# GENERALIZED HANKEL MATRICES

Michael A. Arbib
Computer and Information Science
and
Ernest G. Manes
Department of Mathematics

COINS Technical Report 77-10

(July 1977)

CONTENTS:

# GENERALIZED HANKEL MATRICES AND SYSTEM REALIZATION[1]

Michael A. Arbib
Computer & Information Science

Ernest G. Manes
Mathematics
University of Massachusetts
Amherst, Massachusetts 01003, USA

June 1977

## ABSTRACT

We define the Hankel matrix of an adjoint system. Adjoint systems include linear and bilinear systems, automata, and group systems in both the time-varying and time-invariant cases. Our definition of the Hankel matrix unifies the familiar $H_i^j = CA^{i+j}B$ of linear system theory (e.g. [17]) with the bilinear Hankel matrix of [16], [21] and the Hankel matrix of [12]. The time-varying case is subsumed by regarding a time-varying system as a time-invariant system in a sequence category as in [5]. For minimal realization theory and duality theory in the framework of this paper see [1], [4]. However, we lean much less heavily on category theory than in our earlier works on realization.

We introduce 'adjoint correspondences' as the key algebraic ingredient in generalizing familiar linear system results to the nonlinear case. For example, the linear realizability criterion $H_{i+1}^j = H_i^{j+1}$ does not make sense in the nonlinear setting; the precise condition needed is that '$H_{i+1}^j$ and $H_i^{j+1}$ correspond under adjointness'.

We provide a realizability theorem characterizing when a matrix $H_i^j$ can be the Hankel matrix of a system, and offer partial realization and canonical realization theorems which associate systems with finite blocks of a Hankel matrix. We provide a general theory of 'dimension in a category', and relate it to system realization via a simple recursion principle.

---

1 The research reported in this paper was supported in part by the National Science Foundation under grant DCR72-03733 A01. This paper is the third reference promised in [6]. The authors gratefully acknowledge T. J. Tarn for conversations which initiated the investigations that led to this paper.

## 1. Adjoint Processes and Systems

In what follows, $\mathcal{K}$ denotes an arbitrary category [3], [19]. Further axioms on $\mathcal{K}$ will be added gradually -- a summary appears after Lemma 2.13. In this section, we define adjoint processes and systems, and present a number of examples. Recall that a *functor* $X : \mathcal{K} \longrightarrow \mathcal{K}$ assigns to each object $Q$ of $\mathcal{K}$ another object $QX$ of $\mathcal{K}$ and assigns to each morphism $f : Q \longrightarrow R$ of $\mathcal{K}$ another morphism of form $fX : QX \longrightarrow RX$ subject to the preservation of identities and composition, that is, $id_Q X = id_{QX}$ and, given $f : Q \longrightarrow R$, $g : R \longrightarrow S$, $(gf)X = gX\ fX$. As discussed below, basic examples include tensoring with a fixed vector space in the category of vector spaces and linear maps or assigning to a set $Q$ the set of all functions from a fixed set to $Q$ in the category of sets and functions. Let $\mathcal{K}(Q,R)$ denote the set of morphisms from $Q$ to $R$ in $\mathcal{K}$.

1 DEFINITION: An *adjoint process* in $\mathcal{K}$ is a pair $(X,Z)$ of functors $\mathcal{K} \longrightarrow \mathcal{K}$ together with bijective correspondences $\mathcal{K}(RX,S) \longrightarrow \mathcal{K}(R,SZ)$ (one such for each pair $(R,S)$ of objects) subject to the axiom that, given $f : Q \longrightarrow R$ and $h : S \longrightarrow T$, if $g : RX \longrightarrow S$ and $\psi : R \longrightarrow SZ$ correspond then $h\ g\ fX : QX \longrightarrow T$ and $hZ\ \psi\ f : Q \longrightarrow TZ$ correspond.

In the usual language of category theory, to say $(X,Z)$ is an adjoint process amounts to saying that $Z$ is right adjoint to $X$ and, equivalently, that $X$ is left adjoint to $Z$.

*For the duration of the paper we fix an adjoint process $(X,Z)$ in $\mathcal{K}$.*

A convenient notation is the display

$$\frac{RX \xrightarrow{\ g\ } S}{R \xrightarrow[\ \psi\ ]{} SZ}$$

to indicate that $g$ and $\psi$ correspond. We say $g$ and $\psi$ *correspond under adjointness.*

We may equally well use the notation

$$\frac{R \xrightarrow{\psi} SZ}{RX \xrightarrow[g]{} S}$$

and we will frequently use displays like

$$\begin{array}{c} RX \xrightarrow{g} S \\ \hline R \xrightarrow{\psi} SZ \\ \hline RX \xrightarrow{k} S \end{array}$$

to conclude that $g = k$.

The axiom above is then succinctly displayed as

2 $$\frac{QX \xrightarrow{fX} RX \xrightarrow{g} S \xrightarrow{h} T}{Q \xrightarrow[f]{} R \xrightarrow[\psi]{} SZ \xrightarrow[hZ]{} TZ} .$$

We call attention to the special cases that arise when $f = id_R$ and when $h = id_S$.

3 <u>LEMMA</u>: *For each* B, *define* $\varepsilon_B : BZX \longrightarrow B$ *as the correspondent of* $id_{BZ} : BZ \longrightarrow BZ$. *Then if* $g : AX \longrightarrow B$ *and* $\psi : A \longrightarrow BZ$ *correspond under adjointness, we may recover* $g$ *from* $\psi$ *by*

$$g = \varepsilon_B \cdot \psi X .$$

<u>Proof</u>: Applying 2 we conclude that

$$\begin{array}{c} AX \xrightarrow{\psi X} BZX \xrightarrow{\varepsilon_B} B \\ \hline A \xrightarrow{\psi} BZ \xrightarrow{id_{BZ}} BZ \\ \hline AX \xrightarrow{\quad g \quad} B \end{array}$$

□

4 ADJOINT SYSTEMS: An *adjoint system* is $M = (Q,\delta,I,\tau,Y,\beta)$ where Q, I, Y are objects (the *state object, input object* and *output object* of M) and $\delta : QX \longrightarrow Q$, $\tau : I \longrightarrow Q$ and $\beta : Q \longrightarrow Y$ are morphisms (the *dynamics, input map* and *output map* of M). (Note: 'map' is here a synonym for 'morphism'.) The *codynamics* of M is the map $\Delta : Q \longrightarrow QZ$ which corresponds to $\delta$ under adjointness.

Given two dynamics $\delta : QX \longrightarrow Q$ and $\theta : RX \longrightarrow R$, a *dynamorphism* $h : (Q,\delta) \longrightarrow (R,\theta)$ is a map $h : Q \longrightarrow R$ which 'respects the dynamics':

$$\begin{tikzcd} QX \arrow[r, "hX"] \arrow[d, "\delta"'] & RX \arrow[d, "\theta"] \\ Q \arrow[r, "h"] & R \end{tikzcd}$$

The *time-i reachability map* $r_i : IX^i \longrightarrow Q$ and the *time-j observability map* $\sigma_j : Q \longrightarrow YZ^j$ are defined by

$$r_0 = \tau$$
$$r_{i+1} = IX^{i+1} \xrightarrow{r_i X} QX \xrightarrow{\delta} Q$$
$$\sigma_0 = \beta$$
$$\sigma_{j+1} = Q \xrightarrow{\Delta} QZ \xrightarrow{\sigma_j Z} YZ^{j+1} .$$

The bisequence $H_i^j$, where $H_i^j : IX^i \longrightarrow YZ^j$ is defined by $H_i^j = \sigma_j\, r_i$, is the *Hankel matrix* of M.

Adjoint systems are closely related to the machines studied in [10], [11] and [13]. Realization theory for adjoint systems was developed in [1] and [4]. The Hankel matrix for adjoint systems is new, perhaps because the previous authors were motivated more by automata theory (where the Hankel matrix is not conventionally defined) than by system theory.

We conclude this section with a number of examples of adjoint systems and their Hankel matrices.

5 EXAMPLE: THE DECOMPOSABLE CASE. Here $X = Z$ is the identity functor of $\mathcal{K}$. The realization theory in this special case was studied in [2]. When $\mathcal{K}$ is the category of vector spaces (or of modules over a ring) an adjoint system is just a linear system

$$I \xrightarrow{B} Q \qquad Q \xrightarrow{A} Q \qquad Q \xrightarrow{C} Y .$$

The same system description holds in any category. The adjointness correspondence is just

$$\frac{Q \xrightarrow{g} R}{Q \xrightarrow[g]{} R}$$

so that the codynamics is again A. We have $r_i = A^i B$ and $\sigma_j = CA^j$ so that $H_i^j = CA^{i+j}B$.

6 EXAMPLE: AUTOMATA. Let $\mathcal{K}$ be the category of sets and functions. Let A be a fixed input alphabet. Define $QX = Q \times A$, $QZ = Q^A$, the set of functions from A to Q. For $f : Q \longrightarrow R$, $fX : Q \times A \longrightarrow R \times A$ is defined by $(q,a) \mapsto (f(q),a)$ whereas $fZ : Q^A \longrightarrow R^A$ sends $g : A \longrightarrow Q$ to $fg : A \longrightarrow R$. The adjointness correspondence

$$\frac{Q \times A \xrightarrow{g} R}{Q \xrightarrow[\psi]{} R^A}$$

is the familiar $(\psi q)(a) = g(q,a)$. Let I have one element. Then $\tau$ amounts to an element of Q, the initial state. The dynamics and output map have their usual forms $\delta : Q \times A \longrightarrow Q$, $\beta : Q \longrightarrow Y$. It is easily checked that $r_i : A^i \longrightarrow Q$ sends an i-tuple of input letters to the state reached from the initial state if the letters are inputted in sequence, whereas $\sigma_j : Q \longrightarrow Y^{(A^j)}$ sends q to that function $A^j \longrightarrow Y$ composing $\beta$ with the time-j reachability map if the initial state were q. Thus $H_i^j : A^i \longrightarrow Y^{(A^j)}$ is essentially a way of describing $\beta \cdot r_{i+j}$ with emphasis on i as 'present time'.

7 EXAMPLE: INTERNALLY BILINEAR MACHINES ([12], [16], [21]): Let $\mathcal{K}$ be the category of real vector spaces and linear maps. Define $QX = Q \otimes U$, tensoring with a vector space U, while $QZ = Q^U$, the vector space of linear maps from U to Q. The adjointness correspondence

$$\frac{Q \times U \xrightarrow{g} R}{Q \xrightarrow{\psi} R^U}$$

is then the familiar $\psi q(u) = g(q \otimes u)$. Let I be a vector space. Then $\tau : I \longrightarrow Q$ specifies the space $\tau(I)$ of initial states 'reachable in time 0', the dynamics is then a bilinear map $\delta : Q \otimes U \longrightarrow Q$ while the output is a linear map $\beta : Q \longrightarrow Y$.

It is easily checked that $r_i : I \otimes U^{\otimes i} \longrightarrow Q$ extends the map $I \times U^i \longrightarrow Q$ which sends an i in I and i-tuple of input vectors to the state reached from $\tau(i)$ under that input sequence; whereas $\sigma_j : Q \longrightarrow Y^{U^j}$ sends q to that function $U^j \longrightarrow Y$ composing $\beta$ with the time-j reachability map if the initial state were q. The Hankel matrix $H_i^j : I \otimes U^{\otimes i} \longrightarrow Y^{U^j}$ can be viewed in a more symmetrical way as providing for each initial state label a matrix $U^{\otimes i} \otimes U^{\otimes j} \longrightarrow Y$.

The previous three examples can be subsumed in one very general example, given below as example 11. But first we need to recall, [3, section 1.2] [19, III.3, III.4], that if $(Q_i : i \in I)$ is a family of objects of $\mathcal{K}$ then their *product* $pr_k : \prod Q_i \longrightarrow Q_k$ satisfies the universal property that for all families of form $f_i : Q \longrightarrow Q_i$ $(i \in I)$ there exists unique $f : Q \longrightarrow \prod Q_i$ with $pr_i\, f = f_i$ for all i. If it exists, the product is unique up to isomorphism.

8
$$\begin{tikzcd} \prod Q_i \arrow[rr, "pr_k"] & & Q_k \\ & Q \arrow[ul, "f"] \arrow[ur, "f_k"'] & \end{tikzcd} \qquad \begin{tikzcd} Q_k \arrow[rr, "in_k"] \arrow[dr, "f_k"'] & & \coprod Q_i \arrow[dl, "f"] \\ & Q & \end{tikzcd}$$

The dual notion is the *coproduct* $in_k : Q_k \longrightarrow \coprod Q_i$. As we see in 8, in both cases there is a bijective correspondence between arbitrary families $(f_k : k \in I)$ and morphisms f. In the category of sets, coproducts are constructed as the disjoint union whereas in the category of modules over a ring (or a semiring), coproducts are constructed as weak direct sums. Both categories have products via the usual cartesian product construction.

9 PRESERVATION PRINCIPLE FOR ADJOINT PROCESSES. *X preserves coproducts, that is, if* $in_k : Q_k \longrightarrow \coprod Q_i$ *is a coproduct, so is* $in_k X : Q_k X \longrightarrow (\coprod Q_i)X$. *Similarly, Z preserves products.*

Proof: The result is standard in category theory [3, p. 134] [19, V.5]. To outline the proof, given a family $f_k : Q_k X \longrightarrow Q$, let $g_k : Q_k \longrightarrow QZ$ correspond to $f_k$ under adjointness, inducing the $g : \coprod Q_i \longrightarrow QZ$ whose correspondent is the desired f. The second statement is dual. □

10 BI-INDEX PRINCIPLE: *If* $(Q_i : i \in I)$, $(R_j : j \in J)$ *and* $f_i^j : Q_i \longrightarrow R_j$ *then, so long as the coproduct and product exist, there exists a unique morphism* $f : \coprod Q_i \longrightarrow \prod R_j$ *such that* $pr_j\, f\, in_i = f_i^j$ *for all* i, j.

Proof: Define $f^j : \coprod Q_i \longrightarrow R_j$ by $f^j\, in_i = f_i^j$ and then define f by $pr_j\, f = f^j$. Uniqueness is left as an exercise. □

*For the balance of the paper we assume our category* $\mathcal{K}$ *to be such that every countable family of objects has a product and a coproduct.*

<u>11</u> <u>EXAMPLE</u>: The following very general example of adjoint processes and systems subsumes examples 5, 6 and 7. Let A be a fixed set (usually finite in applications). Then define

$$QX = Q \times A =_{\text{def}} \coprod_{a \in A} Q$$

the coproduct of $|A|$ copies of Q. For $f : Q \longrightarrow R$, fX is defined by the coproduct property

$$\begin{tikzcd} Q \arrow[r, "in_a"] \arrow[d, "f"'] & QX \arrow[d, dashed, "fX"] \\ R \arrow[r, "in_a"] & RX \end{tikzcd} \qquad (a \in A)$$

If we define Z by

$$QZ = Q^A =_{\text{def}} \prod_{a \in A} Q$$

the product of $|A|$ copies of Q, with

$$\begin{tikzcd} Q \arrow[d, "f"'] & QZ \arrow[l, "pr_a"'] \arrow[d, dashed, "fZ"] \\ R & RZ \arrow[l, "pr_a"'] \end{tikzcd}$$

it can easily be verified that (X,Z) is indeed an adjoint process, with the correspondence

$$\frac{QX \xrightarrow{g} R}{Q \xrightarrow{\psi} RZ}$$

being simply given by $g \cdot in_a = pr_a \cdot \psi : Q \longrightarrow R$ for each $a \in A$.

Given a system $(\tau : I \longrightarrow Q,\ \delta : QX \longrightarrow Q,\ \beta : Q \longrightarrow Y)$, we have that $r_i : \coprod_{v \in A^i} I \longrightarrow Q$, $\sigma_j : Q \longrightarrow \prod_{w \in A^j} Y$ and that, by the bi-index principle, the Hankel matrix $H_i^j : \coprod_{v \in A^i} I \longrightarrow \prod_{w \in A^j} Y$ is equivalent to a $|A^i| \times |A^j|$ 'matrix' whose entries are maps $pr_w \cdot H_i^j \cdot in_v : I \longrightarrow Y$.

## 2. Realizability and Realizations

Before stating the next theorem we define the *object of inputs* $\Omega$ and the *observability space* $\Gamma$. The notation follows Kalman's for the linear case [19, 10.3]. In [4] the notation used was $IX^{@}$ for $\Omega$ and $YX_{@}$ for $\Gamma$.

1 We set $\Omega$ to be the coproduct $\coprod(IX^i : i \geq 0)$ where $IX^0 = I$ and $IX^{i+1} = (IX^i)X$. $\Omega$ carries a dynamical structure $\mu_0 : \Omega X \longrightarrow \Omega$ defined by

$$\begin{tikzcd} \Omega X \arrow[rr, "\mu_0"] & & \Omega \\ & IX^iX \arrow[ul, "in_iX"] \arrow[ur, swap, "in_{i+1}"] & \end{tikzcd}$$

Here, we have used the preservation principle 1.9. (The story behind the cumbersome notation $\mu_0$ instead of $\mu$ is found in [20, section 4.2].)

2 $\Gamma$ is defined as the product $\prod(YZ^j : j \geq 0)$ with dynamical structure $L : \Gamma X \longrightarrow \Gamma$ the correspondent under adjointness of the map $\Lambda$ defined by

$$\begin{tikzcd} \Gamma \arrow[rr, "\Lambda"] \arrow[dr, swap, "pr_{j+1}"] & & \Gamma Z \arrow[dl, "pr_jZ"] \\ & YZ^jZ & \end{tikzcd}$$

These definitions coincide with Kalman's (save that he denotes both $\mu_0$ and $\Lambda$ by z) when $\mathcal{K}$ is the category of modules over a ring and when (X,Z) is the identity process.

3 REALIZABILITY THEOREM: *Let* $H_i^j : IX^i \longrightarrow YZ^j$ *be an arbitrary bisequence of morphisms and let* $H : \Omega \longrightarrow \Gamma$ *be the unique morphism with* $pr_j\, H\, in_i = H_i^j$ *as in 1.10. Then the following three conditions are equivalent (and we say* $H_i^j$ *is a* Hankel matrix *if these conditions hold).*

*(i)* $H_i^j$ *is realizable, that is, is the Hankel matrix of some system.*

*(ii)* *(The Hankel crossover condition):* *For all* i, j:

$$\frac{IX^{i+1} \xrightarrow{H_{i+1}^{j}} YZ^{j}}{IX^{i} \xrightarrow[H_i^{j+1}]{} YZ^{j+1}}$$

*Equivalently, by 1.3, the condition states*

$$\begin{tikzcd}
IX^iX \arrow[r, "id"] \arrow[d, "H_i^{j+1}X"'] & IX^{i+1} \arrow[d, "H_{i+1}^{j}"] \\
YZ^{j+1}X \arrow[r, "\varepsilon"] & YZ^{j}
\end{tikzcd}$$

*(iii)* $H : (\Omega,\mu_0) \longrightarrow (\Gamma,L)$ *is a dynamorphism, that is,*

$LHX = H\mu_0 : \Omega X \longrightarrow \Gamma.$

Proof: (i) $\Longrightarrow$ (ii) is immediate from

$$\frac{IX^iX \xrightarrow{r_iX} QX \xrightarrow{\delta} Q \xrightarrow{\sigma_j} YZ^j}{IX^i \xrightarrow[r_i]{} Q \xrightarrow[\Delta]{} QZ \xrightarrow[\sigma_jZ]{} YZ^jZ}$$

For (ii) $\Longrightarrow$ (iii), consult the diagram

$$\begin{tikzcd}
IX^iX \arrow[r, "in_iX"] \arrow[dr, "in_{i+1}"'] & \Omega X \arrow[r, "HX"] \arrow[d, "\mu_0"] & \Gamma X \arrow[d, "L"] & \\
 & \Omega \arrow[r, "H"'] & \Gamma \arrow[r, "pr_j"'] & YZ^j
\end{tikzcd}$$

By principles 1.9 and 1.10, it suffices to prove that the bottom and top paths from $IX^iX$ to $YZ^j$ are equal. But the bottom path is exactly $H_{i+1}^j$, whereas $pr_jL$ corresponds under adjointness to $pr_{j+1} : \Gamma \longrightarrow YZ^{j+1}$ so that the top path corresponds to $H_i^{j+1}$ and is thus also $H_{i+1}^j$.

To complete the proof we show (iii) $\Longrightarrow$ (i). We shall show that if H is a dynamorphism, then the 'free realization' $Q = \Omega$, $\delta = \mu_0$, $\tau = in_0$, $\beta = pr_0 H$ has Hankel matrix H. One checks easily that $r_i = in_i$. To show that $\sigma_j r_i = pr_j H\, in_i$ it suffices to show that $\sigma_j = pr_j H$. This is true by definition for $j = 0$. The inductive step here is given by using the adjointness axiom with $f = id_\Omega$, and where $\Delta$ is now the codynamics of $\mu_0$:

$$\frac{\Omega \xrightarrow{\Delta} \Omega Z \xrightarrow{(pr_j \cdot H)Z} YZ^{j+1}}{\Omega X \xrightarrow[\mu_0]{} \Omega \xrightarrow[(pr_j \cdot H)]{} YZ^j}$$

Again

$$\frac{\Omega \xrightarrow{H} \Gamma \xrightarrow{pr_{j+1}} YZ^{j+1}}{\Omega X \xrightarrow[HX]{} \Gamma X \xrightarrow[(pr_j \cdot L)]{} YZ^j}$$

But by the dynamorphism property, $H \cdot \mu_0 = L \cdot HX$, and so $pr_{j+1} \cdot H = (pr_j \cdot H)Z \cdot \Delta = \sigma_{j+1}$. □

The Hankel crossover condition provides evidence that adjointness arises naturally in system theory. In the decomposable case (example 1.5) we capture the familiar condition '$H^j_{i+1} = H^{j+1}_i$' of linear system theory.

In the general context of example 1.11, $\Omega$ may be identified with $I \otimes A^*$, where $A^*$ is the free monoid generated by A and $\Gamma$ may be identified with $Y^{A^*}$. Now $(- \otimes A^*, (-)^{A^*})$ is again an adjoint process in the category of sets. $H : I \otimes A^* \longrightarrow Y^{A^*}$ corresponds under adjointness to a map $\Theta : I \otimes A^* \otimes A^* \longrightarrow Y$.

In familiar system examples one can discuss the 'subspace of Q reached by time i'. Such a subspace may be constructed by 'taking the image' of the map $f : \coprod (IX^k : 0 \le k \le i) \longrightarrow Q$ defined by $f\, in_k = r_k$. To formalize 'taking the image' we structure $\mathcal{K}$ with an image factorization system.

4 An *image factorization system* for a category $\mathcal{K}$ is a pair $(\mathcal{E}, \mathcal{M})$ where $\mathcal{E}, \mathcal{M}$ are subclasses of morphisms satisfying the following four axioms:

IFS1. $\mathcal{E}$ and $\mathcal{M}$ are each closed under composition.

IFS2. Every isomorphism is both in $\mathcal{E}$ and in $\mathcal{M}$.

IFS3. Every element of $\mathcal{E}$ is an epimorphism and every element of $\mathcal{M}$ is a monomorphism. (A map $f : R \longrightarrow S$ is an *epimorphism* if whenever $g, h : S \longrightarrow T$ satisfy $gf = hf$, then $g = h$; dually, f is a *monomorphism* if whenever $a, b : Q \longrightarrow R$ satisfy $fa = fb$ then $a = b$.)

IFS4. Every morphism $f : Q \longrightarrow R$ admits an $\mathcal{E}, \mathcal{M}$ factorization (e,m) -- that is, $f = m\,e$ with $e \in \mathcal{E}$, $m \in \mathcal{M}$ -- and such factorizations are unique up to isomorphism in the sense that if (e',m') is another one than there exists a unique isomorphism $\psi$ with $\psi e = e'$ and $m'\psi = m$.

The category of sets and the category of modules over a ring both have $\mathcal{E}$ = surjections and $\mathcal{M}$ = injections as unique image factorization system. The same construction works in the category of semigroups but in that category $\mathcal{E}$ = epimorphisms determines (see 8 below) another system; the inclusion of the natural numbers into the integers is a non-surjective epimorphism in that category. Image factorization systems in the category of linearly topologized vector spaces were investigated in a system-theoretic context in [14].

The notion of an image factorization system can be traced to [18]. The version presented here is due to [15]. References in the system literature to the 'Zeiger fill-in lemma' (see 7 below) are historically inaccurate.

We conclude this section by collecting a number of standard results. Proofs of 6-11 appear in [20, section 3.4] although all are easy exercises.

*For the balance of the paper, $(\mathcal{E}, \mathcal{M})$ is a fixed image factorization system in $\mathcal{K}$.*

5 DEFINITION: Let M be an adjoint system. The *reachability map* $r : \Omega \longrightarrow Q$ of M is defined by

$$\begin{tikzcd} \Omega = \coprod_{i\geq 0} IX^i \arrow[r, dashed, "r"] & Q \\ IX^i \arrow[u, "in_i"] \arrow[ur, "r_i"'] & \end{tikzcd}$$

Dually, the *observability map* $\sigma : Q \longrightarrow \Gamma$ of M is defined by

$$\begin{tikzcd} Q \arrow[r, dashed, "\sigma"] \arrow[dr, "\sigma_j"'] & \prod_{j\geq 0} YZ^j = \Gamma \arrow[d, "pr_j"] \\ & YZ^j \end{tikzcd}$$

We say M is *reachable* if M is in $\mathcal{E}$, *observable* if $\sigma$ is in $\mathcal{M}$. M is *reachable in time* i if $(r_k \mid 0 \leq k \leq i) : \coprod(IX^k \mid 0 \leq k \leq i) \longrightarrow Q$ is in $\mathcal{E}$. Correspondingly, M is *observable in time* j if $(\sigma_k \mid 0 \leq k \leq j)$ is in $\mathcal{M}$.

M is *reachable in bounded time* if M is reachable in time i for some i; and M is *observable in bounded time* if M is observable in time j for some j.

Let $\mathcal{K}^{op}$ denote the dual category of $\mathcal{K}$. For a discussion of duality for adjoint systems see [4]. Thus products in $\mathcal{K}$ = coproducts in $\mathcal{K}^{op}$, monomorphisms in $\mathcal{K}$ = epimorphisms in $\mathcal{K}^{op}$ and

6 PROPOSITION: *$(\mathcal{M}, \mathcal{E})$ is an image factorization system in the opposite category $\mathcal{K}^{op}$.*

Proposition 6 clearly plays a role in establishing the duality between results on reachability and corresponding results on observability -- e.g. the fact noted after Proposition 9.

7 PROPOSITION (DIAGONAL FILL-IN): *Given a commutative square* $ge = mf$

$$\begin{tikzcd} Q \arrow[r, "e"] \arrow[d, "f"'] & R \arrow[d, "g"] \arrow[dl, dashed, "\psi"'] \\ S \arrow[r, "m"'] & T \end{tikzcd}$$

*with* $e \in \mathcal{E}$, $m \in \mathcal{M}$ *there exists (necessarily unique)* $\psi$ *with* $\psi e = f$ *and* $m\psi = g$. [Hint for proof: take the images of f and g.]

8 PROPOSITION ($\mathcal{E}$ *determines* $\mathcal{M}$): *The converse of diagonal fill-in holds. That is, if* m *is an arbitrary morphism with the property that whenever* $ge = mf$ *with* $e \in \mathcal{E}$ *there exists* $\psi$ *with* $\psi e = f$ *then necessarily* $m \in \mathcal{M}$. [Hint for proof: factor $m = m'\, e$ and let $f = id_S$.] *Dually,* $\mathcal{M}$ *determines* $\mathcal{E}$.

9 PROPOSITION: *If* $f \in \mathcal{E}$ *and* $f \in \mathcal{M}$ *then* f *is an isomorphism.*

10 PROPOSITION: *If* $f : Q \longrightarrow R$ *and* $g : R \longrightarrow S$ *then* $gf \in \mathcal{E}$ *implies* $g \in \mathcal{E}$ *whereas* $gf \in \mathcal{M}$ *implies* $f \in \mathcal{M}$. [Hint for proof: use 8.]

Reachability in bounded time implies reachable. To prove this, observe that

$$\begin{tikzcd} \Omega \arrow[rr, "r"] & & Q \\ & \coprod(IX^k \mid 0 \le k \le i) \arrow[ul, "\psi"] \arrow[ur, "(r_i)"'] & \end{tikzcd}$$

where $\psi$ is defined by $\psi in_k = in_k$. Thus $(r_i) = r \cdot \psi$ in $\mathcal{E}$ implies r in $\mathcal{E}$. Dually, observability in bounded time implies observable.

11 PROPOSITION: *Given a family* $f_i : Q_i \longrightarrow R_i$ *with each* $f_i \in \mathcal{M}$ *then the unique* $f : \prod Q_i \longrightarrow \prod R_i$ *defined by* $pr_i\, f = f_i\, pr_i$ *is also in* $\mathcal{M}$. *Dually, given a family* $f_i : Q_i \longrightarrow R_i$ *with each* $f_i \in \mathcal{E}$, *the unique* $f : \coprod Q_i \longrightarrow \coprod R_i$ *with* $f\, in_i = in_i\, f_i$ *is again in* $\mathcal{E}$. [Hint for proof: use 8.]

12 PROPOSITION. *X preserves $\mathcal{E}$ if and only if Z preserves $\mathcal{M}$.*

Proof: Assuming X preserves $\mathcal{E}$ we wish to show that $mZ : QZ \longrightarrow RZ \in \mathcal{M}$ given that $m : Q \longrightarrow R \in \mathcal{M}$. This is immediate from 8 and the adjoint correspondences

$$\begin{tikzcd} S \arrow[r, "e"] \arrow[d, "f"'] & T \arrow[d, "g"] \arrow[dl, dashed, "\psi"'] \\ QZ \arrow[r, "mZ"] & RZ \end{tikzcd}$$

---

$$\begin{tikzcd} SX \arrow[r, "eX"] \arrow[d, "F"'] & TX \arrow[d, "G"] \arrow[dl, "\psi"'] \\ Q \arrow[r, "m"] & R \end{tikzcd}$$

(where capital and lower case letters correspond). The converse result is dual. □

We can now state all necessary standing assumptions and summarize them here for convenience. For the balance of the paper we assume that:

$\mathcal{K}$ is a category with products and coproducts of countable families.

(X,Z) is an adjoint process in $\mathcal{K}$.

$(\mathcal{E}, \mathcal{M})$ is an image factorization system in $\mathcal{K}$.

X preserves $\mathcal{E}$ (and hence Z preserves $\mathcal{M}$).

I is a fixed input object. Y is a fixed output object.

It is often the case that $\mathcal{E}$ = all epimorphisms or that $\mathcal{E}$ = all morphisms which are the coequalizer of some pair [3, section 1.3] [19, p. 64]. This is the case for the category of sets and for the category of modules over a ring, the unique $\mathcal{E}$ being the class of epimorphisms = the class of all coequalizers. In these two cases, it is well known that X must preserve $\mathcal{E}$ [19, V.5].

13 DYNAMORPHIC IMAGE LEMMA: *Let* $f : (Q,\delta) \longrightarrow (S,\theta)$ *be a dynamorphism, that is, the perimeter of the diagram below commutes.*

$$\begin{tikzcd}
QX \arrow[r, "eX"'] \arrow[d, "\delta"] \arrow[rr, bend left, "fX"] & RX \arrow[r, "mX"'] \arrow[d, dashed, "\gamma"] & SX \arrow[d, "\theta"] \\
Q \arrow[r, "e"] \arrow[rr, bend right, "f"'] & R \arrow[r, "m"] & S
\end{tikzcd}$$

*Let* $f = m\,e$ *be an* $\mathcal{E}$-$\mathcal{M}$ *factorization of* $f$. *Then there exists a unique dynamics* $\gamma : RX \longrightarrow R$ *rendering the above diagram commutative.*

Proof: Since $eX \in \mathcal{E}$, this is immediate from 7. $\square$

14 DEFINITION: The *canonical realization* $M_H$ of a Hankel matrix $H_i^j$ is the system $(Q_H, \delta_H, \tau_H, \beta_H)$ defined as follows. Let $H : (\Omega,\mu_0) \longrightarrow (\Gamma,L)$ be the dynamorphism defined by $pr_j \cdot H \cdot in_i = H_i^j$. Let

$$\Omega \xrightarrow{r_H} Q_H \xrightarrow{\sigma_H} \Gamma$$

be an $\mathcal{E}$-$\mathcal{M}$ factorization of $H$. By the dynamorphic image lemma, there exists a unique dynamics $\delta_H : Q_H X \longrightarrow Q_H$ rendering $r_H$ and $\sigma_H$ dynamorphisms. Define $\tau_H = r_H\, in_0$ and $\beta_H = pr_0\, \sigma_H$.

It is proved in [4, theorems 2.1, 3.15] that $M_H$ is a realization of $H_i^j$, that the reachability and observability maps of $M_H$ are $r_H$ and $\sigma_H$ (so that $M_H$ is reachable and observable) and that any other reachable and observable realization is isomorphic to $M_H$.

The question of interest here, however, is under what conditions the canonical realization can be found from a finite fragment of the Hankel matrix $(H_i^j \mid 0 \le i \le k,\ 0 \le j \le n)$. We first present a partial realization which tells us when such a fragment lets us define an adjoint system whose behavior is consistent with that portion of the Hankel matrix. Then, in the remaining sections, we present general conditions on m and n under which this partial realization will be isomorphic to the canonical realization.

Let us fix the following notations:

$$\bar n = \{0, 1, \ldots, n\}$$

$$IX^{\bar k} = \coprod_{i\in\bar k} IX^{i} \; ; \quad YX^{\bar n} = \prod_{j\in\bar n} YZ^{j}$$

while

$$H_{\bar k}^{\bar n} : IX^{\bar k} \longrightarrow YZ^{\bar n}$$

is defined by $pr_j \cdot H_{\bar k}^{\bar n} \cdot in_i = H_i^j$ for $i \in \bar k$, $j \in \bar n$.

Define $\tilde\mu : IX^{\bar k}X \longrightarrow IX^{\overline{k+1}}$ by $\tilde\mu \cdot in_i X = in_{i+1}$

$\tilde\varepsilon : YZ^{\overline{n+1}}X \longrightarrow YZ^{\bar n}$ by $pr_j \cdot \tilde\varepsilon = \varepsilon_{YZ^j} \cdot pr_{j+1}X$ .

Then the Hankel crossover condition yields

15

$$\begin{tikzcd}
IX^{\bar k}X \arrow[r, "\tilde\mu"] \arrow[d, "H_{\bar k}^{\overline{n+1}}X"'] & IX^{\overline{k+1}} \arrow[d, "H_{\overline{k+1}}^{\bar n}"] \\
YZ^{\overline{n+1}}X \arrow[r, "\tilde\varepsilon"] & YZ^{\bar n}
\end{tikzcd}$$

(just precede the square by $in_i X$ and follow it by $pr_j$ for $0 \le i \le k$, $0 \le j \le n$ to recapture the square of 3(ii)).

Let $H_{\bar k}^{\overline{n+1}}$ have $\mathcal{E}$-$\mathcal{M}$ factorization $(\bar e, \bar m)$ with image $\bar Q$ while $H_{\overline{k+1}}^{\bar n}$ factors as $(\hat e, \hat m)$ with image $\hat Q$. Then, since X preserves $\mathcal{E}$, we may define $\tilde\delta : \bar Q X \longrightarrow \hat Q$ by diagonal fill-in:

16

$$\begin{tikzcd}
IX^{\bar k}X \arrow[r, "\tilde\mu"] \arrow[d, "\bar e X"'] & IX^{\overline{k+1}} \arrow[d, "\hat e"] \\
\bar Q X \arrow[r, dashed, "\tilde\delta"] \arrow[d, "\bar m X"'] & \hat Q \arrow[d, "\hat m"] \\
YZ^{\overline{n+1}}X \arrow[r, "\tilde\varepsilon"] & YZ^{\bar n}
\end{tikzcd}$$

The important fact is that $\tilde{\delta}$ is completely determined by the $H_i^j$ for $0 \le i \le k+1,\ \ 0 \le j \le n+1$.

To obtain our partial realization theorem, we must establish conditions under which $\tilde{\delta}$ may be viewed as a dynamics. To this end, define

$$\mathrm{in} : IX^{\bar{k}} \longrightarrow IX^{\overline{k+1}} \quad \text{by} \quad \mathrm{in}\cdot\mathrm{in}_i = \mathrm{in}_i$$

$$\mathrm{pr} : YZ^{\overline{n+1}} \longrightarrow YZ^{\bar{n}} \quad \text{by} \quad \mathrm{pr}_j\cdot\mathrm{pr} = \mathrm{pr}_j\ .$$

Then the bi-index principle, 1.10, yields

17

$$\begin{tikzcd}
IX^{\bar{k}} \arrow[r, "H_{\bar{k}}^{\overline{n+1}}"] \arrow[d, "\mathrm{in}"'] \arrow[dr, "H_{\bar{k}}^{\bar{n}}"] & YZ^{\overline{n+1}} \arrow[d, "\mathrm{pr}"] \\
IX^{\overline{k+1}} \arrow[r, "H_{\overline{k+1}}^{\bar{n}}"'] & YZ^{\bar{n}}
\end{tikzcd}$$

Forming the $\mathcal{E},\mathcal{M}$ factorization $(e,m)$ of $H_{\bar{k}}^{\bar{n}}$ with image $R$, we then obtain $t$ and $u$ by diagonal fill-in:

18

$$\begin{tikzcd}
IX^{\bar{k}} \arrow[r, "\bar{e}"] \arrow[dd, "\mathrm{in}"'] \arrow[dr, "e"'] & \bar{Q} \arrow[r, "\bar{m}"] \arrow[d, dashed, "t"] & YZ^{\overline{n+1}} \arrow[dd, "\mathrm{pr}"] \\
 & R \arrow[d, dashed, "u"] \arrow[dr, "m"] & \\
IX^{\overline{k+1}} \arrow[r, "\hat{e}"] & \hat{Q} \arrow[r, "\hat{m}"] & YZ^{\bar{n}}
\end{tikzcd}$$

19 <u>PARTIAL REALIZATION THEOREM</u>: *If $t$, $u$ are isomorphisms in 18, we may define the system* $M = (\bar{Q},\ \delta,\ \tau,\ \beta)$ *by*

20 $$\delta = t^{-1}\cdot u^{-1}\cdot\tilde{\delta} : \bar{Q}X \longrightarrow \bar{Q} \qquad \text{(using 16 and 18)}$$

21 $$\tau = \bar{e}\cdot\mathrm{in}_0 : I \longrightarrow IX^{\bar{k}} \longrightarrow \bar{Q}$$

22 $$\beta = \mathrm{pr}_0\cdot\bar{\bar{m}} : \bar{Q} \longrightarrow YZ^{\overline{n+1}} \longrightarrow Y.$$

*Then the Hankel matrix of* $M$ *agrees with* $H_i^j$ *for* $0 \le i \le n,\ 0 \le j \le k$.

<u>Proof</u>: Let $r_i$, $\sigma_j$ be the i-step reachability and j-step observability maps, respectively, of M. We prove the theorem in two steps:

(i) We show $r_i = IX^i \xrightarrow{in_i} IX^{\bar{k}} \xrightarrow{\bar{e}} \bar{Q}$ for $0 \le i \le k$.

(ii) We show $\sigma_j = \bar{Q} \xrightarrow{\bar{m}} YZ^{\overline{n+1}} \xrightarrow{pr_j} YZ^j$ for $0 \le j \le k$.

It is then immediate that $\sigma_j \cdot r_i = pr_j \cdot \bar{m} \cdot \bar{e} \cdot in_i = pr_j \cdot H_{\bar{k}}^{\overline{n+1}} \cdot in_i = H_i^j$.

<u>Proof of (i)</u>: For $i = 0$, this is 21. Now, for $0 \le i < k$, we have by induction

$$
\begin{aligned}
r_{i+1} &= \delta \cdot r_i X \\
&= t^{-1} u^{-1} \cdot \tilde{\delta} \cdot r_i X \\
&= t^{-1} u^{-1} \cdot \tilde{\delta} \cdot \bar{e} X \cdot in_i X && \text{by induction hypothesis} \\
&= t^{-1} u^{-1} \cdot \hat{e} \cdot \tilde{\mu} \cdot in_i X && \text{by 16} \\
&= t^{-1} u^{-1} \cdot \hat{e} \cdot in_{i+1} && \text{by definition of } \tilde{\mu} \\
&= t^{-1} u^{-1} \cdot u \cdot t \cdot \bar{e} \cdot in_{i+1} && \text{by 18, and definition of in} \\
&= \bar{e} \cdot in_{i+1} && \text{as was to be shown.}
\end{aligned}
$$

<u>Proof of (ii)</u>: By definition $\sigma_0 = \beta = pr_0 \cdot \bar{m}$. But then, for $0 \le j < k$,

$$
\begin{aligned}
\sigma_{j+1} &= \sigma_j Z \cdot \Delta \\
\hline
&\sigma_j \cdot \delta && \text{by 1.2} \\
&= pr_j \cdot \bar{m} \cdot \delta && \text{by induction hypothesis} \\
&= pr_j \cdot \hat{m} \cdot u \cdot t \cdot \delta && \text{by 18, and definition of pr} \\
&= pr_j \cdot \hat{m} \cdot \tilde{\delta} && \text{by definition of } \delta \\
&= pr_j \cdot \tilde{\varepsilon} \cdot \bar{m} X && \text{by 16} \\
&= \varepsilon_{YZ^j} \cdot pr_{j+1} X \cdot \bar{m} X && \text{by definition of } \tilde{\varepsilon} \\
\hline
&pr_{j+1} \cdot \bar{m}
\end{aligned}
$$

□

## 3. Generalizing the Notion of Finite Dimensionality

For a linear system M, the subspaces $Q_i$ generated by the union of the images of $A^kB : I \longrightarrow Q$, $0 \le k \le i$, constitute an ascending chain of subspaces of Q. If Q is finite-dimensional -- or more generally, for modules over a ring rather than vector spaces, if Q is Noetherian -- this chain is eventually stationary, $Q_m = Q_{m+1} = \ldots,$ and M is reachable in time m. In this section, we show how such dimensionality considerations may be extended to our category $\mathcal{K}$ -- with dimension reducing, essentially, to cardinality in the case of <u>Set</u>. The notions of $\mathcal{E}$-height and $\mathcal{M}$-height were introduced in [1]. Further properties of Noetherian objects appear in [7].

<u>1</u> <u>DEFINITIONS</u>: Let Q be an object of $\mathcal{K}$. The set of all pairs (R,m) with $m : R \longrightarrow Q \in \mathcal{M}$ admits a reflexive and transitive order by defining $(R,m) \le (R',m')$ if there exists $\psi$ with $m'\psi = m$

$$\begin{tikzcd} R \arrow[dr, "m"] \arrow[d, dashed, "\psi"'] & \\ R' \arrow[r, "m'"'] & Q \end{tikzcd}$$

(note that such $\psi$ is necessarily unique and is itself in $\mathcal{M}$). Thus $(R,m) \sim (R',m')$ if $(R,m) \le (R',m')$ and $(R',m') \le (R,m)$ is an equivalence relation whose equivalence classes [R,m] are called the *subobjects* of Q. $[R,m] \le [R',m']$ if $(R,m) \le (R',m')$ is a well-defined partial order on the subobjects of Q. It is easily seen that $[R,m] = [R',m']$ if and only if there exists an isomorphism $\psi$ with $m'\psi = m$.

Q is *Noetherian* if every strictly ascending chain of subobjects of Q is finite. Let $h \ge 0$ be an integer. Q *has* $\mathcal{M}$*-height* h if Q admits a strict chain of proper subobjects of length h, but none of length h+1. Q *has finite* $\mathcal{M}$*-height* if Q has $\mathcal{M}$-height h for some h.

The dual concepts relative to $\mathcal{K}$ are formulated by repeating the above definitions in $\mathcal{K}^{op}$ (using 2.6). Thus, the ordering on *quotient objects* of Q is described by

$$
\begin{tikzcd}
R & \\
& Q \arrow[ul, "e"'] \arrow[dl, "e'"] \\
R' \arrow[uu, dashed, "\psi"] &
\end{tikzcd}
\qquad
\begin{array}{l}
[R,e] \le [R',e'] \\
e,\ e' \in \mathcal{E}
\end{array}
$$

(Note that we reverse the arrows, not the ordering.) We say Q is *Artinian* if Q is co-Noetherian, that is, if every strictly ascending chain of quotient objects of Q is finite. The definitions of '*$\mathcal{E}$-height*' and '*finite $\mathcal{E}$-height*' are clear.

<u>2</u> <u>EXAMPLES</u>: In the category of sets, subobjects may be identified with subsets of Q and quotient objects may be identified with the canonical quotient projections induced by equivalence relations on Q. A set with h elements has $\mathcal{M}$-height h+1 and $\mathcal{E}$-height h, except that the empty set has $\mathcal{E}$-height 1. For sets, Noetherian = Artinian = finite. Notice that for both subsets and quotient sets, ascending chains mean increasing cardinality.

In the category of modules over a ring, the passage from a submodule S to its cokernel Q/S establishes an anti-isomorphism of partially ordered sets between subobjects and quotient objects. For this reason, Artinian is equivalent to the descending chain condition on subobjects (the usual definition in module theory) and a module has finite height if and only if it is simultaneously Noetherian and Artinian. These two properties do not hold in a general category where descending chain conditions are not equivalent to the ascending chain conditions defined above, and do not seem to be well motivated in a system context.

In abelian groups, Noetherian = finitely-generated, whereas finite $\mathcal{M}$-height = finite. The group of additive integers is not Artinian. For vector spaces, on the other hand, Noetherian = Artinian = finite height and $\mathcal{M}$-height = $\mathcal{E}$-height = 1 + dimension.

It is unclear how to define 'proper subobject' so that the four finite-height conditions discussed above come out 'right'. One could exclude the proper subobject $[Q, id_Q]$ although from the system point of view this subobject is not the trivial one; it is the *zero* subobject that is trivial from the point of view of building increasing chains. Recall that an object 0 is *initial* if there is a unique morphism $0 \longrightarrow Q$ to every Q and, dually, an object 1 is *terminal* if there is a unique morphism $Q \longrightarrow 1$ for every Q. For sets, 0 is the empty set, 1 is a one-element set and for modules 0 is both initial and terminal. While the unique $0 \longrightarrow Q$ is not always in $\mathcal{M}$, the image factorization of this map produces the least element of the partially ordered set of subobjects of Q. Dually, the image factorization of $Q \longrightarrow 1$ produces the least quotient object of Q. It seems hard to posit a natural procedure to decide when to omit the zero subobject from chains which both preserves duality (i.e., the same procedure must be applied to quotient chains) and works right in the examples above.

Motivated by the sequence $r_i : IX^i \longrightarrow Q$ induced by an adjoint system, we consider an arbitrary sequence of morphisms of form $f_i : P_i \longrightarrow Q$, $i \in \mathcal{N}$ = the set $\{0,1,2,...\}$ of natural numbers. The following four constructions are useful. We fix their notations for the remainder of the paper.

<u>3</u> For each non-empty subset S of $\mathcal{N}$ define

$$P_S = \coprod (P_i : i \in S)$$

$$f_S = P_S \longrightarrow Q, \quad \text{where} \quad f_S\, in_i = f_i \quad (i \in S).$$

4 For $S \subset T \subset \eta$, $in_{ST} : P_S \longrightarrow P_T$ is defined by $in_{ST}\, in_i = in_i$ $(i \in S)$.

5 Fix an $\mathcal{E}$-$\mathcal{M}$ factorization of $f_S$:

$$P_S \xrightarrow{e_S} I_S \xrightarrow{m_S} Q .$$

6 For $S \subset T \subset \eta$, $m_{ST} : I_S \longrightarrow I_T$ is defined by diagonal fill-in (2.7):

$$\begin{tikzcd}
P_S \arrow[rr, "e_S"] \arrow[d, "in_{ST}"'] & & I_S \arrow[dd, "m_S"] \arrow[ddll, dashed, "m_{ST}"'] \\
P_T \arrow[d, "e_T"'] & & \\
I_T \arrow[rr, "m_T"] & & Q
\end{tikzcd}$$

(to prove that the square commutes observe that both paths are $f_i$ when preceded by $in_i$).

We observe at once, using the results of section 2, that $m_{ST} \in \mathcal{M}$, that $m_{TU}\, m_{ST} = m_{SU}$ for $S \subset T \subset U$, and that if $f_S \in \mathcal{E}$ then $f_T \in \mathcal{E}$ whenever $S \subset T$.

Motivated by system theory we should like to prove results such as 'if $Q$ is Noetherian and $M$ is reachable then $M$ is reachable in bounded time' and 'if $f_S$ is onto and if $T$ is the subset of $S$ obtained by deleting those $k$ for which the union of the images of $f_0,\ldots,f_{k-1}$ is the same as the union of the images of $f_0,\ldots,f_k$ then $f_T$ is still onto'. We observe that the reason these results are so easy to obtain in the category of sets is because the passage $S \mapsto I_S$ is union-preserving; $I_S$ is, after all, just the union of the images of the $(f_s : s \in S)$. Our approach below is to show that, in general, this passage is sufficiently supremum-preserving to lift the theory to a category. We present general results about 'dimension in a category' in this section; and turn to their system-theoretic application in section 4.

<u>7</u> <u>LEMMA</u>: *For any object Q, every non-empty countable family of subobjects of Q has a supremum.*

<u>Proof</u>: Given $[R_k, m_k]$ define f by $f\ in_k = m_k$, and consider

$$\begin{tikzcd}
R_k \arrow[r, "in_k"] \arrow[dr, dashed, "\psi_k"] \arrow[ddr, "m_k"'] & \coprod R_k \arrow[d, "e"] \arrow[dd, bend left=60, "f"] \\
 & R \arrow[d, "m"] \\
 & Q
\end{tikzcd}$$

where (e,m) is an $\mathcal{E}$-$\mathcal{M}$ factorization of f. Using diagonal fill-in on the 'square' $m(e\ in_k) = m_k\ id$ induces $\psi_k$, so that $[R_k, m_k] \le [R,m]$ for all k. We will show that [R,m] is the least upper bound. Suppose that $[R_k, m_k] \le [R', m']$. Then there exist $\psi'_k$ as shown

$$\begin{tikzcd}
R_k \arrow[r, "in_k"] \arrow[dr, "\psi'_k"] \arrow[ddr, "m_k"'] & \coprod R_k \arrow[d, dashed, "h"] \arrow[r, "\hat{e}"] & I \arrow[dl, "\hat{m}"] \\
 & R \arrow[d, "m'"] & \\
 & Q &
\end{tikzcd}$$

and hence a unique h with $h\ in_k = \psi'_k$. Clearly $m'h = f$. Hence, if $(\hat{e}, \hat{m})$ is an $\mathcal{E}$-$\mathcal{M}$ factorization of h, $(\hat{e}, m'\hat{m})$ is an $\mathcal{E}$-$\mathcal{M}$ factorization of f so that $[R,m] = [I, m'\hat{m}]$. But then, via $\hat{m}$, $[R,m] \le [R',m']$. □

Given $f : R \longrightarrow Q$, let [f] denote the subobject of Q obtained by taking the image factorization of f.

<u>8</u> <u>LEMMA</u>: *Let* $f_i : R_i \longrightarrow Q$ *be a non-empty countable family of morphisms and let* $f : \coprod R_i \longrightarrow Q$ *be defined by* $f\ in_i = f_i$. *Then* $[f] = \sup([f])$.

Proof: Consider the diagram shown in which $[f_i] = [T_i, m_i]$ and $[I,m] = \sup([f_i])$ according to the construction of Lemma 7.

$$\begin{tikzcd}
R_i \arrow[rrrr, "in_i"] \arrow[dr, "e_i"] \arrow[dddrr, bend right, "f_i"'] & & & & \coprod R_i \arrow[dl, "e"'] \arrow[dddll, bend left, "f"] \\
& T_i \arrow[rr, "in_i"] \arrow[ddr, "m_i"] & & \coprod T_i \arrow[d, "e'"'] & \\
& & & I \arrow[dl, "m"'] & \\
& & Q & &
\end{tikzcd}$$

Define e by $e\ in_i = in_i\ e$. Then the diagram commutes -- that is, $f = m\ e'\ e$ -- because both paths coincide with $f_i$ when preceded by $in_i$. By 2.11, $e \in \mathcal{E}$ so that $[f] = [I,m]$ as desired. □

Before continuing, we introduce the abbreviation $I_S$ for the more cumbersome $[I_S, m_S]$, '$I_S$ is a subobject of Q'.

9 LEMMA: *The passage* $S \mapsto I_S$ *preserves non-empty countable suprema.*

Proof: Let $(S_k : k \in I)$ be a non-empty countable family of non-empty subsets of $\eta$ and set $S = \cup S_k$. We must show that $I_S$ is the supremum of the $I_k$ (where we use the subscript k for the more cumbersome $S_k$ throughout). Consider the map $\theta : \coprod P_k \longrightarrow P_S$ defined by $\theta\ in_k = in_{kS}$ $(k \in I)$. In view of the commutative diagram

$$\begin{tikzcd}
P_k \arrow[rrrrr, "in_k"] \arrow[drrrr, "in_{kS}"] \arrow[ddr, "e_k"'] & & & & & \coprod P_k \arrow[dl, "\theta"] \\
& & & & P_S \arrow[dl, "e_S"] & \\
& I_k \arrow[rr, "m_{kS}"] \arrow[dr, "m_k"'] & & I_S \arrow[dl, "m_S"] & & \\
& & Q & & &
\end{tikzcd}$$

it suffices to show that $\theta \in \mathcal{E}$, for then $(e_S\theta, m_S)$ is precisely the construction of the supremum in Lemma 7. To prove that $\theta \in \mathcal{E}$ we use the dual of 2.8. For $s \in S$ choose $k(s)$ with $s \in S_{k(s)}$. Then consider the diagram

$$\begin{tikzcd}
P_{k(s)} \arrow[d, "in_{k(s)}"'] & P_s \arrow[l, "in_s"'] \arrow[d, "in_s"] \\
\coprod P_k \arrow[r, "\theta"] \arrow[d, "f"'] & P_S \arrow[d, "g"] \arrow[dl, dashed, "\psi"'] \\
A \arrow[r, "m"] & B
\end{tikzcd}$$

where $g$, $f$ and $m$ are only required to satisfy $g\theta = mf$ and $m \in \mathcal{M}$.

We must construct $\psi$ with $m\psi = g$.

Define $\psi$ by $\psi\, in_s = f\, in_{k(s)}\, in_s$ as shown. Since $\theta\, in_{k(s)}\, in_s$ $= in_{k(s)S}\, in_s = in_s$ (see 4) we have $(m\psi) in_s = m\, f\, in_{k(s)}\, in_s =$ $g\, \theta\, in_{k(s)}\, in_s = g\, in_s$ for all $s \in S$, so that $m\psi = g$. □

For the next definition and two propositions we consider an arbitrary non-empty-countable-supremum-preserving map $I : R \longrightarrow L$ where $R$ is the partially ordered set of non-empty subsets of $\mathcal{N}$ and $L$ is an arbitrary partially ordered set. For the general $I$ we write $I(S)$ instead of $I_S$. $I_S$ is not the only application; see [8].

<u>10</u> <u>DEFINITION:</u> Define $\bar{n} = \{0,\ldots,n\} \in R$. $I$ is *stationary* if $I(\bar{n}) = I(\overline{n+1})$ for all $n$. $A \in R$ is *adequate* if $I(A) = I(S)$ whenever $A \subset S$. Equivalently, $A$ is adequate if and only if $I(S) = I(T)$ whenever $A \subset S \subset T$.

<u>11</u> <u>PROPOSITION:</u> *If $A$ is 'one-step adequate' in the sense that $I(A) = I(A \cup \{k\})$ for all $k$ then $A$ is adequate.*

<u>Proof:</u> If $A \subset S$, $S = \cup(A \cup \{k\} : k \in S)$. □

<u>12</u> <u>PROPOSITION:</u> *For each non-empty-countable-supremum-preserving map $I : R \longrightarrow L$, the set*

$$A = \{0\} \cup \{k \in \mathcal{N} \mid I(\overline{k-1}) < I(\bar{k})\}$$

*is adequate.*

<u>Proof</u>: Note that if I is stationary, then $A = \{0\}$, and certainly this A is adequate. Otherwise, it suffices to prove $I(A) = I(A \cup \{0,\ldots,n\})$ for all n. Since $\{0\} \subset A$, this is certainly true for $n = 0$. Suppose now that $I(A) = I(A \cup \{0,\ldots,n-1\})$. Then if $n \in A$, it is certainly true that $I(A) = I(A \cup \{0,\ldots,n\})$. Otherwise, $I(\overline{n-1}) = I(\bar{n})$, and so

$$\begin{aligned} I(A \cup \{0,\ldots,n\}) &= \sup(I(A), I(\bar{n})) \\ &= \sup(I(A), I(\overline{n-1})) \\ &= I(A \cup \overline{n-1}) = A \end{aligned} \qquad \square$$

<u>13</u> <u>COROLLARY</u>: *If Q has $\mathcal{M}$-height h, $f_i : P_i \longrightarrow Q$ has an adequate set with h+1 or fewer elements.* $\square$

## 4. Adequacy for Systems and the Simple Recursion Principle

In this section, we study the implications of Section 3 for an adjoint system M. We introduce all of the notions of Section 3, with $f_i : P_i \longrightarrow Q \;=\; r_i : IX^i \longrightarrow Q$. We write $IX^S$ instead of $P_S$.

1 PROPOSITION: *If* M *is reachable and* Q *is Noetherian,* M *is reachable in bounded time. Dually, if* M *is observable and* Q *is Artinian,* M *is observable in bounded time.*

Proof: By definition, $r_{\eta} = r$ is the reachability map of M and M is reachable in bounded time if and only if $r_S \in \mathcal{E}$ for some finite S. Since Q is Noetherian, there exists a finite one-step adequate set A. Then A is adequate by 3.11 and, in the diagram shown, $m_{S\eta}$ is an isomorphism.

$$\begin{tikzcd}
IX^S \arrow[rrrr, "in_{S\eta}"] \arrow[dr, "e_S"] \arrow[ddrr, bend right, "r_S"'] & & & & \Omega \arrow[dl, "e_{\eta}"'] \arrow[ddll, bend left, "r_{\eta}"] \\
& I_S \arrow[rr, "m_{S\eta}"] \arrow[dr, "m_S"'] & & I_{\eta} \arrow[dl, "m_{\eta}"] & \\
& & Q & &
\end{tikzcd}$$

If M is reachable, $m_{\eta}$ is an isomorphism (2.10 and 2.11) so that $r_S = m_{\eta}\, m_{S\eta}\, e_S \in \mathcal{E}$ (by IFS1 and IFS2). □

2 DEFINITION: Let V be a set. A sequence $v_n$ in V is *defined by simple recursion* if there exists a function $g : V \longrightarrow V$ such that $v_{n+1} = g(v_n)$, that is, $v_n = g^n(v_0)$.

3 EXAMPLE: The sequence $\mathrm{span}(B, AB, \ldots, A^nB)$ of subspaces of the state space of a linear system is defined by simple recursion. Define $g(S) = \mathrm{span}(S \cup F(S))$ for each subspace S. The next result shows that this construction works for arbitrary adjoint systems.

Recall that $\bar{n} = \{0,\ldots,n\}$.

<u>4</u> <u>SIMPLE RECURSION PRINCIPLE FOR ADJOINT SYSTEMS</u>: *Let $I_{\bar n}$ be the subobject $[r_{\bar n}]$ of the state object of an adjoint system 'reachable in time n'. Then the ascending sequence $I_{\bar n}$ is defined by simple recursion. Dually, the ascending sequence of observability quotient objects of the state object is also defined by simple recursion.*

<u>Proof</u>: We define the endomorphism g on the subobjects of Q by $g([R,m]) = \sup([R,m],[\delta\cdot mX])$. To verify that $g(I_{\bar n}) = I_{\overline{n+1}}$, recall that $r_{i+1} = \delta\cdot r_iX$, and that $r_i = r_{\bar n}\cdot in_i$ for $i \le n$, and that we have the coproduct diagram

$$\begin{tikzcd}
IX^{i}X \arrow[d, "in_iX"'] \arrow[dr, "r_iX"] \arrow[drr, bend left, "r_{i+1}"] & & \\
IX^{\bar n}X \arrow[r, "r_{\bar n}X"'] & QX \arrow[r, "\delta"'] & Q
\end{tikzcd}$$

Now, by lemma 3.8, $[\delta\cdot r_{\bar n}X] = \sup([r_1],\ldots,[r_{n+1}]) = \sup(I_1,\ldots,I_{n+1})$. But because X preserves $\mathcal{E}$,

$$[\delta\cdot r_{\bar n}X] = [\delta\cdot m_{\bar n}X\cdot e_{\bar n}X] = [\delta\cdot m_{\bar n}X].$$

Moreover, $I_{\bar n} = \sup(I_0,\ldots,I_n)$ by 3.9. Therefore

$$g(I_{\bar n}) = \sup(\sup(I_0,\ldots,I_n),\ \sup(I_1,\ldots,I_{n+1})) = \sup(I_0,\ldots,I_{n+1}) = I_{\overline{n+1}}. \quad \square$$

An immediate consequence is a better proof of the general version [1, Theorem 4.6] of the 'if you stick you're stuck' result of [9]:

<u>5</u> <u>COROLLARY</u>: *If $I_{\bar n} = I_{\overline{n+1}}$, then $\bar n$ is adequate.*

<u>Proof</u>: $I_{\overline{n+k+1}} = g(I_{\overline{n+k}}) = g(I_{\overline{n+k-1}})$ (induction hypothesis) $= I_{\overline{n+k}}$. $\square$

While the proof of Corollary 5 bypasses Proposition 3.12, the latter is still a useful principle, as we shall see in [8].

6 COROLLARY: *Let* M *be an adjoint system with state object* Q. *If* Q *has* $\mathcal{M}$*-height* h *then* M *is reachable in time* h. *Dually, if* Q *has* $\mathcal{E}$*-height* h, M *is observable in time* h. $\square$

To tie this back to the realization theory of Section 2, and especially the Partial Realization theorem 2.19, we make the

7 OBSERVATION: It is clear from 2.18 and 2.10 that t is in $\mathcal{E}$ and that u is in $\mathcal{M}$. It is then clear that if $\mathcal{E}$-height $(\bar{Q})$ = $\mathcal{E}$-height (R) and both are finite, then t is an isomorphism; while if $\mathcal{M}$-height (R) = $\mathcal{M}$-height $(\hat{Q})$ and both are finite, then u is an isomorphism. Thus the condition 't and u are isomorphisms' in 2.19 may be replaced by '$\mathcal{E}$-height $(\bar{Q})$ = $\mathcal{E}$-height (R) and $\mathcal{M}$-height (R) = $\mathcal{M}$-height $(\hat{Q})$ and both are finite'.

8 COROLLARY: *For adjoint processes in* Vect, *we may obtain a partial realization as soon as*

$$\dim(Q) = \dim(R) = \dim(\hat{Q}) = \text{finite}.$$

This yields both Tether's [22] criterion for partial realization of linear systems, and Isidori's [16] criterion for partial realization of bilinear systems (internal sense).

Combining the argument for Corollary 6 with the Partial Realization Theorem 2.19 and observation 7 we have

9 THE HANKEL REALIZATION THEOREM: *Let* $H_i^j$ *be a Hankel matrix having a realization with state object* Q *having* $\mathcal{E}$*-height* $\le h$ *and* $\mathcal{M}$*-height* $\le \ell$ *with* h *and* $\ell$ *finite. Then the canonical realization of* $H_i^j$ *may be constructed by applying the construction of 2.19 with* $k = h$ *and* $n = \ell$.

Proof outline: The crucial point is that the finite height conditions imply that items (i) and (ii) of the proof of 2.19 -- $r_i = \bar{e} \cdot in_i$ and $\sigma_j = pr_j \cdot \bar{m}$ -- hold for all i and j respectively. But this not only shows that the M of 2.19 has Hankel matrix $H_i^j$, but also that M has reachability map $\bar{e}$ in $\mathcal{E}$ and observability map $\bar{m}$ in $\mathcal{M}$ -- so that M is canonical. □

10 OBSERVATION: As in 8, we note that when $\mathcal{K}$ = Vect the two height conditions in 9 collapse to the single condition 'Q having finite dimension h', and we may then take k = n = h in forming the realization.

For the biadequacy criterion for Hankel realization, see [8, Theorem 4.9].

## REFERENCES


1. B.D.O. Anderson, M. A. Arbib and E. G. Manes: Foundations of System Theory: Finitary and Infinitary Conditions, *Lecture Notes in Economics and Mathematical Systems,* 115, Springer-Verlag (1976).

2. M. A. Arbib and E. G. Manes: Foundations of system theory: decomposable systems, *Automatica* 10 (1974) 285-302.

3. M. A. Arbib and E. G. Manes: *Arrows, Structures and Functors: The Categorical Imperative,* Academic Press (1975).

4. M. A. Arbib and E. G. Manes: Adjoint machines, state-behavior machines and duality, *J. Pure Appl. Alg.* 6 (1975) 313-344.

5. M. A. Arbib and E. G. Manes: Time-varying systems, *SIAM J. Control* 13 (1975) 1252-1270.

6. M. A. Arbib and E. G. Manes: Recurrence and finiteness for systems in a general setting, *Proc. IEEE Conf. Decision and Control* 76CH 1150-2CS (1976) 1180-1183.

7. M. A. Arbib and E. G. Manes: Recurrence conditions for generalized Hankel matrices, (in press).

8. M. A. Arbib and E. G. Manes: On the evolution of generalized Hankel matrices, (in press).

9. M. A. Arbib and H. P. Zeiger: On the relevance of abstract algebra to control theory, *Automatica* 5 (1969) 589-606.

10. L. Budach and H.-J. Hoehnke: *Automaten und Funktoren,* VEB (1975).

11. H. Ehrig, K.-D. Kiermeier, H.-J. Kreowski and W. Kühnel, *Universal Theory of Automata,* Teubner (1974).

12. M. Fliess: Matrices de Hankel, *J. de Math. Pure Appl.* 53 (1974) 197-224.

13. J. A. Goguen: Minimal realizations of machines in closed categories, *Bull. Amer. Math. Soc.* 78 (1972) 777-783.

14. S. J. Hegner: Duality theory for discrete-time linear systems, Technical report 76-7, Department of Computer and Information Science, University of Massachusetts at Amherst, July (1976).

15. J. R. Isbell: Some remarks concerning categories and subspaces, *Canad. J. Math.* 9 (1957) 563-577.

16. A. Isidori: Direct construction of minimal bilinear realizations from nonlinear input-output maps, *IEEE Trans. Automat. Contr.* AC-18 (1973) 626-631.

17. R. E. Kalman, P. L. Falb and M. A. Arbib: *Topics in Mathematical System Theory*, McGraw-Hill (1969).

18. S. Mac Lane: Groups, categories and duality, *Proc. Nat. Acad. Sci. USA* 34 (1948) 263-267.

19. S. Mac Lane: *Categories for the Working Mathematician*, Springer-Verlag (1971).

20. E. Manes: *Algebraic Theories*, Springer-Verlag (1976).

21. T. J. Tarn and S. Nonoyama: Realization of discrete-time internally bilinear systems, *Proc. IEEE Conf. Decision and Control* 76CH 1150-2CS, (1976) 125-133.

22. A. J. Tether: Construction of linear state-variable models from finite input-output data, *IEEE Trans. Automat. Contr.* AC-15 (1970) 427-436.

RECURRENCE CONDITIONS FOR GENERALIZED HANKEL MATRICES[1]

Michael A. Arbib
Computer & Information Science

Ernest G. Manes
Mathematics
University of Massachusetts
Amherst, Massachusetts 01003, USA

June 1977

ABSTRACT

Recurrence and corecurrence of the Hankel matrix of an adjoint system is related to the existence of finite realizations. Adjoint systems include linear and bilinear systems, automata, and group systems in both the time-varying and time-invariant cases.

After a brief review of the setting of [4], we define recurrence of degree m for a Hankel matrix $H_i^j$. Such recurrence implies that $H_i^m$ depends on $H_i^0, \ldots, H_i^{m-1}$ for all i. The dual notion is corecurrence. To relate these conditions to 'finiteness' we investigate ascending chain conditions for subobjects and quotient objects of an object in a category. Recurrence is related to Artinian realizations with injective Artinian input object whereas corecurrence is related to Noetherian realizations with projective Noetherian output object. The scarcity of Artinian injective modules in the context of linear systems over a ring puts the emphasis on corecurrence there. Our theorems unify the familiar linear results [9], [6, XVI.10] with the group machine result of [5, theorem 2].

The 'recurrence polynomials' in the context of modules over a noncommutative ring, as considered by [10], are algebraic operations in the sense of universal algebra [8, section 1.5] as opposed to morphisms. They cannot be subsumed in the discussion here and will be discussed elsewhere.

---

1 The research reported in this paper was supported in part by the National Science Foundation under grant MCS76-84477.

## 1. Generalities

We quickly record some definitions and facts concerning the Hankel matrix of an adjoint system. For details, see [4] and the references cited there.

We work in an arbitrary category $\mathcal{K}$ with countable products and coproducts and provided with an image factorization system $(\mathcal{E},\mathcal{M})$ (generalizing $\mathcal{E}$ = surjections, $\mathcal{M}$ = injections in familiar categories such as sets and functions, or vector spaces and linear maps).

We fix an adjoint process (X,Z). This means that X and Z are functors from $\mathcal{K}$ to itself with Z right adjoint to X. Thus there are bijective correspondences

$$\frac{RX \xrightarrow{g} S}{R \xrightarrow[\psi]{} SZ}$$

(one to each pair of $\mathcal{K}$-objects (R,S)) subject to the naturality condition

$$\frac{QX \xrightarrow{fX} RX \xrightarrow{g} S \xrightarrow{h} T}{Q \xrightarrow[f]{} R \xrightarrow[\psi]{} SZ \xrightarrow[hZ]{} TZ}$$

whenever $f : Q \longrightarrow R$, $h : S \longrightarrow T$. In this notation, $\psi$ may alternatively appear on the top with g on the bottom. We also assume that X preserves $\mathcal{E}$ (and hence [4, theorem 2.12] that Z preserves $\mathcal{M}$).

For example, let $\mathcal{K}$ be the category of sets and functions. Let A be a fixed input alphabet. Define $QX = Q \times A$, $QZ = Q^A$, the set of functions from A to Q. For $f : Q \longrightarrow R$, $fX : Q \times A \longrightarrow R \times A$ is defined by $(q,a) \longmapsto (f(q),a)$ whereas $fZ : Q^A \longrightarrow R^A$ sends $g : A \longrightarrow Q$ to $fg : A \longrightarrow R$.

The adjointness correspondence

$$\frac{Q \times A \xrightarrow{\;g\;} R}{Q \xrightarrow[\psi]{} R^A}$$

is the familiar $(\psi q)(a) = g(q,a)$. Here, let $\mathcal{E}$ = surjections and let $\mathcal{M}$ = injections. It is indeed true that fX is surjective when X is and that gZ is injective when g is.

An *adjoint system* is $M = (Q,\delta,I,\tau,Y,\beta)$ where Q, I, Y are objects (the *state object, input object* and *output object* of M) and $\delta : QX \longrightarrow Q$, $\tau : I \longrightarrow Q$ and $\beta : Q \longrightarrow Y$ are morphisms (the *dynamics, input map* and *output map* of M). (Note: 'map' is here a synonym for 'morphism'.) The *codynamics* of M is the map $\Delta : Q \longrightarrow QZ$ which corresponds to $\delta$ under adjointness.

Given two dynamics $\delta : QX \longrightarrow Q$ and $\theta : RX \longrightarrow R$, a *dynamorphism* $h : (Q,\delta) \longrightarrow (R,\theta)$ is a map $h : Q \longrightarrow R$ which 'respects the dynamics':

$$\begin{tikzcd} QX \arrow[r, "hX"] \arrow[d, "\delta"'] & RX \arrow[d, "\theta"] \\ Q \arrow[r, "h"] & R \end{tikzcd}$$

The *time-i reachability map* $r_i : IX^i \longrightarrow Q$ and the *time-j observability map* $\sigma_j : Q \longrightarrow YZ^j$ are defined by:

$$r_0 = \tau$$

$$r_{i+1} = IX^{i+1} \xrightarrow{\;r_iX\;} QX \xrightarrow{\;\delta\;} Q$$

$$\sigma_0 = \beta$$

$$\sigma_{j+1} = Q \xrightarrow{\;\Delta\;} QZ \xrightarrow{\;\sigma_jZ\;} YZ^{j+1}.$$

The bisequence $H_i^j$, where $H_i^j : IX^i \longrightarrow YZ^j$ is defined by $H_i^j = \sigma_j\, r_i$, is the *Hankel matrix* of M. The Hankel matrix for adjoint systems was introduced in [4], which also contains further references on related studies.

We continue our earlier example to describe 'automata theory'. (For the other examples mentioned in the abstract we refer the reader to [4].) Let I have a single element so that $\tau$ amounts to an element of $Q$, the initial state. The dynamics and output map have their usual forms $\delta : Q \times A \longrightarrow Q$, $\beta : Q \longrightarrow Y$. It is easily checked that $r_i : A^i \longrightarrow Q$ sends an i-tuple of input letters to the state reached from the initial state if the letters are inputted in sequence, whereas $\sigma_j : Q \longrightarrow Y^{(A^j)}$ sends q to that function $A^j \longrightarrow Y$ with $\beta$ following the time-j reachability map if the initial state were q. Thus $H_i^j : A^i \longrightarrow Y^{(A^j)}$ is essentially a way of describing $\beta \cdot r_{i+j}$ with emphasis on i as 'present time'.

As in [4] we define the object of inputs $\Omega$ to be the coproduct $\coprod(IX^i : i \geq 0)$ and we define the observability space $\Gamma$ to be the product $\prod(YZ^j : j \geq 0)$. Then $\Omega$ and $\Gamma$ carry canonical X-dynamical structure [4, 2.1 and 2.2]. The following realizability theorem is proved in [4, 2.3].

<u>REALIZABILITY THEOREM</u>: *Let* $H_j^i : IX^i \longrightarrow YZ^j$ *be an arbitrary bisequence of morphisms and let* $H : \Omega \longrightarrow \Gamma$ *be the unique morphism with* $pr_j\ H\ in_i = H_i^j$. *Then the following three conditions are equivalent (and we say* $H_i^j$ *is a* <u>*Hankel matrix*</u> *if these conditions hold).*

*(i)* $H_i^j$ *is realizable, that is, is the Hankel matrix of some system.*

*(ii)* *(The Hankel crossover condition). For all* i, j:

$$\frac{IX^{i+1} \xrightarrow{\;H_{i+1}^{j}\;} YZ^{j}}{IX^{i} \xrightarrow[\;H_{i}^{j+1}\;]{} YZ^{j+1}}$$

*(iii)* $H : \Omega \longrightarrow \Gamma$ *is a dynamorphism.*

Let M be an adjoint system. The *reachability map* $r : \Omega \longrightarrow Q$ of M is defined by $r\ in_i = r_i : IX^i \longrightarrow Q$. The *observability map* $\sigma : Q \longrightarrow \Gamma$ of M is defined by $pr_j \cdot \sigma = \sigma_j : Q \longrightarrow YZ^j$. We say M is *reachable* if $r \in \mathcal{E}$, and *observable* if $\sigma \in \mathcal{M}$.

To conclude this section we recall that the *canonical realization* $M_H$ of the Hankel matrix $H_i^j$ is the system $(Q_H, \delta_H, \tau_H, \beta_H)$ defined as follows.

Let $H : (\Omega, \mu_0) \longrightarrow (\Gamma, L)$ be the above dynamorphism. Let

$$\Omega \xrightarrow{r_H} Q_H \xrightarrow{\sigma_H} \Gamma$$

be an $\mathcal{E}$-$\mathcal{M}$ factorization of H. By the dynamorphic image lemma [4, 2.13], there exists a unique dynamics $\delta_H : Q_H X \longrightarrow Q_H$ rendering $r_H$ and $\sigma_H$ dynamorphisms. Define $\tau_H = r_H\ in_0$ and $\beta_H = pr_0\ \sigma_H$.

It is proved in [2, theorems 2.1, 3.15] that $M_H$ is a realization of $H_i^j$, that the reachability and observability maps of $M_H$ are $r_H$ and $\sigma_H$ (so that $M_H$ is reachable and observable) and that any other reachable and observable realization is isomorphic to $M_H$.

## 2. Recurrence for Matrices

For the duration of the paper fix an input object I and an output object Y.

**1 DEFINITION:** Given $H_i^j : IX^i \longrightarrow YX^j$ and an integer $m > 0$, a *recurrence morphism of degree* m is a morphism p such that

$$\begin{tikzcd}
IX^i \arrow[rr, "(H_i^j)"] \arrow[dr, "H_i^m"'] & & \prod(YZ^j : 0 \le j < m) \arrow[dl, "p"] \\
& YZ^m &
\end{tikzcd}$$

holds for all i. By $(H_i^j)$ we mean the unique morphism f such that $pr_j\, f = H_i^j$ for all $0 \le j < m$. If such p exists, $H_i^j$ is said to be *(morphically) recurrent* of degree m. Dually, a *corecurrence morphism of degree* m is a morphism $\rho$ such that

$$\begin{tikzcd}
\coprod(IX^i : 0 \le i < m) \arrow[rr, "(H_i^j)"] & & YZ^j \\
& IX^m \arrow[ul, "\rho"] \arrow[ur, "H_m^j"'] &
\end{tikzcd}$$

holds for all j. If such $\rho$ exists, $H_i^j$ is *corecurrent of degree* m.

**2 RECURRENCE THEOREM FOR MATRICES:** *Let $H_i^j$ be a Hankel matrix which is recurrent of degree* m. *Then $H_i^j$ has a realization with state object* $Q = \prod(YZ^j : 0 \le j < m)$.

**Proof:** Define $\beta = pr_0 : Q \longrightarrow Y$, define $\tau : I \longrightarrow Q$ by $pr_j\, \tau = H_0^j : I \longrightarrow YZ^j$ and let $\delta : QX \longrightarrow Q$ correspond under adjointness to the $\Delta$ defined by

$$\begin{tikzcd} & YZ^{j+1} \\ Q \arrow[ur, "pr_{j+1}"] \arrow[r, "\Delta"] \arrow[dr, "p"'] & QZ \arrow[u, "pr_j Z"'] \arrow[d, "pr_{m-1}Z"] \\ & YZ^m \end{tikzcd} \qquad (j < m-1)$$

(where we make use of the fact [4, 1.9] that Z preserves products). We must show that $H_i^j = \sigma_j r_i$. We first show that $r_i = (H_i^j : 0 \le j < m) : IX^i \longrightarrow Q$ by induction on i. For i = 0 this is the definition of $\tau$. For the inductive step, if $0 \le j < m-1$ we have

$$\begin{array}{l} IX^{i+1} \xrightarrow{r_i X} QX \xrightarrow{\delta} Q \xrightarrow{pr_j} YZ^j \\ \hline IX^i \xrightarrow{r_i} Q \xrightarrow{\Delta} QZ \xrightarrow{pr_j Z} YZ^{j+1} \\ = IX^i \xrightarrow{r_i} Q \xrightarrow{pr_{j+1}} YZ^{j+1} \\ = H_i^{j+1} \\ \hline H_{i+1}^j \end{array}$$

whereas, for $j = m-1$ we have the related argument that $pr_{m-1}Z \cdot \Delta \cdot r_i = p\, r_i$ $= p\,(H_i^j : 0 \le j < m)$ (by the induction hypothesis) $= H_i^m$ (by the definition of recurrence morphism) which corresponds under adjointness to $H_{i+1}^{m-1}$ as desired. In particular, we have shown that $pr_0\, r_i = H_0^i$ which is the case '$j = 0$' in showing that $H_i^j = \sigma_j r_i$. But the inductive step on j is immediate from the fact that both $H_i^j$ and $\sigma_j r_i$ satisfy the Hankel crossover condition. □

<u>3</u> <u>CORECURRENCE THEOREM FOR MATRICES</u>: *Let* $H_i^j$ *be a Hankel matrix which is corecurrent of degree* m. *Then* $H_i^j$ *has a realization with state object* $Q = \coprod(IX^i : 0 \le i < m)$.

<u>Proof</u>: The proof is dual to that of 2 but we record the construction. Define $\tau = in_0$, define $\beta$ by $\beta\, in_i = H_i^0$ and define $\delta$ by

$$(i < m-1) \qquad \begin{tikzcd}
QX^{i+1} \arrow[d, "in_i X"'] \arrow[dr, "in_{i+1}"] & \\
QX \arrow[r, "\delta"] & Q \\
QX^m \arrow[u, "in_{m-1} X"] \arrow[ur, "\rho"'] &
\end{tikzcd}$$

□

4 EXAMPLE: THE DECOMPOSABLE CASE. Here $X = Z$ is the identity functor of $\mathcal{K}$. The realization theory in this special case was studied in [1]. When $\mathcal{K}$ is the category of vector spaces (or of modules over a ring) an adjoint system is just a linear system

$$I \xrightarrow{B} Q \qquad Q \xrightarrow{A} Q \qquad Q \xrightarrow{C} Y .$$

The same system description holds in any category. The adjointness correspondence is just

$$\frac{Q \xrightarrow{g} R}{Q \xrightarrow[g]{} R}$$

so that the codynamics is again A. We have $r_i = A^i B$ and $\sigma_j = CA^j$ so that $H_i^j = CA^{i+j}B$. A recurrence map takes the form $p : Y^m \longrightarrow Y$ (where $Y^m$ denotes the m-fold product of copies of Y). If $\mathcal{K}$ is a category such as modules over a ring or groups in which objects are 'sets with structure', morphisms are determined as functions (though not all functions may be admissible as morphisms) and the underlying set of a finite product is the product of the underlying sets, the recurrence condition literally takes the form

$$p(H_i^0(u),\ldots,H_i^{m-1}(u)) = H_i^m(u)$$

for all $i \geq 0$, $u \in I$. When $\mathcal{K}$ is the category of groups, theorem 2 above is a direct generalization of the sufficiency proof of [5, theorem 2] (for the necessity part see 3.4 below). Owing to the fact that the coproduct is the free product in the category of groups (see [3, 3.2.3] for the details) corecurrence for group systems may not be a very useful notion. When $\mathcal{K}$ is the category of modules over a ring (or semiring), however, a corecurrence map has the form $\rho : I \longrightarrow I^m$ and the corecurrence condition is

$$H_m^j(u) = \sum_{k=0}^{m-1} H_k^j \rho_k(u)$$

where $\rho(u) = (\rho_0(u),\ldots,\rho_{m-1}(u))$ and $j \geq 0$. This essentially recaptures the recurrence polynomials familiar in linear system theory (see, e.g., [6, XVI.9.1] and [9, p. 3405]).

## 3. Recurrence for Systems

**1** DEFINITIONS: Let M be an adjoint system. M is *reachable in time* i if $(r_k : 0 \le k \le i) : \coprod(IX^k : 0 \le k \le i) \longrightarrow Q \in \mathcal{E}$ and, dually, M is *observable in time* j if $(\sigma_k : 0 \le k \le j) \in \mathcal{M}$. M is *reachable in bounded time* if M is reachable in time i for some i, and M is *observable in bounded time* if M is observable in time j for some j.

Reachability in bounded time implies reachable [4, 2.10]. Dually, observability in bounded time implies observable.

**2** RECURRENCE THEOREM FOR SYSTEMS: *Let M be an adjoint system with Hankel matrix* $H_i^j = \sigma_j r_i$. *Let* $p : \prod(YZ^j : 0 \le j < m) \longrightarrow YZ^m$ *be a morphism. Consider the diagram*

$$\begin{tikzcd} Q \arrow[rr, "(\sigma_j \,:\, 0 \le j < m)"] \arrow[dr, "\sigma_m"'] & & \prod(YZ^j : 0 \le j < m) \arrow[dl, "p"] \\ & YZ^m & \end{tikzcd}$$

*Then*

*(i) If the diagram commutes,* $H_i^j$ *is recurrent.*

*(ii) If M is reachable and* $H_i^j$ *is recurrent with recurrence map p, the diagram commutes.*

Proof: Consider the diagram

$$\begin{tikzcd} \Omega \arrow[r, "r"] & Q \arrow[r, "(\sigma_j)"] \arrow[dr, "\sigma_m"'] & \prod(YZ^j : 0 \le j < m) \arrow[d, "p"] \\ IX^i \arrow[u, "in_i"] \arrow[ur, "r_i"'] & & YZ^m \end{tikzcd}$$

In the notation below, $0 \le j < m$. From the definition of a product, we have that $r_i(\sigma_j) = (r_i\sigma_j) = (H_i^j)$. Thus, if $p(\sigma_j) = \sigma_m$, $p(H_i^j) = p(\sigma_j r_i) = p(\sigma_j)r_i = \sigma_m r_i = H_i^m$ for all i, and thus p is a recurrence map. To prove (ii), if p is a recurrence map then for all i, $[p(\sigma_j)r]in_i = p(H_i^j) = H_i^m = [\sigma_m r]in_i$. By the definition of a coproduct, $p(\sigma_j)r = \sigma_m r$. As M is reachable, r is an epimorphism so that $p(\sigma_j) = \sigma_m$ as desired. □

Theorem 2 reduces the problem of finding a recurrence morphism of degree m to the filling-in of a single diagram. Before exploiting the result, we pause to state the dual theorem:

<u>3</u> <u>CORECURRENCE THEOREM FOR SYSTEMS</u>: *Let* M *be an adjoint system with Hankel matrix* $H_i^j$ *and let* $\rho : IX^m \longrightarrow \coprod(IX^i : 0 \le i < m)$ *be a morphism. Then*

*(i) If* $(r_i : 0 \le i < m)\rho = r_m$ $H_i^j$ *is corecurrent.*

*(ii) If* M *is observable and* $H_i^j$ *is corecurrent with corecurrence map* $\rho$*, then* $(r_i : 0 \le i < m)\rho = r_m$. □

Our first application of 2 is to finite-state decomposable systems, fully recapturing theorem 2 of [5] by choosing $\mathcal{K}$ to be the category of groups and $\mathcal{F}$ to be the finite groups. We note that our proof specializes to theirs in all essentials. The generalization to any class of universal algebras is immediate.

<u>4</u> <u>RECURRENCE THEOREM FOR FINITE-STATE DECOMPOSABLE SYSTEMS</u>: *Let* X *and* Z *be the identity functor of* $\mathcal{K}$ *as in example 2.4. Let* $\mathcal{F}$ *be a class of 'finite' objects in the sense that the following two axioms hold:*

*(i)* $Y^m \in \mathcal{F}$ *for all* $m > 0$.

*(ii) For all* $Q \in \mathcal{F}$, *the set* $\mathcal{K}(Q,Y)$ *of all morphisms from* Q *to* Y *is finite.*

*Let* $H_i^j : I \longrightarrow Y$ *be an arbitrary Hankel matrix. Then* $H_i^j$ *has a realization with state object in* $\mathcal{F}$ *if and only if* $H_i^j$ *is recurrent.*

<u>Proof</u>: One direction is clear from 2.2. Conversely, Let M realize $H_i^j$ with state object $Q \in \mathcal{F}$. Since $\sigma_j : Q \longrightarrow Y$ and there are only finitely many $\mathcal{K}$-morphisms from Q to Y, there exists $0 \leq k < m$ with $\sigma_k = \sigma_m$. But then $pr_k(\sigma_j : 0 \leq j < m) = \sigma_k = \sigma_m$ so that $pr_k : Y^m \longrightarrow Y$ is a recurrence map by 2.(i). □

The reader may easily formulate the dual theorem to 4.

<u>5</u> <u>INJECTIVE AND PROJECTIVE OBJECTS</u>: An object Q is *injective* if each diagram of form

$$\begin{tikzcd} R \arrow[rr, "m"] \arrow[dr, "f"'] & & S \arrow[dl, dashed, "g"] \\ & Q & \end{tikzcd}$$

may be completed (not necessarily uniquely) -- that is, $gm = f$ for some g -- whenever $m \in \mathcal{M}$. Dually, Q is *projective* if

$$\begin{tikzcd} R & & S \arrow[ll, "e"'] \\ & Q \arrow[ul, "f"] \arrow[ur, dashed, "g"'] & \end{tikzcd}$$

whenever $e \in \mathcal{E}$.

The following result is immediate from 2 and 3. It is trivially checked that any product of injectives is injective and any coproduct of projectives is projective so that, e.g., the condition that $YZ^m$ be injective for all m is often reducible to showing that Y is injective.

<u>6</u> <u>OBSERVATION</u>: *Let* M *be an adjoint system with Hankel matrix* $H_i^j$. *If* $YZ^m$ *is injective and if* M *is observable in time* m *then* $H_i^j$ *is recurrent of degree* m. *If* $IX^m$ *is projective and if* M *is reachable in time* m *then* $H_i^j$ *is corecurrent of degree* m. □

**7** **EXAMPLE:** In the category of sets, all non-empty sets are projective and all sets with at least two elements are injective. It follows that, in the context of the example of section 1, even if I, Y or A are infinite, observability in bounded time implies recurrence, and reachability in bounded time implies corecurrence.

**8** **EXAMPLE:** In the linear case of example 2.4 with $\mathcal{K}$ the category of modules over a ring (or semiring), if I is a free (or projective) module then $IX^m = I^m$ is projective for all m. Using 6, we recapture the familiar result that if $(A^iB : 0 \leq i < m) : I^m \longrightarrow Q$ has the same image as $(A^iB : i \geq 0)$ then $CA^{i+j}B$ is recurrent of degree m. This result does not generalize to arbitrary universal algebras because -- unlike the module case where finite products are also finite coproducts -- there is no reason for $I^m$ to be projective when I is free.

A converse result to 6 is a trivial extension of 2.2:

**9** **PROPOSITION:** *If $H_i^j$ is recurrent of degree m, $H_i^j$ has a realization which is reachable in time m. If $H_i^j$ is corecurrent of degree m, $H_i^j$ has a realization which is observable in time m.*

Proof: Use the realizations of 2.2, 2.3. It is trivial to check that $(\sigma_j : 0 \leq j < m)$ and $(r_i : 0 \leq i < m)$ are, respectively, the identity map. □

## 4. Noetherian and Artinian Objects

Noetherian and Artinian objects were defined in [4, 3.1]. We briefly recall the essentials. An object Q of $\mathcal{K}$ is *Noetherian* if every structly ascending chain of subobjects of Q is finite. Here a subobject is an equivalence class [R,m] of pairs (R,m) with $m : R \longrightarrow Q$ in $\mathcal{M}$. The length of the longest strictly ascending chain of subobjects of Q is called the *$\mathcal{M}$-height* of Q. Thus finite $\mathcal{M}$-height implies Noetherian, but the converse is false. Dually Q is *Artinian* if every strictly ascending chain of quotient objects of Q is finite. The length of the longest strictly ascending chain of quotient objects of Q is called the *$\mathcal{E}$-height* of Q.

In the category of sets, subobjects and quotient objects take on their usual meanings. We note that the partial orderings have been chosen so that ascending chains -- of either subsets or quotient sets -- have increasing cardinality.

In the category of modules over a ring, the passage from a submodule S to its cokernel Q/S establishes an anti-isomorphism of partially ordered sets between subobjects and quotient objects. For this reason, Artinian is equivalent to the descending chain condition on subobjects (the usual definition in module theory) and a module has finite height if and only if it is simultaneously Noetherian and Artinian.

In abelian groups, Noetherian = finitely-generated whereas finite $\mathcal{M}$-height = finite. The additive integers is not Artinian. For vector spaces, on the other hand, Noetherian = Artinian = finite height and $\mathcal{M}$-height = $\mathcal{E}$-height = 1 + dimension.

1 A class $\mathcal{F}$ of objects is *closed under subobjects* if whenever $m : Q \longrightarrow F \in \mathcal{M}$ with $F \in \mathcal{F}$, also $Q \in \mathcal{F}$. Similarly, $\mathcal{F}$ is *closed under quotients* if $e : F \longrightarrow Q \in \mathcal{E}$ and $F \in \mathcal{F}$ imply $Q \in \mathcal{F}$.

A class of 'finite' objects might well be expected to be closed under subobjects and quotients. It is obvious that 'Noetherian' is closed under subobjects and that 'Artinian' is closed under quotients. We conclude this section with a proposition that concludes that 'Noetherian' is closed under quotients, leaving the statement of the dual theorem as an exercise for the reader.

**2** LEMMA [8, 3.4.12]: *Given a pullback square*

$$\begin{tikzcd} P \arrow[r, "e'"] \arrow[d, "m'"'] & S \arrow[d, "m"] \\ Q \arrow[r, "e"] & R \end{tikzcd}$$

*with* $m \in \mathcal{M}$, *also* $m' \in \mathcal{M}$. *It is easily verified that* $[P,m']$ *is uniquely determined by* $[S,m]$; $[P,m']$ *is the* *inverse image* *of* $[S,m]$ *under* $e$.

*We say that* $\mathcal{E}$ *is* *closed under inverse images* *if* $e' \in \mathcal{E}$ *whenever* $e \in \mathcal{E}$ *and* $m \in \mathcal{M}$ *in a pullback square as above.*

**3** PROPOSITION: *If* $\mathcal{E}$ *is closed under inverse images then 'Noetherian' is closed under quotients.*

Proof: Let $Q$ be Noetherian, let $e : Q \longrightarrow R \in \mathcal{E}$ and let $[R_n, m_n]$ be an ascending chain of subobjects of $R$. Let $(P_n, e'_n, m'_n)$ be pullbacks of

$$\begin{tikzcd} P_n \arrow[r, "e'_n"] \arrow[d, "\psi'_n"'] \arrow[dd, bend right=60, "m'_n"'] & R_n \arrow[d, "\psi_n"] \arrow[dd, bend left=60, "m_n"] \\ P_{n+1} \arrow[r, "e'_{n+1}"] \arrow[d, "m'_{n+1}"'] & R_{n+1} \arrow[d, "m_{n+1}"] \\ Q \arrow[r, "e"] & R \end{tikzcd}$$

$(e,m_n)$ as shown. If $m_{n+1}\psi_n = m_n$ defines $\psi_n$ then, since $m_{n+1}(\psi_n e'_n) = m_n e'_n = e\, m'_n$, there exists unique $\psi'_n$ as shown (by the definition of a pullback). Thus $[P_n, m'_n]$ is an ascending chain of subobjects of $Q$ and there exists $N$ such that $\psi'_n$ is an isomorphism if $n \geq N$. But for $n \geq N$ we have two $\mathcal{E}$-$\mathcal{M}$ factorizations of the same map:

$$\begin{tikzcd}
 & R_n \arrow[dr, "\psi_n"] \arrow[dd, dashed, "\psi_n"] & \\
P_n \arrow[ur, "e'_n"] \arrow[dr, "e'_{n+1}\psi'_n"'] & & R_{n+1} \\
 & R_{n+1} \arrow[ur, "id"'] &
\end{tikzcd}$$

This induces an isomorphism as shown which must be $\psi_n$. □

In both the category of sets and the category of modules over a ring, both 'Noetherian' and 'Artinian' are closed under subobjects and under quotients. This may be shown to be a corollary of the proposition although, of course, these results are well-known.

## 5. Noetherian Corecurrence

**1** LEMMA: *Let M realize $H_i^j$ with state object Q. Then there exists a diagram*

$$Q_H \xleftarrow{\;e\;} R \xrightarrow{\;m\;} Q$$

*with* $m \in \mathcal{M}$, $e \in \mathcal{E}$. *R is the 'reachable part' of Q.*

Proof: Recall the canonical realization $H_H$ of section 1 and consider the diagram

$$\begin{tikzcd}
\Omega \arrow[r, "e_1"] \arrow[d, "r_H"'] & R \arrow[r, "m"] \arrow[dl, dashed, "e"] & Q \arrow[d, "\sigma"] \\
Q_H \arrow[rr, "\sigma_H"] & & \Gamma
\end{tikzcd}$$

where $(e_1, m)$ is an $\mathcal{E}$-$\mathcal{M}$ factorization of r. The outside commutes by the bi-index principle [4, 1.10] inducing the diagonal fill-in e [4, 2.7] which is in $\mathcal{E}$ by [4, 2.10]. □

**2** THEOREM: *Let $\mathcal{F}$ be a class of Noetherian objects which is closed under $\mathcal{E}$, closed under $\mathcal{M}$, closed under the formation of finite coproduces and which contains $IX^i$ for every $i \geq 0$. Assume that $IX^i$ is projective for all i. Then the following three conditions on the Hankel matrix $H_i^j$ are equivalent:*

*(i)* $H_i^j$ *is corecurrent.*

*(ii)* $H_i^j$ *has a realization with state object in* $\mathcal{F}$.

*(iii) The canonical state object* $Q_H$ *is in* $\mathcal{F}$.

Proof: For (i) $\Longrightarrow$ (ii) use 2.3. For (ii) $\Longrightarrow$ (iii) use lemma 1. That (iii) $\Longrightarrow$ (ii) is obvious. To see that (iii) $\Longrightarrow$ (i) combine 3.6 with the fact [4, Proposition 4.1] that if M is reachable and Q is Noetherian then M is reachable in bounded time. □

3 COROLLARY: *Let* R *be an arbitrary ring and let* $H_i^j : I \longrightarrow Y$ *be* R-*linear where* I *is a projective Noetherian* R-*module. Then* $H_i^j$ *has a realization with a Noetherian state module if and only if* $H_i^j$ *is corecurrent* (see example 2.4). □

4 COROLLARY: *Let* R *be an arbitrary ring and let* $H_i^j : I \longrightarrow Y$ *be* R-*linear where* I *is a projective* R-*module. Then* $H_i^j$ *has a realization whose state module has finite homological dimension if and only if* $H_i^j$ *is corecurrent.*

Proof: This is theorem 2 with $\mathscr{F}$ the class of modules which have finite homological dimension. See [7, VII.1] noting exercise 3 on page 204. □

Corollary 3 may also be stated when $\mathscr{F}$ = modules with finite $\mathcal{M}$-height and when $\mathscr{F}$ = finite modules (although the latter is also clear from 3.4). The duals of these theorems are also of some interest. We state the dual of corollary 3:

5 COROLLARY: *Let* R *be an arbitrary ring and let* $H_i^j : I \longrightarrow Y$ *be* R-*linear where* Y *is an injective Artinian* R-*module. Then* $H_i^j$ *has a realization with an Artinian state-module if and only if* $H_i^j$ *is recurrent.* □

# REFERENCES


1. M. A. Arbib and E. G. Manes: Foundations of system theory: decomposable systems, *Automatica* 10 (1974) 285-302.

2. M. A. Arbib and E. G. Manes: Adjoint machines, state-behavior machines and duality, *J. Pure Appl. Alg.* 6 (1975) 313-344.

3. M. A. Arbib and E. G. Manes: *Arrows, Structures and Functors: The Categorical Imperative*, New York: Academic Press (1975).

4. M. A. Arbib and E. G. Manes: Generalized Hankel matrices and system realization, in press.

5. R. W. Brockett and A. S. Willsky: Finite-state homomorphic sequential machines, *IEEE Trans. Aut. Control* AC-17 (1973) 483-490.

6. S. Eilenberg: *Automata, Languages, and Machines*, Volume A, New York: Academic Press (1974).

7. S. Mac Lane: *Homology*, Berlin-Heidelberg: Springer-Verlag (1973).

8. E. G. Manes: *Algebraic Theories*, Berlin-Heidelberg: Springer-Verlag (1976).

9. Y. Rouchaleau, B. F. Wyman and R. E. Kalman: Algebraic structures of linear mathematical systems III, Realization theory over a commutative ring, *Proc. Nat. Acad. Sci. USA* 69 (1972) 3404-3406.

10. E. D. Sontag: On linear systems and non-commutative rings, *Mathematical System Theory* 9 (1976) 327-344.

# ON THE EVOLUTION OF GENERALIZED HANKEL MATRICES[1]

Michael A. Arbib
Computer & Information Science

Ernest G. Manes
Mathmatics
University of Massachusetts
Amherst, Massachusetts 01003, USA

June 1977

ABSTRACT

For the Hankel matrix of an adjoint system, the image of each submatrix is a subquotient of the minimal state space. While the simple recursion principle fails, the evolution of the Hankel matrix, as input length and observation time increase, follows a supremum-preserving principle. The theory applies to linear and bilinear systems, automata, and group systems in both the time-varying and time-invariant cases, and is new even for linear systems.

1 The research reported in this paper was supported in part by the National Science Foundation under grant MCS76-84477.

## 1. Motivation

If $B : I \longrightarrow Q$, $A : Q \longrightarrow Q$, $C : Q \longrightarrow Y$ is a reachable and observable linear system, let $Q^J_S$, for $J$, $S$ non-empty subsets of the set $\mathcal{N}$ of natural numbers, be the image of the partial Hankel matrix $(CA^{i+j}B : i \in S,\ j \in J)$ considered as a linear map from $I^S$ to $Y^J$. Thus $Q^{\mathcal{N}}_S$ is the set of states reachable by inputs applied at times in $S$, and then $Q^J_S$ is obtained by merging those states that are indistinguishable by observations made at times in $J$. Hence, given $T$, $K$ finite subsets of $\mathcal{N}$ with $S \subset T$ and $J \subset K$ there is an obvious commutative diagram

$$
\begin{tikzcd}
Q^K_T \arrow[rr, tail] \arrow[dd, two heads] & & Q^K_{\mathcal{N}} \arrow[dd, two heads] & \\
& & & Q^{\mathcal{N}}_{\mathcal{N}} \cong Q \arrow[ul, two heads] \arrow[dl, two heads] \\
Q^J_T \arrow[rr, tail] & & Q^J_{\mathcal{N}} & \\
& Q^J_S \arrow[ul, tail] \arrow[ur, tail] & &
\end{tikzcd}
$$

where $\twoheadrightarrow$ denotes a projection whereas $\rightarrowtail$ denotes an injection which fills in unused slots with zeroes. Thus each $Q^J_S$ is a 'subquotient' of $Q$, that is, is a subobject of a quotient object of $Q$. The diagram above is used to motivate the definition of a partial order on the set of subquotients of an object in a category. The main abstract result is that, replacing the $CA^{i+j}B$ by a much more general bisequence of morphisms, the passage $(J,S) \mapsto Q^J_S$ preserves countable suprema. In particular, the theorem applies to the Hankel matrix of any adjoint system [2, 3] with projective input object.

Recall that a sequence $v_n$ in $V$ is *defined by simple recursion* if $v_n = g^n(v_0)$ for some $g : V \longrightarrow V$. The ascending chain $\mathrm{span}(B, AB, \ldots, A^nB)$ of reachability subspaces of a linear system is defined by simple recursion:

let V be the set of subspaces of Q and set $g(S) = \mathrm{span}(S \cup A(S))$. More generally, the ascending chain of reachability subobjects of the state object (and, dually, the ascending chain of observability quotient objects of the state object) of any adjoint system is defined by simple recursion [2, 4.4]. Examples abound to show that for the bisequence $Q_S^J$ of a linear system, none of the sequences

$$Q_S^{\bar{n}} \qquad \text{(fixed S)}$$

$$Q_{\bar{n}}^J \qquad \text{(fixed J)}$$

$$Q_{\bar{n}}^{\bar{n}}$$

(where $\bar{n} = \{1,\ldots,n\}$) need be defined by simple recursion. In particular, $Q_{\bar{n}}^{\bar{n}} = Q_{\overline{n+1}}^{\overline{n+1}}$ does not imply $Q_{\overline{n+1}}^{\overline{n+1}} = Q_{\overline{n+2}}^{\overline{n+2}}$ (as is well known in the study of partial realizations). This paper, then, was motivated by the desire to find a more positive result for $Q_S^J$.

## 2. Generalities

Let $\mathcal{K}$ be a category and let $(\mathcal{E},\mathcal{M})$ be an image factorization system in $\mathcal{K}$, that is, $\mathcal{E}, \mathcal{M}$ are subclasses of morphisms satisfying the following four axioms:

IFS1. $\mathcal{E}$ and $\mathcal{M}$ are each closed under composition.

IFS2. Every isomorphism is both in $\mathcal{E}$ and in $\mathcal{M}$.

IFS3. Every element of $\mathcal{E}$ is an epimorphism and every element of $\mathcal{M}$ is a monomorphism. (A map $f : R \longrightarrow S$ is an *epimorphism* if whenever $g, h : S \longrightarrow T$ satisfy $gf = hf$, then $g = h$; dually, $f$ is a *monomorphism* if whenever $a, b : Q \longrightarrow R$ satisfy $fa = fb$ then $a = b$.)

IFS4. Every morphism $f : Q \longrightarrow R$ admits an $\mathcal{E}$-$\mathcal{M}$ factorization $(e,m)$ -- that is, $f = m\, e$ with $e \in \mathcal{E}$, $m \in \mathcal{M}$ -- and such factorizations are unique up to isomorphism in the sense that if $(e',m')$ is another one then there exists a unique isomorphism $\psi$ with $\psi e = e'$ and $m'\psi = m$.

The category of sets and the category of modules over a ring both have $\mathcal{E}$ = surjections and $\mathcal{M}$ = injections as unique image factorization system. For the theory of image factorization systems see [5, 3.4]. We record two general lemmas here. The first is standard. The second, pointed out to us by J. R. Isbell, improves [5, 3.4.14].

1 DIAGONAL FILL-IN LEMMA: *The following two statements about a morphism* $e : Q \longrightarrow R$ *are equivalent.*

*(i)* $e \in \mathcal{E}$.

*(ii) For every commutative square* $ge = mf$ *with* $m \in \mathcal{M}$,

$$\begin{tikzcd} Q \arrow[r, "e"] \arrow[d, "f"'] & R \arrow[d, "g"] \arrow[dl, dashed, "\psi"'] \\ S \arrow[r, "m"'] & T \end{tikzcd}$$

*there exists unique* $\psi$ *with* $\psi e = f$ *and* $m\psi = g$. □

Recall that, given $f, g : U \longrightarrow V$, their *coequalizer* is a morphism $h : V \longrightarrow W$ such that $hf = hg$, and such that whenever $h'f = h'g$ there exists unique $\psi$ with $\psi h = h'$.

$$\begin{tikzcd} U \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & V \arrow[r, "h"] \arrow[dr, "h'"'] & W \arrow[d, dashed, "\psi"] \\ & & W' \end{tikzcd}$$

The category $\mathcal{K}$ *has coequalizers* if every pair f, g has a coequalizer. See [1, pp. 20, 34] and [4, pp. 64-65]. The category of modules over a ring has coequalizers -- let h be the canonical projection to the cokernel of $f - g$. The category of sets has coequalizers -- let h be the canonical projection to V/R where R is the smallest equivalence relation containing $\{(f(x), g(x)) \mid x \in U\}$.

2 LEMMA: *If* h *is the coequalizer of some* f, g *then* $h \in \mathcal{E}$.

Proof: We use lemma 1. Consider the diagram shown where $ma = bh$ and

$$\begin{tikzcd} U \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & V \arrow[r, "h"] \arrow[d, "a"'] & W \arrow[d, "b"] \arrow[dl, dashed, "\psi"'] \\ & A \arrow[r, "m"'] & B \end{tikzcd}$$

$m \in \mathcal{M}$. Since m is a monomorphism $af = ag$ and the desired $\psi$ is induced by the coequalizer property. □

Let Q be an object of $\mathcal{K}$. The set of all pairs (R,m) with $m : R \longrightarrow Q$ in $\mathcal{M}$ admits a reflexive and transitive order by defining $(R,m) \le (R',m')$ if there exists $\psi$ with $m'\psi = m$

$$\begin{tikzcd} R \arrow[dr, "m"] \arrow[dd, dashed, "\psi"'] & \\ & Q \\ R' \arrow[ur, "m'"'] & \end{tikzcd}$$

(note that such $\psi$ is necessarily unique and is itself in $\mathcal{M}$). Thus $(R,m) \sim (R',m')$ if $(R,m) \le (R',m')$ and $(R',m') \le (R,m)$ is an equivalence relation whose equivalence classes [R,m] are the *subobjects* of Q. $[R,m] \le [R',m']$ if $(R,m) \le (R',m')$ is a well-defined partial order on the subobjects of Q. It is easily seen that $[R,m] = [R',m']$ if and only if there exists an isomorphism $\psi$ with $m'\psi = m$. Q is *Noetherian* if every strictly ascending chain of subobjects of Q is finite.

The dual concepts relative to $\mathcal{K}$ are formulated by repeating the above definitions in the dual category $\mathcal{K}^{op}$. Thus, the ordering on *quotient objects* of Q is described by

$$\begin{tikzcd} R & \\ & Q \arrow[ul, "e"'] \arrow[dl, "e'"] \\ R' \arrow[uu, dashed, "\psi"] & \end{tikzcd} \qquad \begin{array}{l} [R,e] \le [R',e'] \\ e,\ e' \in \mathcal{E} \end{array}$$

(Note that we reverse the arrows, but not the ordering.) We say Q is *Artinian* if Q is co-Noetherian, that is, if every strictly ascending chain of quotient objects of Q is finite. Facts about Noetherian and Artinian objects appear in [2, 3].

We assume that every countable family $Q_i$ of objects of $\mathcal{K}$ has a product $pr_k : \prod Q_i \longrightarrow Q_k$ and a coproduct $in_k : Q_k \longrightarrow \coprod Q_i$ [1, 1.2], [4, III.3, III.4]. Hence there are bijective correspondences

$$\begin{tikzcd} \prod Q_i \arrow[rr, "pr_k"] & & Q_k \\ & Q \arrow[ul, "f"] \arrow[ur, "f_k"'] & \end{tikzcd} \qquad \begin{tikzcd} Q_k \arrow[rr, "in_k"] \arrow[dr, "f_k"'] & & \coprod Q_i \arrow[dl, "f"] \\ & Q & \end{tikzcd}$$

between morphisms f and families $f_i$ as shown in the diagram. An immediate consequence is the

<u>3</u> <u>BI-INDEX PRINCIPLE</u>: *If* $(Q_i : i \in I)$, $(R_j : j \in J)$ *and* $f_i^j : Q_i \longrightarrow R_j$ *then, so long as the coproduct and product exist, there exists a unique morphism* $f : \coprod Q_i \longrightarrow \prod R_j$ *such that* $pr_j\ f\ in_i = f_i^j$ *for all* $i, j$.

<u>Proof</u>: Define $f^j : \coprod Q_i \longrightarrow R_j$ by $f^j\ in_i = f_i^j$ and then define f by $pr_j\ f\ f^j$. Uniqueness is left as an exercise. □

An object Q is *projective* if each diagram of the form

$$\begin{tikzcd} R \arrow[rr, two heads, "e"] & & S \\ & Q \arrow[ul, dashed] \arrow[ur, "f"'] & \end{tikzcd}$$

may be completed (not necessarily uniquely) as shown. Clearly, a coproduct of projective objects is projective. In the category of sets all objects are projective. In the category of modules over a ring, every free module is projective.

## 3. Subquotients

Fix an object Q. Consider (R,m,S,e) where

$$R \overset{m}{\rightarrowtail} S \overset{e}{\twoheadleftarrow} Q$$

Motivated by the diagram of section 1, say that $(R,m,S,e) \le (R',m',S',e')$ if there exists a commutative diagram

$$\begin{tikzcd}
R' \arrow[rr, tail, "m'"] \arrow[dd] & & S' \arrow[dd] & \\
 & & & Q \arrow[ul, two heads, "e'"'] \arrow[dl, two heads, "e"] \\
I \arrow[rr, tail] & & S & \\
 & R \arrow[ul, tail] \arrow[ur, tail, "m"'] & &
\end{tikzcd}$$

An equivalent way to say this is that $[S,e] \le [S',e']$ as quotient objects of Q and that $[R,m]$ is contained in the image of $R' \longrightarrow S' \longrightarrow S$. It is clear that $\le$ is reflexive. We also have

1 LEMMA: *$\le$ is transitive.*

Proof: Suppose that $(R,m,S,e) \le (R',m',S',e') \le (R'',m'',S'',e'')$. This is expressed in the diagram

$$\begin{tikzcd}
R'' \arrow[rr, tail, "m''"] \arrow[d] & & S'' \arrow[dd, "\alpha''"] & \\
I_1 \arrow[rrd, tail, "\tilde{m}"] \arrow[ddr, two heads] & & & \\
R' \arrow[u, tail] \arrow[rr, tail, "m'"] \arrow[d] & & S' \arrow[dd, "\alpha'"] & Q \arrow[uul, two heads, "e''"'] \arrow[l, two heads, "e'"'] \arrow[ddl, two heads, "e"] \\
I_2 \arrow[r, dashed, tail] \arrow[rrd, tail] & I_3 \arrow[rd, tail] & & \\
R \arrow[u, tail] \arrow[rr, tail, "m"] & & S &
\end{tikzcd}$$

where $I_1 \twoheadrightarrow I_3 \rightarrowtail S$ is obtained by $\mathcal{E}$-$\mathcal{M}$ factorization of $\alpha'\tilde{m}$, and where $I_2 \rightarrowtail I_3$ is then obtained by diagonal fill-in, Lemma 2.1. But this shows that $(R,m,S,e) \leq (R'',m'',S'',e'')$. □

$\underline{\underline{2}}$ DEFINITION: A *subquotient object of* $Q$ is an antisymmetry class $[R,m,S,e]$ arising from the reflexive and transitive order above. The subquotients of $Q$ are partially ordered by $[R,m,S,e] \leq [R',m',S',e']$ if $(R,m,S,e) \leq (R',m',S',e')$. The following result shows that the ordering relation is not "too abstract".

$\underline{\underline{3}}$ OBSERVATION: $[R,m,S,e] = [R',m',S',e']$ *if and only if there exist isomorphisms* $\alpha$ *and* $\beta$ *as shown.*

$$\begin{tikzcd}
R \arrow[r, tail, "m"] \arrow[dd, "\beta"'] & S \arrow[dd, "\alpha"'] & \\
& & Q \arrow[ul, two heads, "e"'] \arrow[dl, two heads, "e'"] \\
R' \arrow[r, tail, "m'"] & S' &
\end{tikzcd}$$

Proof: If $[S,e] = [S',e']$ as quotient objects of $Q$ then $\alpha$ is an isomorphism. But then $[R,m] = [R', \alpha^{-1}m']$ as subobjects of $S$ so that $\beta$ is an isomorphism. □

$\underline{\underline{4}}$ PROPOSITION: *Consider the two conditions*

*(i) Every strictly ascending chain of subquotient objects of* $Q$ *is finite.*

*(ii)* $Q$ *is Noetherian and Artinian.*

*Then (i)* $\Longrightarrow$ *(ii). If* $\mathcal{E}$ *is closed under inverse images [3, 4.2] then (ii)* $\Longrightarrow$ *(i).*

Proof: If $[R_i,m_i]$ is a strictly ascending chain of subobjects then $[R_i,m_i,Q,id]$ is a strictly ascending chain of subquotient objects. If $[S_j,e_j]$ is a strictly ascending chain of quotient objects then $[S_j,id,S_j,e_j]$ is a strictly ascending chain of subquotient objects. Conversely, if $[R_n,m_n,S_n,e_n]$ is a strictly ascending chain of subquotients then, as $Q$ is Artinian, we eventually have $[S_n,e_n] = [S_{n+1},e_{n+1}] = \ldots$ and $[R_{n+k},m_{n+k}]$ is an ascending chain of subobjects of $S_n$. Now use [3, 4.3] which asserts that 'Noetherian' is closed under quotients providing $\mathcal{E}$ is closed under inverse images. □

5 COROLLARY: *In the category of modules over a ring, a module is Noetherian and Artinian if and only if its every strictly ascending chain of its subquotients is finite.* □

## 4. Biadequacy

We consider an arbitrary bisequence $f_i^j : P_i \longrightarrow R_j$, $j$ being just a superscript.

1 For each pair $(S,J)$ of non-empty subsets of $\mathcal{N}$ define

$P_S = \coprod(P_i : i \in S)$

$R^J = \prod(R_j : j \in J)$

$f_S^J : P_S \longrightarrow R^J$ is defined, using the bi-index principle 2.3, by $pr_j\, f_S^J\, in_i = f_i^j$ $(i \in S,\ j \in J)$.

2 Pairs of non-empty subsets of $\mathcal{N}$ constitute a partially ordered set with ordering $(S,J) \le (T,K)$ if $S \subset T$ and $J \subset K$. If $(S,J) \le (T,K)$ we have a commutative diagram

$$\begin{tikzcd} P_T \arrow[r, "f_T^K"] \arrow[dr, "f_T^J"] & R^K \arrow[d, "pr^{KJ}"] \\ P_S \arrow[u, "in_{ST}"] \arrow[r, "f_S^J"'] & R^J \end{tikzcd}$$

where $in_{ST}\, in_i = in_i$ $(i \in S)$ and $pr_j\, pr^{KJ} = pr_j$ $(j \in J)$. The diagram commutes by the bi-index principle.

3 Fix an $\mathcal{E}$-$\mathcal{M}$ factorization of $f_S^J$

$$P_S \xrightarrow{e_S^J} Q_S^J \xrightarrow{m_S^J} R^J .$$

4 Diagonal fill-in on the diagram of (2) yields

$$\begin{tikzcd}
P_T \arrow[r, two heads, "e_T^K"] \arrow[dr, "e_T^J"] & Q_T^K \arrow[r, tail, "m_T^K"] \arrow[d, dashed, "e_T^{KJ}"] & R^K \arrow[dd, "pr^{KJ}"] \\
& Q_T^J \arrow[dr, tail, "m_T^J"] & \\
P_S \arrow[uu, "in_{ST}"] \arrow[r, two heads, "e_S^J"'] & Q_S^J \arrow[u, dashed, tail, "m_{ST}^J"] \arrow[r, tail, "m_S^J"'] & R^J
\end{tikzcd}$$

$\underline{\underline{5}}$ Whenever $(S,J) \le (T,K)$ we have

$$\begin{tikzcd}
Q_T^K \arrow[r, tail, "m_{T\mathcal{N}}^K"] \arrow[dd, "e_T^{KJ}"'] & Q_{\mathcal{N}}^K \arrow[dd, "e_{\mathcal{N}}^{KJ}"] & \\
& & Q = Q_{\mathcal{N}}^{\mathcal{N}} \arrow[ul, two heads, "e_{\mathcal{N}}^{\mathcal{N}K}"'] \arrow[dl, two heads, "e_{\mathcal{N}}^{\mathcal{N}J}"] \\
Q_T^J \arrow[r, tail, "m_{T\mathcal{N}}^J"] & Q_{\mathcal{N}}^J & \\
& Q_S^J \arrow[ul, tail, "m_{ST}^J"] \arrow[u, tail, "m_{S\mathcal{N}}^J"'] &
\end{tikzcd}$$

This diagram recaptures that of section 1. Here $Q$ abbreviates $Q_{\mathcal{N}}^{\mathcal{N}}$. Thus $Q_S$ is a subquotient object of $Q$ (we use $Q_S^J$ *qua* subquotient object for the more cumbersome $[Q_S^J, m_{S\mathcal{N}}^J, Q_{\mathcal{N}}^J, e_{\mathcal{N}}^{\mathcal{N}J}]$) and the passage from $(S,J)$ to $Q_S^J$ is order-preserving.

$\underline{\underline{6}}$ <u>DEFINITION</u>: Let $A$, $B$ be non-empty subsets of $\mathcal{N}$. $(A,B)$ is *biadequate* if $(A,B) \le (S,J)$ implies $Q_A^B = Q_S^J$. By 3.3, $Q_A^B = Q_S^J$ if and only if $e_{\mathcal{N}}^{KJ}$, $e_T^{KJ}$ and $m_{ST}^J$ are isomorphisms.

The following result is immediate from [2, 3.12] applied to the passage $S \mapsto Q_S^S$.

$\underline{\underline{7}}$ <u>THEOREM</u>: *Let* $\bar{i} = \{0,\ldots,i\}$. *Define* $A = \{i \in \mathcal{N} \mid i = 0 \text{ or } Q_{\overline{i-1}}^{\overline{i-1}} < Q_{\bar{i}}^{\bar{i}}\}$. *Then* $(A,A)$ *is biadequate.* $\square$

We now turn to the main result of this paper.

8 **THEOREM:** *In the context of (1) - (5), assume that $P_i$ is projective for all $i \in \mathcal{N}$ and assume that $\mathcal{K}$ has coequalizers. Then the passage from $(S,J)$ to the subquotient object $Q_S^J$ of $Q$ preserves suprema of non-empty countable families.*

Proof: Let $S_\alpha$, $J_\beta$ be non-empty countable families of non-empty subsets of $\mathcal{N}$ and set $S = \cup S_\alpha$, $J = \cup J_\beta$. We must show that $Q_S^J$ is the supremum, in the partially ordered set of subquotient objects of $Q$, of the $Q_\alpha^\beta$ (we will use the subscript $\alpha$ for $S_\alpha$ and the superscript $\beta$ for $J_\beta$). That $Q_S^J$ is an upper bound follows from (5). Now let $(F,m,G,e)$ be another upper bound, and consider the diagram shown. To explain the diagram, we first apply the dual

$$\begin{tikzcd}
F \arrow[rr, tail, "m"] \arrow[dr] & & G \arrow[dd] & \\
& I \arrow[dr, tail] \arrow[dd] & & \\
Q_S^J \arrow[ur, dashed] \arrow[rr, tail] & & Q_{\mathcal{N}}^J \arrow[dd] & Q \arrow[l, two heads] \arrow[uul, "e"'] \arrow[ddl] \\
& I_\beta \arrow[dr, tail] & & \\
Q_\alpha^\beta \arrow[ur] \arrow[rr, tail] & & Q_{\mathcal{N}}^\beta &
\end{tikzcd}$$

of [2, lemma 3.9] to the singly-indexed sequence $f_{\mathcal{N}}^j : Q \longrightarrow R^j$ to conclude that $Q_{\mathcal{N}}^J$ is the supremum of the $Q_{\mathcal{N}}^\beta$ as quotient objects of $Q$. This is why $G$ factors through $Q_{\mathcal{N}}^J$. Let $I$ be the image of $F \longrightarrow Q_{\mathcal{N}}^J$ and then let $I_\beta$ be the image of the various $I \longrightarrow Q^\beta$. Then $I_\beta$ is the image of $F \longrightarrow Q_{\mathcal{N}}^\beta$ so that, by hypothesis, $Q_\alpha^\beta$ factors through $I_\beta$ for all $\alpha$ and $\beta$. We must show that $Q_S^J$ factors through $I$. The argument is outlined in the second diagram. The desired factorization $Q_S^J \longrightarrow I$ is constructed by diagonal fill-in.

$$\begin{tikzcd}
\coprod P_\alpha \arrow[rrr, two heads, "\omega"] \arrow[drr, dashed, "\psi"] & & & Q^J_S \arrow[dd, tail] \arrow[ddl, dashed] \\
& & F \arrow[d] & \\
& & I \arrow[r, tail] \arrow[d] & Q^J_{\mathcal{N}} \arrow[d, "e^{J\beta}_{\mathcal{N}}"] \\
P_\alpha \arrow[uuu, "in_\alpha"] \arrow[r, two heads] \arrow[uurr, dashed, "\psi_\alpha"] & Q^\beta_\alpha \arrow[r, tail] & I_\beta \arrow[r, tail] & Q^\beta_{\mathcal{N}}
\end{tikzcd}$$

This breaks down into three steps: (i) defining $\omega \in \mathcal{E}$; (ii) defining $\psi$; (iii) proving that the two maps $t = \coprod P_\alpha \longrightarrow F \longrightarrow Q^J_{\mathcal{N}}$ and $u = \coprod P_\alpha \longrightarrow Q^J_S \longrightarrow Q^J_{\mathcal{N}}$ are equal.

Step (i). Define $\theta : \coprod P_\alpha \longrightarrow P_S$ by $\theta\, in_\alpha = in_{\alpha S}$. Then $\theta \in \mathcal{E}$. To prove it use 2.1 and the fact that S is the union of the $S_\alpha$; proof details (with the same notations) appear in [2, proof of lemma 3.9]. Then $\omega = e^J_S\, \theta : \coprod P_\alpha \twoheadrightarrow P_S \twoheadrightarrow Q^J_S \in \mathcal{E}$.

Step (ii). Any coproduct of projectives is again projective, so each $P_\alpha$ is projective. Thus there exist $\psi_\alpha$ and hence $\psi$ as shown.

Step (iii). It is routinely checked that $e^{J\beta}_{\mathcal{N}}\, t\, in_\alpha = m^\beta_{\alpha\mathcal{N}}\, e^\beta_\alpha = e^{J\beta}_{\mathcal{N}}\, t\, in_\alpha$ for all $\alpha$ and $\beta$. By the coproduct property it follows that $e^{J\beta}_{\mathcal{N}}\, t = e^{J\beta}_{\mathcal{N}}\, u$ for all $\beta$. From this we conclude that $t = u$ as follows. Let $h : Q^J_{\mathcal{N}} \longrightarrow W$ be the coequalizer of t and u. Then $h \in \mathcal{E}$ by 2.2. Thus, as a quotient object of Q, $Q^J_{\mathcal{N}} \geq W$. On the other hand, $e^{J\beta}_{\mathcal{N}}$ factors through h because $e^{J\beta}_{\mathcal{N}}\, t = e^{J\beta}_{\mathcal{N}}\, u$ and h is the coequalizer of t and u, so that $W \geq Q^\beta_{\mathcal{N}}$ for each $\beta$. But as $Q^J_{\mathcal{N}}$ is the supremum of the $Q^\beta_{\mathcal{N}}$, $Q^J_{\mathcal{N}} = W$ and h is an isomorphism. Since $th = uh$ we have $t = thh^{-1} = uhh^{-1} = u$. □

As a corollary, we obtain:

9 THEOREM: *If* (A,B) *is 'one-step biadequate' in the sense that* $Q^{B}_{A\cup\{i\}} = Q^{B}_{A} = Q^{B\cup\{j\}}_{A}$ *for all* i, j $\in \mathcal{N}$ *then* (A,B) *is biadequate.*

Proof: If $(A,B) \leq (S,J)$ then $(S,J) = \sup\{A \cup \{i\},\ B \cup \{j\} \mid i \in S,\ j \in J\}$. □

## REFERENCES


1. M. A. Arbib and E. G. Manes: *Arrows, Structures and Functors: The Categorical Imperative,* New York: Academic Press (1975).

2. M. A. Arbib and E. G. Manes: Generalized Hankel matrices and system realization, In Press.

3. M. A. Arbib and E. G. Manes: Recurrence conditions for generalized Hankel matrices, In Press.

4. S. Mac Lane: *Categories for the Working Mathematician,* Berlin-Heidelberg: Springer-Verlag (1971).

5. E. G. Manes: *Algebraic Theories,* Berlin-Heidelberg: Springer-Verlag (1976).