A CATEGORICAL APPROACH

TO

CONTINUOUS-TIME LINEAR SYSTEMS

by

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COINS Technical Report 76-8
July, 1976

COMPUTER AND INFORMATION SCIENCE

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This research is part of the author's doctoral dissertation at the University of Massachusetts at Amherst. The author wishes to thank Dr. M. A. Arbib and Dr. E. G. Manes for their help and advice during its preparation. He also wishes to thank Dr. T. A. Cook for many helpful discussions on topological vector spaces. This research was supported by the National Science Foundation under grant DCR72-03733 A01. The University of California, Lawrence Livermore Laboratory, provided typing services.

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ABSTRACT

Three basic problems of system theory are (a) specification of external behavior from internal dynamics, (b) realization of internal dynamics from external behavior, and (c) description of system characteristics in terms of characteristics of its dual. All of these problems are investigated for linear, continuous-time systems within a categorical framework.

The crucial observation for (a) is that, for many categories K of locally-convex spaces, the forgetful functor from the category of differentiable semigroups over K to K has both a left and a right adjoint. This gives a categorical construction of the input-output behavior of any linear-differential system whose state-transition map is the infinitesimal generator of a differentiable semigroup.

Realization is done within the context of an image-factorization system for the category K. Canonical realizations may be obtained for a wide variety of such image-factorization systems.

The categories of differentiable semigroups over the categories of Mackey spaces and weak spaces provide examples for which the general duality theory applies. The usual results of duality between reachability and observability are obtained within this context.

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INTRODUCTION

In Kalman's algebraic theory of discrete-time linear systems over a field K [28], a system with input space I, output space Y, and state space Q all finite-dimensional vector spaces, described by equations of the form

$$q(t+1) = f(q(t)) + g(i(t))$$
$$y(t) = h(q(t)) \tag{1}$$

with $f\colon Q \to Q$ the state-transition map, $g\colon I \to Q$ the input map, and $h\colon Q \to Y$ the output map is given. t (an integer) represents the current time, q(t), i(t), and y(t) represent, respectively, the state, input, and output at time t. The input-output behavior of such a system can be described by a K[z]-module homomorphism between a free module Ω (the space of input sequences) and a cofree module Γ (the space of output sequences). Conversely, a K[z]-module homomorphism between a free module Ω over I and a cofree module Γ over Y defines uniquely (up to isomorphism) a reachable and observable discrete-time linear system specified by equations as those above (although the state space need not be finite-dimensional).

A main purpose of this report is to show that it possible to develop analogous techniques for continuous-time linear systems. The internal behavior for such an approach would logically be described by equations of the form

$$\frac{dq(t)}{dt} = f(q(t)) + g(i(t))$$

$$y(t) = h(x(t)) \quad , \tag{2}$$

where I, Y, and Q are now finite-dimensional vector spaces over the real field or the complex field. However, just as later work (see Arbib and Manes [3]) showed that the assumptions of finite-dimensionality were not necessary to the Kalman approach, so too it is possible to consider differential equations over vector spaces which are not finite-dimensional. The appropriate starting point to such an extension is to view f (in the finite-dimensional case) as the infinitesimal generator of the one-parameter semigroup $T(t) = e^{ft}$. If an input signal drives the system to state q_o at time $t = 0$, the state of the system at each $t \geq 0$ (for zero input after $t = 0$) is just $e^{ft}(q_o)$. There are basically two ways to extend this to infinite-dimensional spaces, and these are in turn grounded in the two great theories of locally-convex spaces, the theory of normed spaces and the theory of nuclear spaces. These extensions will be entirely disjoint, since the only nuclear spaces which are normable are the finite-dimensional ones. The theory of one-parameter analytic semigroups on Banach spaces is well-developed (see [19], [26], [41], and [51]), and it certainly seems possible to develop an algebraic theory of continuous-time systems along these lines. However, this approach requires dealing with densely-defined linear maps which are not continuous, and all spaces involved in theory must be normable, as most of the useful properties of normable spaces are lost when they interact with other locally-convex spaces. On the other hand, there is an elegant theory of tensor products of nuclear spaces with arbitrary locally-convex spaces (including normable spaces)(see Grothendieck's monograph [22] for details). Also, spaces of infinitely-differentiable functions and spaces of distributions are generally nuclear (and generally not normable), so that derivation may be interpreted as a continuous operator, rather than a densely-defined discontinuous operator. Unfortunately, the theory of one-parameter analytic semigroups on spaces of infinitely-differentiable functions, spaces of distributions, and related spaces is not as well developed as its Banach-space counterpart. Waelbroeck's abstract [49] appears to be the only work in print. However, the development of a complete theory is reasonably straightforward, and this task is undertaken in §1 of this report. Such semigroups are termed differentiable semigroups.

Attention in this paper will be restricted to systems based in the theory of differentiable semigroups. The structure of a differential system over a subcategory K of the category of locally-convex spaces is that of a 6-tuple (Q, f, I, g, Y, h), where Q, I, and Y are K objects, and $f:Q \to Q$, $g:I \to Q$, and $h:Q \to Y$ are K morphisms, with f the infinitesimal generator of a unique differentiable semigroup T on Q. The system may be thought of as being governed by the equations (2) (in a sense to be made precise in §2); the state trajectory for time $t \geq 0$ with state q_0 at

$t = 0$ and zero input after $t = 0$ will be just $T(t)(q_o)$, in harmony with the finite-dimensional case (all differentiable semigroups on finite-dimensional spaces are of the form $t \mapsto e^{ft}$).

Returning now to the problem of producing an external behavior from an internal description such as (2), it is useful te recall the approach used in the discrete-time theory of Arbib and Manes ([1]-[6]). A category K and an endofunctor X (called the process functor) are fixed. The category Dyn(X) of X-dynamics is then formed, with objects pairs (Q, δ) with $Q \in \mathcal{O}bj(K)$ and δ: $X(Q) \to Q$ a K morphism, and morphisms $f: (Q,\delta) \to (Q',\delta')$ just K morphisms $f: Q \to Q'$ such that $\delta' \circ f = f \circ \delta$. The basic results of the theory rest upon the hypothesis that the forgetful functor U: Dyn(X) $\to K : (Q,\delta) \mapsto Q$ has a left adjoint and a right adjoint; a process having both such adjoints is called a state-behavior process. A large number of processes are state-behavior; in particular, $1_{\underline{R\text{-mod}}}$, the identity functor on the category of left R modules (R a ring) is state-behavior, and its use as a process recaptures the theory of discrete-time linear systems.

In the approach of this paper, the category Dyn(X) is repalced by the category $\mathrm{DSG}(K)$ of differentiable semigroups over K (K is a subcategory of the category of all locally-convex spaces). The crucial hypothesis here is that the forgetful functor $\mathfrak{Z}_K$: $\mathrm{DSG}(K) \to K$ taking each semigroup to its underlying locally-convex space has both a left adjoint and a right adjoint. This is shown to be the case for several important categories of locally-convex spaces. The space of input signals (corresponding to Ω above) turns out to essentially a space of I-valued distributions. The space of output signals (corresponding to Γ above) turns out to be exactly $E(\mathbb{R}_+,Y)$, the space of all infintely-differentiable functions from the non-negative reals into Y. The details are all developed in §2.

Conversely, given a differentiable-semigroup morphism k which represents the external behavior of a continuous-time system, it is shown in §3 that a unique (up to isomorphism) canonical realization of k can be obtained (with respect to a particular image-factorization system, of course). Thus, a complete theory for going from internal to external behavior, and from external to internal behavior is given in this paper.

Another mainstay of modern system theory is the duality between reachability and observability. In §4, a duality theory for continuous-time linear systems is presented which recaptures the classical results.

§0 Terminology and Notation

The purpose of this section is to explain the notation and terminology used in this paper. In general, only notation which is less-than-standard or particular to this paper is included. References to works where the reader may find discussions of standard results and terminology are provided.

Within each section, formal facts are numbered consecutively, starting with 1. Within a section, reference to a fact is made by giving its number. Reference to a fact in another section is made by giving its section followed by its number.

Category Theory

For category theory, the references [25], [37], and [43] are used.

Let K be a category. $Obj(K)$ denotes the class of K objects, and $Mor(K)$ denotes the class of K morphisms. $Mor_K(X,Y)$ denotes the set of K morphisms from the K object X to the K object Y.

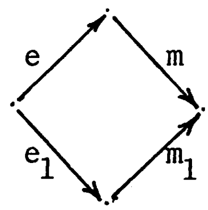
Let F: $K \to H$ and G: $K \to H$ be functors, and let η: F $\to$ G be a natural transformation. If H: $J \to K$ is a functor, η*H is the natural transformation given by $\eta*H(j) = \eta(H(j))$. If K: $H \to J$ is a functor, K*η is the natural transformation given by $K*\eta(h) = K(\eta(h))$.

A functor F: $K \to H$ is called an embedding if it is injective on morphisms (caution: MacLane [37] called a faithful functor an embedding). F is called representative if each $h \in Obj(H)$ is isomorphic to F(k) for some $k \in Obj(K)$. For any two K objects X and Y, $F_{X,Y}$: $Mor_K(X,Y) \to Mor_H(F(X),F(Y))$ denotes the restricted morphism function.

The following useful terminology is introduced for this report. Let F be a functor. A two-sided adjoint situation for F is a 5-tuple (F,L,R,η,ζ), where L and R are functors and η and ζ are natural transformations, such that L is a left-adjoint to F with unit η and R is right-adjoint to F with counit ζ.

K^{op} denotes the opposite of the category K. The convention that K^{op} has the *same* objects and morphisms as K is followed.

Because of its importance, the definition of image-factorization system is repeated here. Let K be a category. An image-factorization system for K is an ordered pair $(\mathbb{E},\mathbb{M})$, where $\mathbb{E}$ is a class of K epimorphisms and $\mathbb{M}$ is class of K monomorphisms such that both $\mathbb{E}$ and $\mathbb{M}$ are closed under composition and contain all isomorphisms, and such that every $\mathcal{K}$ morphism f has a factorization $m \circ e$ with $e \in \mathcal{E}$ and $m \in \mathcal{M}$, which is unique up to isomorphism of the middle element in the sense that if $m_1 \circ e_1$ is another such factorization, then there is an isomorphism i such that



commutes. (e,m) is called an $(\mathcal{E},\mathcal{M})$ factorization of f. It should be noted that in [25], a slightly different definition is used, but the two definitions can be shown to be equivalent ([24], App. 2).

The symbol $\cong$ is used as an abbreviation for "is isomorphic to".

Topology

The reference for topology is [13].

$\mathbb{R}_+$ denotes the non-negative reals, with the usual topology.

$\mathbb{Q}_+$ denotes the non-negative rationals.

$\mathbb{N} =$ 0,1,2,... is the set of natural numbers.

Let X and Y be sets, and let H be a set of maps from X into Y. If U is a subset of X, $H(U) = \bigcup_{h \in H} h(U)$, and if V is a subset of Y, $H^{-1}(V) = \bigcap_{h \in H} h^{-1}(V)$.

Locally-Convex Spaces

The references for the theory of locally-convex spaces include [12], [23], [27], [33], [36], [39], [40], [42], and [48].

$\mathbb{K}$ denotes either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers, each with their usual topology. $\mathbb{K}$ is to be fixed in any particular context.

l.c.s. is an abbreviation for locally-convex, *separated*, topological vector space over $\mathbb{K}$. $\mathbb{LCS}$ denotes the category whose objects are the l.c.s.'s over $\mathbb{K}$, and whose morphisms are the continuous linear maps.

$\mathcal{U}(E)$ denotes the set of all convex neighborhoods of 0 of the l.c.s. E.

A subset U of a l.c.s. E is called a barrel if it is closed, absolutely convex, and absorbing; and U is called bornivorous if it is convex and it absorbs every bounded subset of E. E is called barreled (resp. quasi-barreled, resp. bornological) if every barrel (resp. bornivorous barrel, resp. bornivorous set) is in $\mathcal{U}(E)$.

E' denotes the dual of E.

$\langle E,F\rangle$ denotes that E and F form a dual pair which separates points. Polars in dual pairs are denoted by the symbol °.

A continuous linear map f: E → F of l.c.s.'s is called a dense map if $\overline{f(E)} = F$, a homomorphism if it transforms neighborhoods of 0 in E into neighborhoods of 0 in f(E), a quotient map if it is a surjective homomorphism, an embedding if it is an injective homomorphism, and closed if f(E) is closed in F.

$\mathbb{MS}$ (resp. $\mathbb{WS}$) denotes the full subcategory of $\mathbb{LCS}$ consisting of the Mackey (resp. weak) l.c.s.'s, with $\mathfrak{Y}$: $\mathbb{WS} \to \mathbb{MS}$ the natural isomorphism.

$\mathcal{CS}$ denotes the full subcategory of $\mathcal{LCS}$ consisting of the complete spaces. The inclusion functor $\mathcal{CS} \hookrightarrow \mathcal{LCS}$ has a left adjoint, the completion functor, which is denoted by $^\wedge$. $^\wedge$ is only unique up to isomorphism, but a standard choice of completion will be assumed to be used, unless otherwise noted. (Note: A completion of E is denoted by $\hat{E}$ or $(E)^\wedge$, not $^\wedge(E)$.)

An l.c.s. is <u>quasi-complete</u> if each of its closed, bounded subsets is complete. $\mathcal{QC}$ denotes the full subcategory of $\mathcal{LCS}$ consisting of the quasi-complete spaces. The inclusion functor $\mathcal{QC} \hookrightarrow \mathcal{LCS}$ has a left adjoint, the quasi-completion functor, which is denoted $\tilde{}$. The same remarks apply to this case as to completion.

<u>Special Locally-Convex Spaces</u>

Consult, in particular, [23] and [36] for discussions of the concepts below.

An $(\mathcal{F})$ <u>space</u> is a l.c.s. which is metrizable and complete (also called a <u>Fréchet space</u>).

A $(\mathcal{DF})$ <u>space</u> is a l.c.s. which admits a fundamental sequence of bounded sets, and for which every bounded set of its strong dual which is the union of a sequence of equicontinuous sets is equicontinuous.

The strong dual of an $(\mathcal{F})$ (resp. $(\mathcal{DF})$) space is a $(\mathcal{DF})$ (resp. $(\mathcal{F})$) space.

A (B) <u>space</u> is a complete l.c.s. whose topology is defined by a single norm. *An l.c.s. is a* (B) *space if and only if it is simultaneously an* $(\mathcal{F})$ *space and a* $(\mathcal{DF})$ *space.*

An (S) space (also called a Schwartz space) is an l.c.s. for which every continuous linear mapping into a (B) space is compact.

Spaces of Linear Mappings

Consult [12], [23], and [42] for detailed discussions of these concepts.

Let E and F be l.c.s's. $L(E,F)$ denotes the linear space of all continuous linear maps from E into F. Let $\mathfrak{S}$ be a set of bounded subsets of E which is covering in E (i.e., $\cup\mathfrak{S} = E$). On $L(E,F)$, the topology of $\mathfrak{S}$ convergence is defined to have as fundamental neighborhoods of 0 sets of the form $\{f|f(A) \subset U\}$ where $A \in \mathfrak{S}$ and $U \in \mathcal{U}(F)$. The topology is denoted $L_{\mathfrak{S}}(E,F)$. The neighborhood $\{f|f(A) \subset U\}$ is denoted $N(A,U)$.

The saturated hull of $\mathfrak{S}$ is the smallest class of subsets of E containing $\mathfrak{S}$ along with arbitrary subsets, scalar multiples, and closed absolutely-convex hulls of finite families of its elements. The saturated hull is denoted $\bar{\mathfrak{S}}$; $L\ (E,F) = L_{\bar{\mathfrak{S}}}(E,F)$ always.

Three particularly important cases of the above are when $\mathfrak{S}$ = all finite sets, $\mathfrak{S}$ = all weakly-compact convex sets, and $\mathfrak{S}$ = all compact sets. They are denoted, respectively $L_s(E,F)$, $L_k(E,F)$, and $L_c(E,F)$, and are called the topology of simple (or weak or pointwise) convergence, the topology of Mackey convergence, and the topology of compact convergence.

When $F = \mathbb{K}$, $L_{\mathfrak{S}}(E,F)$ is denoted E'. E_s (resp. E_k) denotes $L_s(E'_s,\mathbb{K})$ (resp. $L_k(E'_k,\mathbb{K})$).

When $E = F$, $\mathcal{L}(E,F)$ (resp. $\mathcal{L}_{\mathfrak{S}}(E,F)$) is denoted $\mathcal{L}(E)$ (resp. $\mathcal{L}_{\mathfrak{S}}(E)$).

<u>Bilinear Maps and Tensor Products</u>

Consult [12], [22], [23], [44], [45], and [48] for detailed treatment.

Let E, F, and G be l.c.s.'s and let $F\colon E\times F \to G$ be a continuous linear map. For the purposes of this paper f is called <u>hypocontinuous</u> if it is hypocontinuous with respect to the bounded sets of each space, i.e., for each $V \in \mathcal{U}(G)$, $A \subset E$ bounded, there is a $U \in \mathcal{N}(F)$ such that $f(A\times U) \subset V$, and for each $V \in \mathcal{U}(G)$, $B \subset F$ bounded, there is a $W \in \mathcal{U}(E)$ such that $f(W\times B) \subset V$. A family of <u>equihypocontinuous</u> maps is defined similarly.

(1) *Let* E, F, *and* G *be l.c.s.'s with* G *quasi-complete (resp. conplete), and let* $f\colon E \times F \to G$ *be bilinear and hypocontinuous. If every bounded set of* $\widehat{E}$ *(resp.* $\hat{E}$*) is contained in the closure of a bounded set of* E, *and if every bounded set of* $\widehat{F}$ *(resp.* $\hat{F}$*) is contained in the closure of a bounded set of* F, *then* f *has a unique extension* $\widehat{f}\colon \widehat{E} \times \widehat{F} \to G$ *(resp.* $\hat{f}\colon \hat{E} \times \hat{F} \to G$*) which is hypocontinuous.*

Proof: Consult [23], Ch. 3, Prop 10. ∎

The notations $\widehat{f}$ and $\hat{f}$ introduced in (1) will be used in the course of the text.

On $E\otimes F$, there are several topologies of importance. The π (or <u>projective</u>) <u>topology</u> is the strongest making the canonical surjection $p\colon E \times F \to E\otimes F$ continuous, and is denoted $E\otimes_{\pi}F$. The β (or <u>hypocontinuous</u>) topology is the strongest making $p\colon E \times F \to E\otimes F$ hypocontinuous, and is denoted $E\otimes_{\beta}F$. The ι (or <u>inductive</u>) topology is the strongest making $p\colon E \times F \to E\otimes F$ separately-continuous, and is denoted $E\otimes_{\iota}F$.

Differentiable Functions and Distributions

The theory used here is essentially that developed by Schwartz [44]-[47].

$\mathcal{E}^m(\mathbb{R}_+,E)$ ($m \in \mathbb{N}$ or $m = \infty$) denotes the space of all m-times continuously-differentiable functions on the l.c.s. E. The symbol D always denotes the differentiation operator in this space. The topology of $\mathcal{E}^m(\mathbb{R}_+,E)$ is defined by the family of seminorms of the form $\phi \mapsto \sup_{\substack{t\in K \\ q\leq p}} \alpha(D^q\phi(t))$ where α is a continuous seminorm on E, $K \subset \mathbb{R}_+$ is compact, and $p \in \mathbb{N}$ with $p \leq m$. If $m = \infty$, $\mathcal{E}^m(\mathbb{R}_+,E)$ is denoted simply $\mathcal{E}(\mathbb{R}_+,E)$.

The mean-value theorem extends easily to $\mathcal{E}^1(\mathbb{R}_+,E)$.

(2) *Let* E *be a l.c.s.,* $f \in \mathcal{E}^1(\mathbb{R}_+,E)$, α *a continuous seminorm on* E, *and* $I = [a,b] \subset \mathbb{R}_+$ *a compact, connected interval with* $b > a$.

(a) $\alpha(f(b)-f(a)) \leq (b-a) \sup_{t\in I} \alpha(Df(t))$.

(b) $\alpha\left(\frac{f(b)-f(a)}{b-a} - Df(a)\right) \leq \sup_{t\in I} \alpha(Df(x)-Df(a))$.

Proof: The proof is essentially the same as the case in which E is a normed space. See for example [11], Ch. 1, 2, Th. 2 and Cor. 3., or [17], 8.5.2 and 8.6.2. ∎

In case $E = \mathbb{K}$, the space $\mathcal{E}(\mathbb{R}_+,E)$ is denoted $\mathcal{E}(\mathbb{R}_+)$. $\mathcal{E}(\mathbb{R}_+)$ may be identified as the quotient of $\mathcal{E}(\mathbb{R})$ (= infinitely differentiable functions defined on all of $\mathbb{R}$) modulo the subspace consisting of the functions which are 0 on $]-\infty,0]$. Indeed, this quotient may be identified (topologically) with subspace of $\mathcal{E}(\mathbb{R}_+)$ which contains at least the restrictions of all polynomials. However, these are clearly dense in $\mathcal{E}(\mathbb{R}_+)$ (use Stone-Weienstrass), whence the result.

The strong dual of $\mathcal{E}(\mathbb{R}_+)$ may thus be identified with $\mathcal{E}'(\mathbb{R}_+)$, the space of all distribution on $\mathbb{R}$ with their support compact and contained in $\mathbb{R}_+$. $\mathcal{E}'(\mathbb{R}_+)$ is a bornological, complete, reflexive (hence barreled) ($\mathcal{F}$) space, since $\mathcal{E}(\mathbb{R}_+)$ is an ($\mathcal{F}$) and (S) space (see [23], Ch. 4, Part 3, §4 for details).

§1 General Properties of Differentiable Semigroups

Basic Concepts

The idea of the differentiable semigroup is originally due to Waelbroeck [49]. However, his axioms are weakened here somewhat, in order to allow a larger class of semigroups.

Let E be a l.c.s. A map T: $\mathbb{R}_+ \to L_s(E)$ is called a differentiable semigroup (abbreviated d.s.g.) on E if it satisfies the following three properties.

(s_1) $T(0) = 1_E$.
(s_2) $T(t_1+t_2) = T(t_1) \circ T(t_2)$ for each $t_1, t_2 \in \mathbb{R}_+$.
(s_3) $T \in \mathcal{E}^1(\mathbb{R}_+, L_s(E))$.

The element $\lim_{t\to 0} \frac{T(t)-1}{t} \in L_s(E)$ is called the infinitesimal generator of T, and is denoted g_T. Note that the condition (s_3) says precisely that $\lim_{h\to 0} \frac{T(t+h)-T(t)}{h}(x)$ exists for each $t \in \mathbb{R}_+$, and $x \in E$, and that the assignment is linear and continuous.

(1) *Let* E *be a l.c.s., and let* T *be a d.s.g. on* E.

(a) $T \in \mathcal{E}(\mathbb{R}_+, L_s(E))$

(b) $D^pT(t) = (g_T)^p \circ T(t) = T(t) \circ (g_T)^p$, for each $p \in \mathbb{N}$.

Proof:

$$DT(t) = \lim_{\substack{h\to 0\\ h>0}} \frac{T(t+h)-T(t)}{h} = \lim_{h\to 0} T(t)\circ\frac{T(h)-1}{h} = T(t)\circ\lim_{h\to 0}\frac{T(h)-1}{h} = T(t)\circ g_t .$$

Clearly $T(t)\circ g_T = g_T\circ T(t)$. The proof is completed by induction. ∎

Let S: $\mathbb{R}_+ \to L_s(E)$ and T: $\mathbb{R}_+ \to L_s(e)$ be d.s.g.'s. A <u>d.s.g. morphism</u> from S to T is a continuous linear map f: E → F such that

$$\begin{array}{ccc} E & \xrightarrow{\;S(t)\;} & E \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ F & \xrightarrow{\;T(t)\;} & F \end{array}$$

commutes for each $t \in \mathbb{R}_+$. When no confusion can result, the d.s.g. morphism which f represents will be identified with f.

Let K be a subcategory of **LCS**. The category whose objects are d.s.g.'s on objects of K and whose morphisms are d.s.g. morphisms represented by morphisms of K is denoted **DSG**(K). **DSG**(**LCS**) is denoted by just **DSG**. Note that f is an isomorphism in **DSG**(K) if and only if it is an isomorphism in K.

A very important characterization of $E(\mathbb{R}_+,E)$ and hence d.s.g.'s will now be developed. Denote by $\Delta(\mathbb{R}_+)$ the subspace of $E'(\mathbb{R}_+)$ consisting of the distributions of which have finite support. The elements of $\Delta(\mathbb{R}_+)$ are just linear combinations of elements of the form $D^p\delta_t$, where $p \in \mathbb{N}$ and δ_t is the Dirac measure at $t \in \mathbb{R}_+$ (see [47], Ch. III, Th. XXXV). The next results gives some important properties of $\Delta(\mathbb{R}_+)$.

(2)(a) $\Delta(\mathbb{R}_+)$ *is dense in* $E'(\mathbb{R}_+)$

(b) *Every bounded subset of* $E'(\mathbb{R}_+)$ *is contained in the closure of a bounded subset of* $\Delta(\mathbb{R}_+)$.

(c) *The strong dual of* $\Delta(\mathbb{R}_+)$ *is* $E(\mathbb{R}_+)$.

(d) $\Delta(\mathbb{R}_+)$ *is a* $(\mathcal{DF})$ *space.*

(e) $\Delta(\mathbb{R}_+)$ *is quasi-barreled.*

(f) $E'(\mathbb{R}_+)$ *is a quasi-completion and a completion of* $\Delta(\mathbb{R}_+)$.

Proof: (a) is obvious.

(b) Let $U_0, U_1, U_2, \ldots$ be the fundamental sequence of barreled neighborhoods of 0 in $E(\mathbb{R}_+)$ given by $U_k = \{\phi \in E(\mathbb{R}_+) \mid \sup_{\substack{x\in[0,k]\\ q\le k}} |D^q\phi(x)| \le 1/k\}$. $E'(\mathbb{R}_+)$ is the strong dual of $E(\mathbb{R}_+)$, so $(U_0)^\circ, (U_1)^\circ, (U_2)^\circ, \ldots$ is a fundamental sequence of bounded sets in $E'(\mathbb{R}_+)$. Denote by B_k the restriction of $(U_k)^\circ$ to $\Delta(\mathbb{R}_+)$. B_k is surely bounded in $\Delta(\mathbb{R}_+)$, and it suffices to show that $(B_k)^\circ \subset U_k$, for then $(B_k)^\circ = U_k$ (all polar taken in $\langle E'(\mathbb{R}_+), E(\mathbb{R}_+)\rangle$). However, it is clear that $kD^j\delta_t \in B_k$ for each $j \le k$, $t \in [0,k]$, so that $\phi \in (B_k)^\circ \Rightarrow |D^j\phi(t)| \le 1/k$ for each $j \le k$, $t \in [0,k]$, i.e., $\phi \in U_k$.

(c) This follows immediately from (b).

(d) This follows immediately from the definition of $(\mathcal{DF})$ space since the strong dual of $\Delta(\mathbb{R}_+)$ is an (F) space.

(e) If $f \in E(\mathbb{R}_+)$ is zero at each $t \in \mathbb{Q}_+$, then clearly $f = 0$. Hence $B = \{\delta_t \mid t \in \mathbb{Q}_+\}$ is dense in $E'(\mathbb{R}_+)$, since $(B^\circ)^\circ = 0^\circ = E'(\mathbb{R}_+)$ for the pair $\langle E'(\mathbb{R}_+), E(\mathbb{R}_+)\rangle$. Hence B is *a fortiori* dense in $\Delta(\mathbb{R}_+)$. Now every separable $(\mathcal{DF})$ space is quasi-barreled ([36], §29, 3.(12)), and since B is clearly countable, it follows from (d) that $\Delta(\mathbb{R}_+)$ is quasi-barreled.

(f) $\mathcal{D}'(\mathbb{R}_+)$ is complete. Hence it is a completion of $\Delta(\mathbb{R}_+)$ since $\Delta(\mathbb{R}_+)$ is dense in $E'(\mathbb{R}_+)$ by (a). $E'(\mathbb{R}_+)$ is also a quasi-completion of $\Delta(\mathbb{R}_+)$ since $\Delta(\mathbb{R}_+)$ is a $(\mathcal{DF})$ space, by (d), and a quasi-complete $(\mathcal{DF})$ space is complete ([23], Ch. 4, Part 3, Prop. 4, Cor. 2). ∎

Let E be a l.c.s. Define the map Φ_E: $\Delta(\mathbb{R}_+) \times \mathcal{E}(\mathbb{R}_+,E) \to E$ by $(D^p\delta_t,\phi) \to D^p\phi(t)$. This map is clearly bilinear. Much more is true, in fact. First, a notation is given. If $\phi \in \mathcal{E}(\mathbb{R}_+,E)$ and $e' \in E'$, $<\phi, e'>$ denotes the function $t \mapsto <\phi(t),e'>$, a slight abuse of notation. $t \mapsto <\phi(t),e'>$ is clearly an element of $\mathcal{E}(\mathbb{R}_+)$.

(3) *Let* E *be a l.c.s.*

(a) *For* $x \in \Delta(\mathbb{R}_+)$ $\phi \in \mathcal{E}(\mathbb{R}_+,e)$, and $e' \in E'$, $<\Phi_E(x,\phi),e'> = x(<\phi,e'>)$.

(b) Φ_E *is hypocontinuous.*

Proof: (a) is immediate.

(b) Let $V \in \mathcal{U}(E)$ be a barrel. Let A be an absolutely-convex closed bounded subset of $\mathcal{E}(\mathbb{R}_+,E)$, and let B be an absolutely-convex closed bounded subset of $\Delta(\mathbb{R}_+)$. It suffices to find $U \in \mathcal{U}(\Delta(\mathbb{R}_+))$ and $W \in \mathcal{U}(\mathcal{E}(\mathbb{R}_+,E))$ such that $\Phi_E(U\times A) \subset V$ and $\Phi_E(B\times W) \subset V$. It suffices also to assume that V is the closed unit semiball of a continuous seminorm α on E.

A is bounded, hence for any compact $K \subset \mathbb{R}_+$, $p \in \mathbb{N}$,

$$\sup_{\phi\in A}\ \sup_{\substack{x\in K\\ q\le p}} \alpha(D^q\phi(x)) = M < \infty \quad .$$

Hence,

$$\sup_{\substack{\phi\in A\\ e'\in V}}\ \sup_{\substack{x\in K\\ q\le p}} |<\phi^q(x),e'>| \le M,$$

i.e., $\{<\phi,e'>|\phi \in A,\ e' \in V\}$ is bounded in $\mathcal{E}(\mathbb{R}_+)$. Put

$$U = \{\langle\phi,e'\rangle \mid \phi \in A,\ e' \in V^\circ\}^\circ .$$

$$\sup_{\substack{x\in U\\ \phi\in A\\ e'\in V^\circ}} |\langle\Phi_E(x,\phi),e'\rangle| = \sup_{\substack{x\in U\\ \phi\in A\\ e'\in V^\circ}} |x(\langle\phi,e'\rangle)| = \sup_{\substack{x\in U\\ y\in U^\circ}} |x(y)| = 1 .$$

Hence $\Phi_E(U\times A) \subset V$.

Next, B is the polar of a neighborhood of 0 in $\mathcal{E}(\mathbb{R}_+)$ (polar for the pair $\langle\Delta(\mathbb{R}_+), \mathcal{E}(\mathbb{R}_+)\rangle$). Say $B^\circ = \{\phi \in \mathcal{E}(\mathbb{R}_+) \sup_{\substack{x\in K\\ q\leq p}} |D^p\phi(x)|<\xi\}$ for some $K \subset \mathbb{R}_+$ compact, $p \in \mathbb{N}$, $\xi > 0$, without loss of generality. Put $W = \{\phi \in \mathcal{E}(\mathbb{R}_+,E) \sup_{\substack{x\in K\\ q\leq p}}\alpha(D^p\phi(x)) \leq \xi\}$. Now

$$\sup_{\substack{x\in B\\ \phi\in W\\ e\in V^\circ}} |\langle\Phi_E(x,\phi),e'\rangle| = \sup_{\substack{x\in B\\ \phi\in W\\ e'\in V^\circ}} |x(\langle\phi,e'\rangle)| = 1 ,$$

since $\phi \in W$, $e' \in V \Rightarrow \langle\phi,e'\rangle \in B^\circ$. Hence $\Phi_E(B\times W) \subset V$. ∎

$\Phi_E(x,\phi)$ is called the <u>extension of</u> x <u>to vector-valued functions</u>. If E is quasi-complete (for example), then Φ_E has a unique (hypocontinuous) extension Φ_E: $\mathcal{E}'(\mathbb{R}_+) \times \mathcal{E}(\mathbb{R}_+,E) \to E$, in view of (2b and e). It is easy to see that this extension is exactly the extension of distributions to vector-valued functions, as developed (in an entirely different manner) by Schwartz [45].

With these preliminary results, the characterization theorem for $\mathcal{E}(\mathbb{R}_+,E)$ may be stated and proved.

(4) *Let* E *be a l.c.s. There is an isomorphism* i: $\mathcal{E}(\mathbb{R}_+,E) \to L_s(\Delta(\mathbb{R}_+),E)$ *given by* $\phi \mapsto (x \mapsto \Phi_E(x,\phi))$. *The inverse of this map is* $f \mapsto (t \mapsto f(\delta_t))$.

Proof: It is easy to see that $f \mapsto (t \mapsto f(\delta_t))$ is both a left and a right inverse to i, so that i is a bijection. The continuity of i is an immediate consequence of the hypocontinuity of Φ_E. To show that i^{-1} is continuous, let $U \in N(\mathcal{E}(\mathbb{R}_+,E))$. Without loss of generality, it suffices to assume that $U = \{\phi \in \mathcal{E}(\mathbb{R}_+,E) \mid \sup_{\substack{t\in K\\ q\le p}} \alpha(D^p\phi(t)) \le 1\}$ for some compact $K \subset \mathbb{R}_+$, $p \in \mathbb{N}$, and α a continuous seminorm on E. Now set V equal to the unit ball of α, and let $A = \{D^q\delta_t \in \Delta(\mathbb{R}_+) \mid q \le p \text{ and } t \in K\}$. Clearly $i^{-1}(\{f \mid f(A)\subset V\}) \subset U$, so it suffices to show that A is bounded. However $A^\circ = \{\psi \in \mathcal{E}(\mathbb{R}_+) \ \sup_{\substack{x\in K\\ q\le p}} \mid D^q\psi(z)\mid \le 1\}$, which is in $U(\mathcal{E}(\mathbb{R}_+))$. Hence A is bounded, and so i^{-1} is continuous. ∎

Besides giving a characterization of $\mathcal{E}(\mathbb{R}_+,E)$, (4) is very useful in characterizing d.s.g.'s. By (1a), each d.s.g. T on the l.c.s. E is in $\mathcal{E}(\mathbb{R}_+,L_s(E))$; hence by (4), in $L_b(\Delta(\mathbb{R}_+),L_s(E))$. However, not every $T \in L_b(\Delta(\mathbb{R}_+),L_s(E))$ is a d.s.g., for (s_1) and (s_2) must also be satisfied. However, if $\Delta(\mathbb{R}_+)$ is regarded as an algebra under convolution $D^{p_1}\delta_{t_1} * D^{p_2}\delta_{t_2} = D^{p_1+p_2}\delta_{t_1+t_2}$, and $L_s(E)$ as an algebra under composition, then it is easy to see that $T \in L_b(\Delta(\mathbb{R}_+),L_s(E))$ is a d.s.g. if and only if it is an algrebra homomorphism.

Differential Equations

The problem of recovering a d.s.g. from its infinitesimal generator will now be investigated. As will be seen, this can always be accomplished.

(5) *Let* E *be a l.c.s., and let* T *be a d.s.g. on* E.

(a) *For every* $e \in E$, *the map* $f: \mathbb{R}_+ \to E$ *given by* $t \mapsto T(t)x$ *is in* $\mathcal{E}(\mathbb{R}_+,E)$.

(b) *The canonical map* $\Lambda_{T,E}: E \to \mathcal{E}(\mathbb{R}_+,E)$ *given by* $x \mapsto (t \mapsto T(t)x)$ *is continuous.*

Proof: The proofs are immediate. ∎

The notation $\Lambda_{T,E}$ introduced in (4) will be used as a definition later in the paper.

For purposes of this paper, a linear differential equation on the l.c.s. E is an equation of the form Du = Au, with $A \in \mathcal{L}(E)$. A solution to this linear differential equation with initial condition $u(0) = x$ ($x \in E$) is a $f \in \mathcal{E}^1(\mathbb{R}_+,E)$ such that $f(0) = x$, and for each $t \in \mathbb{R}_+$, $(Df)(t) = A(f(t))$. A differential equation for which A is the infinitesimal generator of a d.s.g. has the unique-solution property, and will now be shown.

(6) *Let* $E[\mathcal{T}]$ *be a l.c.s.,* $f \in \mathcal{E}(\mathbb{R}_+,E[\mathcal{T}])$. *If* $\mathcal{T}_1$ *is another topology on* E *which has a base of neighborhoods of* 0 *consisting of bornivorous barrels of* $\mathcal{T}$, *then* $f \in \mathcal{E}(\mathbb{R}_+,E[\mathcal{T}_1])$.

Proof: In view of (4), f may be regarded as an element of $\mathcal{L}_b(\Delta(\mathbb{R}_+), E[\mathcal{T}])$. However, since $f: \Delta(\mathbb{R}_+) \to E[\mathcal{T}]$ is continuous, the inverse image by f of a bornivorous barrel is a bornivorous barrel. However, $\Delta(\mathbb{R}_+)$ is quasi-barreled, by (2e), and so $f: \Delta(\mathbb{R}_+) \to E[\mathcal{T}_1]$ is continuous. ∎

(7) *Let* E *and* F *be l.c.s.'s and let* $\mathcal{B}$ *be a covering set of bounded subsets of* E. *For every* $A \subset E$ *and every barrel* $U \in N(F)$, $N(A,U)$ *is closed in* $L_{\mathcal{B}}(E,F)$.

Proof: It suffices to show that for each $x \in E$ and $U \in U(F)$ barreled, $N(\{x\},U)$ is closed in $L_s(E,F)$, for if this is true, then for any $A \subset E$, $N(A,U) = \bigcap_{x \in A} N(\{x\},A)$ is closed in $L_s(E,F)$, since the intersection of closed sets is closed. Since $L_{\mathcal{B}}(E,F)$ is finer than $L_s(E,F)$, $N(A,U)$ will be closed in $L_{\mathcal{B}}(E,F)$ also. Hence, let $x \in E$, and let $U \in N(F)$ be a barrel. Let α be the Minkowski functional of U, i.e., $U = \{y \in F | \alpha(y) \leq 1\}$. Suppose $f \in \overline{N(\{x\},U))}$ (closure in $L_s(E,F)$). For each $n \in N\backslash\{0\}$, there is a $f_n \in N(\{x\},U)$ such that $f-f_n \in N(\{x\}, 1/n\, U)$. Hence $f = (f-f_n) + f_n \in (\{x\}, 1/n \cdot U) + N(\{x\},U) = N(\{x\}, \frac{n+1}{n} \cdot U)$. Hence $f \in \bigcap_{n=1}^{\infty} (\{x\}, \frac{n+1}{n} U) = N(\{x\}, \bigcap_{n=1}^{\infty} \frac{n+1}{n} U)$. However $\bigcap_{n=1}^{\infty} \frac{n+1}{n} \cdot U = \{y \in F \mid \frac{n+1}{n} \cdot \alpha(x) \leq 1$ for each $n \in \mathbb{N}\backslash\{0\} = \{y \in F | \alpha(x) \leq 1\} = U$. Hence $f \in N(\{x\},U)$, as was to be shown. ∎

(8) *If* E *is a l.c.s.,* T *a d.s.g. on* E, *and* $\mathcal{B}$ *a total set of convex, circled, and complete bounded subsets of* E, *then* $T \in E(\mathbb{R}_+, L_{\mathcal{B}}(E))$. *In particular,* $T \in E(\mathbb{R}_+, L_c(E))$.

Proof: First assume $\mathcal{B}$ is covering. $L_s(E)$ and $L_{\mathcal{B}}(E)$ have the same bounded sets ([42], Ch. III, 3.4) and each barreled neighborhood of 0 in $L_{\mathcal{B}}(E)$ is a barrel in $L_s(E)$, by (7). Hence $L_{\mathcal{B}}(E)$ has a base of neighborhoods of 0 consisting of bornivorous barrels in $L_s(E)$. Hence, by (6), $T \in E(\mathbb{R}_+, L_{\mathcal{B}}(E))$.

If $\mathcal{B}$ is not covering, set $\tilde{\mathcal{B}} = \mathcal{B} \cup \mathcal{S}$. By the above, $T \in \mathcal{E}(\mathbb{R}_+, L_{\tilde{\mathcal{B}}}(E))$, and since the $\tilde{\mathcal{B}}$ topology is finer than the $\mathcal{B}$ topology $T \in \mathcal{E}(\mathbb{R}_+, L_{\mathcal{B}}(E))$ also. ∎

The unique solution property may now be stated.

(9) *Let* E *be a l.c.s.,* T *a d.s.g. on* E, *and* $x \in E$. *The differential equation* $Du = \mathcal{G}_T U$ *has a unique solution with* $u(0) = x$, *which is given by* $t \mapsto T(t)x$.

Proof: Clearly f is a solution of $Du = \mathcal{G}_T u$ with $u(0) = x$. It remains to verify the uniqueness of the solution. Let $g \in \mathcal{E}^1(\mathbb{R}_+, E)$ be a solution with $g(0) = x$. Let $y \in \mathbb{R}_+ \backslash \{0\}$ and define g^y: $[0,x] \to E$ by $t \mapsto T(y-t)g(t)$. For $t \in [0,y]$, $t + h \in [0,y]$,

$$\frac{g^y(t+h) - g^y(t)}{h} = \frac{T(y-(t+h))g(t+h) - T(y-t)g(t)}{h}$$

$$= \frac{(T(y-(t-h)) - T(y-t))g(t)}{h} + \frac{T(y-t)(g(t+h)-g(t))}{h}$$

$$- \frac{(T(y-(t+h)) - T(y-t))(g(t+h - g(t))}{h} .$$

As $h \to 0$, the first term tends to $-(\mathcal{G}_T \circ T(y-t))g(t)$, while the second term tends to $T(y-t)\circ Dg(t)$. The third term approaches 0 since $\{g(t+h) - g(t) | 0 \le h \le \varepsilon\}$ is compact, and $\frac{T(y-(t+h)) - T(y-t)}{h}$ converges uniformly on each compact set, by (8). Hence $Dg^y(t) = -(D\circ T(y-t))g(t) + T(y-t)\circ Dg(t) = -T(x-t)\circ \mathcal{G}_T g(t) + T(x-t)\circ Dg(t)$. Thus $Dg^y(0) = 0$, since $\mathcal{G}_T g(t) = Dg(t)$. Now by the mean-value theorem, for any seminorm α

continuous on E, $\alpha(g^y(y) - g^y(0)) \leq \sup_{0 \leq z \leq x} \alpha(Dg^y(z))$, so $g^y(y) = g^y(0)$. Hence $S(0)g(y) = S(y)g(0)$, and since y is arbitrary, $g(t) = S(t)g(0) = S(t)x$ for all $t \in \mathbb{R}_+$. Hence the solution is unique. ∎

A consequence of the above is that an infinitesimal generator uniquely determines a d.s.g.

(10) *Let* E *and* F *be a l.c.s.'s with* S *a d.s.g. on* E *and* T *a d.s.g. on* F. *For a continuous linear map* $f\colon E \to F$ *to be a d.s.g. morphism from* S *and* T, *it is necessary and sufficient that the diagram*

$$\begin{array}{ccc} E & \xrightarrow{\;g_S\;} & E \\ f \downarrow & & \downarrow f \\ F & \xrightarrow{\;g_T\;} & F \end{array}$$

commutes. In particular, if $g_S = g_T$, *then* $S = T$.

Proof: The condition is clearly necessary. Conversely, let $x \in E$, and suppose $f \circ g_S = g_T \circ f$. The functions $t \mapsto f(T(t)x)$ and $t \mapsto T(t)(f(x))$ are each solutions to the differential equation $Du = g_T u$ with $u(0) = f(x)$, as is readily verified. Hence, by (9), they are equal, so that $f \circ S(t) = T(t) \circ f$ for all $t \in \mathbb{R}_+$. ∎

The d.s.g. determined by the infinitesimal generator g will be denoted T_g.

There does not seem to be any reason to believe that given a l.c.s. E, each $f \in L(E)$ is the infinitesimal generator of some d.s.g. on E. However, it will be shown later that this is the case if E is a (B) space.

Examples

Some important examples of d.s.g.'s will now be given. Let E be a l.c.s. If $t \in \mathbb{R}_+$ and $f \in \mathcal{E}(\mathbb{R}_+,E)$, the left-shift map $f_t\colon \mathbb{R}_+ \to E$ is defined by $r \mapsto f(t+r)$. It is clear that $f_t \in \mathcal{E}(\mathbb{R}_+E)$, and that the map $\mathfrak{S}_E(t)\colon f \mapsto f_t$ is in $L(\mathcal{E}(\mathbb{R}_+,E))$. Thus, define $\mathfrak{S}_E\colon \mathbb{R}_+ \to L(\mathcal{E}(\mathbb{R}_+,E))$ by $t \mapsto (f \mapsto f_t)$. The verification of the following is trivial.

(11) *Let* E *be a l.c.s.* $\mathfrak{S}_E$ *is a d.s.g. on* $\mathcal{E}(\mathbb{R}_+,E)$ *with* $g_{\mathfrak{S}_E} = D$. ∎

Dualization provides the next example of a d.s.g. construction. Let E be a l.c.s. and let T be a d.s.g. on E. The dual of S is the map $T'\colon \mathbb{R}_+ \to L(E')$ given by $t \mapsto T(t)'$.

Let E be a l.c.s. and let $\mathcal{B}$ be a total set of bounded subsets of E. $\mathcal{B}$ is called transposable if for every $A \in \mathcal{B}$ and every $f \in L(E)$, there is a $B \in \mathcal{B}$ such that $f(A) \subset B$.

(12) *Let* E *be a l.c.s.,* T *a d.s.g. on* E, *and* $\mathcal{B}$ *a total set of bounded subsets of* E. *If* $\mathcal{B}$ *is transposable, then* $S(\mathbb{R}_+) \subset L(E'_{\mathcal{B}})$, *and if further,* $\overline{\mathcal{B}}$ *has a base of complete sets, then* S' *is a d.s.g. on* $E'_{\mathcal{B}}$, *with* $(g_T)' = g_{T'}$.

Proof: Clearly transposability implies $S'(\mathbb{R}_+) \subset L(E'_{\mathcal{B}})$. Suppose, further that $\mathcal{B}$ has a base consisting of complete sets. Let $x \in E'$ and let $V \in \mathcal{U}(E'_{\mathcal{B}})$. It is sufficient to show that there is an $\varepsilon > 0$ such that $(\frac{T'(t)-1}{t} - (g_T)')(x) \subset V$ whenever $t \leq \varepsilon$. However, by transposition for the pair $\langle E',E\rangle$, this amounts to $(\frac{T(t)-1}{t} - g_T)(V^\circ) \subset \{x\}^\circ$ for $t \leq \varepsilon$. Now $V^\circ \in \overline{\mathcal{B}}$, and $\{x\}^\circ$ is certainly a neighborhood of 0 in E, since it is a weak neighborhood, hence $(\frac{T(t)-1}{t} - g_T)(V^\circ) \subset \{x\}^\circ$ by (8). Hence T' is a d.s.g. on $E'_{\mathcal{B}}$ with infinitesimal generator $(g_T)'$. ∎

(13) *Let* E *be a l.c.s., and let* T *be a d.s.g. on* E.

(a) T' *is a d.s.g. on* E'_s *and* E'_k *with infinitesimal generator* $(g_T)'$.

(b) T *is a d.s.g. on* E *for every locally-convex topology on* E *which is weaker than the Mackey topology for which each* S(t), $t \in \mathbb{R}_+$ *is continuous, with the same infinitesimal generator.*

Proof: For (a), it suffices to apply (12). In the weak case, $\overline{\mathcal{B}}$ = all bounded, finite-dimensional sets, and a closed finite-dimensional set is always complete. In the Mackey case, $\overline{\mathcal{B}}$ has a base all weakly-compact, absolutely convex sets, and these sets are complete ([23], Ch. 2, Prop. 36, Cor. 1).

For (b), it follows from (a) that by double transposition, T is a d.s.g. on E_k. Clearly, then, it is also a d.s.g. for any coarser locally-convex topology for which each S(t) is continuous. ∎

An important application of the above is the following. The space $\mathcal{E}(\mathbb{R}_+)$ is reflexive (see §0). Hence, its strong dual $\mathcal{E}'(\mathbb{R}_+)$ is the same as its Mackey dual. The d.s.g. on $E'(\mathbb{R}_+)$ which is dual to $\mathfrak{S}$ is denoted $\mathfrak{S}'$ (the subscript k dropped). $\mathfrak{S}'(t)$ is just right shift by t, in the sense of distributions, and $g_{\mathfrak{S}'}$ is just D, the derivative operator, also in the sense of distributions.

The next result, whose proof is clear, allows the construction of an important d.s.g. from $E'(\mathbb{R}_+)$.

(14) *Let* E *be a l.c.s., F a linear subspace of* E, *and* T *a d.s.g. on* E. *Suppose also that for each* $t \in \mathbb{R}_+$, $T(t)(F) \subset F$, *and denote this restriction by* $S(t): F \to F$. S *is a d.s.g. on* F *if and only if* $g_T(F) \subset F$. ∎

From the above, it is obvious that the restriction of $\mathfrak{S}'$ to $\Delta(\mathbb{R}_+)$ defines a d.s.g. This d.s.g. will be denoted $\mathfrak{S}^\Delta$, and will be of fundamental importance in §2.

It is not necessarily true that every d.s.g. T on the l.c.s. E can be extended to a d.s.g. on $\widetilde{E}$ or $\hat{E}$. However, there is an important special case when this can be done.

(15) *Let* E *be a l.c.s., and let* T *be a d.s.g. on* E. *If* $\{\frac{T(t)-1}{t} \mid 0 \le t \le \varepsilon\}$ *is equicontinuous for some* $\varepsilon > 0$, *then* T *has a unique extension* $\widetilde{T}$ *(resp.* $\hat{T}$*) to a d.s.g. on* E *(resp.* $\hat{E}$*). This extension is given by* $\widetilde{T}(t)(x) = (T(t))^{\frown}(x)$ *(resp.* $\hat{T}(t)(x) = (T(t))^{\wedge}(x)$*), and its infinitesimal generator is* $(g_T)^{\frown}$ *(resp.* $(g_T)^{\wedge}$*).*

Proof: Only the completion case will be given; the quasi-completion case is entirely analogous. (s_1) and (s_2) are trivial. Extensions of equicontinuous families are equicontinuous ([13], Ch. X, §2, Prop. 4), so $\{\frac{\hat{T}(h)-1}{h} \mid 0 < h \le \varepsilon\}$ is equicontinuous. Let $V \in \mathcal{U}(\hat{E})$, and let $t \in R_+$. $\{\frac{\hat{T}(t+h)-\hat{T}(h)}{h} \mid |h| < \frac{\varepsilon}{2}, t + h \in R_+\}$ is also equicontinuous, and so there is a $U \in N(\hat{E})$ such that

$$\bigcup_{\substack{|h|<\varepsilon/2 \\ t+h\in R_+}} \left(\frac{\hat{T}(t+h)-\hat{T}(t)}{h} - \hat{T}(t)\circ g_{\hat{T}}\right)(U) \subset V .$$

Let $x \in \hat{E}$. Since E is dense in $\hat{E}$, there is a $y \in E$ such that $x-y \in U$. Thus

$$\left(\bigcup_{\substack{|h|\le\varepsilon/2 \\ t+h\in R_+}} \left(\frac{\hat{T}(t+h)-\hat{T}(t)}{h} - T(t)\circ g_{\hat{T}}\right)(x-y) \in V .$$

Hence $\lim_{h\to 0} \left(\frac{\hat{T}(t+h)-\hat{T}(t)}{h} - \hat{T}(t)\circ g_{\hat{T}}\right)(x-y) = 0$, and since $\lim_{h\to 0} \left(\frac{\hat{T}(t+h)-\hat{T}(t)}{h} - \hat{T}(t)\circ g_{\hat{T}}\right)(y) = 0$, it follows that $\lim_{h\to 0} \left(\frac{\hat{T}(t+h)-\hat{T}(t)}{h} - \hat{T}(t)\circ g_{\hat{T}}\right)(x) = 0$. Hence $\hat{T} \in \mathcal{E}^1(R_+, \mathcal{L}_s(E))$, so (s_3) is satisfied, and T is a d.s.g. ∎

Note that the above result holds in particular in the case in which E is barreled, for then every simply-bounded set is equicontinuous ([42], Ch. III, 4.2), and $\{\frac{T(t)-1}{t} \mid 0 < t \le \varepsilon\} \cup \{g_T\}$ is certainly bounded since it is compact, being the image of [0,t] under the continuous mapping $t \mapsto \frac{T(t)-1}{t}$, extended by continuity.

Relationship to (C_o)-Semigroups

The theory of equicontinuous semigroups of class (C_o) is an alternate approach to semigroups of operators on locally-convex spaces. The theory will now be compared to that of differentiable semigroups. Let E be a sequentially-complete l.c.s., and let $T: \mathbb{R}_+ \to L(E)$ be a map which satisfies axioms (s_1) and (s_2). T is called an equicontinuous semigroup of class (C_o) (abbreviated e.s.g.) if the following two axioms are also satisfied ([51], Ch. IX, 2).

(e_1) $\lim_{t\to t_o} T(t)x = T(t_o)x$ for any $t_o \in \mathbb{R}_+$, $x \in X$.

(e_2) $\{ S(t) \mid t \in \mathbb{R}_+\}$ is equicontinuous.

Let $A = \{x \in E \mid \lim_{t\to 0} \frac{T(t)-1}{t}(x)$ exists$\}$, and define $A: A \to E$ by $x \mapsto \lim_{t\to 0} \frac{T(t)-1}{t}(x)$. A is called the infinitesimal generator of T, and A is dense in E ([52], Ch. IX, §3, Th. 1).

(16) *Let* E *be a l.c.s., and let* $T: \mathbb{R}_+ \to L(E)$.

(a) *If* T *is a d.s.g., then* T *is an e.s.g. if and only if* E *is sequentially-complete and* $\{S(t) \mid t \in \mathbb{R}_+\}$ *is equicontinuous.*

(b) *If* E *is sequentially-complete and* T *is an e.s.g., then* T *is a d.s.g. if and only if the domain of the infinitesimal generator is* E.

Proof: The proof is an immediate consequence of the definitions. ∎

$\{ \mathfrak{S}(t) \mid t \in \mathbb{R}_+\}$ is easily seen to be *not* equicontinuous, so not every d.s.g. is an e.s.g. Conversely, let $C(\mathbb{R}_+)$ denote the space of bounded uniformly-continuous $\mathbb{K}$-valued functions on $\mathbb{R}_+$ with the sup-norm topology, and define $T(t)\colon \mathbb{R}_+ \to L(C(\mathbb{R}_+))$ by $(T(t)f)(x) = f(x+t)$. T is an e.s.g. ([51], Ch. IX, §2, Example 2), but not a d.s.g., since not every uniformly-continuous function is differentiable. Hence, not every e.s.g. is a d.s.g.

If E is a (B) space, then property (e_2) is not necessary. A map $T\colon \mathbb{R}_+ \to L(E)$ (E a (B) space) is called a (C_0) semigroup if (s_1), (s_2), and (e_1) are satisfied. The infinitesimal generator of a (C_0) semigroup is defined as for an e.s.g., and is densely defined. (C_0) semigroups have the following important property.

(17) *Let* E *be a* (B) *space, and let* $A \in L(E)$. A *is the infinitesimal generator of a unique* (C_0) *semigroup (which is also a d.s.g.) given by* $t \mapsto e^{At}$.

Proof: Consult ([41], 13.36). ▮

Further properties of e.g.s.'s and (C_0) semigroups will not be given here. For further discussions, consult [19], [26], [41], and [51].

§2 Input-Output Behavior

General Theory

In this section, the problem of recovering the external (input-output) behavior of a linear differential system from its internal behavior will be investigated.

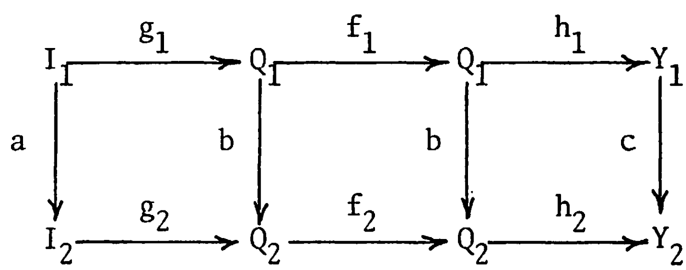
Let K be a subcategory of $\mathbf{LCS}$. A differential system in K is a 6-tuple $M = (Q,f,I,g,Y,h)$ where Q, I, and Y are K objects, and $f: Q \to Q$, $g: I \to Q$, and $h: Q \to Y$ are K morphisms, with f the infinitesimal generator of a (unique) d.s.g. on K (denoted T_f). Q is called the state space, I the input space, and Y the output space of M, with f the state-transition map, g the input map, and h the output map. The system may be thought of as governed by the equations

$$\frac{dq(t)}{dt} = f(q(t)) + g(i(t))$$

$$y(t) = h(q(t)) \quad ,$$

although they must be properly interpreted. This will be discussed in more detail later.

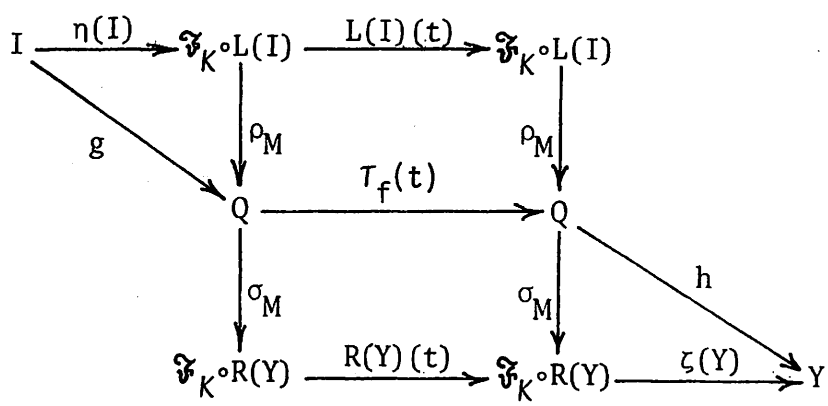
A morphism of differential systems $M_1 = (Q_1,f_1,I_1,g_1,Y_1,h_1)$ to $M_2 = (Q_2,f_2,I_2,g_2,Y_2,h_2)$ is a 3-tuple of K morphisms (a,b,c) such that



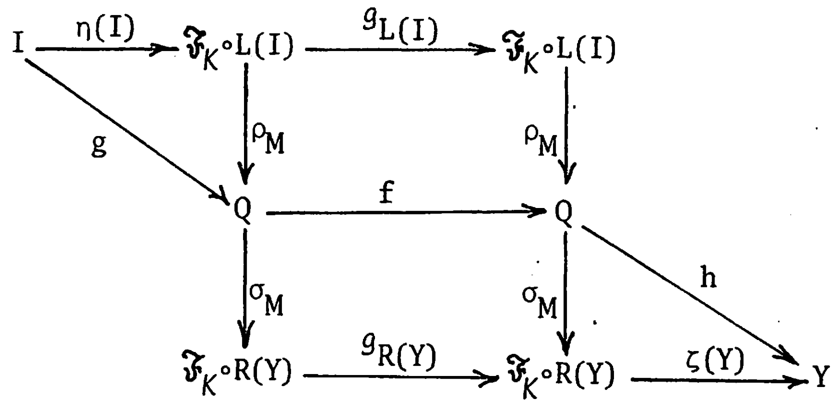
commutes. With this notation, the differential systems over K may be made into a category $\mathcal{D}if(K)$. The objects and morphisms are as just described. Note that a $\mathcal{D}i\!f(K)$ morphism (a,b,c) is an isomorphism if and only if a, b, and c are each isomorphisms.

Maintaining K as above, define $\mathfrak{F}_K$: DSG(K) $\to K$ to be the functor which sends the d.s.g. T: $\mathbb{R}_+ \to L(E)$ to the underlying l.c.s. E, and which sends each d.s.g. morphism to its underlying continuous linear map. K is called a <u>differential-behavior category</u> if $\mathfrak{F}_K$ has both a left and a right adjoint. K is <u>full</u> if it is a full subcategory of $\mathbb{LCS}$.

(1) *Let K be a differential-behavior category, and let* A = $(\mathfrak{F}_K, L, R, N, \zeta)$ *be a two-sided adjoint situation for* $\mathfrak{F}_K$. *Let* M = (Q,f,I,g,Y,h) *be a differential system in* K. *There are unique* K *morphisms* ρ_M *and* σ_M *such that*



commutes for each $t \in \mathbb{R}_+$. *The commutativity of these diagrams is equivalent to the commutativity of the single diagram*



Proof: The first part is a standard characterization of adjunctions ([37], Ch. IV, §1, Th. 2). The second part follows form 1.(10). ∎.

In the situation above, ρ_M is called the reachability map of M, σ_M the observability map of M, and $\sigma_M \circ \rho_M$ is called the total response or input-output map of M. It should be noted that ρ_M and σ_M are only unique within the framework of a specific adjoint situation A for $\mathfrak{F}_K$. Of course, all such adjoint situations are unique up to isomorphism, and so ρ_M and σ_M are unique up to composition with a isomorphism, regardless of choice of A.

Some important examples of differential-behavior categories will now be developed.

Examples of Left Adjoints to $\mathfrak{F}_K$

A left ajoint to $\mathfrak{F}_K$ for K equal to $\mathbb{LCS}$ (all locally-convex spaces), $\mathbb{QC}$ (quasi-complete l.c.s.'s), $\mathbb{CS}$ (complete l.c.s.'s), $\mathbb{MS}$ (Mackey l.c.s.'s), and $\mathbb{WS}$ (weakly-topologized l.c.s.'s) will be constructed. The constructions for each of these five cases are quite similar, so it is advantageous to develop them somewhat in parallel. If I is the input space, then the free input d.s.g. will be the induced shift on $\Delta(\mathbb{R}_+)\otimes_\iota I$, or a suitable completion or retopologization of this space.

Recall that $\mathfrak{S}^\Delta$ is the d.s.g. on $\Delta(\mathbb{R}_+)$ given by the right-shift operator, i.e., $\mathfrak{S}^\Delta(t)(D^p\delta_s) = D^p\delta_{t+s}$. The infinitesimal generator of this d.s.g. is just the distributional derivation operator D restricted to $\Delta(\mathbb{R}_+)$. Let I be a l.c.s. An $\mathbb{R}_+$-induced semigroup of operators $\mathfrak{S}^\Delta\otimes I$ may be induced on $\Delta(\mathbb{R}_+)\otimes I$ by defining $(\mathfrak{S}^\Delta\otimes I)(t) = \mathfrak{S}^\Delta(t)\otimes 1_I$. When $\Delta(\mathbb{R}_+)\otimes I$ is given the ι topology, this semigroup becomes a d.s.g., as is now shown.

(2) *Let* I *be a l.c.s.* $\mathfrak{S}^\Delta\otimes I$ *is a d.s.g. on* $\Delta(\mathbb{R}_+)\otimes_\iota I$, *with infinitesimal generator* $D\otimes 1_I$.

Proof: First of all note that $(\mathfrak{S}^\Delta\otimes I)(t) \in L(\Delta(\mathbb{R}_+)\otimes_\iota I)$, by definition. Hence, all that need be shown is that $\mathfrak{S}^\Delta\otimes I$ is differentiable. Let $x \in \Delta(\mathbb{R}_+)$ and $e \in I$, and let $U \in \mathcal{U}(\Delta(\mathbb{R}_+)\otimes_\iota I)$. Since the canonical projection $p\colon \Delta(\mathbb{R}_+)\times I \to \Delta(\mathbb{R}_+)\otimes I$ is separately-continuous, $\{y \in \Delta(\mathbb{R}_+) \mid y\otimes e \in U\} \in \mathcal{U}(\Delta(\mathbb{R}_+))$. Fix $t \in \mathbb{R}_+$. Pick $\varepsilon > 0$ such that $\left(\frac{\mathfrak{S}^\Delta(t+h)-\mathfrak{S}^\Delta(t)}{h} - D\circ\mathfrak{S}^\Delta(t)\right)(x) \in \{y \in \Delta(\mathbb{R}_+) \mid y\otimes e \in \mathcal{U}(\mathbb{R}_+))\}$ whenever

$|h| \leq \varepsilon$ and $t + h \in \mathbb{R}_+$. This is possible since $\mathfrak{S}^{\Delta}$ is differentiable. However, This implies $(\frac{\mathfrak{S}^{\Delta}(t+h)-\mathfrak{S}^{\Delta}(t)}{t} \otimes 1_I - D\circ\mathfrak{S}^{\Delta}(t)\otimes 1_I)(x\otimes e) \in U$ whenever $|h| \leq \varepsilon$ and $t + h \in \mathbb{R}_+$. Hence $\lim_{h\to 0} (\frac{\mathfrak{S}^{\Delta}(t+h)-\mathfrak{S}^{\Delta}(t)}{h} \otimes 1_I - D\circ\mathfrak{S}^{\Delta}(t)\otimes 1_I)$ $(x\otimes e) = 0$. Since every element of $\Delta(\mathbb{R}_+)\otimes_\iota I$ is of the form $\sum_{k=1}^{n} x_k\otimes e_k$, it follows that $\mathfrak{S}^{\Delta}\otimes I$ is differentiable. ∎

To avoid confusion, $\mathfrak{S}^{\Delta}\otimes I$ will be denoted $\mathfrak{S}^{\Delta}\otimes_\iota I$ when it operates on $\Delta(\mathbb{R}_+)\otimes I$.

To show that $\mathfrak{S}^{\Delta}\otimes_\iota I$ is a universal d.s.g. over I, it is necessary to find a d.s.g. morphism from $\mathfrak{S}^{\Delta}\otimes_\iota I$ to each d.s.g. T, which satisfies the universal property given at the beginning of this section. To do this, let T be a d.s.g. on the l.c.s. Q, and let g: I → Q be a continuous linear map. Recall that Φ_Q: $\Delta(\mathbb{R}_+) \times \mathcal{E}(\mathbb{R}_+,Q) \to Q$ is defined by $(D^p\delta_t,\phi) \mapsto D^p\phi(t)$, and is bilinear and hypocontinuous (see 1.(3)). Define $\Lambda_{T,Q}$: $Q \to \mathcal{E}(\mathbb{R}_+,Q)$ by $e \mapsto (t \mapsto T(t)e)$. By 1.(5), $\Lambda_{T,Q}$ is continuous. Define the map $\tilde{g}_T$: $\Delta(\mathbb{R}_+) \times I \to Q$ by

$$\tilde{g}_T = \Delta(\mathbb{R}_+) \times I \xrightarrow{1\times\Lambda_{T,Q}\circ g} \Delta(\mathbb{R}_+) \times \mathcal{E}(\mathbb{R}_+,Q) \xrightarrow{\Phi_Q} Q \quad .$$

(3) *Let* I *and* Q *be l.c.s.'s, let* g: I → Q *be a continuous linear map, and let* T *be a d.s.g. on* Q. *The map* $\tilde{g}_T$: $\Delta(\mathbb{R}_+) \times I \to Q$ *is bilinear and hypocontinuous.*

Proof: Clearly $\tilde{g}_T$ is bilinear. Since 1 and $\Lambda_{T,Q}$ are continuous, they each map bounded sets into bounded sets, whence the hypocontinuity of $\tilde{g}_T$ is immediate from the hypocontinuity of Φ_Q (see 1.(3)). ∎

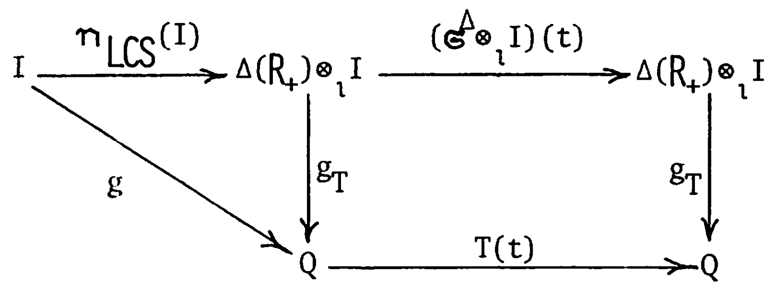
With the situation as above, the hypocontinuous bilinear map $\tilde{g}_T$ induces a linear map g_T: $\Delta(R_+)\otimes_\beta I \to Q$ via the diagram

$$\begin{array}{ccc} \Delta(R_+)\times I & \xrightarrow{\quad p \quad} & \Delta(R_+)\otimes_\beta I \\ & \searrow^{\tilde{g}_T} & \downarrow g_T \\ & & Q \end{array}$$

with p the canonical projection. Now, since the ι topology is finer than the β topology, g_T: $\Delta(R_+)\otimes_\iota I \to Q$ is also continuous. g_T is the desired reachability map, as will now be shown.

For I a l.c.s., define $\eta_{LCS}(I)$: $I \to \Delta(R_+)\otimes_\iota I$ by $e \mapsto \delta_0 \otimes e$. $\eta_{LCS}(I)$ may be regarded as the restriction of the canonical projection p: $\Delta(R_+)\times I \to \Delta(R_+)\otimes_\iota I$ to $\{\delta_0\}\times I$, and also is continuous, since p is separately-continuous.

(4) *Let* I *be a l.c.s.* $(\overset{\Delta}{\mathfrak{S}}\otimes_\iota I, \eta_{LCS}(I))$ *is a universal map for* I *with respect to* $\mathfrak{F}_{LCS}$. *Specifically, if* Q *is a l.c.s.,* T *a d.s.g. on* F, *and* g: $I \to Q$ *a continuous linear map,* g_T *is the unique continuous linear map from* $\Delta(R_+)\otimes_\iota I \to Q$ *which makes the diagram*



commute for each $t \in R_+$.

Proof: The triangle commutes by definition of (I) and g_T. The square commutes because $g_T \circ (\mathcal{G}^{\Delta}(t)\otimes_{\iota} 1_I)(D^p\delta_s \otimes e) = g_T(D^p\delta_{t+s}\otimes e) = (g_T)^p \circ T(t+s)g(e) = T(t)\circ (g_T)^p \circ T(t+s)g(e) = T(t)\circ g_t(D^p\delta_t \otimes e)$ (recall 1.(10)). Hence this entire diagrams commutes. It remains to shown that g_T is unique. If the above diagram were to commute with k replacing g_T, then $k\circ (D^p\delta_t\otimes e) = k\circ (D\otimes 1_I)^p(\mathcal{G}^{\Delta}(t)\otimes 1_I)(\delta_0\otimes e) = (g_t)^p \circ T(t)\circ k\circ (\delta_0 \otimes e) = (g_T)^p T(t)\eta_{LCS}(I)(e) = (g_T)^p T(t)g(e) = g_T(D^p\delta_t\otimes e)$, so $k = g_T$, and so g_T is unique. ∎

Define the functor $\mathcal{L}_{LCS}$: LCS → DSG by $I \mapsto {}^{\Delta}\otimes_{\iota} I$ on objects and $f \mapsto 1\otimes f$ on morphisms. (4) may be restated as follows.

(5) $\mathcal{L}_{LCS}$ *is left-adjoint to* $\mathcal{F}_{LCS}$, *and* $\eta_{LCS} : 1_{LCS} \to \mathcal{F}_{LCS}\circ\mathcal{L}_{LCS}$ *is a natural transformation which is the unit of this adjunction.*

Proof: Follows from (4), as this is a standard characterization of adjunctions in terms of universal maps (see [25], 27.3). ∎

The system theoretic interpretation of the above is the following. Each input signal in $\Delta(\mathbb{R}_+)\otimes_\iota I$ is of the form $\sum_{k=1}^{n} (D^{p_k}\delta_{t_k}\otimes i_k)$, with $p_k \in \mathbb{N}$, $t_k \in \mathbb{R}_+$, and $i_k \in I$ for each $k = 1, \ldots, n$. That is, an input signal is just an I-valued distribution with finite support. In terms of the equation

$$\frac{dq(t)}{dt} = (g_T)(q(t)) + g(i(t)) \quad ,$$

the input $i(t)$ ($t \in \mathbb{R}_-$, where $\mathbb{R}_-$ denotes the nonpositive reals), to the system is 0 except at $-t_k$ for $k = 1, \ldots, n$ (note the minus sign), when the input is $D^p\delta_{t_k}\otimes i_k$. The equations above thus make sense for all $t \in \mathbb{R}_-$. The minus sign is necessary because inputs are interpreted to start at negative time and continue until 0. Thus, in the $\mathbb{LCS}$ case, the system is driven only by impulses. To get a righer input structure, it is necessary to deal with categories of l.c.s.'s which have some completeness properties.

The cases $K = \mathbb{QC}$ (quasi-complete l.c.s.'s) and $\mathbb{CS}$ (complete l.c.s.'s) will now be investigated. Since the underlying theory for them is virtually identical, they will be discussed entirely in parallel.

It will be necessary to extend the d.s.g. $\mathfrak{S}^\Delta\otimes_\iota I$ to the quasi-completion and completion of $\Delta(\mathbb{R}_+)\otimes I$. As indicated in §1, this is not automatic, but depends upon certain properties, which are fortunately valid in the case being considered.

Before proceeding further, it is useful to recall that $\mathcal{E}'(\mathbb{R}_+)$ may be identified as both a quasi-completion and a completion of $\Delta(\mathbb{R}_+)$. (1.(2f)). However, one may not immediately assert that $\mathcal{E}'(\mathbb{R}_+)\tilde{\otimes}_\iota I \cong \Delta(\mathbb{R}_+)\tilde{\otimes}_\iota I$ and $\mathcal{E}'(\mathbb{R}_+)\hat{\otimes}_\iota I \cong \Delta(\mathbb{R}_+)\hat{\otimes}_\iota I$, since the extension of a separately-continuous map need not be separately-continuous. However, in this particular case, the necessary results can be shown.

(6) *For every l.c.s.* I, $\Delta(\mathbb{R}_+)\otimes_\iota I = \Delta(\mathbb{R}_+)\otimes_\beta I$, *i.e., every separately-continuous bilinear map on* $\Delta(\mathbb{R}_+)\times I$ *is hypocontinuous.*

Proof: By (3), for any l.c.s. Q, d.s.g. T on Q, and continuous linear map $g: I \to Q$, $\tilde{g}_T$ is hypocontinuous, and so $g_T: \Delta(\mathbb{R}_+)\otimes_\beta I \to Q$ is continuous. Since the β topology of $\Delta(\mathbb{R}_+)\otimes I$ is coarser than the ι topology, it follows that $\mathfrak{S}^\Delta\otimes I$ is also a d.s.g. on $\Delta(\mathbb{R}_+)\otimes_\beta I$ (denoted $\mathfrak{S}^\Delta\otimes_\beta I$). Hence, as in (4), $(\mathfrak{S}^\Delta\otimes_\beta I, \eta_{LCS}(I))$ is a universal map for I with respect to $\mathfrak{F}_{LCS}$. Since universal objects are unique up to isomorphism, it follows that $\Delta(\mathbb{R}_+)\otimes_\beta I = \Delta(\mathbb{R}_+)\otimes_\iota I$. ∎

(7) *Let* I *be a quasi-complete (resp. complete) l.c.s.*

(a) *Every hypocontinuous bilinear map on* $\Delta(\mathbb{R}_+)\times I$ *into a quasi-complete (resp. complete) l.c.s. extends uniquely to a hypocontinuous bilinear map on* $\mathcal{E}'(\mathbb{R}_+)\times I$.

(b) $\mathcal{E}'(\mathbb{R}_+)\otimes_\beta I$ *is a dense subspace of* $\Delta(\mathbb{R}_+)\tilde{\otimes}_\beta I$ *(resp.* $\Delta(\mathbb{R}_+)\hat{\otimes}_\beta I$).

(c) $\mathcal{E}'(\mathbb{R}_+)\otimes_\iota I = \mathcal{E}'(\mathbb{R}_+)\otimes_\beta I$, *i.e., every separately-continuous bilinear map on* $\mathcal{E}'(\mathbb{R}_+)\times I$ *is hypocontinuous.*

Proof: (a) follows immediately from 0.(1) and 1.(2b).

(b) By fixing any $i \in I$, $\Delta(\mathbb{R}_+)$ may be regarded as a closed subspace subspace of $\Delta(\mathbb{R}_+)\otimes_\beta I$. Since a closed subspace of a quasi-complete (resp. complete) space is quasi-complete (resp. complete), the result follows from (a) and 1.(2f).

(c) Let f be a separately-continuous bilinear map from $E'(\mathbb{R}_+)\times I$ into a l.c.s. G, which is assumed to be quasi-complete (resp. complete), without loss of generality. By (6), f is hypocontinuous when restricted to $\Delta(\mathbb{R}_+)\times I$, and so, by (a), extends uniquely to a hypocontinuous bilinear map on $E'(\mathbb{R}_+)\times I$, which must be f. Hence every separately-continuous bilinear map on $E'(\mathbb{R}_+)\times I$ is hypocontinuous. ∎

From now on, if I is a quasi-complete (resp. complete) l.c.s., $E'(\mathbb{R}_+)\tilde{\otimes}_\iota I$ and $\Delta(\mathbb{R}_+)\tilde{\otimes}_\iota I$ (resp. $E'(\mathbb{R}_+)\hat{\otimes}_\iota I$ and $\Delta(\mathbb{R}_+)\hat{\otimes}_\iota I$) shall be identified with each other. An analogous statement holds with β replacing ι.

Let I be a l.c.s. On $E'(\mathbb{R}_+)\otimes_\iota I$, the semigroup $\mathfrak{S}'\otimes_\iota I$ is defined by $(\mathfrak{S}'\otimes_\iota I)(t) = \mathfrak{S}'(t)\otimes_\iota 1_I$. The next statements shows that $\mathfrak{S}'\otimes_\iota I$ is a d.s.g.

(8) *For any l.c.s.* I *and any* $\varepsilon > 0$, $\{\frac{(\mathfrak{S}'\otimes_\iota I)(t)-1}{t} \mid 0 < t \le \varepsilon\}$ *and* $\{\frac{(\overset{\Delta}{\mathfrak{S}}\otimes_\iota I)(t)-1}{t} \mid 0 < t \le \varepsilon\}$ *are equicontinuous.*

Proof: $\{\frac{\mathfrak{S}'(t)-1}{t} \mid 0 < t \le \varepsilon\} \cup \{D\}$ is bounded for any $\varepsilon > 0$, since it is compact, being the image of $[0,t]$ under the continuous map $t \mapsto \frac{\mathfrak{S}'(t)-1}{t}$. Now $E'(\mathbb{R}_+)$ is barreled (see §0), and so every simply-bounded subset of $L(E(\mathbb{R}_+))$ is equicontinuous, hence $\{\frac{\mathfrak{S}'(t)-1}{t} \mid 0 < t \le \varepsilon\}$ is equicontinuous

([42], Ch. III, 4.2). Since restrictions of equicontinuous sets are equicontinuous ([13], Ch. X, §2, Prop. 4), $\{\frac{\overset{\Delta}{\mathfrak{S}}(t)-1}{t} \mid 0 < t \leq \varepsilon\}$ is also equicontinuous.

Now $E'(R_+)\otimes_\iota I = E'(R_+)\otimes_\beta I$ by (7), and so to show that $\{\frac{(\mathfrak{S}'\otimes_\iota I)(t)-1}{t} \mid 0 < t \leq \varepsilon\}$ is hypocontinuous, it suffices to show that the family of maps $\{p\circ(\frac{\mathfrak{S}'(t)-1}{t}\times 1) \mid 0 < t \leq \varepsilon\}$ is equihypocontinuous from $E'(R_+)\times I \to E'(R_+)\otimes_\beta I$, where $p: E'(R_+)\times I \to E'(R_+)\otimes_\beta I$ is the canonical projection. Let $V\in \mathcal{U}(E'(R_+)\otimes_\beta I)$, A be a bounded subset of $E'(R_+)$ and B a bounded subset of E. If suffices to find $U \in \mathcal{U}(E'(R_+))$ and $W \in \mathcal{U}(E)$ such that $\bigcup_{0<t\leq\varepsilon} p\circ(\frac{\mathfrak{S}'(t)-1}{t}\times 1)(A\times W) \subset V$ and $\bigcup_{0<t\leq\varepsilon} p\circ(\frac{\mathfrak{S}'(t)-1}{t}\times 1)(U\times B) \subset V$.

Since $\{\frac{\mathfrak{S}'(t)-1}{t} \mid 0 < t \leq \varepsilon\}$ is equicontinuous, $C = \bigcup_{0<t\leq\varepsilon} \frac{\mathfrak{S}'(t)-1}{t}(A)$ is bounded. Put $W = \{e \in I \mid (x,e) \in p^{-1}(V) \text{ and } x \in C\}$. $W \in \mathcal{U}(E)$, since p is hypocontinuous. Hence $\bigcup_{0<t\leq\varepsilon} p\circ(\frac{\mathfrak{S}'(t)-1}{t}\times 1)(B\times U) = p(C\times W) = p(p^{-1}(V)) \subset V$.

Next, put $Y = \{x \in E'(R_+) \mid (x,e) \in p^{-1}(V) \text{ and } e \in B\}$. $Y \in \mathcal{U}(E'(R_+))$, since p is hypocontinuous. Let $U \in \mathcal{U}(E'(R_+))$ such that $\bigcup_{0<t\leq\varepsilon} \frac{\mathfrak{S}'(t)-1}{t}(u) \subset Y$; U exists since $\{\frac{\mathfrak{S}'(t)-1}{t} \mid 0 < t \leq \varepsilon\}$ is hypocontinuous. Now $\bigcup_{0<t\leq\varepsilon} p\circ(\frac{\mathfrak{S}'(t)-1}{t}\times 1)(U\times B) = p(Y\times B) = p(p^{-1}(V)) \subset V$. Hence $\{\frac{(\mathfrak{S}'\otimes_\iota I)(t)-1}{t} \mid 0 < t \leq \varepsilon\}$ is hypocontinuous.

By (7), $\Delta(R_+)\otimes_\iota I$ is a dense subspace of $E'(R_+)\otimes_\iota I$. Hence $\{\frac{(\overset{\Delta}{\mathfrak{S}}\otimes_\iota I)-1}{t} \mid 0 < t \leq \varepsilon\}$ is equicontinuous, since restrictions of equicontinuous families are equicontinuous. ∎

Now, in view of 1.(15), if I is a quasi-complete (resp. complete) l.c.s., $\mathfrak{S}'\otimes_\iota I$ extends uniquely to a d.s.g. on $E'(R_+)\tilde{\otimes}_\iota I$ (resp. $E'(R_+)\hat{\otimes}_\iota I$). This extension will be denoted $\mathfrak{S}'\tilde{\otimes}_\iota I$ (resp. $\mathfrak{S}'\hat{\otimes}_\iota I$).

(9) *Let* I *be a quasi-complete (resp. complete) l.c.s.* $(\mathfrak{S}'\tilde{\otimes}_\iota I, (\eta_{LCS}(I))^{\sim})$ *(resp.* $(\mathfrak{S}'\hat{\otimes}_\iota I, (\eta_{LCS}(I))^{\wedge})$*) is a universal map for* I *with respect to* $\mathfrak{F}_{QC}$ *(resp.* $\mathfrak{F}_{CS}$*). Specifically, if* Q *is a quasi-complete (resp. complete) l.c.s.,* T *a d.s.g. on* Q*, and* g: I → Q *a continuous linear map,* $\tilde{f}_T$ *(resp.* $\hat{f}_T$*) is the unique continuous linear map from* $E'(R_+)\tilde{\otimes}_\iota I \to Q$ *(resp.* $E'(R_+)\hat{\otimes}_\iota I \to Q$*) which makes the diagram*

$$\begin{array}{ccccc} I & \xrightarrow{\eta_{LCS}(I)} & E'(R_+)\otimes_\iota I & \xrightarrow{(\mathfrak{S}'\otimes_\iota I)(t)} & E'(R_+)\otimes_\iota I \\ & \searrow^{g} & \downarrow g_T & & \downarrow g_T \\ & & Q & \xrightarrow{T(t)} & Q \end{array}$$

(resp.

$$\begin{array}{ccccc} I & \xrightarrow{\eta_{LCS}(I)^{\wedge}} & E'(R_+)\hat{\otimes}_\iota I & \xrightarrow{(\mathfrak{S}'\hat{\otimes}_\iota I)(t)} & E'(R_+)\otimes_\iota I \\ & \searrow^{g} & \downarrow \hat{g}_T & & \downarrow \hat{g}_T \\ & & Q & \xrightarrow{T(t)} & Q \end{array}$$

)

commute for each $t \in \mathbb{R}_+$.

Proof: Follows from (4) and (6)-(8). ∎

Define the functor $\mathfrak{L}_{QC}: \mathbf{QC} \to \mathbf{DSG}(\mathbf{QC})$ (resp. $\mathfrak{L}_{CS}: \mathbf{CS} \to \mathbf{DSG}(\mathbf{CS})$) by $I \mapsto \mathfrak{S}'\tilde{\otimes}_\iota I$ (resp. $I \mapsto \mathfrak{S}'\hat{\otimes}_\iota I$) on objects and $f \mapsto 1\tilde{\otimes}f$ (resp. $f \mapsto 1\otimes f$) on morphisms. If I is a quasi-complete (resp. complete) l.c.s., define $\eta_{QC}(I)$ (resp. $\eta_{CS}(I)$) by $\eta_{QC}(I) = (\eta_{LCS}(I))^{\sim}$ (resp. $\eta_{CS}(I) = (\eta_{LCS}(I))^{\wedge}$). (9) may be restated as follows.

(10) $\mathfrak{L}_{QC}$ *(resp.* $\mathfrak{L}_{CS}$*) is left-adjoint to* $\mathfrak{F}_{QC}$ *(resp.* $\mathfrak{F}_{CS}$*), and* $\eta_{QC}: 1_{QC} \to \mathfrak{F}_{QC}\circ\mathfrak{L}_{QC}$ *(resp.* η_{CS} $1_{CS} \to \mathfrak{F}_{CS}\circ\mathfrak{L}_{CS}$*) is a natural transformation which is the unit of this adjunction.*

Proof: Similar to (5). ∎

The system-theoretic interpretation of the above results is essentially the unique extension of the discussion following (5). The space of input signals ($E'(\mathbb{R}_+)\tilde{\otimes}_\iota I$ in the $\mathbf{QC}$ case and $E'(\mathbb{R}_+)\hat{\otimes}_\iota I$ in the $\mathbf{CS}$ case) is, of course, much richer than the space $\Delta(\mathbb{R}_+)\otimes_\iota I$ in the $\mathbf{LCS}$ case. In each case, the space of input signals contains at least $E'(\mathbb{R}_+)\otimes_\iota I$, which may be interpreted (algebraically) as the set of all functions in $L(E(\mathbb{R}_+),I)$ whose images are contained in a finite-dimensional subspace of I (If $\phi \in E(\mathbb{R}_+)$, $(\sum_{i=1}^{n} (x_i\otimes e_i))(\phi) = \sum_{i=1}^{n} (x_i(\phi)\, e_i)$). If I is finite dimensional, say $I \cong (\mathbb{K})^n$, this is the entire input space, and $L(E(\mathbb{R}_+),I)$ may be identified with $(E'(\mathbb{R}_+))^n$ (algebraically).

A distribution $\lambda \in E'(\mathbb{R}_+)$ has compact support, and may be interpreted as a generalized-function input signal which starts at (is zero before) $t = -\sup(\mathrm{supp}(\lambda))$ (supp means "support of") and ends at time $t = 0$. Recall that the sapce of all I-valued distributions on $\mathbb{R}_+$ with scalarly-compact support is defined to be $L_b(E(\mathbb{R}_+),I)$ (see [45], Ch. 1, p. 52). Each element of $E'(\mathbb{R}_+)\otimes_\iota I$ may be identified with an element of $L_b(E(\mathbb{R}_+),I)$, as shown above. Unfortunately, the topology which $L_b(E(\mathbb{R}_+),I)$ induces on $E'(\mathbb{R}_+)\otimes I$ is (by definition) the ε topology and not the ι topology. Of course, the ε topology corresponds to the π topology, since $E'(\mathbb{R}_+)$ is nuclear ([22], Ch. II, Ref. 4), and by (7c), the ι and β topologies coincide on $E'(\mathbb{R}_+)\otimes I$. Hence the problem reduces to determining when the β and π topologies coincide. The next result answers this question for $(\mathcal{DF})$ spaces.

(11) *Let* I *be a quasi-complete $(\mathcal{DF})$ space.*

(a) I *is complete.*

(b) $E'(\mathbb{R}_+)\otimes_\iota I = E'(\mathbb{R}_+)\otimes_\pi I$.

(c) $E'(\mathbb{R}_+)\widehat{\otimes}_\iota I = E'(\mathbb{R}_+)\hat{\otimes}_\iota I \cong L_b(E(\mathbb{R}_+),I)$, *the last isomorphism being the completion of the canonical injection.*

(d) *Every element of* $L_b(E(\mathbb{R}_+),I)$ *has compact support.*

Proof: (a) Consult [23], Ch. 4, Part 3, Prop. 4, Cor. 2.

(b) By (7c), it suffices to show that every hypocontinuous bilinear map on $E'(\mathbb{R}_+)\times I$ is continuous, which is true, since $E'(\mathbb{R}_+)$ and I are $(\mathcal{DF})$ spaces ([23], Ch. 4, Part 3, Th. 1).

(c) The projective (π) tensor product of two $(\mathcal{DF})$ spaces is a $(\mathcal{DF})$ space, as is its completion ([22], Ch. I, Prop. 5). Hence the first equality follows from (a). $L_b(E(\mathbb{R}_+),\mathrm{I})$ is complete ([23], Ch. 3, Prop. 3), and so the isomorphism follows from the nuclearity of $E'(\mathbb{R}_+)$ (which makes the π and ε topologies equal).

(d) This is a theorem of Schwartz ([46], Ch. I, pp. 62-63). ∎

The above shows that in case the iput space I is a $(\mathcal{DF})$ space, then in both the $\mathbb{QC}$ and $\mathbb{CS}$ constructions, the space of input signals may be identified with $L_b(E(\mathbb{R}_+),\mathrm{I})$, and each element has compact support. While this is admittedly a special case it does cover many important applications. For example, every normable space (and in particular every (B) space) is a $(\mathcal{DF})$ space.

The question of whether, in the case of I a general quasi-complete (resp. complete) l.c.s., $E'(\mathbb{R}_+)\tilde{\otimes}_\iota \mathrm{I}$ (resp. $E'(\mathbb{R}_+)\hat{\otimes}_\iota \mathrm{I}$) can be regarded (algebraically) as a subspace of $L(E(\mathbb{R}_+),\mathrm{I})$ is an injectivity problem, typical to topological-tensor-product theory. The author does not know the answer to this problem. See [22], Ch. I, §3, n°2, for a discussion of related problems.

The left adjunctions of $\mathfrak{F}_K$ for $K = \mathbb{MS}$ and $K = \mathbb{WS}$ will now quickly be investigated.

(12) *Let* I *be a l.c.s. If* I *is a Mackey space, then* $\Delta(\mathbb{R}_+)\otimes_\iota \mathrm{I}$ *is also a Mackey space.*

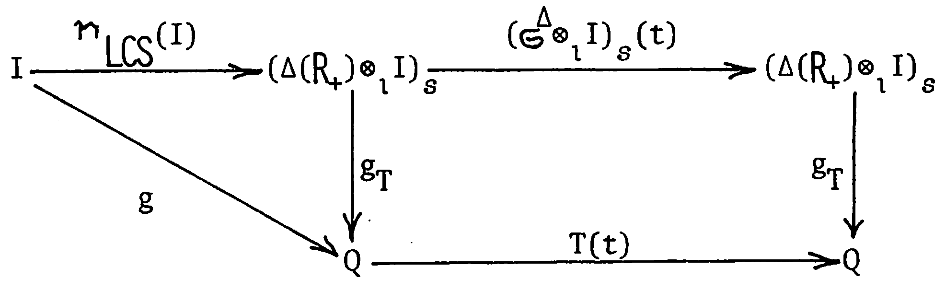
Proof: By 1.(2d), $\Delta(\mathbb{R}_+)$ is quasi-barreled. Hence it is a Mackey space ([27], Ch. 3, §6, Prop. 8). Thus $\Delta(\mathbb{R}_+)\otimes_\iota I$ is a locally-convex hull of Mackey spaces, which is a Mackey space ([36],§22,7.(8)). ∎

Thus, the construction of (4) also applies in the category $\mathcal{MS}$ of Mackey spaces. Define the functor $\mathfrak{L}_{\mathcal{MS}}: \mathcal{MS} \to \mathcal{DSG}(\mathcal{MS})$ as the restriction of $\mathfrak{L}_{\mathcal{LCS}}$, and $\eta_{\mathcal{MS}}(I) = \eta_{\mathcal{LCS}}(I)$ for each Mackey space I. The next result is thus immediate.

(13) *Let* I *be a Mackey space.* $(\overset{\Delta}{\mathfrak{S}}\otimes_\iota I, \eta_{\mathcal{MS}}(I))$ *is a universal map for* I *with respect to* $\mathfrak{F}_{\mathcal{MS}}$. $\mathfrak{L}_{\mathcal{MS}}$ *is left-adjoint to* $\mathfrak{F}_{\mathcal{MS}}$, *and* $\eta_{\mathcal{MS}}$: $1_{\mathcal{MS}} \to \mathfrak{F}_{\mathcal{MS}}\circ\mathfrak{L}_{\mathcal{MS}}$ *is a natural transformation which is the unit of this adjunction.* ∎

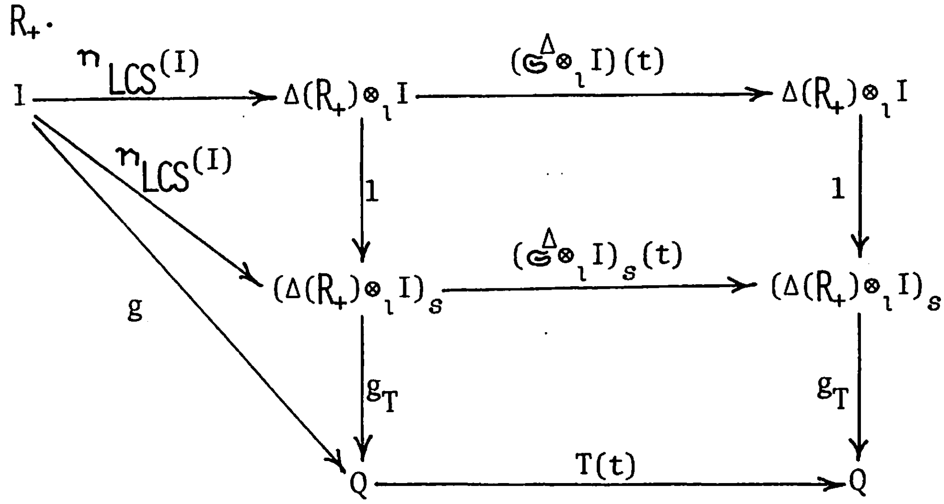
Unfortunately, if I is a weak l.c.s., it is not necessarily true that $(\Delta(\mathbb{R}_+))_s\otimes_\iota I$ is a weak l.c.s. However, by 1.(13a), $(\overset{\Delta}{\mathfrak{S}}\otimes_\iota I)_s$ is a d.s.g. on $(\Delta(\mathbb{R}_+)\otimes_\iota I)_s$, which leads to the following result.

(14) *Let* I *be a weak l.c.s.* $((\overset{\Delta}{\mathfrak{S}}\otimes_\iota I)_s, \eta_{\mathcal{LCS}}(I))$ *is a universal map for* I *with respect to* $\mathfrak{F}_{\mathcal{WS}}$. *Specifically, if* Q *is a weak l.c.s.,* T *a d.s.g. on* Q, *and* g: I → Q *a continuous linear map,* f_T *is the unique continuous linear map from* $(\Delta(\mathbb{R}_+)\otimes_\iota I)_s \to Q$ *which makes the diagram*



commute for each $t \in \mathbb{R}_+$.

Proof: Using (4), the following commutative diagram may be constructed for each $t \in \mathbb{R}_+$.



The uniqueness of g_T now follows immediately, since 1 is surjective. The rest of the proof is exactly like (4). ∎

Define the functor $\mathcal{L}_{WS}$: $WS \to DSG(WS)$ by $I \mapsto (\overset{\Delta}{\mathfrak{S}}\otimes_\iota I)_s$ on objects and as $\mathcal{L}_{LCS}$ on morphisms. Define $\eta_{WS}(I)\colon I \mapsto (\Delta(\mathbb{R}_+)\otimes_\iota I)_s$ for each weak l.c.s. I by $e \mapsto \delta_o \otimes e$. As with (5), the following is valid.

(15) $\mathfrak{L}_{WS}$ *is left-adjoint to* $\mathfrak{F}_{WS}$, *and* η_{WS}: $1_{WS} \to \mathfrak{F}_{WS} \circ \mathfrak{L}_{WS}$ *is a natural isomorphims which is the unit of this adjunction.* ∎

This completes the constructions of left-adjoint examples for this paper. Of course, there are several other examples of categories K of l.c.s.'s for which $\mathfrak{F}_K$ has a left adjoint.

Examples of Right Adjoints to $\mathfrak{F}_K$

For the same categories as above, a right adjoint to $\mathfrak{F}_K$ will now be constructed. If Y is the output space, the cofree output d.s.g. will be the left-shift semigroup on $E(\mathbb{R}_+,Y)$, or a suitable retopologization of this space. The following preliminary result will serve to combine three cases.

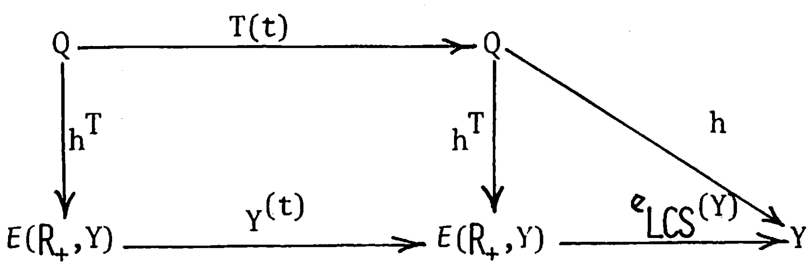
(16) *Let* Y *be a l.c.s. If* Y *is quasi-complete (resp. complete), then* $E(\mathbb{R}_+,Y)$ *is also quasi-complete (resp. complete).*

Proof: By 1.(3), $E(\mathbb{R}_+,Y)$ may be identified with $L_b(\Delta(\mathbb{R}_+),Y)$. If Y is quasi-complete, this space may be identified with $L_b(E'(\mathbb{R}_+),Y)$, since by 1.(2), every bounded subset of $E'(\mathbb{R}_+)$ is contained in the closure of a bounded subset of $\Delta(\mathbb{R}_+)$, and $E'(\mathbb{R}_+)$ is a quasi-completion of $\Delta(\mathbb{R}_+)$. Now $E'(\mathbb{R}_+)$ is barreled, (§0), and so $L_b(E'(\mathbb{R}_+),Y)$ is quasi-complete ([42], Ch. III, 4.4 Cor.). Since $E'(\mathbb{R}_+)$ is also bornological (§0), $L_b(E'(\mathbb{R}_+),Y)$ is complete whenever Y is complete ([48], Th. 32.2, Cor.). ∎

Given a l.c.s. Y, denote by $\mathcal{e}_{LCS}(Y)$: $\mathcal{E}(\mathbb{R}_+,Y) \to Y$ the map given by $\phi \mapsto \phi(0)$. $\mathcal{e}_{LCS}$ is linear and continuous because it is $\phi \mapsto \Phi_Y(\delta_o,\phi)$. The main cofree construction is as follows. Recall that $\mathfrak{S}_Y$ is the left-shift d.s.g. on $\mathcal{E}(\mathbb{R}_+,Y)$.

Given l.c.s.'s Q and Y, a d.s.g. T on Q, and a continuous linear map h: $Q \to Y$, define h^T: $Q \to \mathcal{E}(\mathbb{R}_+,Y)$ by $q \mapsto (t \mapsto h\circ T(t)q)$.

(17) *Let* Y *be a l.c.s. (resp. quasi-complete l.c.s., resp. complete l.c.s.).* $(\mathfrak{S}_Y, \mathcal{e}_{LCS}(Y))$ *is a couniversal map for* Y *with respect to* $\mathfrak{F}_{LCS}$ *(resp.* $\mathfrak{F}_{QC}$*, resp.* $\mathfrak{F}_{CS}$*). Specifically, if* Q *is a l.c.s. (resp. quasi-complete l.c.s., resp. complete l.c.s.),* T *a d.s.g. on* Q, *and* h: $Q \to Y$ *a continuous linear map, then* h^T *is the unique continuous linear map from* $Q \to \mathcal{E}(\mathbb{R}_+,Y)$ *which makes the diagram*



commute for each $t \in \mathbb{R}_+$.

Proof: By (16), the proof is identical in each of three cases. In the diagram, the triangle commutes by definition of h^T and $\mathcal{e}_{LCS}(I)$. The square commutes for each $t \in \mathbb{R}_+$, because $\mathfrak{S}_Y(t)\circ h^T(x) = \mathfrak{S}_Y(t)(s \mapsto h\circ T(s)x) = s \mapsto h\circ T(t+s)(x) = s \mapsto h\circ T(s)(T(t)x) = h^T\circ T(t)(x)$. It remains to show that h^T is unique. If the diagram were to commute for each $t \in \mathbb{R}_+$ with k replacing h^T, then $k(x)(t) = \mathfrak{S}_Y(t)(k(x))(0) = h \circ T(t)x = h^T(x)(t)$. Hence $k = f^T$, so h^T is unique. ∎

Define the functor $\mathfrak{R}_{LCS}: \mathrm{LCS} \to \mathrm{DSG}$ by $Y \mapsto \mathfrak{S}_Y$ on objects and $f \mapsto (\phi \mapsto f \circ \phi)$ on morphisms. Define $\mathfrak{R}_{QC}: \mathrm{QC} \to \mathrm{DSG}(\mathrm{QC})$ (resp. $\mathfrak{R}_{CS}: \mathrm{CS} \to \mathrm{DSG}(\mathrm{CS})$) to be the appropriate restriction of $\mathfrak{R}_{LCS}$. Define $e_{QC}(Y)$ (resp. $e_{CS}(Y)$) to be just $e_{LCS}(Y)$ for each quasi-complete (resp. complete) l.c.s. Y. The following is dual to (5).

(18) *$\mathfrak{R}_{LCS}$ (resp. $\mathfrak{R}_{QC}$, resp. $\mathfrak{R}_{CS}$) is right-adjoint to $\mathfrak{F}_{LCS}$ (resp. $\mathfrak{F}_{QC}$, resp. $\mathfrak{F}_{CS}$) and $e_{LCS}: \mathfrak{F}_{LCS} \circ \mathfrak{R}_{LCS} \to 1_{LCS}$ (resp. $e_{QC}: \mathfrak{F}_{QC} \circ \mathfrak{R}_{QC} \to 1_{QC}$, resp. $e_{CS}: \mathfrak{F}_{CS} \circ \mathfrak{R}_{CS} \to 1_{CS}$) is a natural transformation which is the counit of this adjunction.* ∎

The system-theoretic interpretation of the above is extremely simple, and is expressed entirely by the equation

$$y(t) = h(q(t)) \ .$$

The signal $\phi \in E(R_+,Y)$, $\phi = h^T(q)$, gives for each $t \in R_+$, the value of the output of the system at time t caused by state $q \in Q$ at $t = 0$, with the zero input applied after $t = 0$.

The cases of MS and WS are approached similarly to the left-adjoint case. $E(R_+,Y)$ need not be a Mackey (resp. weak) l.c.s. when Y is, so retopologization is necessary. In a manner similar to (14), the following result may be proved.

(19) *Let* Y *be a Mackey (resp. weak l.c.s.).* $((\mathfrak{S}_Y)_k, e_{LCS}(Y))$ *(resp.* $((\mathfrak{S}_Y)_s, e_{LCS}(Y))$ *is a couniversal map for* Y *with respect to* $\mathfrak{F}_{MS}$ *(resp.* $\mathfrak{F}_{WS}$*), with the details of the construction as in* (17). ∎

Define $\mathfrak{R}_{MS}: MS \to DSG(MS)$ (resp. $\mathfrak{R}_{WS}: WS \to DSG(WS)$) by $Y \mapsto (\mathfrak{S}_Y)_k$ (resp. $Y \mapsto (\mathfrak{S}_Y)_s$) on objects and as $\mathfrak{R}_{LCS}$ on morphisms. Define $e_{MS}: E(R_+,Y)_k \to Y$ (resp. $e_{WS}: E(R_+,Y)_s \to Y$) by $\phi \mapsto \phi(0)$. The following is similar to (13) and (15).

(20) $\mathfrak{R}_{MS}$ *(resp.* $\mathfrak{R}_{WS}$*) is right-adjoint to* $\mathfrak{F}_{MS}$ *(resp.* $\mathfrak{F}_{WS}$*), and* $e_{MS}: \mathfrak{F}_{MS} \circ \mathfrak{R}_{MS} \to 1_{MS}$ *(resp.* $e_{WS}: \mathfrak{F}_{WS} \circ \mathfrak{R}_{WS} \to 1_{WS}$*) is a natural transformation which is the counit of this adjucntion.* ∎

It is difficult, in general, to determine when $E(R_+,Y)$ is a Mackey (resp. weak) l.c.s. However, the following result covers many cases which may arise for the MS case.

(21) *Let* Y *be a l.c.s. If* Y *is metrizable, then* $E(R_+,Y)$ *is also metrizable, and each is a Mackey space.*

Proof: A metrizable l.c.s. is always a Mackey space ([23], Ch. III, Th. 3, Cor). Assume Y is metrizable, and identify $E(R_+,Y)$ with $L_b(\Delta(R_+),Y)$ (1.(4)). $\Delta(R_+)$ is a (DF) space (1.(2)), and so has a fundamental increasing sequence $B_0, B_1, B_2, \ldots$ of bounded sets. Since Y is metrizable, it has a fundamental sequence of decreasing neighborhoods of 0, $U_0, U_1, U_2, \ldots$ Clearly $V_0, V_1, V_2, \ldots$, where $V_k = N(B_k, U_k)$ forms a fundamental sequence of neighborhoods of 0 for $L_b(E(R_+),Y)$, so $E(R_+,Y)$ is metrizable. ∎

The above result holds in particular when Y is a (B) space.

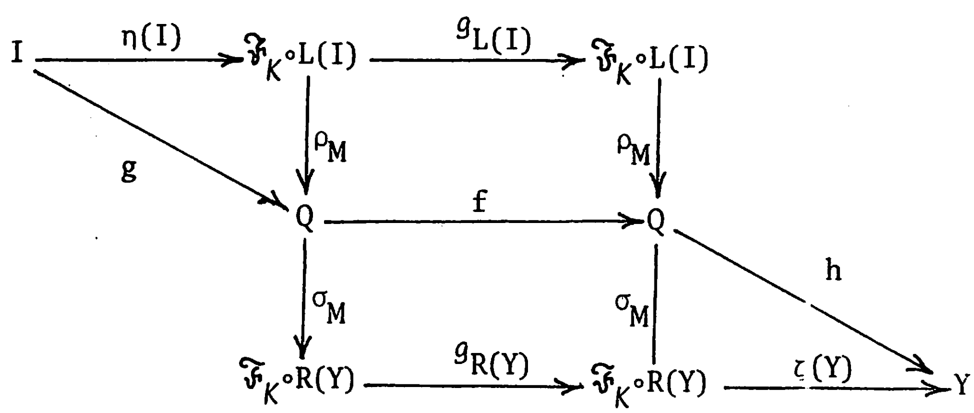
<u>Summary</u>

(22) *For* K = LCS, QC, CS, MS, *and* WS, $(\mathfrak{F}_K, \mathfrak{L}_K, \mathfrak{R}_K, \eta_K, \epsilon_K)$ *is a two-sided adjoint situation.* ▮

§3 Realization, Controllability, and Observability

<u>General Principles</u>

Let K be a differential-behavior category, and let A = $(\mathfrak{F}_K, L, R, N, \zeta)$ be a two-sided adjoint situation for $\mathfrak{F}_K$. Let M = (Q,f,I,g,Y,h) be a differential system in K. As given in §2, M has a reachability map ρ_M and an observability map σ_M defined by the diagram



Now let $(\mathcal{E},\mathcal{M})$ be an image-factorization system for K. M is $\mathcal{E}$-<u>reachable</u> if $\rho_M \in \mathcal{E}$ and $\mathcal{M}$-<u>observable</u> if $\sigma_M \in \mathcal{M}$. M is $(\mathcal{E},\mathcal{M})$-canonical if it is both $\mathcal{E}$-reachable and $\mathcal{M}$-observable. Note that since adjunctions are unique up to isomorphism, *the concepts of* E*-reachability,* M*-observability, and* (E,M)*-canonicity are independent of the choice* A *of two sided adjoint situation for* $\mathfrak{F}_K$.

This motivates a problem, the realization problem, which is in some sense a converse of the problem studied in §2. Again, let K be a differential-behavior category and let $A = (\mathfrak{F}_K, L, R, N, \zeta)$ be a two-sided adjoint situation for $\mathfrak{F}_K$. Let I and Y be K objects and let k: $\mathfrak{F}_K \circ L(I) \to \mathfrak{F}_K \circ R(Y)$ be the underlying map of a d.s.g. morphism from L(I) to R(Y). k is called a <u>total-response map</u>. A <u>realization</u> of k is a differential system M in K such that $\sigma_M \circ \rho_M = k$. Given an image-factorization system (E,M) for K, the <u>realization problem</u> for k is to find an (E,M)-canonical realization of k. (E,M) called a <u>compatible</u> image-factorization system for the differential-behavior category K if every total response in K has an (E,M)-canonical realization. A canonical realization is unique up to isomorphism in $\mathcal{D}i\mathfrak{f}(K)$, if it exists.

As a starting point, epimorphisms and monomorphisms in K are characterized.

(1) *Let* K *be a subcategory of* LCS*, and let* f *be a* K *morphism.*

(a) *If* f *is dense (resp. injective) then* f *is an epimorphism (resp. monomorphims).*

(b) *If* K *is full and contains* K *(or an isomorphic copy), then* f *is an epimorphism (resp. monomorphism) implies* f *is dense (resp. injective).*

Proof: (a) Suppose f: $E \to F$ is dense, and g and h are K morphisms such that $g \circ f = h \circ f$. By a usual characterization of surjectivity, $g(x) = h(x)$ for all $x \in f(E)$. However $\{x \in f \mid g(x) = h(x)\}$ is closed in F ([13], Ch. I, §8.1, Prop. 2, Cor. 1). Hence $g(x) = h(x)$.

Every injection is clearly a monomorphism.

(b) Assume K is full and f: $E \to F$ is a K morphism which is not dense. Pick $x \in F \setminus \overline{f(E)}$, and let $g \in F'$ with $g(x) = 1$ and $g(\overline{f(E)}) = 0$ (the existance of such a function is guaranteed by the Hahn-Banach theorem). Now let $h: \mathbb{K} \to E$ be any nonzero linear map (necessarily continuous). $h \circ g$ is a K morphism, and $(h \circ g) \circ f = 0 \circ f$, yet $h \circ g \neq 0$. Hence f is not an epimorphism.

Let f: $E \to F$ be a monomorphism which is not injective. Pick $x \in \mathit{ker}(f)$, and define $g: \mathbb{K} \to E$ by $a \mapsto a \cdot x$. g is necessarily continuous. Now let $h \in F' \setminus 0$. Clearly $f \circ (g \circ h) = f \circ 0$, yet $g \circ h \neq 0$. Hence f is not a monomorphism.∎

A continuous linear map f: $E \to F$ is called a <u>near quotient</u> if $\overline{f(U)}$ is a neighborhood of 0 in F whenever U is a neighborhood of 0 in E. $\{\overline{f(U)} \mid U \in \mathcal{U}(E)\}$ is thus a fundamental system of neighborhoods of 0 for F, with $\overline{f(E)} \subset F$, so f is dense.

The fundamental realization theorem is the following.

(2) *Let* K *be a differential-behavior cateory,* $A = (\mathfrak{F}_K, L, R, N, \zeta)$ *a two-sided adjoint situation for* $\mathfrak{F}_K$, *and* (E, M) *an image-factorization system for* K. *Let* I *and* Y *be* K *objects,* $k: \mathfrak{F}_K \circ L(I) \to \mathfrak{F}_K \circ R(Y)$ *a total*

response map, and $\mathfrak{F}_K \circ L \overset{e}{\to} Q \overset{m}{\to} \mathfrak{F}_K \circ R(Y)$ *an* $(\mathcal{E},\mathcal{M})$ *factorization for* k. *There is a unique* $T\colon \mathbb{R}_+ \to \mathcal{L}(Q)$ *satisfying* (s_1) *and* (s_2) *such that the diagram*

$$\begin{array}{ccc}
\mathfrak{F}_K \circ L(I) & \xrightarrow{\;L(I)(t)\;} & \mathfrak{F}_K \circ L(I) \\
\downarrow e & & \downarrow e \\
Q & \xrightarrow{\;T(t)\;} & Q \\
\downarrow m & & \downarrow m \\
\mathfrak{F}_K \circ R(Y) & \xrightarrow{\;R(Y)(t)\;} & \mathfrak{F}_K \circ R(Y)
\end{array}$$

commutes for each $t \in \mathbb{R}_+$. *Furthermore, there is a unique* $f \in \mathcal{L}(Q)$ *such that*

$$\begin{array}{ccc}
\mathfrak{F}_K \circ L(I) & \xrightarrow{\;g_{L(I)}\;} & \mathfrak{F}_K \circ L(I) \\
\downarrow e & & \downarrow e \\
Q & \xrightarrow{\;f\;} & Q \\
\downarrow m & & \downarrow m \\
\mathfrak{F}_K \circ R(Y) & \xrightarrow{\;g_{R(Y)}\;} & \mathfrak{F}_K \circ R(Y)
\end{array}$$

commutes. If either e *is surjective or* m *is an embedding, then* T *is a d.s.g. on* Q, *and* $g_T = f$. *Hence, if either* $\mathcal{E} \subset$ *surjections or* $\mathcal{M} \subset$ *embeddings, then* $(\mathcal{E},\mathcal{M})$ *is compatible. Furthermore, if* $\{\frac{L(I)-t}{t} \mid 0<t\leq\varepsilon\}$ *is equicontinuous for some* $\varepsilon>0$, *then it suffices that* e *be a near quotient for* T *to be a d.s.g. with* $g_T = f$, *and so for* $(\mathcal{E},\mathcal{M})$

to be compatible under these conditions, it suffices that $\mathsf{E} \subset$ *near quotients.*

Proof: T and f are uniquely provided by the fill-in property of image-factorizaton systems ([25], 33.5). Clearly T satisfies (s_1) and (s_2). If e is surjective, the continuity of e immediately transfers the d.s.g. properties of L(I) down to T. If m is an embedding, it suffices to apply 1.(14). Now suppose e is a near quotient and $\{\frac{L(I)-1}{t} | 0<t\leq\varepsilon\}$ is equicontinuous. This clearly implies that $\{\frac{\overline{T}(t)-1}{t} | 0<t\leq\varepsilon\}$ is equi-continuous, where $\overline{T}(t)$ is T(t) restricted to $e(\mathfrak{F}_K \circ L(I))$. However, by the preceding, $\overline{T}$ is a d.s.g. on $e(\mathfrak{F}_K \circ L(I))$, and so by 1.(15), extends uniquely to a d.s.g. on $\hat{Q}$, which must then restrict back to a d.s.g. on Q, by 1.(14). Uniqueness of the fill-in proves that f must be the infinitesimal generator of this d.s.g. ∎

Examples

The LCS case will be treated first.

(3) *Each of the following is a compatible image-factorizaton system for* LCS.

(a) *(quotient maps, injections) = (extermal epimorphisms, monomorphisms)*

(b) *(surjections, injections)*

(c) *(dense maps, closed embeddings) = (epimorphisms, extremal monomorphisms).*

Proof: Each of the factorizations is standard ([36], §15,4(3 and 4)). Let f: E → F be a continuous linear map. $E \overset{f_1}{\to} E/ker(f) \overset{f_2}{\to} F$ with f_1 the canonical quotient map and f_2: $[x] \mapsto f(x)$ is a factorization for (a). $E \overset{f_3}{\to} f(e) \overset{f_4}{\to} F$ with f_3: $x \mapsto f(x)$ and f_4 the inclusion map is a factorization for (b). Finally, $E \overset{f_5}{\to} \overline{f(E)} \overset{f_6}{\to} F$ with f_5: $x \mapsto f(x)$ and f_6: $x \mapsto x$ is a factorization for (c). The compatibility in each case follows from (2). By (1), injections = monomorphisms and dense maps = epimorphisms. By [25], 33.7, in an image-factorizaton system $(\mathbb{E},\mathbb{M})$, $\mathbb{E}$ = epimorphisms implies $\mathbb{M}$ = extremal monomorphisms and $\mathbb{M}$ = monomorphisms implies $\mathbb{E}$ = extremal epimorphisms, whence the characterization of (a) and (c). ∎

Hence, canonical realizations exist in $\mathbb{LCS}$ for each of the image-factorization systems of (3) above.

An obvious technique for extending the image-factorization systems of (4) to $\mathbb{QC}$ (quasi-complete l.c.s.'s)(resp. $\mathbb{CS}$ (complete l.c.s.'s)) is to construct the quasi-completion (resp. completion) of an image-factorization system for $\mathbb{LCS}$. That is, if f: E → F is a continuous linear map of quasi-complete (resp. complete) l.c.s.'s, and $E \overset{e}{\to} G \overset{m}{\to} F$ is an $(\mathbb{E},\mathbb{M})$ factorization of f (where $(\mathbb{E},\mathbb{M})$ is an image-factorization system for $\mathbb{LCS}$), regard the factorization $E \overset{\hat{e}}{\to} \dot{G} \overset{\hat{m}}{\to} F$ (resp. $E \overset{\hat{e}}{\to} G \overset{\hat{m}}{\to} M$) as the induced factorization in $\mathbb{QC}$ (resp. $\mathbb{CS}$). This technique works with each of the image-factorization systems given in (3), although the proof for the (3a) case is not trivial.

Since a closed subspace of a quasi-complete (resp. complete) l.c.s. is itself quasi-complete (resp. complete), QC (resp. CS) inherits the image-factorization system of (3c) from LCS.

(4) *(dense maps, closed embeddings) is an image-factorization system for* QC *(resp.* CS*).* ∎

Since the completion of a linear subspace of a complete l.c.s. is closed, it is immediate that the extension of the LCS image-factorization system (surjections, embeddings) to CS is just (dense maps, closed embeddings). The case for QC requires a new definition. Call a QC morphism $f: E \to F$ a quasi-surjection if F is isomorphic to a quasi-completion of f(E) (when f(E) has the relative topology induced by F), and call f a quasi-closed embedding if it is an embedding and if f(E) is quasi-closed in F.

(5) *(quasi-surjections, quasi-closed embeddings) is an image-factorization system for* QC.

Proof: The factorization is just the quasi-completion of the (surjections, embeddings) factorization in LCS. That is, if $f: E \to F$ is a continuous linear map of quasi-complete l.c.s.'s and $E \overset{e}{\to} G \overset{m}{\to} F$ is a (surjections, embeddings) factorization of f in LCS, then $E \overset{\hat{e}}{\to} G \overset{\hat{m}}{\to} F$ is a (quasi-surjections, quasi-closed embeddings) factorization in QC. The rest of the image-factorization system properties follow from the functoriality of $\widehat{\ }$. The compatibility follows from (2) and 2.(8). ∎

The extension of (quotient maps, injections) is substantially more difficult to handle than the other two. The less-than-obvious part of the construction is showing that the quasi-completion (resp. completion) of the monomorphic part of the factorization is monomorphic. That is, if f: E → F is a continuous linear map of quasi-complete (resp. complete) l.c.s.'s, show that $\widetilde{m}$ (resp. $\hat{m}$) is injective. An approach using a factorization theorem from category theory provides the proof.

A $\mathfrak{QC}$ morphism is called a <u>quasi-quotient</u> if it is a homomorphism and F is a quasi-completion of f(E) (regarded as a subspace of F).

(6) *Let* E *and* F *be a quasi-complete (resp. complete) l.c.s.'s, let* f: E → F *be a continuous linear map, and let* $G = \{(x,y) \in E \times E \mid x - y \in \ker(f)\}$.

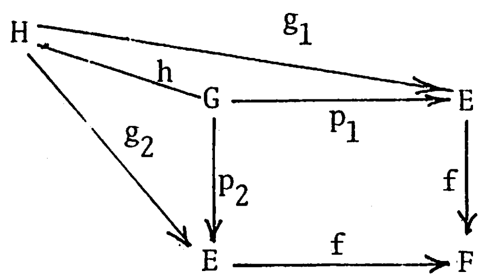
(a) G *is a quasi-complete (resp. complete) l.c.s., regarded as a linear subspace of* E×E.

(b) $G \overset{p_2}{\underset{p_1}{\rightrightarrows}} E$ *is a kernel pair of* E×E, *where* $p_1\colon (x,y) \mapsto x$ *and* $p_2\colon (x,y) \mapsto y$.

(c) f *is a coequalizer in* $\mathfrak{QC}$ *(resp.* $\mathfrak{CS}$*) if and only if it is a quasi-quotient (resp. near quotient), and in this case it is a coequalizer of* $G \overset{p_1}{\underset{p_2}{\rightrightarrows}} E$.

Proof: (a) Since the product of the quasi-complete (resp. complete) l.c.s.'s is quasi-complete (resp. complete), it suffices to show that G is closed in E×E. However, G is the kernel of the continuous linear map $E \times E \overset{f\times f}{\to} F \times F \overset{(-)}{\to} F$, whence it is closed.

(b) Clearly $f \circ p_1 = f \circ p_2$. Let H be a complete (resp. quasi-complete) l.c.s. and let g_1: H → E and g_2: H → E be continuous linear maps such that $f \circ g_1 = f \circ g_2$. Define h: H → G by $x \mapsto (g_1(x), g_2(x))$. h is clearly linear and continuous, and it follows that the diagram

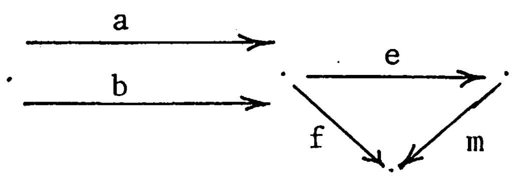


commutes. The uniqueness of h is clear. Hence $G \underset{p_2}{\overset{p_1}{\rightrightarrows}} E$ is a kernel pair of f in QC (resp. CS).

(c) It is easy to verify that the quotient maps are precisely the coequalizers in LCS, and that the quotient map q: $E \to E/\overline{(p_1 - p_2)(G)}$ is a coequalizer of $G \underset{p_2}{\overset{p_1}{\rightrightarrows}} E$ in LCS. Now the functor $\frown$ (resp $\hat{}$) has a right adjoint, namely the inclusion map : QC ↪ LCS (resp : CS ↪ LCS) (see §0), and so it preserves all colimits, particularly coequalizers ([37], Ch. V, §8, Th. 2). Hence f is a coequalizer if and only if it is the image under $\frown$ (resp. $\hat{}$) of a quotient map. ∎

The next statement summarizes the necessary categorical results for the image-factorization system sought.

(7) *Let* K *be a category which has kernel pairs and coequalizers of kernel pairs, and suppose that the class of all coequalizers is closed under composition. (coequalizers, monomorphisms) is an image-factorization system for* K, *and if* f *is a* K *morphism, a (coequalizers, monomorphisms) factorization of* f *is given by* $f = m \circ e$, *where* e *is a coequalizer of a kernel pair* (a,b) *of* f, *and* m *is the unique morphism making the diagram*



commute.

Proof: Consult [43], 18.4.7. ∎

(8) (a) *(quasi-quotients, injections) is a compatible image-factorization system for* QC.

(b) *(near quotients, injections) is a compatible image-factorization system for* CS.

Proof: That each is an image-factorization system follows from (6) and (7). The compatibility follows from (2) and 2.(8). ∎

The image-factorization systems of LCS restrict perfectly to WS (weakly-topologized l.c.s.'s).

(9) *Each of the following is a compatible image-factorization system for* $\mathbb{WS}$.

(a) *(quotient maps, injections)*

(b) *(surjections, injections)*

(c) *(dense maps, closed embeddings).*

Proof: It suffices to note that a quotient of a weak l.c.s. is weak ([36], §22,2.(3)), and a subspace of a weak l.c.s. is weak ([36], §22, 2.(2)), and then to apply (3). The compatibility follows again from (2). ∎

The case of $\mathbb{MS}$ (Mackey l.c.s.'s) is not quite as easy, since while quotients of Mackey spaces are Mackey spaces ([36], §22,2.(4)), a subspace of a Mackey space need not be a Mackey space. Call an injection $f: E \to F$ of Mackey spaces a <u>Mackey injection</u> if f is a weak embedding.

(10) *Each of the following is a compatible image-factorization system for* $\mathbb{MS}$.

(a) *(quotient maps, injections)*

(b) *(surjections, Mackey injections)*

(c) *(dense maps, closed Mackey injections).*

Proof: That each is an image-factorization system follows immediately from (9) and the isomorphism $\mathfrak{Y}: \mathbb{WS} \to \mathbb{MS}$ (see §0). The compatibility of (a) and (b) follow follow from (2). For (c), it suffices to note that the fill-in semigroup of the induced factorization in $\mathbb{DSG}(\mathbb{MS})$ is a d.s.g. when its underlying space carries its weak topology, in view of (9) and 2.(19). Hence, by 1.(13), it is also a d.s.g. when it carries its Mackey topology. ∎

Finite-Dimensional Systems

Let K be a full differential-behavior category, and let M = (Q,f,I,g,Y,h) be a differential system in K. M is called finite-dimensional if Q has finite dimension as a linear space (i.e., $Q \cong \mathbb{K}^n$ for some finite n). If $(\mathbb{E},\mathbb{M})$ is a compatible image-factorization system for K and k is a total response in K, then an $(\mathbb{E},\mathbb{M})$-canonical realization for k depends, in general, upon $(\mathbb{E},\mathbb{M})$ as well as k. However, if k has a finite-dimensional canonical realization for *some* image-factorization system, then this realization is canonical for *every* image-factorization system for K. The existence of some finite-dimensional realization for k (not necessarily canonical) is enough to guarantee the existence of such a universal canonical realization.

(11) *Let* K *be a full differential-behavior category which contains the subcategory of* $\mathbb{LCS}$ *consisting of the finite-dimensional spaces, let* $A = (\mathfrak{F}_K, L, R, N, \zeta)$ *be an adjoint situation for* $\mathfrak{F}_K$*, let* I *and* Y *be* K *objects, and let* $k\colon \mathfrak{F}_K \circ L(I) \to \mathfrak{F}_K \circ R(Y)$ *be a total response map. Suppose there is a factorization* $\mathfrak{F}_K \circ L(I) \xrightarrow{k_1} P \xrightarrow{k_2} \mathfrak{F}_K \circ R(Y)$ *of* k *with* P *finite-dimensional.*

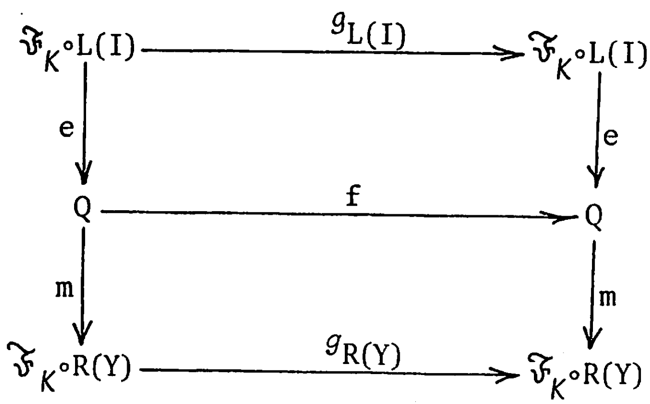
(a) *There is a factorization* $\mathfrak{F}_K \circ L(I) \xrightarrow{e} Q \xrightarrow{m} \mathfrak{F}_K \circ R(Y)$ *of* k *with* e *surjective,* m *injective, and* Q *finite-dimensional.*

(b) *There is a unique differential machine* M *in* $\mathcal{K}$ *with input space* I, *output space* Y, *and state space* Q, *such that* k *is the total response of* M.

(c) M *is an* $(\mathcal{E},\mathcal{M})$*-canonical realization of* k *in* $\mathcal{K}$ *for every image-factorization system* $(\mathcal{E},\mathcal{M})$ *in* $\mathcal{K}$, *compatible or not.*

Proof: (a) Let $\mathfrak{F}_K \circ L(I) \overset{e_1}{\to} E \overset{m_1}{\to} P$ be a (quotient maps, injections) factorization of k_1 in LCS. E is finite-dimensional since m_1 is injective and P is finite-dimensional. Let $E \overset{e_2}{\to} Q \overset{m_2}{\to} \mathfrak{F}_K \circ R(Y)$ be a (dense maps, closed embeddings) factorization of $k_2 \circ m_1$ in LCS. It follows that e_2 is surjective, since every finite-dimensional l.c.s. is complete, hence relatively closed, and so Q is finite-dimensional. Hence $\mathfrak{F}_K \circ L(I) \overset{e}{\to} Q \overset{m}{\to} \mathfrak{F}_K \circ R(Y)$ is a factorization of k, with $e = e_2 \circ e_1$ and $m = m_2$. e is surjective and m is injective by construction.

(b) e is a quotient map, since Q carries its finest locally-convex topology, being finite-dimensional. Hence $m \circ e$ is a (quotient maps, injections) factorization of k in LCS. By (2), there is a unique $f \in L(Q)$ such that



commutes. Since (quotient maps, injections) is compatible, by (3a), there is a unique d.s.g. T on Q such that $f = g_T$. Put M = (Q,f,I, $e \circ N(I)$,Y,$\zeta(Y) \circ m$). k is the total response of M, so that M is a realization of k. The uniqueness of M is a consequence of the fact that (quotient maps, injections) is compatible for $\mathbb{LCS}$.

(c) As shown above, e is a quotient map and m is a closed embedding. Hence, by (3), e is an extremal epimorphism in $\mathbb{LCS}$ and m is an extremal monomorphism in $\mathbb{LCS}$. Hence e is also an extremal epimorphism in K and m is also an extremal monomorphism in K. Thus $m \circ e$ is an $(\mathbb{E},\mathbb{M})$ factorization of k for any image-factorization system of K, since extremal epimorphisms $\subset \mathbb{E}$ and extremal monomorphisms $\subset \mathbb{M}$ always ([25], 33.6 and 33.7). Hence M is an $(\mathbb{E},\mathbb{M})$-canonical realization. ∎

This shows that while the number of distinct concepts of canonical realization in the full differential-behavior category K is at least as large as the number of distinct compatible image-factorization systems for K, there is only one concept of canonicity for finite-dimensional linear systems, in harmony with the classical theory of finite dimensional linear systems. The problem of characterizing those differential systems in K, all of whose canonical realizations coincide, (for each image-factorization system for K), is certainly a valid question, but is not investigated in this paper.

§4 DUALITY THEORY

LCS Categories

In this section, a general duality theory for differential systems is developed. The examples of the duality will be devleoped within the category of Mackey l.c.s.'s and within the category of weak l.c.s.'s.

In treating duality theory, it is convenient to deal with categories relative to LCS rather than ordinary categories. For a complete treatment of relative categories, consult [20]. For now, the following special case suffices. A category K is called an LCS category if for each pair of K objects (E,F), $Mor_K(E,F)$ has the structure of a l.c.s., and for any 3-tuple of K objects (E,F,G), the conposition map $\circ$: $Mor_K(E,F)\times Mor_K(F,G) \to Mor_K(E,G)$ is a separately-continuous bilinear map.

Each subcategory of LCS may be made into an LCS category in several ways. Let K be a subcategory of LCS. Assign to $Mor_K(E,F) \subset L(E,F) \subset L(E_s,F_s)$ the topology inherited from $L_s(E_s,F_s)$. It is trivially verified that the composition map $\circ$: $L_s(E_s,F_s)\times L_s(F_s,G_s) \to L_s(E_s,G_s)$: $(f,g)\mapsto g\circ f$ is separately-continuous, so that K is indeed an LCS category under this structure. Under this *same* structure, K^{op} is also an LCS category, i.e., $Mor_{K^{op}}(E,F)$ carries the topology induced by $L_s(F_s,E_s)$. *Throughout this section, whenever a subcategory K of LCS or its dual K^{op} is regarded as an LCS category, the above structure will always be implied.*

A functor P: $K \to H$ between LCS categories is called an LCS functor if each induced function $P_{E,F}$: $Mor_K(E,F) \to Mor_H(P(E),P(F))$ is a continuous linear map. To emphasize the special case being considered, call an

LCS category <u>usual</u> if it is either a subcategory of LCS, or else the opposite of a subcategory of LCS. An LCS functor P: $K \to H$ between usual LCS categories is called a differential functor if for each pair of K objects (E,F), $P_{E,F}$ is continuous when the morphism classes carry the topologies indicated in the preceding paragraph.

It is now convenient to generalize the definitions of DSG(K) and $\mathcal{D}if(K)$ to include those cases for which K is the opposite of a subcategory of LCS. The definitions are the obvious ones. If K is the opposite of a subcategory of LCS, DSG(K) is defined to be (DSG(K^{op}))^{op}, so that DSG(K) has the same objects as DSG(K^{op}), but a DSG(K) morphism from S to T is a DSG(K^{op}) morphism form T to S. Similarly, M = (Q,f,I,g,Y,h) is an object of $\mathcal{D}if(K)$ if and only if M^{op} = (Q,f,Y,h,I,g) (M^{op} is called the opposite system of M) is an object of $\mathcal{D}if(K^{op})$. (a,b,c) is a morphism from M_1 to M_2 if and only if (c,b,a) is a morphism from M_2^{op} to M_1^{op} in $\mathcal{D}if(K^{op})$. These conventions will be used throughout the rest of this paper without further mention.

The reason for the terminology differential functor is justified by the following fact.

(1) *Let* K *and* H *be usual* LCS *categories, and let* P: $K \to H$ *be a differential functor. Let* E *be a* K *object. If* T *is a d.s.g. on* E, *then* $P_{E,E} \circ T$ *is a d.s.g. on* P(E) *with infinitesimal generator* $P(\mathcal{G}_T)$.
Proof: Since P is a functor, $P_{E,E} \circ T$ clearly satisfies conditions (s_1) and (s_2) of §1. Now by 1.(13b), T is a d.s.g. on E . Hence $P_{E,E} \circ T$ is a d.s.g. on $P(E)_s$ with infinitesimal generator $P(\mathcal{G}_T)$, since $P_{E,E}$ is linear and continuous. Since $(P_{E,E} \circ T)(t)$ is continuous on E, it follows from 1.(13b) that $P_{E,E} \circ T$ is a d.s.g. on E. ∎

By abuse of notation, the d.s.g. $P_{E,F} \circ T$ will be denoted by just $P \circ T$.

Let K and H be usual LCS categories, and let $P\colon K \to H$ be a differential functor. If S is a d.s.g. on the K object E and T is a d.s.g. on the K object F, and $f\colon E \to F$ is a K morphism, the commutativity of

$$\begin{array}{ccc} E & \xrightarrow{\ S(t)\ } & E \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ F & \xrightarrow{\ T(t)\ } & F \end{array}$$

implies the commutativity of

$$\begin{array}{ccc} P(E) & \xrightarrow{\ (P \circ S)(t)\ } & P(E) \\ {\scriptstyle P(f)}\downarrow & & \downarrow{\scriptstyle P(f)} \\ P(F) & \xrightarrow{\ (P \circ T)(t)\ } & P(F) \end{array}$$

Hence P induces a functor from $DSG(K)$ to $DSG(H)$, given by $T \mapsto P \circ T$ on objects and $f \mapsto P(f)$ on underlying morphisms. This functor is denoted $\check{P}$ or $(P)^{\vee}$. Similarly, if $M_i = (Q_i, f_i, I_i, g_i, Y_i, h_i)$ for $i = 1,2$ are differential systems in K, and $(a,b,c)\colon M_1 \to M_2$ is a $\mathcal{D}if(K)$ morphism, the diagram

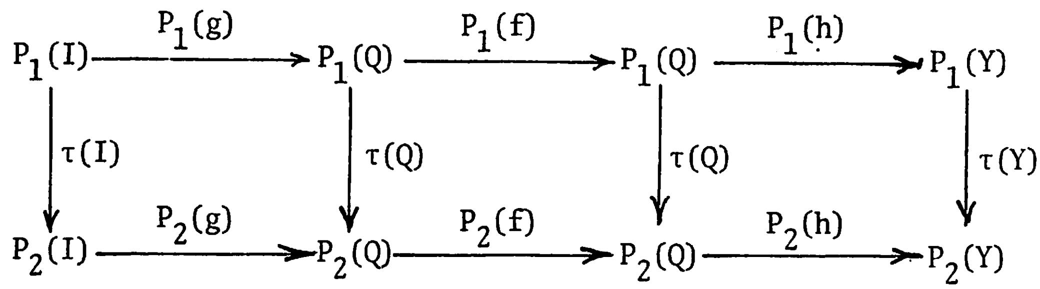
$$\begin{array}{ccccccc}
P(I_1) & \xrightarrow{P(g_1)} & P(Q_1) & \xrightarrow{P(f_1)} & P(Q_1) & \xrightarrow{P(h_1)} & P(Y_1) \\
\downarrow{\scriptstyle P(a)} & & \downarrow{\scriptstyle P(b)} & & \downarrow{\scriptstyle P(b)} & & \downarrow{\scriptstyle P(c)} \\
P(I_2) & \xrightarrow{P(g_2)} & P(Q_2) & \xrightarrow{P(f_2)} & P(Q_2) & \xrightarrow{P(h_2)} & P(Y_2)
\end{array}$$

commutes, so P induces a functor from $\mathcal{Dif}(K)$ to $\mathcal{Dif}(H)$ given by $(Q,f,I,g,Y,h) \mapsto (P(Q),P(f),P(I),P(g),P(Y),P(h))$ on objects and $(a,b,c) \mapsto (P(a),P(b),P(c))$ on morphisms. Thus functor is denoted $\check{P}$ or $(P)^{\vee}$.

The ideas above extend to natural transformations of differential functors. Specifically, let K and H be usual LCS categories, $P_i\colon K \to H$ for $i = 1,2$ be functors, and $\tau\colon P_1 \to P_2$ be a natural transformation. If T is a d.s.g. on the K object E, the diagram

$$\begin{array}{ccc}
P_1(E) & \xrightarrow{(P_1 \circ S)(t)} & P_1(E) \\
{\scriptstyle \tau(E)}\downarrow & & \downarrow{\scriptstyle \tau(E)} \\
P_2(E) & \xrightarrow{(P_2 \circ S)(t)} & P_2(E)
\end{array}$$

commutes for each $t \in \mathbb{R}_+$. This defines a map from $Obj(\mathrm{DSG}(K))$ to $Mor(\mathrm{DSG}(H))$ given by $(T\colon \mathbb{R}_+ \to Mor(E)) \mapsto \tau(E)$. Denote this map by $\check{\tau}$ or $(\check{\tau})$. Similarly, if $M = (Q,f,I,g,Y,h)$ is a $\mathcal{Dif}(K)$ object, the diagram



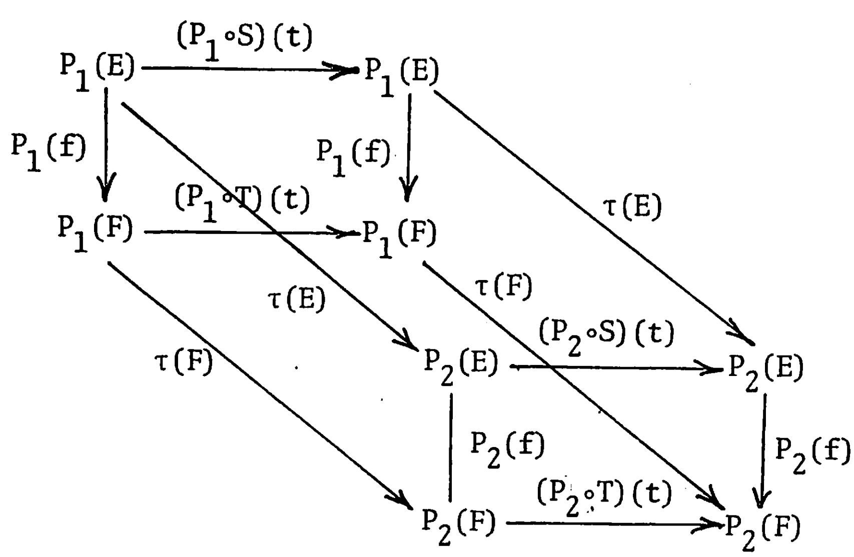
commutes. This defines a map from $Obj(\mathcal{D}if(K))$ to $Mor(\mathcal{D}if(H))$ given by $(Q,f,I,g,Y,h) \mapsto (\tau(I),\tau(Q),\tau(Y))$. Denote this map by $\check{\tau}$ or $(\tau)^{\vee}$.

(2) *Let* K *and* H *be usual* LCS *categories, let* $P_i\colon K \to H$ *for* $i = 1,2$ *be functors, and let* $\tau\colon P_1 \to P_2$ *be a natural transformation.*

(a) $\check{\tau}\colon \check{P}_1 \to \check{P}_2$ *and* $\breve{\tau}\colon \breve{P}_1 \to \breve{P}_2$ *are natural transformations.*

(b) $\check{\tau}$ *(resp.* $\breve{\tau}$*) is a natural isomorphism if and only if* τ *is a natural isomorphism.*

Proof: (a) The case of $\check{\tau}$ will be given; the case of $\breve{\tau}$ is similar. It suffices to show that given DSG(K) objects $S\colon \mathbb{R}_+ \to Mor_K(E)$ and $T\colon \mathbb{R}_+ \to Mor_K(F)$, with $f \in Mor_K(E,F)$ a DSG(K) morphism form S to T, the diagram



commutes for each $t \in \mathbb{R}_+$. The front (resp. back) commutes because P_1 (resp. P_2) is a functor. The other four faces commute because τ is a natural transformation. Hence the entire diagram commutes.

(b) This follows immediately from the characterization of isomorphisms in the categories concerned. ∎

Some of the more useful properties of these induced functors are now given. The proofs are easy and are omitted.

(3) *Let K and H be usual* LCS *categories, and let* $P: K \to H$ *be a differential functor.*

(a) $\check{P}$ *(resp.* $\breve{P}$*) is faithful if and only if* P *is.*

(b) $\check{P}$ *(resp.* $\breve{P}$*) is an embedding if only if* P *is.*

(c) $\check{P}$ *(resp.* $\breve{P}$*) is an isomorphism if and only if* P *is bijective on objects and for each pair of K objects* (E,F), $P_{E,F}$ *is an* LCS *isomorphism.*

(d) $\check{P}$ *(resp.* $\breve{P}$*) is an equivalence if* P *is representative and for each pair of K objects* (E,F), $P_{E,F}$ *is an* LCS *isomorphism.* ∎

With the above notation, if $\check{P}$ is an isomorphism (resp. equivalence), then P is called a <u>differential isomorphism</u> (resp. <u>differential equivalence</u>).

(4) *Let K, H, and J be usual* LCS *categories, and let* $P_1: K \to H$ *and* $P_2: K \to J$ *be differential functors.* $P_2 \circ P_1$ *is differential, and*

(a) $(P_2 \circ P_1)^{\vee} = \check{P}_2 \circ \check{P}_1$.

(b) $(P_2 \circ P_1)^{\cup} = \breve{P}_2 \circ \breve{P}_1$. ∎

Duality Theory for Differential Systems

Using the concept of differential equivalence, it is possible to develop a general duality theory for differential systems.

Recall that if K is a subcategory of LCS, the functor $\mathfrak{F}_K$: $\mathsf{DSG}(K) \to K$ sends each d.s.g. to its underlying l.c.s. and sends each d.s.g. morphism to its underlying K morphism. To extend this to the case for which K is the opposite of a subcategory of LCS it is only necessary to define $\mathfrak{F}_K$ to be $(\mathfrak{F}_{K^{op}})^{op}$. This convention will be assumed from now on.

The following result is very useful; its proof is clear.

(5) *Let* K *and* H *be usual* LCS *categories.*

(a) *If* P: $K \to H$ *is a differential functor, then* $\mathfrak{F}_H \circ P = \check{P} \circ \mathfrak{F}_K$.

(b) *If* P_i: $K \to H$ *for* $i = 1,2$ *is a differential functor and* τ: $P_1 \to P_2$ *is a natural transformation, then* $\mathfrak{F}_H * \check{\tau} = \tau * \mathfrak{F}_K$. ∎

The next result characterizes the invertibility properties of differential functors.

(6) *Let* K *and* H *be usual* LCS *categories, and let* P: $K \to H$ *be a differential equivalence which is also an embedding. There is a differential equivalence* N: $H \to K$ *and a natural isomorphism* τ: $1_H \to P \circ N$ *such that*

(a) $N \circ P = 1_K$

(b) $X \in Obj(P(K)) \Rightarrow \tau(X) = 1_X$.

Proof: Chose a function τ: $Obj(H) \to Mor(H)$ such that for each $X \in Obj(H)$, $\tau(X)$ is an isomorphism with $dom(\tau(X)) = X$ and $cod(\tau(X)) \in Obj(P(K))$ with $\tau(x) = 1_x$ if $x \in Obj(P(K))$. The existence of such a function is guaranteed by the axiom of choice for classes. For each $X \in Obj(H)$, put $N(X) = Y$, where Y is the unique element of $P^{-1}(cod(\tau(X)))$. For each $f \in Mor_H(X,Y)$, put $N(f) = g$, where g is the unique element of $P^{-1}(\tau(Y) \circ f \circ \tau^{-1}(X))$. $P^{-1}(\tau(Y) \circ f \circ \tau^{-1}(X))$ is nonempty since an equivalence is full. From these definitions, it is immediate that

$$\begin{array}{ccc} X & \xrightarrow{\quad \tau(X) \quad} & P \circ N(X) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle P \circ N(f)} \\ X & \xrightarrow{\quad \tau(X) \quad} & P \circ N(X) \end{array}$$

commutes for each $f \in Mor_H(X,Y)$. If f is an identiy, then N(f) is also an identity, since embeddings reflect identities. If also $g \in Mor_H(Y,Z)$, $Z \in Obj(H)$, then it is easy to see that $N(g \circ f) = N(g) \circ N(f)$, from the commutativity of the above diagram and the fact that $\tau(X)$ and $\tau(Z)$ are isomorphisms. Hence N is a functor, and τ: $1_H \to P \circ N$ is a natural isomorphism. (a) and (b) are satisfied by construction, and N is an equivalence inverse to P. Finally, N is a differential functor; this is also implicit in the construction. ∎

Let K and H be differential-behavior categories. A <u>duality specifier</u> from K to H is a 3-tuple (P,N,τ), where P: $K^{op} \to H$ is a differential functor which is an equivalence and an embedding, N: $H \to K^{op}$ is an equivalence with $N \circ P = 1_{K^{op}}$, and $\tau\colon 1_H \circ P \to N$ is a natural isomorphism with $\tau(X) = 1_X$ for each $X \in Obj(P(K^{op}))$. (6) says that given a differential equivalence and embedding $P\colon K^{op} \to H$, N and τ always exist such that (P,N,τ) is a duality specifier from K to H. Duality specifiers define a specific dual adjunction and dual machine, as shown by the next result.

(7) *Let* K *and* H *be differential-behavior categories and let* $A = (\mathfrak{F}_K, L, R, N, \zeta)$ *be a two-sided adjoint situation for* $\mathfrak{F}_K$. *Each duality specifier from* K *to* H (P,N,τ) *uniquely determines a two-sided adjoint situation* $A' = (\mathfrak{F}_H, L', R', \eta', \zeta')$ *for* $\mathfrak{F}_H$ *with*

(a) $L' = P \circ R^{op} \circ N$;

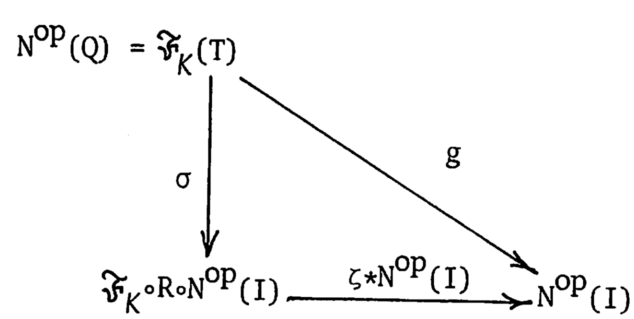
(b) $R' = P \circ L^{op} \circ N$;

(c) $\eta' = (P * \zeta^{op} * N) \circ \tau$;

(d) $\zeta' = \tau^{-1} \circ (P * N^{op} * N)$.

If M *is a differential system in* K *with reachability map* ρ_M *and observability map* σ_M *with respect to the adjoint situation* A, *then* P(M) *has a reachability map* $P(\sigma_M)$ *and observability map* $P(\rho_M)$ *with respect to the adjoint situation* A'.

Proof: ((a) and (c)) From the adjoint situation A, given H objects I and Q, a continuous linear map $g\colon N^{op}(Q) \to N^{op}(I)$, and a d.s.g. T on $N^{op}(Q)$, there is a unique d.s.g. morphism $\sigma\colon N^{op}(Q) \to R \circ N^{op}(I)$ such that



commutes. Applying P to the opposite of this diagram and using (5) yields

$$\begin{array}{ccc}
P\circ N(I) & \xrightarrow{P*\zeta^{op}*N(I)} & \mathfrak{F}_H\circ\check{P}\circ R^{op}\circ N(I) \\
 & \searrow^{P\circ N(g)} & \downarrow P(\sigma) \\
 & & \mathfrak{F}_H\circ\check{P}(T) = P\circ N(Q)
\end{array}$$

Next, consider the following diagram.

$$\begin{array}{ccccc}
I & \xrightarrow{\tau(I)} & P\circ N(I) & \xrightarrow{P*\zeta^{op}*N(I)} & \mathfrak{F}_H\circ\check{P}\circ R^{op}\circ N(I) \\
 & \searrow^{g} & & \searrow^{P\circ N(g)} & \downarrow P(\sigma) \\
 & & \mathfrak{F}_H(T) & \xrightarrow{\mathfrak{F}_H\circ\check{\tau}(T)} & \mathfrak{F}_H\circ\check{P}(T) = P\circ N(Q)
\end{array}$$

The triangle commutes by the preceding. The square commutes in view of (5) and the fact that τ is a natural transformation. This combines to yield the diagram

$$\begin{array}{ccc}
I & \xrightarrow{P*\zeta^{op}*N(I)} & \mathfrak{F}_H\circ\check{P}\circ R^{op}\circ N(I) \\
 & \searrow^{g} & \downarrow \tau(Q)^{-1}\circ P(\sigma) \\
 & & \mathfrak{F}_H\circ\check{P}(T) = P\circ N(Q)
\end{array}$$

using $\mathfrak{F}_H\circ\check{\tau}(T) = \tau(Q)$. $\tau(Q)^{-1}\circ P(\sigma)$ is clearly a d.s.g. morphism from $\check{P}\circ R^{op}\circ N(I)$ to T; it remains to verify that it is unique. However, this is an immediate consequence of the fact that the diagram is, up to isomorphism, the dual of an adjunction diagram (P and N are both differential equivalences). Thus, (a) and (c) and satisfied; (b) and (d) are dual to (a) and (c), respectively.

Now suppose $M = (Q,f,I,g,Y,h)$ is a differential machine in K. The reachability of the dual machine $\breve{P}(M) = (P(Q),P(f),P(I),P(g),P(Y),P(h))$ for the two-sided adjoint situation A' is given by $\tau(P(Q))^{-1} \circ P(\sigma_M) = P(\sigma_M)$, since $\tau(P(Q))^{-1} = 1_{P(Q)}$, since (P,N,τ) is a duality specifier. Hence the reachability map of $\breve{P}(M)$ is $P(\sigma_M)$. The characterization of the output map is dual. ∎

The two-sided adjoint situation A' of (7) is called the two-sided adjoint situation derived from A by (P,N,τ).

The next, rather lengthy, theorem states that the dual of the dual of a differential machine is isomorphic to the machine itself.

(8) *Let K and H be differential-behavior categories, and let* $A = (\mathfrak{F}_K,L,R,\eta,\zeta)$ *be an adjoint situation for* $\mathfrak{F}_K$. *Let* (P_1,N_1,τ_1) *be a duality specifier from K to H, and let* (P_2,N_2,τ_2) *be a duality specifier from H to K. Let A' be the adjoint situation for* $\mathfrak{F}_H$ *derived from A by* (P_1,N_1,τ_1) *and let A'' be the adjoint situation for* $\mathfrak{F}_K$ *derived from A by* (P_2,N_2,τ_2). *Let M be a differential system in K with reachability map* ρ_M *and observability map* σ_M *with respect to A. Under these conditions,* $\breve{P}_2 \circ \breve{P}_1(M)$ *is isomorphic to M in $\mathcal{D}if(K)$, and with respect to A'', the reachability of* $\breve{P}_2 \circ \breve{P}_1(M)$ *is* $\breve{P}_2 \circ \breve{P}_1(\rho_M)$ *and its observability map is* $\breve{P}_2 \circ \breve{P}_1(\sigma_M)$.

Proof: $P_2 \circ P_1$ is differential by (4), and a differential equivalence by (3c). Hence there is a natural isomorphism $\kappa\colon 1_K \to P_2 \circ P_1$ which extends to a differential-system preserving isomorphism $\breve{\kappa}$, by (2). Thus $\breve{P}_2 \circ \breve{P}_1(M)$ is isomorphic to M. The structure of the reachability and observability maps follows by applying (7) twice. ∎

The next two results show that the general duality theory presented here gives the usual duality between controllability and observability, provided the duality of image-factorization systems is carefully noted.

Let $\mathcal{F}$ be a class of morphisms in a category K. The smallest class of K morphisms which contains all isomorphisms as well as $\mathcal{F}$ and which is closed under composition is called the <u>closure</u> of $\mathcal{F}$ and is denoted $\overline{\mathcal{F}}$.

(9) *Let* $P_1\colon K \to H$ *and* $P_2\colon H \to K$ *be equivalences of categories with* $P_2 \circ P_1 \cong 1_K$, $P_1 \circ P_2 \cong 1_H$, *and let* $(\mathcal{E},\mathcal{M})$ *be an image-factorization system for* K.

(a) $(\overline{P_1(\mathcal{E})}, \overline{P_1(\mathcal{M})})$ *is an image-factorization system for* H.

(b) $(P_2(\overline{P_1(\mathcal{E})}), P_2(\overline{P_1(\mathcal{M})}) = (\mathcal{E},\mathcal{M})$

Proof: Consult [24], App. 2. ∎

(10) *Let* K *and* H *be differential-behavior categories, let* (P_1, N_1, τ_1) *be a duality specifier from* K *to* H, *let* $(\mathcal{E},\mathcal{M})$ *be an image-factorization system for* K, *and let* M *be a differential system in* K.

(a) $\breve{P}_1(M)$ *is* $\overline{P_1(\mathcal{M})}$*-reachable if and only if* M *is* $\mathcal{M}$*-observable.*

(b) $\breve{P}_2(M)$ *is* $\overline{P_2(\mathcal{E})}$*-observable if and only if* M *is* $\mathcal{E}$*-reachable.*

(c) $P_1(M)$ *is* $(\overline{P_1(\mathcal{M})}, \overline{P_1(\mathcal{E})})$*-canonical if and only if* M *is* $(\mathcal{E},\mathcal{M})$ *canonical.*

(d) *If* $(P_2, N_2, {}_2)$ *is a duality specifier from* $\mathcal{H}$ *to* $\mathcal{K}$*, then* $P_2 \circ P_1(M)$ *is* $\mathcal{E}$*-reachable (resp.* $\mathcal{M}$*-observable, resp.* $(\mathcal{E},\mathcal{M})$*-canonical) if and only if* M *has the same property.*

Proof: Follows from (7), (8), and (9). ∎

In the above, recall that reachability, observability, and canonicity are independent of the particular adjoint situation, depending only upon the image-factorization system. Also recall that $(\mathcal{E},\mathcal{M})$ is an image-factorization system for $\mathcal{K}$ if and only if $(\mathcal{M},\mathcal{E})$ is an image-factorization system for $\mathcal{K}^{op}$.

Examples

Examples of duality theory in the cases of Mackey l.c.s.'s and Weak l.c.s.'s are very easy to construct. All that need be shown is that the duality functors are differential and embeddings. The behavior of the various image-factorization systems under the duality will also be shown.

Let $\mathcal{D}_{\mathcal{MS}}: (\mathcal{MS})^{op} \to \mathcal{MS}$ (resp. $\mathcal{D}_{\mathcal{WS}}: (\mathcal{WS})^{op} \to \mathcal{WS}$ denote the functor defined by $E \mapsto E'_k$ (resp $E \mapsto E'_s$) on objects and $(f: E \to F) \mapsto (f': F' \to E')$ on morphisms ($f: E \to F$ here is interpreted to mean f is a continuous linear map, so $f \in \mathit{Mor}_{(\mathcal{MS})^{op}}(F,E)$ (resp. $f \in \mathit{Mor}_{(\mathcal{WS})^{op}}(F,E)$)).

(11) $\mathfrak{D}_{\mathsf{MS}}$ *(resp.* $\mathfrak{D}_{\mathsf{WS}}$*) is a differential equivalence and an embedding.*

Proof: It is immediate that $\mathfrak{D}_{\mathsf{MS}}$ (resp. $\mathfrak{D}_{\mathsf{WS}}$) is a differential equivalence. Now E' determines E as a set, since any f $\in$ E' has domain E. The l.c.s. structure of E may now be determined by the natural identification of E and E". ∎

(12) *The duality of the compatible image-factorization systems for* WS *is expressed below.*

(a) $(\overline{\mathfrak{D}_{\mathsf{WS}}(\textit{injections})}, \overline{\mathfrak{D}_{\mathsf{WS}}(\textit{quotient maps})})$ = *(dense maps, closed embeddings)*

(b) $(\overline{\mathfrak{D}_{\mathsf{WS}}(\textit{embeddings})}, \overline{\mathfrak{D}_{\mathsf{WS}}(\textit{surjections})})$ = *(surjections, embeddings)*

(c) $(\overline{\mathfrak{D}_{\mathsf{WS}}(\textit{closed embeddings})}, \overline{\mathfrak{D}_{\mathsf{WS}}(\textit{dense maps})})$ = *(quotient maps, injections).*

Proof: This follows imeediately from the characterizations of weakly continuous linear maps in terms of their transposes (see [23], Ch. 2, §16). ∎

In the above, recall that that domain of $\mathfrak{D}_{\mathsf{WS}}$ is $(\mathsf{WS})^{op}$, so that the order of (E,M) is reversed, for each image-factorization system of WS.

Using the natural isomorphism $\mathfrak{Y}\colon \mathsf{WS} \to \mathsf{MS}$, (12) immediately implies the following.

(13) *The duality of the compatible image-factorization systems for* $\mathbb{MS}$ *is expressed below.*

(a) $(\overline{\mathfrak{D}_{\mathbb{MS}}(\text{injections})}, \overline{\mathfrak{D}_{\mathbb{MS}}(\text{quotient maps})})$ = *(dense maps, closed Mackey injections)*

(b) $(\overline{\mathfrak{D}_{\mathbb{MS}}(\text{Mackey embeddings})}, \overline{\mathfrak{D}_{\mathbb{MS}}(\text{surjections})})$ = *(surjections, Mackey embeddings)*

(c) $(\overline{\mathfrak{D}_{\mathbb{MS}}(\text{closed Mackey embeddings})}, \overline{\mathfrak{D}_{\mathbb{MS}}(\text{dense maps})})$ = *(quotient maps, injections)).* ∎

§5 Remarks

Remarks on the Literature

Other workers in the algebraic theory of continuous-time linear systems have used an $E'(\mathbb{R}_+)$-module approach rather than a semigroup approach. In order to compare their work to the present report, it is necessary to recast the d.s.g. approach in a module framework. Let T be a d.s.g. on the l.c.s. E. T may be regarded as a $\Delta(\mathbb{R}_+)$ module by using $(\tilde{1}_E)_T$: $\Delta(\mathbb{R}_+) \times E \to E$, where 1_E is the identity on E (~ is defined in §2). Also, as shown in §2, if E is quasi-complete, then this may be extended to a map $(\tilde{1}_E)_T$: $E'(\mathbb{R}_+) \times E \to E$. *This module action is hypo-continuous, but not in general continuous* (again see §2).

The approach of Kalman and Hautus [29] may now be reinterpreted as a special case of the realization results of this report. Their $E'(\mathbb{R}_+)$-module homomorphism f: $(E'(\mathbb{R}_+))^m \to (E(\mathbb{R}_+))^p$ defines a total response k: $\mathfrak{F}_{QC} \circ \mathcal{C}_{QC}(K^m) \to \mathfrak{F}_{QC} \circ \mathfrak{R}_{QC}(K^P)$. Factor this response using the image-factorization system (quasi-quotients, injections). This corresponds to the Kalman and Hautus (quotient map, injection) factorization, since a quotient of $(E(\mathbb{R}_+))^m$ is complete (it is metrizable). The differential equation of Kalman and Hautus corresponds to the d.s.g. and differential system obtained on the quotient space, using the techniques of §3. The Kalman and Hautus paper does not deal with the problem of going from internal to external behavior.

Part of the Bensoussan, Delfour, and Mitter approach [9] also may be regarded as a special case of the realization problem of this report. Their input and output spaces, I and Y, are fixed to be reflexive separable (B) spaces. Their external system representation is given to be an $E'(\mathbb{R}_+)$-module homomorphism between $L_1(E(\mathbb{R}_+),I)$ (the space of nuclear operators from $E(\mathbb{R}_+)$ into I) and $E(\mathbb{R}_+,Y)$. Since $E(\mathbb{R}_+)$ is nuclear, $L_1(E(\mathbb{R}_+)I) = L(E(\mathbb{R}_+),I) \cong E'(\mathbb{R}_+)\hat{\otimes}_\pi I = E'(\mathbb{R}_+)\hat{\otimes}_\pi Y$. Once again, the external representation may be regarded as a total response k: $\mathfrak{F}_{QC} \circ \mathcal{C}_{QC}(I) \to \mathfrak{F}_{QC} \circ \mathfrak{R}_{QC}(Y)$. They factor the map with (quotient maps, injections), although this is not an image-factorization system for QC. The semigroup which Bensoussan, Delfour, and Mitter get for the internal representation corresponds exactly to the d.s.g. obtained using the theory of this report, with the factorization modification mentioned above. They do not treat the converse problem of recovering an external behavior from a given semigroup of a certain class (corresponding to §2 of this report).

Bensoussan, Delfour, and Mitter also present a duality theory, in which they go to the weak topology for the dual system (the original system being described as above). As such, their approach differs from §4 of this report. In particular, they factor both the original system and the dual system with (quotient maps, injections), with the consequence that the state space of the dual need not be the dual of the original state space (refer to 4.(12) of this report).

The Benosoussan, Delfour, and Mitter paper also contains interesting approaches to continuous-time systems using Sobolev spaces and Banach algebras, the latter being closely realted to the paper by Bensoussan and Kamen [10]. No similar points of view are taken in the present paper.

Kamen [30], [31], and [32] also has investigated continuous-time systems within the algebraic framework. While he also uses spaces of distributions extensively, his emphasis is quite different form that of the present work.

Carlson [15] is the only other worker who has used category theory in an approach to continuous time systems. His abstract is very different in emphasis and content from the present report, dealing with an adjunction between behavior and realization for simple differential machines.

Extensions to this Report

There are many ways in which the current report could be extended. Differential systems using other types of analytic semigroups certainly should be investigated, as should an approach which takes into account more general systems, such as delay-differential equations. Relative category seems to be a solid framework in which to conduct these investigations (see Appendix).

Some of the ideas of distributions and C^{∞}-function algebras have been extended to topological vector spaces which are not locally-convex (see [50]). These ideas could possibly be used to extend the present theory.

APPENDIX RELATIVE CATEGORY-THEORY APPROACH

As this report was being completed, it was discovered that many of the ideas presented have elegant formulations within the context of categories realtive to LCS. It is not the purpose of this appendix to give this formulation, but rather only to indicate the general idea. Full details will appear in a forthcoming report. For this appendix only, it is assumed that the reader is familiar with the ideas of relative category theory ([14], [16], [18], [20], [34], and [35]).

LCS is a monoidal category under the bifunctor $-\otimes_l-$. The right adjoint to $-\otimes_l E$ is $L_s(E,-)$. Thus it makes sense to talk about categories relative to LCS.

Regard any (full) subcategory of LCS as an LCS category by putting the l.c.s. structure $L_s(E,F)$ on the set of morphisms from E to F. Regard $\Delta(\mathbb{R}_+)$ as a one-object LCS category whose morphism l.c.s. is $\Delta(\mathbb{R}_+)$; composition is convolution. DSG(K) then reinterprets as the LCS-functor category $K^{\Delta(\mathbb{R}_+)}$ (see the discussion following 1.(4)). K itself may be regarded as the LCS-functor category K^1, where 1 is the one-object LCS category whose morphism l.c.s. is K; composition is multiplication. The natural injection $1 \hookrightarrow \Delta(\mathbb{R}_+): k \mapsto k\cdot\delta_0$ induces an LCS functor $\mathfrak{F}_K: K^{\Delta(\mathbb{R}_+)} \to K^1$, which corresponds to the $\mathfrak{F}_K$ defined in §2. The problem of finding a left (resp. right) adjoint for $\mathfrak{F}_K$ amounts to finding, for each $E \in Obj(K^{\Delta(\mathbb{R}_+)})$, a left (resp. right) relative Kan extension of E along $1 \hookrightarrow \Delta(\mathbb{R}_+)$. In case K is tensored (resp. contensored), these extensions have formulations in terms of ends (resp. coends), and the d.s.g.'s found in §2 are produced immediately by these formulations. All of the subcategories of LCS which are considered in this report are both tensored and cotensored. This functor category appraoch to external behavior specification is essentially that which is employed by Bainbridge in this thesis [7] for ordinary automata theory; he worked relative to the category SET of sets and functions (i.e., in ordinary category theory) rather than LCS.

It appears that realization and duality may also be treated within this framework, although the details have not yet been completed.

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