rangle$ *is given by* $\langle E^{\S},E_{\S}\rangle$. *The injections are the linear-space injections.*

Proof: (a) is obvious. To verify (b), it must be shown that $\langle E^{\S},E_{\S}\rangle \cong \langle E_{\S},(E_{\S})^{*}\rangle$. Since 4.(3) already shows that $\langle E^{\S},E_{\S}\rangle$ is a dual pair, it remains to show that $\tilde{E}_{\S} = (E^{\S})^{*}$. Let $u \in (E^{\S})^{*}$. Define $u_i \in F$ by $\langle e,u_i\rangle = \langle in_i(e),u\rangle$. Clearly, for $(e_o,e_1,e_2,\ldots) \in E^{\S}$, $\langle (e_o,e_1,e_2,\ldots),u\rangle = \sum_{i=o}^{\infty} \langle e_i,u_i\rangle$. Since only finitely many of the e_i's are nonzero, the u_i's may be arbitrary. The map $u \mapsto (u_o,u_1,u_2,\ldots)$ is thus the required isomorphism from $(E^{\S})^{*}$ to $E_{\S}$. ∎

Duality immediately produces the following result.

(8) K-DM *has countable powers and copowers. Let* $\langle E,F\rangle$ *be a minimal dual pair.*

(a) *A countable power of* $\langle E,F\rangle$ *is given by* $\langle E_{\S},F^{\S}\rangle$. *The projections are just the linear-space projections.*

A countable copower of $\langle E,F\rangle$ *is given by* $\langle (F_{\S})^{*},F_{\S}\rangle$. *The injections are the transposes of the linear-space injections.* ∎

In terms of cK-LTS, the translation is as follows.

(9) cK-LTS *has countable powers and countable copowers. Let* E *be a linearly-compact l.t.s.*

(a) *A countable power of* E *is given by* $E_{\S}[\mathfrak{T}_{ls}(E^{\S})]$. *The projections are the usual linear-space projections.*

(b) *A countable copower of* E *is given by* $((E')_{\S})^{*}[\mathfrak{T}_{ls}(E')_{\S}])$. *The injections are the transposes of the usual linear-space projections.* ∎

The analysis of image-factorization systems is similar. It is assumed known that (surjections, injections) is an image-factorization system for $\mathcal{K}$-$\mathcal{LS}$, and that surjections = epimorphisms and injections = monomorphisms (consult [1]). Since $\mathcal{E} \subset$ epimorphisms and $\mathcal{M} \subset$ monomorphisms for any image-factorization system $(\mathcal{E},\mathcal{M})$, (surjections, injections) is clearly the only image-factorization system of $\mathcal{K}$-$\mathcal{LS}$. To obtain image-factorization systems for c$\mathcal{K}$-$\mathcal{LTS}$, the same chain of reasoning as for products and coproducts is used.

(10) $\mathcal{K}$-$\mathcal{MD}$ *has* (surjections, injections) *as its only image-factorization system.*

Proof: By 2.(2) and 2.(3), the only image-factorization system of $\mathcal{K}$-$\mathcal{MD}$ is $(\mathcal{L}_2$ (surjections), $\mathcal{R}_2$(injections)), which is routinely verified to be (surjections, injections). ∎

(11) *Let* g: $\langle E_1,F_1\rangle \to \langle E_2,F_2\rangle$ *be a compatible injection. If* $\langle E_2,F_2\rangle$ *is maximal, then* $\langle E_1,E_1\rangle$ *is also maximal and* g *is an embedding.*

Proof: Each linear functional on $g(E_1)$ is of the form $x \mapsto \langle x,f_2\rangle$ for some $f_2 \in F_2$, since $\tilde{F}_2 = E_2^*$ and $g(E_1) \subset E_2$. Furthermore, $f_2 \in F_2$ vanishes on $g(E_1)$ if and only if $f_2 \in g(E_1)^\perp$. Hence, since g is injective, the map i: $g(E_1) \to E_1$ defined by $x \mapsto g^{-1}(x)$ is compatible, i: $\langle g(E_1),F_2/g(E_1)^\perp\rangle \to \langle E_1,F_1\rangle$ is an isomorphism, and $g\circ i$ is the canonical injection. Hence g is an embedding and $\langle E_1,F_1\rangle$ is maximal. ∎

(12) $\mathcal{K}$-$\mathcal{DM}$ *has* (surjections, injections) *as its only image-factorization system.*

Proof: By the duality, (1) and 4.(6) and 4.(8), it suffices to show that (dense maps, embeddings) is an image-factorization system for K-MD. By (10), this amounts to showing that dense maps = surjections and embeddings = injections in K-MD. embeddings = injections follows at once from (11). It remains to show that every dense map is surjective. However, every epimorphism is surjective by (10) and 2.(2), and every dense map is epimorphic by 4.(6). Hence every dense map is surjective in K-MD. ∎

(13) cK-LTS *has as its only image-factorization system* (surjections, injections).

Proof: Similar to (10).∎

It should be noted that each of the categories of this section has only one image-factorization system, so transformation under equivalence and dual equivalence is unambiguous, and need not be explicitly noted.

One final formality is the K-LS-concreteness of cK-LTS. Define the functor $\mathfrak{U}_c$: cK-LTS $\to$ K-LS by $\mathfrak{U}_c = \mathfrak{U}_{DP} \circ I \circ \mathfrak{S}_2$, where I: K-MD $\hookrightarrow$ K-DP is the inclusion functor.

(14) (cK-LTS,$\mathfrak{U}_c$) *is a* K-LS-*concrete category.* ∎

Thus cK-LTS is a concrete model of $(\text{K-LS})^{op}$.

§8 EXAMPLES

In this section, various examples illustrating the duality theory just developed are given. The following notation will be fixed. I always denotes the input space, Y the output space, Q the state space, G the input map, H the output map, F the state transition map, r the reachability map, σ the observability map, and $f^{\blacktriangle}$ the total response of the system currently under consideration. Primes will be used to denote the transposes corresponding to the dual system.

First, it is necessary to develop some facts which are great aids in simplifying examples involving finite-dimensional linear systems.

(1) *Let* $\langle E,F\rangle$ *be a dual pair, and suppose* E *is of finite dimension* n.

(a) $\langle E,F\rangle$ *is maximal, and so* E *is also of dimension* n.

(b) *The only linear topology on* E *is the discrete topology.*

Proof: (a) Suppose dim F = k < n. Since *dim*F^* = k also, F^* is a proper subset of $\tilde{E}$, which is impossible. Hence *dim*F = n.

(b) Follows immediately, since a linear topology is separated and the intersection of finitely many neighborhoods of 0 is again a neighborhood. ∎

(2) *Let* $\langle E,F\rangle$ *be a dual pair. Every finite-dimensional subspace* G *of* E *is orthogonally closed. If* G *is* n*-dimensional,* $G^{\perp}$ *has codimension* n *in* F.

Proof: Let G be a finite-dimensional linear subspace of E. $\langle G,F/G^{\perp}\rangle$ is a dual pair by 4.(7). By (1), this pair is maximal, so G represents all linear functionals which vanish on $G^{\perp}$. Hence $G = G^{\perp\perp}$. The pairing $\langle G,F/G^{\perp}\rangle$ shows that $G^{\perp}$ has codimension n in F, since $F/G^{\perp} \cong G^*$. ∎

(3) *Let* g: $\langle E_1,F_1\rangle \to \langle E_2,F_2\rangle$ *be a* $\mathbb{K}$-$\mathbb{DP}$ *morphism. Every factorization of* g

$$\langle E_1,F_1\rangle \xrightarrow{e} \langle G,H\rangle \xrightarrow{m} \langle E_2,F_2\rangle$$

with e *an epimorphism,* m *a monomorphism, and* G *finite-dimensional is unique up to isomorphism of the middle element. Furthermore,* e *is always a surjection.*

Proof: e must be surjective since an epimorphism is dense by 4.(6) and every subspace of G is orthogonally-closed by (2): Hence G is determined up to isomorphism by the usual $\mathbb{K}$-$\mathbb{LS}$ (surjection, injection) factorization. By (1a), H is determined up to isomorphism by G, so the proof is complete. ∎

(4) *If* $\langle E^*,E\rangle$ *is maximal, then* E *is finite-dimensional.*

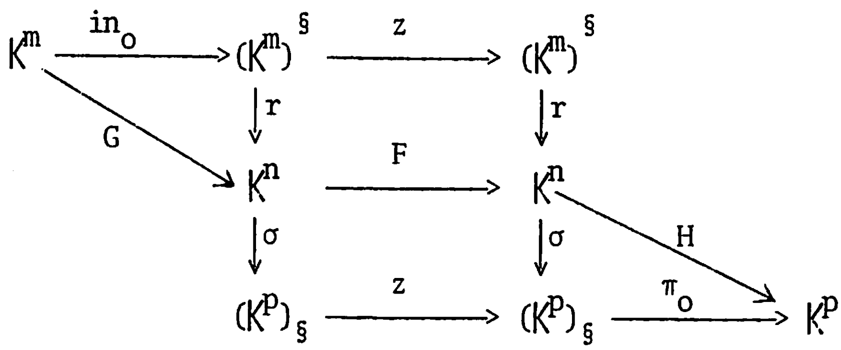
Proof: If E^*,E is maximal, then $\tilde{E} = E^{**}$. However, since E is isomorphic to a coproduct of *dim*(E) copies of $\mathbb{K}$ while E^* is isomorphic to a product of *dim*(E) copies of $\mathbb{K}$ (consult [11] for details), 3.(1) shows that $dim(E^{**}) > dim(E)$ if E is infinite-dimensional. ∎

The first example to be considered is the standard finite-dimensional linear system, governed by the equations

$$q(t+1) = F(q(t)) + G(i(t))$$

$$y(t) = H(q(t)).$$

Here q(t), i(t), y(t) are the values of the state, input, output at time t. The input space $I = \mathbb{K}^m$, the output space $Y = \mathbb{K}^p$, and the state space $Q = \mathbb{K}^n$, where m, p, and n are positive integers. Thus $F = \mathbb{K}^n \to \mathbb{K}^n$, G: $\mathbb{K}^m \to \mathbb{K}^n$, and H: $\mathbb{K}^n \to \mathbb{K}^p$. Using 1.(1), the system diagram is



Since $(\mathbb{K}^m)^{\S}$ and $(\mathbb{K}^p)_{\S}$ are each infinite-dimensional, neither of these spaces is reflexive, by (4). Thus, even in this special case, the operation of taking purely algebraic duals will not work, since the bidual will not be isomorphic to the original machine. It will now be shown how each of the duality theories previously developed overcomes this difficulty.

For the cases of s$\mathbb{K}$-$\mathbb{LTS}$ and k$\mathbb{K}$-$\mathbb{LTS}$, it is easiest to work directly in terms of dual pairs, and to transfer over to linear topologies later. The spaces $\mathbb{K}^m$, $\mathbb{K}^n$, and $\mathbb{K}^p$ are each finite-dimensional, and so by (1a) must be paired with spaces isomorphic to themselves. By 4.(13), $(\mathbb{K}^m)^{\S}$ should be paired with $(\mathbb{K}^m)_{\S}$, and $(\mathbb{K}^p)_{\S}$ with $(\mathbb{K}^p)^{\S}$. Since both $\langle(\mathbb{K}^m)^{\S},(\mathbb{K}^m)_{\S}\rangle$ (by 7.(7)) and $\langle\mathbb{K}^n,\mathbb{K}^n\rangle$ (by (1a)) are maximal, r and σ are compatible by 7.(2).

Thus, the entire diagram below is commutative in $\mathbb{K}$-$\mathbb{DP}$, with each map compatible.

$$\begin{array}{ccccccc}
\langle \mathbb{K}^m, \mathbb{K}^m \rangle & \xrightarrow{in_o} & \langle (\mathbb{K}^m)^{\S}, (\mathbb{K}^m)_{\S} \rangle & \xrightarrow{z} & \langle (\mathbb{K}^m)^{\S}, (\mathbb{K}^m)_{\S} \rangle & & \\
& \searrow^{G} & \downarrow r & & \downarrow r & & \\
& & \langle \mathbb{K}^n, \mathbb{K}^n \rangle & \xrightarrow{F} & \langle \mathbb{K}^n, \mathbb{K}^n \rangle & \searrow^{H} & \\
& & \downarrow \sigma & & \downarrow \sigma & & \\
& & \langle (\mathbb{K}^p)_{\S}, (\mathbb{K}^p)^{\S} \rangle & \xrightarrow{z} & \langle (\mathbb{K}^p)_{\S}, (\mathbb{K}^p)^{\S} \rangle & \xrightarrow{\pi_o} & \langle \mathbb{K}^p, \mathbb{K}^p \rangle.
\end{array}$$

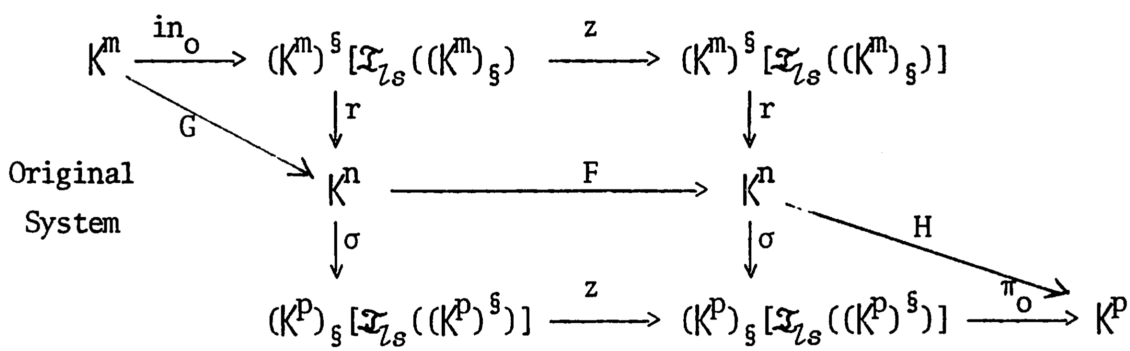
Working directly in $\mathbb{K}$-$\mathbb{DP}$, the dual of this system is obtained by transposition of both dual pairs and maps. The following diagram gives the dual of this system in $\mathbb{K}$-$\mathbb{DP}$ (Note that $z' = z$).

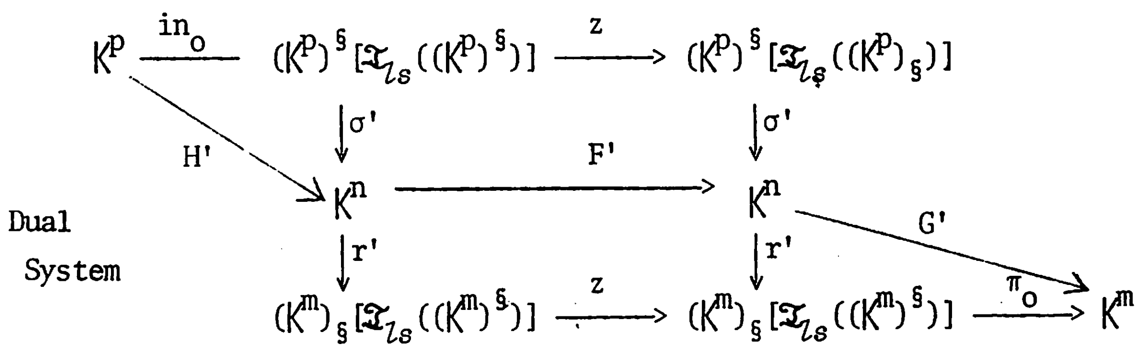
$$\begin{array}{ccccccc}
\langle \mathbb{K}^p, \mathbb{K}^p \rangle & \xrightarrow{in_o} & \langle (\mathbb{K}^p)_{\S}, (\mathbb{K}^p)^{\S} \rangle & \xrightarrow{z} & \langle (\mathbb{K}^p)^{\S}, (\mathbb{K}^p)_{\S} \rangle & & \\
& \searrow^{H'} & \downarrow \sigma' & & \downarrow \sigma' & & \\
& & \langle \mathbb{K}^n, \mathbb{K}^n \rangle & \xrightarrow{F'} & \langle \mathbb{K}^n, \mathbb{K}^n \rangle & \searrow^{G'} & \\
& & \downarrow r' & & \downarrow r' & & \\
& & \langle (\mathbb{K}^m)_{\S}, (\mathbb{K}^m)^{\S} \rangle & \xrightarrow{z} & \langle (\mathbb{K}^m)_{\S}, (\mathbb{K}^m)^{\S} \rangle & \xrightarrow{\pi_o} & \langle \mathbb{K}^m, \mathbb{K}^m \rangle.
\end{array}$$

The next step is to convert these dual pairs (via equivalence) to the proper l.t.s.'s. First note that in either case (s$\mathbb{K}$-LTS or k$\mathbb{K}$-LTS), the spaces $\mathbb{K}^m$, $\mathbb{K}^n$, and $\mathbb{K}^p$ must each carry the discrete topology by (1b).

Case 1 - (s$\mathbb{K}$-LTS). The topologies $\mathfrak{T}_{ls}((\mathbb{K}^m)_{\S})$ on $(\mathbb{K}^m)^{\S}$ and $\mathfrak{T}_{ls}((\mathbb{K}^p)^{\S})$ on $(\mathbb{K}^p)_{\S}$ must be determined. The corresponding topologies for the dual system are entirely analogous. $\langle (\mathbb{K}^m)^{\S}, (\mathbb{K}^m)_{\S} \rangle$ is a maximal dual pair, so it follows from (2) that the linear weak topology on $(\mathbb{K}^m)^{\S}$ has as a basis of neighborhoods of 0 all subspaces of finite codimension. Since each $\mathbb{K}^m$ is discrete, this may be further reduced to a basis consisting of all sets of the form $(U_o, U_1, U_2, \ldots) \subset (K^m)^§$, where $U_i = K^m$ for all but finitely many i, for which $U_i = 0$. Similarly, since a basis for $(K^p)^§$ is given by elements of the form $(0,0,\ldots,0,x_i,0,\ldots)$ where x_i ranges over the elements in a fixed basis $\{x_1,\ldots,x_p\}$ of K^p, the linear weak topology on $(K^p)_§$ has as a basis of neighborhoods of 0 sets of the form $(U_o, U_1, U_2, \ldots)$ where $U_i = K^p$ for all but finitely many i, for which $U_i = 0$. This is just the product topology.

Diagramatically, the duality of these two systems is given below. Only the nondiscrete topologies are indicated.





Note that while the dual system is not exactly that which is specified by the duality of §5, it is certainly isomorphic to it. For this example, this particular isomorphic copy was easier to characterize.

Finally, the questions of reachability and observability must be treated. By (3), the system is reachable (respectively, observable) for some image-factorization system of s$\mathbb{K}$-LTS if and only if it is reachable (respectively, observable) for every such image-factorization system . (Finite-dimensionality of the state space is crucial in this argument). The requirement is the surjectivity of r (respectively, the injectivity of σ), just as in the $\mathbb{K}$-LS case. Note, however, that if some other realization of $\sigma \circ r$ which is not finite-dimensional (such as the free realization) is considered, the properties of reachability and observability may indeed depend upon the image-factorization system under consideration.

Case 2 - (k$\mathbb{K}$-LTS). The analysis is exactly the same as in case 1, except that the *lk* rather than the *ls* topologies are used. Again, these need only be determined for the infinite-dimensional spaces. To do this, certain results which cannot be proved here are used. They are stated below; proofs can be found in [11].

(5) *Let* <E,F> *be a dual pair.*

(a) $\mathfrak{T}_{lk}(F)$ *is the strongest linear topology on* E *such that* $\tilde{F} = E$.

(b) *The topology* $\mathfrak{T}_{lk}(F^{\S})$ *on* $E_{\S}$ *is the product topology* $\prod_{i=0}^{\infty} \mathfrak{T}_{lk}(F)$. ∎

From the above, it follows that the topology $\mathfrak{T}_{lk}((\mathbb{K}^m)_{\S})$ on $(\mathbb{K}^m)^{\S}$ is the discrete topology, since $<(\mathbb{K}^m)^{\S},(\mathbb{K}^m)_{\S}>$ is maximal, by 7.(7b). The topology $\mathfrak{T}_{lk}((\mathbb{K}^p)^{\S})$ on $(\mathbb{K}^p)_{\S}$ is just the product topology (which is the same as $\mathfrak{T}_{ls}((\mathbb{K}^p)^{\S})$). The dual topologies are computed similarly. The rest of the analysis is the same as case 1.

Case 3 - ($\mathbb{K}$-LS and c$\mathbb{K}$-LTS). The system will be regarded as in $\mathbb{K}$-LS and its dual in c$\mathbb{K}$-LTS. Thus, referring to §7 and in particular 7.(7), the duality amounts to transposition of the diagram

$$
\begin{array}{ccccccc}
\mathbb{K}^m & \xrightarrow{in_o} & \langle(\mathbb{K}^m)^{\S},(\mathbb{K}^m)_{\S}\rangle & \xrightarrow{z} & \langle(\mathbb{K}^m)^{\S},(\mathbb{K}^m)_{\S}\rangle & & \\
& \searrow^{G} & \downarrow r & & \downarrow r & & \\
& & \langle\mathbb{K}^n,\mathbb{K}^n\rangle & \xrightarrow{F} & \langle\mathbb{K}^n,\mathbb{K}^n\rangle & \searrow^{H} & \\
& & \downarrow \sigma & & \downarrow \sigma & & \\
& & \langle(\mathbb{K}^p)^{\S},((\mathbb{K}^p)_{\S})^*\rangle & \xrightarrow{z} & \langle(\mathbb{K}^p)^{\S},((\mathbb{K}^p)_{\S})^*\rangle & \xrightarrow{\pi_o} & \langle\mathbb{K}^p,\mathbb{K}^p\rangle
\end{array}
$$

In terms of linear topologies, the dual system is as follows.

$$
\begin{array}{ccccccc}
\mathbb{K}^p & \xrightarrow{(\pi_o)'} & ((\mathbb{K}^p)_{\S})^*[\mathfrak{T}_{ls}((\mathbb{K}^p)_{\S})] & \xrightarrow{z'} & ((\mathbb{K}^p)_{\S})^*[\mathfrak{T}_{ls}((\mathbb{K}^p)_{\S})] & & \\
& \searrow^{H'} & \downarrow \sigma' & & \downarrow \sigma' & & \\
& & \mathbb{K}^n & \xrightarrow{F'} & \mathbb{K}^n & \searrow^{G'} & \\
& & \downarrow r' & & \downarrow r' & & \\
& & (\mathbb{K}^m)_{\S}[\mathfrak{T}_{ls}((\mathbb{K}^m)^{\S})] & \xrightarrow{z} & (\mathbb{K}^m)_{\S}[\mathfrak{T}_{ls}((\mathbb{K}^m)^{\S})] & \xrightarrow{\pi_o} & \mathbb{K}^m.
\end{array}
$$

Only the nondiscrete topologies are shown. The topology $\mathfrak{T}_{ls}((\mathbb{K}^m)^{\S})$ on $(\mathbb{K}^m)_{\S}$ is just the product topology, as shown in case 1. The topology $\mathfrak{T}_{ls}((\mathbb{K}^p)_{\S})$ on $((\mathbb{K}^p)_{\S})^*$ has no easy exemplification. Note that $((\mathbb{K}^p)_{\S})^*$ is not even algebraically isomorphic to $(\mathbb{K}^p)^{\S}$, so that the input space of this dual machine does not have the algebraic copower structure of $\mathbb{K}$-LS. Thus, this dual may be regarded as somewhat inferior to the other two. Its main purpose is to show exactly what the dual of a discretely-topologized linear machine must look like. The result below (which is proved in [11]) combined with (5b), shows that $c\mathbb{K}$-$\mathbb{LTS}$ is the only category of l.t.s.'s for which the duality with $\mathbb{K}$-$\mathbb{LS}$ works.

(6) *Let <E,F> be a dual pair.* $\mathfrak{T}_{ls}(F)$ *is the weakest linear topology on* E *making* $\tilde{F} = E'$. *If* $E[\mathfrak{T}_{ls}(F)]$ *is linearly compact, then* $\mathfrak{T}_{ls}(F) = \mathfrak{T}_{lk}(F)$. ∎

The second example to be considered is designed to show that there is a system total response $f^{\blacktriangle}\colon I^{\S} \to Y_{\S}$ such that both I and Y are finite-dimensional, yet every canonical realization of $f^{\blacktriangle}$ has an infinite-dimensional state space. *For this example only, it is assumed that* $\mathbb{K}$ *has characteristic zero.*

Let $I = Y = \mathbb{K}$, and define $f^{\blacktriangle}\colon I^{\S} \to Y_{\S}$ by

$$(i_o, i_1, i_2, \ldots) \mapsto \left(\sum_{j=o}^{\infty} \frac{1}{j+1} i_j, \sum_{j=o}^{\infty} \frac{1}{j+2} i_j, \sum_{j=o}^{\infty} \frac{1}{j+3} i_j, \ldots\right).$$

Suppose $i = (i_o, i_1, i_2, \ldots) \in \mathbb{K}^{\S}$. The equation $f^{\blacktriangle}(i) = 0$ is equivalent to the matrix equation

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \cdots \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \cdots \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \cdots \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix} \begin{bmatrix} i_o \\ i_1 \\ i_2 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} = 0 .$$

However, the $n \times n$ matrix (a_{ij}) with $a_{ij} = \frac{1}{i+j}$ is nonsingular, being a special case of Cauchy's matrix. Consult [10], p. 36 for details. Hence $i = 0$, since only finitely many of the i_k's are nonzero. This means $f^\blacktriangle$ is injective. Thus $f^\blacktriangle$ is a monomorphism in each of the categories considered, regardless of topologies or pairings. This means that for any factorization $f^\blacktriangle = m \circ e$, e is a monomorphism ($m \circ e$ monomorphic ==> e monomorphic, see [5], 6.5). Hence, for any realization of $f^\blacktriangle$, the reachability map r is injective, so the dimension of the state space Q is at least as great as the dimension of $I^\S$, which is $\aleph_0$. Hence *every* realization of $f^\blacktriangle$ has state-space dimension of at least $\aleph_0$.

Since $\langle \mathcal{K}^\S, \mathcal{K}_\S \rangle$ is maximal, by 7.7(b), $f^\blacktriangle$ is compatible by 7.(2), for the pairs $f^\blacktriangle: \langle \mathcal{K}^\S, \mathcal{K}_\S \rangle \to \langle \mathcal{K}_\S, \mathcal{K}^\S \rangle$. Furthermore, since the above matrix is symmetric, it is easy to see that $f^{\blacktriangle\prime} = f^\blacktriangle$ for these pairings.

The topologies on $I^\S$ and $Y_\S$ are exactly as developed in the first example, and so need not be repeated. The structure of the canonical realizations for s$\mathcal{K}$-LTS and k$\mathcal{K}$-LTS, however, are quite interesting and will be analyzed in detail. The analysis will be done in $\mathcal{K}$-DP first.

The problem is to factor $f^\blacktriangle: \langle \mathcal{K}^\S, \mathcal{K}_\S \rangle \to \langle \mathcal{K}_\S, \mathcal{K}^\S \rangle$ in each of the three image-factorization systems (open surjections, injections), (surjections, embeddings), and (dense maps, closed embeddings). The factorization

$$\langle \mathcal{K}^\S, \mathcal{K}_\S \rangle \xrightarrow{\;1\;} \langle \mathcal{K}^\S, \mathcal{K}_\S \rangle \xrightarrow{\;f^\blacktriangle\;} \langle \mathcal{K}_\S, \mathcal{K}^\S \rangle$$

is clearly a (open surjections, injections) factorization. The factorization $f^\Delta \circ 1$ is also a (surjections, embeddings) factorization. However, the middle pairing must be adjusted so that $f^\blacktriangle$ is an embedding. Since $f^\blacktriangle$ is injective, $f^{\blacktriangle'}$ is dense. Since $f^{\blacktriangle'} = f^{\blacktriangle}$, $<K^{\S}, f^{\blacktriangle}(K^{\S})>$ is a dual pair under the operations induced by $<K^{\S}, K_{\S}>$. Furthermore, $f^{\blacktriangle}: <K^{\S}, f^{\Delta}(K^{\S})> \to <K_{\S}, K^{\S}>$, and is clearly an embedding by 4.(8), since its transpose is a surjection by design. Since $1: <K^{\S}, K_{\S}> \to <K^{\S}, f^{\blacktriangle}(K^{\S})>$ is compatible by 7.(2), it follows that

$$<K^{\S}, K_{\S}> \xrightarrow{1} <K^{\S}, f^{\Delta}(K^{\S})> \xrightarrow{f^{\blacktriangle}} <K^{\S}, K_{\S}>$$

is a (surjections, embeddings) factorization of $f^{\blacktriangle}$. Finally, since $f^{\Delta'} = f^{\Delta}$, it follows that $f^{\blacktriangle}$ is dense. Thus

$$<K^{\S}, K_{\S}> \xrightarrow{f^{\blacktriangle}} <K_{\S}, K^{\S}> \xrightarrow{1} <K_{\S}, K^{\S}>$$

is a (dense maps, closed embeddings) factorization of f^{Δ}.

Thus, there are at least three distinct concepts of canonical realization for this system. Since $f^{\blacktriangle'} = f^{\blacktriangle}$ for the pairs $<K^{\S}, K_{\S}>$, $<K_{\S}, K^{\S}>$ it is easy to describe the duals of this system. In view of 4.(10) -4.(12), a dual of the (open surjections, injections) factorization is the (dense maps, closed embeddings) factorization and conversely. A dual of the (surjections, embeddings) factorization is just that factorization itself. Note that in two of the three factorizations, the definitions of reachability and observability change, even though the category remains the same.

The topology of the state space in each of these cases is easy to determine. In the (open surjections, injections) case, it is the same as the input space, and in the (dense maps, closed embeddings) case, it is the same as the output space. Only in the (surjections, embeddings) case is additional analysis necessary.

In the s$\mathcal{K}$-$\mathcal{LTS}$ case, the topology $\mathfrak{T}_{ls}(f^{\blacktriangle}(\mathcal{K}^{\S}))$ on $\mathcal{K}^{\S}$ is just the subspace topology of $\mathcal{K}_{\S}[\mathfrak{T}_{ls}(\mathcal{K}^{\S})]$, by 5.(7c). However, note that this subspace topology is induced via the map $f^{\blacktriangle}$, and not the canonical injection.

In the k$\mathcal{K}$-$\mathcal{LTS}$ case, the topology $\mathfrak{T}_{lk}(f^{\blacktriangle}(\mathcal{K}^{\S}))$ on $\mathcal{K}^{\S}$ is not necessarily the subspace topology of $\mathcal{K}_{\S}[\mathfrak{T}_{lk}(\mathcal{K}^{\S})]$, by 6.(6a). The characterization of this topology is not simple and will not be treated here.

The rest of the details of this example are similar to those of the first example and will not be repeated here.

The realization in $\mathcal{K}$-$\mathcal{LS}$ and its dual in c$\mathcal{K}$-$\mathcal{LTS}$ raise no new ideas over those discussed in the first example, since each category has only one image-factorization system. Hence this model will not be analyzed here.

Needless to say, the full power of this theory has not been illustrated by these examples, since both I and Y may be infinite-dimensional. The general theory handles such cases as easily as it handles the most basic case. However, examples of such, while simple in principle, are extremely complicated in terms of illustrating the topologies.

§9 Hilbert-Space Duality

This section outlines a technique which gives a duality theory for discrete-time linear systems in the category of Hilbert spaces. It is assumed that the reader knows the terminology and classical results of Hilbert-space theory.

In this section, let $\mathcal{K}$ denote either the real field $\mathcal{K}$ or the complex field $\mathcal{C}$, each with its usual nondiscrete valuated topology. $\mathcal{K}$ - $\mathfrak{H}$ denotes the category whose objects are Hilbert spaces over $\mathbb{K}$ and whose morphisms are continuous linear maps f: E → F which map the unit ball of E into the unit ball of F. In this case, the adjoint f': F → E maps the unit ball of F into the unit ball of E. Hence, there is an automorphism $\mathfrak{Y}$ of $\mathbb{K}$ - $\mathfrak{H}$ given by the identity on objects and f ↦ f' on morphisms. Furthermore, $\mathfrak{Y}$ is its own inverse, so that $(\mathfrak{Y},\mathfrak{Y})$ is a dual isomorphism of $\mathbb{K}$ - $\mathfrak{H}$. Hence, as outlined in §3, it suffices to exhibit a construction for countable power and copower, and some image-factorization systems, in order that $\mathbb{K}$ - $\mathfrak{H}$ have a duality theory for linear systems.

To construct a countable copower in $\mathbb{K}$ - $\mathfrak{H}$, let $\ell^2(E)$ denote the space of all sequences $(e_o,e_1,e_2,\ldots)$ on the Hilbert space E for which $\sum_{i=o}^{\infty}|e_i|^2$ exists. $\ell^2(E)$ is a Hilbert space with the inner product $<(e_o,e_1,e_2,\ldots),(f_o,f_1,f_2,\ldots)> = \sum_{i=o}^{\infty} <e_i,f_i>$. Denote by in_k: $E \to \ell^2(E)$ $(k\epsilon w)$ the map $e_o \mapsto (0,0,\ldots 0,e,0,\ldots)$, with the e in the i^{th} place. $(\ell^2(E),\{in_i|i\epsilon w\})$ is a countable copower for E. By duality, $(\ell^2(E)$, $\{(in_i)'|i\epsilon w\})$ is a countable power for E.

$\mathbb{K}$ - $\mathfrak{H}$ has two interesting image-factorization systems, (quotient isometries, injections) and (dense maps, closed isometries). The functor $\mathfrak{Y}$ transforms each of these to the other. The author does not know of an image-factorization system for $\mathbb{K}$ - $\mathfrak{H}$ which is self-dual.

§10 Remarks Concerning the Literature

[1], [8], [9], and [14] each contain arrow-theoretic approaches to duality theory for discrete-time linear systems, each restricted to the case in which the input, state, and output spaces are finite-dimensional.

In [8] and [9], Kalman works directly with the total response $f^{\blacktriangle}: I^{\S} \to Q \to Y_{\S}$, algebraically transposing it to get $(f^{\blacktriangle})^*$: $(Y_{\S})^* \to Q^* \to (I^{\S})^*$ for the total response of the dual system. Unfortunately, neither $I^{\S}$ nor $Y_{\S}$ are reflexive, unless they are 0, so that this approach is in error. In [14], Rissanen and Wyman note this fact, and constructed from scratch a topology T for $Y_{\S}$ which amounts to $\mathfrak{T}_{lk}((Y^*)^{\S})$, and then take the topological rather than algebraic dual of $Y_{\S}$. Since $(I^{\S})^* = (I^*)_{\S}$ (recall that I is finite-dimensional), the transposition to $(f^{\blacktriangle})'$: $(Y_{\S}[T])' \to Q^* \to (I^{\S})^*$ does yield a total response for a finite-dimensional system. Thus, in a sense, [14] may be interpreted to be a special case (finite-dimensional) of the Mackey duality (§6) presented in this paper.

In [1], Arbib and Manes present a categorical approach to duality *within* a category for finite-dimensional systems. They, of course, use the decomposable-system framework, rather than working directly with f^{Δ}. They postulate that the category K has a subclass F of "finite-dimensional" objects and a "transposition rule" $*$: $K(A,B) \to K(B,A)$ for all $A,B \in F$. The machine $M = (Q,F,I,G,Y,H)$ is dualized to $M^* = (Q,F^*,Y,H^*,I,G^*)$ and the reachability and observability maps (which operate partially on the "infinite-dimensional" spaces $I^{\S}$ and $Y_{\S}$) are cleverly constructed using the universal properties of $I^{\S}$ and $Y_{\S}$. Their dual machine M^* is *algebraically* isomorphic to the dual machine which is constructed using the Weak or Mackey dualities of this paper. Their approach, however, does not appear to be readily extendible to infinite-dimensional systems.

Linearly-topologized spaces have been around since at least 1942 ([3] and [12]). However, they appear to be relatively unknown outside of a few special areas of mathematics. [11] is the only known systematic treatment of this topic to the author; [6], however, contains a few results on infinite-dimensional linear spaces. The present paper certainly appears to be the first to use l.t.s.'s in system theory.

Finally, it should be noted that the categorical theory of decomposable systems has recently been extended to the time-varying case by Arbib and Manes [2]. It would certainly appear that the duality theory presented in this paper can be extended to the time-verying case using the approach in [2], although the details have not been worked out.

APPENDIX 1 COMPARISON OF LINEARLY-TOPOLOGIZED AND TOPOLOGICAL VECTOR SPACES

The purpose of this section is to briefly show the relationship between the theory of linearly-topologized spaces and topological vector spaces over $\mathbb{R}$ (real numbers) and $\mathbb{C}$ (complex numbers). For the purposes of this section, a topological vector space over $\mathbb{R}$ or $\mathbb{C}$ will mean a vector space E, together with a separated topology T such that the operations of addition and scalar multiplication are continuous when the field is given its usual *nondiscrete* valuated topology. Let $E[T]$ be a topological vector space over $\mathbb{R}$ or $\mathbb{C}$, and let $\mathcal{U}$ denote the neighborhood base at 0 for $E[T]$. By [11], §15, 1.(2), one of the conditions which must satisfy is:

For each $U \in \mathcal{U}$ *and each* $x \in E$ *there is a positive integer* n *such that* $x \in nU$.

In words, this condition says that each neighborhood of 0 is absorbent (radial at 0). However, the only linear subspace of E which has this property is clearly E itself. Thus,

(1) *If* $E[T]$ *is both a l.t.s and a topological vector space over* $\mathbb{R}$ *or* $\mathbb{C}$, *then* $E \cong 0$. ∎

Hence, for all practical purposes, the two concepts are disjoint. The reason for choosing linear topologies for the basis of this paper lies in the fact that they apply to linear spaces over any field. The theory of topological vector spaces applies only to $\mathbb{R}$ and $\mathbb{C}$.

Also, the theory developed in this paper shows that the essence of the theory is algebraic and not topological, and so the introduction of topological vector spaces would be a needless tangent. However, the reader familiar with the duality theory of locally-convex spaces will note that when the field is $\mathbb{R}$ or $\mathbb{C}$, the linear topologies used in sections 5-7 may be replaced by their locally-convex counterparts, thus yielding isomorphic theories. In the notation of [11], $\mathfrak{T}_{ls}$ is replaced by $\mathfrak{T}_s$ and $\mathfrak{T}_{lk}$ by $\mathfrak{T}_k$.

APPENDIX 2 IMAGE FACTORIZATIONS AND EQUIVALENCES

Recall that in a category K, an image-factorization system is a pair $(\mathcal{E},\mathcal{M})$ satisfying the following axioms:

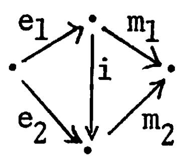
A1. $\mathcal{E}$ consists of epimorphisms and $\mathcal{M}$ consists of monomorphisms.

A2. Both $\mathcal{E}$ and $\mathcal{M}$ contain all isomorphisms.

A3. Both $\mathcal{E}$ and $\mathcal{M}$ are closed under composition.

A4. Each morphism f has a factorization $f = m \circ e$, with $e \in \mathcal{E}$ and $m \in \mathcal{M}$.

A5. If $f = m_1 \circ e_1 = m_2 \circ e_2$ with $e_1, e_2 \in \mathcal{E}$ and $m_1, m_2 \in \mathcal{M}$, there is an isomorphism i such that



commutes.

In [5], A2 is replaced by the following axiom:

A2'. Both $\mathcal{E}$ and $\mathcal{M}$ are closed under composition with isomorphisms.

To use the results of [5], it must be shown that these axiom systems are equivalent.

(1) *The axioms* A1,A2,A3,A4,A5 *are equivalent to* A1,A2',A3,A4,A5.

Proof: Clearly A2 and A3 imply A2'. Conversely, suppose A1,A2',A3,A4,A5 hold. Let 1 be an identity, and let $1 = m \circ e$ with $e \in \mathcal{E}$ and $m \in \mathcal{M}$. $e = e \circ m \circ e$ and since e is an epimorphism, $e \circ m = 1$. Hence both e and m are isomorphisms. Now if i is an isomorphism, $i = i \circ e \circ e^{-1} = i \circ m \circ m^{-1}$. Hence, by A2', $i \in \mathcal{E}$ and $i \in \mathcal{M}$. Hence A2 holds. ∎

Let F: $K \to K$ be a functor, and let P be a property of objects, morphisms, or diagrams. F <u>preserves</u> P provided that for each object, morphism, or diagram k in K with property P, F(k) also has property P. F <u>reflects</u> P provided that for each object, morphism or diagram h in H with property P, F(k) = h implies that k has property P.

Recall that a functor F: $K \to H$ is an <u>equivalence</u> provided that there is a functor G: $H \to K$ and natural isomorphisms η: $1_K \to G \circ F$ and τ: $1_H \to F \circ G$.

A functor F: $K \to H$ is <u>faithful</u> provided that for each pair of K-objects (A,B), the restricted morphism function $F_{A,B}$: $K(A,B) \to H(F(A),F(B))$ is injective, and <u>full</u> if the same function is surjective. F is <u>representative</u> (<u>dense</u> in [5]) if every H-object C is isomorphic to F(D) for some K object D.

(2) *A functor is an equivalence if and only if it is faithful, full, and representative.*

Proof: Consult [5], 14.11. ∎

(3) *Every equivalence preserves and reflects epimorphisms, monomorphisms, and isomorphisms.*

Proof: Consult [4], 12.10. ∎

Let $\mathcal{F}$ be any class of K morphisms. The smallest class of K morphisms which contains $\mathcal{M}$ and all isomorphisms and which is closed under composition is called the closure of $\mathcal{F}$ and is denoted $\overline{\mathcal{F}}$.

(4) *Let* E: $K \to H$ *be an equivalence, and let* $\mathcal{F}$ *be a class of* K*-morphisms. If* $\mathcal{F} = \overline{\mathcal{F}}$, *then each* f $\in \overline{E(\mathcal{F})}$ *is of the form* $i \circ E(g) \circ j$, *with* g $\in \mathcal{E}$, *and* i *and* j *isomorphisms.*

Proof: Let $f \in \overline{E(\mathcal{F})}$. f is clearly of the form $i_1 \circ E(g_1) \circ i_2 \circ E(g_2) \circ \ldots \circ i_n \circ E(g_n) \circ i_{n+1}$, where each i_k is an isomorphism and each $g_i \in \mathcal{F}$. However, each element of the form $E(g_k) \circ i_{k+1} \circ E(g_{k+1})$ can be written as $E(h)$ for some $h \in \mathcal{F}$, since E is full. The assertion now follows easily by induction. ∎

(5) *Let* $E: K \to H$ *be an equivalence.*

(a) *If* $\mathcal{E}$ *is the class of all* K *epimorphisms, then* $\overline{E(\mathcal{E})}$ *is the class of all* H *epimorphisms.*

(b) *If* $\mathcal{M}$ *is the class of all* K *monomorphisms, then* $\overline{E(\mathcal{M})}$ *is the class of all* H *monomorphisms.*

Proof: (a) Clearly $\mathcal{E} = \overline{\mathcal{E}}$, and $\mathcal{E}$ contains all isomorphisms. Hence (4) shows that each $f \in \overline{E(\mathcal{E})}$ is an epimorphism, since E preserves epimorphisms, by (3).

Conversely, suppose $f: A \to B$ is an H epimorphism. Since E is representative, there are K objects C and D and isomorphisms $i: E(C) \to A$ and $j: B \to E(D)$. Since isomorphisms are epimorphisms, $j \circ f \circ i$ is an epimorphism and since E is full, $j \circ f \circ i = E(g)$ for some K morphism g. g is an epimorphism since E reflects epimorphisms, and $f = j^{-1} \circ E(g) \circ i^{-1}$. By (4), $f \in \overline{E(\mathcal{E})}$.

(b) is dual to (a). ∎

(6) *Let* $F: K \to H$ *and* $G: H \to K$ *be equivalences of categories,* $\eta: 1_K \to G \circ F$ *and* $\tau: 1_H \to F \circ G$ *natural isomorphisms, and* $(\mathcal{E},\mathcal{M})$ *an image-factorization system for* K*.*

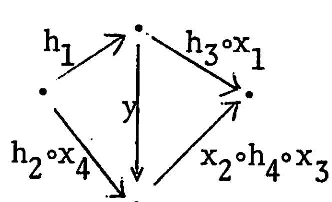
(a) $(\overline{F(\mathcal{E})},\overline{F(\mathcal{M})})$ *is an image-factorization system for* H*.*

(b) $(\overline{G(\overline{F(\mathcal{E})})},\overline{G(\overline{F(\mathcal{M})})}) = (\mathcal{E},\mathcal{M})$.

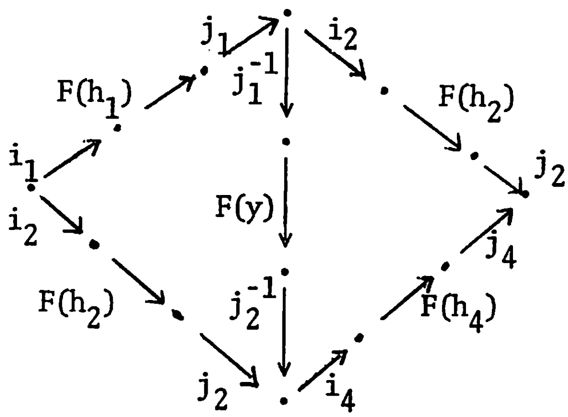
Proof: (a) By (5), $\overline{F(\mathcal{E})}$ consists of epimorphisms and $\overline{F(\mathcal{M})}$ consists of monomorphisms. By definition, each is closed under composition and contains all isomorphisms.

Let $f\colon X \to Y$ be a H morphism. Since F is representative, there are K objects A and B and isomorphisms $i\colon F(A) \to X$ and $j\colon Y \to F(B)$, and since F is full, there is a K morphism $g\colon A \to B$ with $F(g) = j \circ f \circ i$. Let $g = m \circ e$ with $e \in \mathcal{E}$ and $m \in \mathcal{M}$. $F(g) = F(m) \circ F(e)$, so $f = (j^{-1} \circ F(m)) \circ (F(e) \circ i^{-1})$ with $j^{-1} \circ F(m) \in \overline{F(\mathcal{M})}$, $F(e) \circ i^{-1} \in \overline{F(\mathcal{E})}$.

Suppose $f = m_1 \circ e_1 = m_2 \circ e_2$ with $e_1, e_2 \in \overline{F(\mathcal{E})}$ and $m_1, m_2 \in \overline{F(\mathcal{M})}$. Using (4), these can be written as $e_1 = j_1 \circ F(h_1) \circ i_1$, $e_2 = j_2 \circ F(h_2) \circ i_2$, $m_1 = j_3 \circ F(h_3) \circ i_3$, $m_2 = j_4 \circ A(h_4) \circ i_4$, where each i_k and j_k is an isomorphism, and $h_1, h_2 \in \mathcal{E}$, $h_3, h_4 \in \mathcal{M}$. By assumption, $j_3 \circ F(h_3) \circ i_3 \circ j_1 \circ F(h_1) \circ i_1 = j_4 \circ F(h_4) \circ i_4 \circ j_2 \circ F(h_2) \circ i_2$, or $F(h_3) \circ i_3 \circ j_1 \circ F(h_1) = j_3^{-1} \circ j_4 \circ F(h_4) \circ i_4 \circ j_2 \circ F(h_2) \circ i_2 \circ i_1^{-1}$. Since F is full and reflects isomorphisms, there are K isomorphisms x_1, x_2, x_3, x_4 with $F(x_1) = i_3 \circ j_1$, $F(x_2) = j_3^{-1} \circ j_4$, $F(x_3) = i_4 \circ j_2$, and $F(x_4) = i_2 \circ i_1^{-1}$. Hence $F(h_3) \circ F(x_1) \circ F(h_1) = F(x_2) \circ F(h_4) \circ F(x_3) \circ F(h_2) \circ F(x_4)$, or $F(h_3 \circ x_1 \circ h_1) = F(x_2 \circ h_4 \circ x_3 \circ h_2 \circ x_4)$. Since F is faithful, $h_3 \circ x_1 \circ h_1 = x_2 \circ h_4 \circ x_3 \circ h_2 \circ x_4$. By definition of image-factorization system, there is an isomorphism y such that



commutes, since $h_1, h_2 \circ x_4 \in \mathcal{E}$, and $h_3 \circ x_1, x_2 \circ h_4 \circ x_3 \in \mathcal{M}$. Finally, applying F to the above diagram and reshaping,



commutes. Since F preserves isomorphisms, F(y) is an isomorphism, so the above diagram shows that the factorization is unique up to isomorphism.

(b) By (a), $(\overline{G(\overline{F(\mathcal{E})})}, \overline{G(\overline{F(\mathcal{M})})})$ is an image-factorization system for K. In particular, $\overline{G(\overline{F(\mathcal{E})})}$ contains all isomorphisms and is closed under composition. By (4), $f \in \overline{G(\overline{F(\mathcal{E})})}$ is of the form $f = i_1 \circ G(j_1 \circ F(g) \circ j_2) \circ i_2$ for some $g \in \mathcal{E}$, j_1, j_2 H isomorphisms and i_1, i_2 K isomorphisms. Thus, $f = i_1 \circ G(j_1) \circ G(F(g)) \circ G(j_2) \circ i_2$. From the diagram

$$
\begin{array}{ccc}
A & \xrightarrow{\ \eta_A\ } & G\circ F(A) \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle G\circ F(g)} \\
B & \xrightarrow{\ \eta_B\ } & G\circ F(B)
\end{array}
$$

with η_A, η_B isomorphisms, it follows that $G \circ F(g) = \eta_B \circ g \circ \eta_A^{-1}$. Hence $f = i_1 \circ G(j_1) \circ \eta_B\, g \circ \eta_A^{-1} \circ G(j_2) \circ i_2$. Now $G(j_1)$ and $G(j_2)$ are K isomorphisms, since G preserves isomorphisms. Hence $f \in \mathcal{E}$, so $\overline{G(\overline{F(\mathcal{E})})} \subset \mathcal{E}$. The opposite inclusion is obvious, so that $\overline{G(\overline{F(\mathcal{E})})} = \mathcal{E}$.

The proof that $\overline{G(\overline{F(\mathcal{M})})} = \mathcal{M}$ is dual. ∎

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