

GENERALIZED HANKEL MATRICES

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GENERALIZED HANKEL MATRICES AND SYSTEM REALIZATION¹

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ABSTRACT

We define the Hankel matrix of an adjoint system. Adjoint systems include linear and bilinear systems, automata, and group systems in both the time-varying and time-invariant cases. Our definition of the Hankel matrix unifies the familiar $H_i^j = CA^{i+j}B$ of linear system theory (e.g. [17]) with the bilinear Hankel matrix of [16], [21] and the Hankel matrix of [12]. The time-varying case is subsumed by regarding a time-varying system as a time-invariant system in a sequence category as in [5]. For minimal realization theory and duality theory in the framework of this paper see [1], [4]. However, we lean much less heavily on category theory than in our earlier works on realization.

We introduce 'adjoint correspondences' as the key algebraic ingredient in generalizing familiar linear system results to the nonlinear case. For example, the linear realizability criterion $H_{i+1}^j = H_i^{j+1}$ does not make sense in the nonlinear setting; the precise condition needed is that ' H_{i+1}^j and H_i^{j+1} correspond under adjointness'.

We provide a realizability theorem characterizing when a matrix H_i^j can be the Hankel matrix of a system, and offer partial realization and canonical realization theorems which associate systems with finite blocks of a Hankel matrix. We provide a general theory of 'dimension in a category', and relate it to system realization via a simple recursion principle.

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1. Adjoint Processes and Systems

In what follows, \mathcal{K} denotes an arbitrary category [3], [19]. Further axioms on \mathcal{K} will be added gradually -- a summary appears after Lemma 2.13. In this section, we define adjoint processes and systems, and present a number of examples. Recall that a *functor* $X: \mathcal{K} \rightarrow \mathcal{K}$ assigns to each object Q of \mathcal{K} another object QX of \mathcal{K} and assigns to each morphism $f: Q \rightarrow R$ of \mathcal{K} another morphism of form $fX: QX \rightarrow RX$ subject to the preservation of identities and composition, that is, $\text{id}_Q X = \text{id}_{QX}$ and, given $f: Q \rightarrow R$, $g: R \rightarrow S$, $(gf)X = gX fX$. As discussed below, basic examples include tensoring with a fixed vector space in the category of vector spaces and linear maps or assigning to a set Q the set of all functions from a fixed set to Q in the category of sets and functions. Let $\mathcal{K}(Q,R)$ denote the set of morphisms from Q to R in \mathcal{K} .

1. DEFINITION: An *adjoint process* in \mathcal{K} is a pair (X,Z) of functors $\mathcal{K} \rightarrow \mathcal{K}$ together with bijective correspondences $\mathcal{K}(RX,S) \rightarrow \mathcal{K}(R,SZ)$ (one such for each pair (R,S) of objects) subject to the axiom that, given $f: Q \rightarrow R$ and $h: S \rightarrow T$, if $g: RX \rightarrow S$ and $\psi: R \rightarrow SZ$ correspond then $h g fX: QX \rightarrow T$ and $hZ \psi f: Q \rightarrow TZ$ correspond.

In the usual language of category theory, to say (X,Z) is an adjoint process amounts to saying that Z is right adjoint to X and, equivalently, that X is left adjoint to Z .

For the duration of the paper we fix an adjoint process (X,Z) in \mathcal{K} .

A convenient notation is the display

$$\begin{array}{ccc} RX & \xrightarrow{g} & S \\ \hline R & \xrightarrow{\psi} & SZ \end{array}$$

to indicate that g and ψ correspond. We say g and ψ *correspond under adjointness*.

We may equally well use the notation

$$\frac{R \xrightarrow{\psi} SZ}{RX \xrightarrow{g} S}$$

and we will frequently use displays like

$$\frac{RX \xrightarrow{g} S}{\frac{R \xrightarrow{\psi} SZ}{RX \xrightarrow{k} S}}$$

to conclude that $g = k$.

The axiom above is then succinctly displayed as

$$\frac{2 \quad \frac{QX \xrightarrow{fX} RX \xrightarrow{g} S \xrightarrow{h} T}{Q \xrightarrow{f} R \xrightarrow{\psi} SZ \xrightarrow{hZ} TZ}}{\quad}$$

We call attention to the special cases that arise when $f = \text{id}_R$ and when $h = \text{id}_S$.

3 LEMMA: For each B , define $\epsilon_B : BZX \rightarrow B$ as the correspondent of $\text{id}_{BZ} : BZ \rightarrow BZ$. Then if $g : AX \rightarrow B$ and $\psi : A \rightarrow BZ$ correspond under adjointness, we may recover g from ψ by

$$g = \epsilon_B \circ \psi X .$$

Proof: Applying 2 we conclude that

$$\frac{\frac{AX \xrightarrow{\psi X} BZX \xrightarrow{\epsilon_B} B}{A \xrightarrow{\psi} BZ \xrightarrow{\text{id}_{BZ}} BZ}}{AX \xrightarrow{g} B}$$

□

4 ADJOINT SYSTEMS: An *adjoint system* is $M = (Q, \delta, I, \tau, Y, \beta)$ where Q, I, Y are objects (the *state object*, *input object* and *output object* of M) and $\delta: QX \rightarrow Q$, $\tau: I \rightarrow Q$ and $\beta: Q \rightarrow Y$ are morphisms (the *dynamics*, *input map* and *output map* of M). (Note: 'map' is here a synonym for 'morphism'.) The *codynamics* of M is the map $\Delta: Q \rightarrow QZ$ which corresponds to δ under adjointness.

Given two dynamics $\delta: QX \rightarrow Q$ and $\theta: RX \rightarrow R$, a *dynamorphism* $h: (Q, \delta) \rightarrow (R, \theta)$ is a map $h: Q \rightarrow R$ which 'respects the dynamics':

$$\begin{array}{ccc} QX & \xrightarrow{hX} & RX \\ \delta \downarrow & & \downarrow \theta \\ Q & \xrightarrow{h} & R \end{array}$$

The *time- i reachability map* $r_i: IX^i \rightarrow Q$ and the *time- j observability map* $\sigma_j: Q \rightarrow YZ^j$ are defined by

$$\begin{aligned} r_0 &= \tau \\ r_{i+1} &= IX^{i+1} \xrightarrow{r_i X} QX \xrightarrow{\delta} Q \\ \sigma_0 &= \beta \\ \sigma_{j+1} &= Q \xrightarrow{\Delta} QZ \xrightarrow{\sigma_j Z} YZ^{j+1} . \end{aligned}$$

The *bisquence* H_i^j , where $H_i^j: IX^i \rightarrow YZ^j$ is defined by $H_i^j = \sigma_j r_i$, is the *Hankel matrix* of M .

Adjoint systems are closely related to the machines studied in [10], [11] and [13]. Realization theory for adjoint systems was developed in [1] and [4]. The Hankel matrix for adjoint systems is new, perhaps because the previous authors were motivated more by automata theory (where the Hankel matrix is not conventionally defined) than by system theory.

We conclude this section with a number of examples of adjoint systems and their Hankel matrices.

5 EXAMPLE: THE DECOMPOSABLE CASE. Here $X = Z$ is the identity functor of \mathcal{K} . The realization theory in this special case was studied in [2]. When \mathcal{K} is the category of vector spaces (or of modules over a ring) an adjoint system is just a linear system

$$I \xrightarrow{B} Q \quad Q \xrightarrow{A} Q \quad Q \xrightarrow{C} Y .$$

The same system description holds in any category. The adjointness correspondence is just

$$\frac{Q \xrightarrow{g} R}{Q \xrightarrow{g} R}$$

so that the codynamics is again A . We have $r_i = A^i B$ and $\sigma_j = CA^j$ so that $H_i^j = CA^{i+j} B$.

6 EXAMPLE: AUTOMATA. Let \mathcal{K} be the category of sets and functions. Let A be a fixed input alphabet. Define $QX = Q \times A$, $QZ = Q^A$, the set of functions from A to Q . For $f: Q \rightarrow R$, $fX: Q \times A \rightarrow R \times A$ is defined by $(q, a) \mapsto (f(q), a)$ whereas $fZ: Q^A \rightarrow R^A$ sends $g: A \rightarrow Q$ to $fg: A \rightarrow R$. The adjointness correspondence

$$\frac{Q \times A \xrightarrow{g} R}{Q \xrightarrow{\psi} R^A}$$

is the familiar $(\psi q)(a) = g(q, a)$. Let I have one element. Then τ amounts to an element of Q , the initial state. The dynamics and output map have their usual forms $\delta: Q \times A \rightarrow Q$, $\beta: Q \rightarrow Y$. It is easily checked that $r_i: A^i \rightarrow Q$ sends an i -tuple of input letters to the state reached from the initial state if the letters are inputted in sequence, whereas $\sigma_j: Q \rightarrow Y^{(A^j)}$ sends q to that function $A^j \rightarrow Y$ composing β with the time- j reachability map if the initial state were q . Thus $H_i^j: A^i \rightarrow Y^{(A^j)}$ is essentially a way of describing $\beta \cdot r_{i+j}$ with emphasis on i as 'present time'.

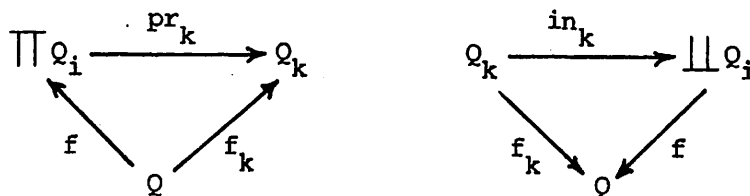
7 EXAMPLE: INTERNALLY BILINEAR MACHINES ([12], [16], [21]): Let \mathcal{K} be the category of real vector spaces and linear maps. Define $QX = Q \otimes U$, tensoring with a vector space U , while $QZ = Q^U$, the vector space of linear maps from U to Q . The adjointness correspondence

$$\frac{Q \times U \xrightarrow{g} R}{Q \xrightarrow{\psi} R^U}$$

is then the familiar $\psi q(u) = g(q \otimes u)$. Let I be a vector space. Then $\tau: I \rightarrow Q$ specifies the space $\tau(I)$ of initial states 'reachable in time 0', the dynamics is then a bilinear map $\delta: Q \otimes U \rightarrow Q$ while the output is a linear map $\beta: Q \rightarrow Y$.

It is easily checked that $r_i: I \otimes U^{\otimes i} \rightarrow Q$ extends the map $I \times U^i \rightarrow Q$ which sends an i in I and i -tuple of input vectors to the state reached from $\tau(i)$ under that input sequence; whereas $\sigma_j: Q \rightarrow Y^{U^j}$ sends q to that function $U^j \rightarrow Y$ composing β with the time- j reachability map if the initial state were q . The Hankel matrix $H_i^j: I \otimes U^{\otimes i} \rightarrow Y^{U^j}$ can be viewed in a more symmetrical way as providing for each initial state label a matrix $U^{\otimes i} \otimes U^{\otimes j} \rightarrow Y$.

The previous three examples can be subsumed in one very general example, given below as example 11. But first we need to recall, [3, section 1.2] [19, III.3, III.4], that if $(Q_i: i \in I)$ is a family of objects of \mathcal{K} then their *product* $pr_k: \prod Q_i \rightarrow Q_k$ satisfies the universal property that for all families of form $f_i: Q \rightarrow Q_i$ ($i \in I$) there exists unique $f: Q \rightarrow \prod Q_i$ with $pr_i f = f_i$ for all i . If it exists, the product is unique up to isomorphism.

8

The dual notion is the *coproduct* $\text{in}_k : Q_k \longrightarrow \coprod Q_i$. As we see in 8, in both cases there is a bijective correspondence between arbitrary families $(f_k : k \in I)$ and morphisms f . In the category of sets, coproducts are constructed as the disjoint union whereas in the category of modules over a ring (or a semiring), coproducts are constructed as weak direct sums. Both categories have products via the usual cartesian product construction.

9 PRESERVATION PRINCIPLE FOR ADJOINT PROCESSES. X preserves coproducts, that is, if $\text{in}_k : Q_k \longrightarrow \coprod Q_i$ is a coproduct, so is $\text{in}_k X : Q_k X \longrightarrow (\coprod Q_i) X$. Similarly, Z preserves products.

Proof: The result is standard in category theory [3, p. 134] [19, V.5]. To outline the proof, given a family $f_k : Q_k X \longrightarrow Q$, let $g_k : Q_k \longrightarrow QZ$ correspond to f_k under adjointness, inducing the $g : \coprod Q_i \longrightarrow QZ$ whose correspondent is the desired f . The second statement is dual. \square

10 BI-INDEX PRINCIPLE: If $(Q_i : i \in I)$, $(R_j : j \in J)$ and $f_i^j : Q_i \longrightarrow R_j$ then, so long as the coproduct and product exist, there exists a unique morphism $f : \coprod Q_i \longrightarrow \prod R_j$ such that $\text{pr}_j f \text{in}_i = f_i^j$ for all i, j .

Proof: Define $f^j : \coprod Q_i \longrightarrow R_j$ by $f^j \text{in}_i = f_i^j$ and then define f by $\text{pr}_j f = f^j$. Uniqueness is left as an exercise. \square

For the balance of the paper we assume our category \mathcal{K} to be such that every countable family of objects has a product and a coproduct.

11 EXAMPLE: The following very general example of adjoint processes and systems subsumes examples 5, 6 and 7. Let A be a fixed set (usually finite in applications). Then define

$$QX = Q \times A =_{\text{def}} \coprod_{a \in A} Q$$

the coproduct of $|A|$ copies of Q . For $f : Q \rightarrow R$, fX is defined by the coproduct property

$$\begin{array}{ccc} Q & \xrightarrow{\text{in}_a} & QX \\ f \downarrow & & \downarrow fX \\ R & \xrightarrow{\text{in}_a} & RX \end{array} \quad (a \in A)$$

If we define Z by

$$QZ = Q^A =_{\text{def}} \prod_{a \in A} Q$$

the product of $|A|$ copies of Q , with

$$\begin{array}{ccc} Q & \xleftarrow{\text{pr}_a} & QZ \\ f \downarrow & & \downarrow fZ \\ R & \xleftarrow{\text{pr}_a} & RZ \end{array}$$

it can easily be verified that (X, Z) is indeed an adjoint process, with the correspondence

$$\frac{QX \xrightarrow{g} R}{Q \xrightarrow{\psi} RZ}$$

being simply given by $g \cdot \text{in}_a = \text{pr}_a \cdot \psi : Q \rightarrow R$ for each $a \in A$.

Given a system $(\tau : I \rightarrow Q, \delta : QX \rightarrow Q, \beta : Q \rightarrow Y)$, we have that $r_i : \coprod_{v \in A}^i I \rightarrow Q$, $\sigma_j : Q \rightarrow \prod_{w \in A}^j Y$ and that, by the bi-index principle, the Hankel matrix $H_i^j : \coprod_{v \in A}^i I \rightarrow \prod_{w \in A}^j Y$ is equivalent to a $|A|^i \times |A|^j$ 'matrix' whose entries are maps $\text{pr}_w \cdot H_i^j \cdot \text{in}_v : I \rightarrow Y$.

2. Realizability and Realizations

Before stating the next theorem we define the *object of inputs* Ω and the *observability space* Γ . The notation follows Kalman's for the linear case [19, 10.3]. In [4] the notation used was $IX^{\textcircled{a}}$ for Ω and $YX_{\textcircled{a}}$ for Γ .

1 We set Ω to be the coproduct $\coprod (IX^i : i \geq 0)$ where $IX^0 = I$ and $IX^{i+1} = (IX^i)X$. Ω carries a dynamical structure $\mu_0 : \Omega X \rightarrow \Omega$ defined by

$$\begin{array}{ccc} \Omega X & \xrightarrow{\mu_0} & \Omega \\ \text{in}_i X \swarrow & & \nearrow \text{in}_{i+1} \\ & IX^i X & \end{array}$$

Here, we have used the preservation principle 1.9. (The story behind the cumbersome notation μ_0 instead of μ is found in [20, section 4.2].)

2 Γ is defined as the product $\prod (YZ^j : j \geq 0)$ with dynamical structure $L : \Gamma X \rightarrow \Gamma$ the correspondent under adjointness of the map Λ defined by

$$\begin{array}{ccc} \Gamma & \xrightarrow{\Lambda} & \Gamma Z \\ \text{pr}_{j+1} \swarrow & & \searrow \text{pr}_j Z \\ & YZ^j Z & \end{array}$$

These definitions coincide with Kalman's (save that he denotes both μ_0 and Λ by z) when \mathcal{K} is the category of modules over a ring and when (X, Z) is the identity process.

3 REALIZABILITY THEOREM: Let $H_i^j : IX^i \rightarrow YZ^j$ be an arbitrary bisequence of morphisms and let $H : \Omega \rightarrow \Gamma$ be the unique morphism with $\text{pr}_j H \text{in}_i = H_i^j$ as in 1.10. Then the following three conditions are equivalent (and we say H_i^j is a Hankel matrix if these conditions hold).

- (i) H_i^j is realizable, that is, is the Hankel matrix of some system.
- (ii) (The Hankel crossover condition): For all i, j :

$$\frac{IX^{i+1} \xrightarrow{H_{i+1}^j} YZ^j}{IX^i \xrightarrow{H_i^{j+1}} YZ^{j+1}}$$

Equivalently, by 1.3, the condition states

$$\begin{array}{ccc} IX^i X & \xrightarrow{\text{id}} & IX^{i+1} \\ H_i^{j+1} X \downarrow & & \downarrow H_{i+1}^j \\ YZ^{j+1} X & \xrightarrow{\epsilon} & YZ^j \end{array}$$

- (iii) $H : (\Omega, \mu_0) \longrightarrow (\Gamma, L)$ is a dynamorphism, that is,
 $LHX = H\mu_0 : \Omega X \longrightarrow \Gamma$.

Proof: (i) \implies (ii) is immediate from

$$\frac{IX^i X \xrightarrow{r_i X} QX \xrightarrow{\delta} Q \xrightarrow{\sigma_j} YZ^j}{IX^i \xrightarrow{r_i} Q \xrightarrow{\Delta} QZ \xrightarrow{\sigma_j Z} YZ^j Z}$$

For (ii) \implies (iii), consult the diagram

$$\begin{array}{ccccc} IX^i X & \xrightarrow{\text{in}_i X} & \Omega X & \xrightarrow{HX} & \Gamma X \\ & \searrow \text{in}_{i+1} & \downarrow \mu_0 & & \downarrow L \\ & & \Omega & \xrightarrow{H} & \Gamma \xrightarrow{\text{pr}_j} YZ^j \end{array}$$

By principles 1.9 and 1.10, it suffices to prove that the bottom and top paths from $IX^i X$ to YZ^j are equal. But the bottom path is exactly H_{i+1}^j , whereas $\text{pr}_j L$ corresponds under adjointness to $\text{pr}_{j+1} : \Gamma \longrightarrow YZ^{j+1}$ so that the top path corresponds to H_i^{j+1} and is thus also H_{i+1}^j .

To complete the proof we show (iii) \implies (i). We shall show that if H is a dynamorphism, then the 'free realization' $Q = \Omega$, $\delta = \mu_0$, $\tau = \text{in}_0$, $\beta = \text{pr}_0 H$ has Hankel matrix H . One checks easily that $r_i = \text{in}_i$. To show that $\sigma_j r_i = \text{pr}_j H \text{in}_i$ it suffices to show that $\sigma_j = \text{pr}_j H$. This is true by definition for $j = 0$. The inductive step here is given by using the adjointness axiom with $f = \text{id}_\Omega$, and where Δ is now the codynamics of μ_0 :

$$\frac{\Omega \xrightarrow{\Delta} \Omega Z \xrightarrow{(\text{pr}_j \cdot H)Z} YZ^{j+1}}{\Omega X \xrightarrow{\mu_0} \Omega \xrightarrow{(\text{pr}_j \cdot H)} YZ^j}$$

Again

$$\frac{\Omega \xrightarrow{H} \Gamma \xrightarrow{\text{pr}_{j+1}} YZ^{j+1}}{\Omega X \xrightarrow{HX} \Gamma X \xrightarrow{(\text{pr}_j \cdot L)} YZ^j}$$

But by the dynamorphism property, $H \cdot \mu_0 = L \cdot HX$, and so

$$\text{pr}_{j+1} \cdot H = (\text{pr}_j \cdot H)Z \cdot \Delta = \sigma_{j+1} \quad \square$$

The Hankel crossover condition provides evidence that adjointness arises naturally in system theory. In the decomposable case (example 1.5) we capture the familiar condition ' $H_{i+1}^j = H_i^{j+1}$ ', of linear system theory.

In the general context of example 1.11, Ω may be identified with $I \otimes A^*$, where A^* is the free monoid generated by A and Γ may be identified with Y^{A^*} . Now $(- \otimes A^*, (-)^{A^*})$ is again an adjoint process in the category of sets. $H : I \otimes A^* \longrightarrow Y^{A^*}$ corresponds under adjointness to a map $\theta : I \otimes A^* \otimes A^* \longrightarrow Y$.

In familiar system examples one can discuss the 'subspace of Q reached by time i '. Such a subspace may be constructed by 'taking the image' of the map $f : \coprod (IX^k : 0 \leq k \leq i) \longrightarrow Q$ defined by $f \text{in}_k = r_k$. To formalize 'taking the image' we structure \mathcal{K} with an image factorization system.

4 An *image factorization system* for a category \mathcal{K} is a pair $(\mathcal{E}, \mathcal{M})$ where \mathcal{E}, \mathcal{M} are subclasses of morphisms satisfying the following four axioms:

IFS1. \mathcal{E} and \mathcal{M} are each closed under composition.

IFS2. Every isomorphism is both in \mathcal{E} and in \mathcal{M} .

IFS3. Every element of \mathcal{E} is an epimorphism and every element of \mathcal{M} is a monomorphism. (A map $f : R \rightarrow S$ is an *epimorphism* if whenever $g, h : S \rightarrow T$ satisfy $gf = hf$, then $g = h$; dually, f is a *monomorphism* if whenever $a, b : Q \rightarrow R$ satisfy $fa = fb$ then $a = b$.)

IFS4. Every morphism $f : Q \rightarrow R$ admits an \mathcal{E}, \mathcal{M} factorization (e, m) -- that is, $f = m e$ with $e \in \mathcal{E}$, $m \in \mathcal{M}$ -- and such factorizations are unique up to isomorphism in the sense that if (e', m') is another one than there exists a unique isomorphism ψ with $\psi e = e'$ and $m' \psi = m$.

The category of sets and the category of modules over a ring both have \mathcal{E} = surjections and \mathcal{M} = injections as unique image factorization system. The same construction works in the category of semigroups but in that category \mathcal{E} = epimorphisms determines (see 8 below) another system; the inclusion of the natural numbers into the integers is a non-surjective epimorphism in that category. Image factorization systems in the category of linearly topologized vector spaces were investigated in a system-theoretic context in [14].

The notion of an image factorization system can be traced to [18]. The version presented here is due to [15]. References in the system literature to the 'Zeiger fill-in lemma' (see 7 below) are historically inaccurate.

We conclude this section by collecting a number of standard results. Proofs of 6-11 appear in [20, section 3.4] although all are easy exercises.

For the balance of the paper, $(\mathcal{E}, \mathcal{M})$ is a fixed image factorization system in \mathcal{K} .

5 DEFINITION: Let M be an adjoint system. The *reachability map*

$r: \Omega \longrightarrow Q$ of M is defined by

$$\begin{array}{ccc} \Omega = \coprod_{i \geq 0} IX^i & \xrightarrow{\quad r \quad} & Q \\ \text{in}_i \uparrow & \nearrow r_i & \\ IX^i & & \end{array}$$

Dually, the *observability map* $\sigma: Q \longrightarrow \Gamma$ of M is defined by

$$\begin{array}{ccc} Q & \xrightarrow{\quad \sigma \quad} & \prod_{j \geq 0} YZ^j = \Gamma \\ & \searrow \sigma_j & \downarrow \text{pr}_j \\ & & YZ^j \end{array}$$

We say M is *reachable* if M is in \mathcal{E} , *observable* if σ is in \mathcal{M} . M is *reachable in time i* if $(r_k \mid 0 \leq k \leq i) : \coprod (IX^k \mid 0 \leq k \leq i) \longrightarrow Q$ is in \mathcal{E} . Correspondingly, M is *observable in time j* if $(\sigma_k \mid 0 \leq k \leq j)$ is in \mathcal{M} .

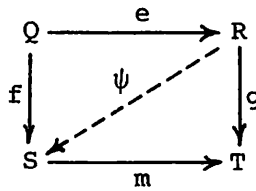
M is *reachable in bounded time* if M is reachable in time i for some i ; and M is *observable in bounded time* if M is observable in time j for some j .

Let \mathcal{K}^{op} denote the dual category of \mathcal{K} . For a discussion of duality for adjoint systems see [4]. Thus products in $\mathcal{K} =$ coproducts in \mathcal{K}^{op} , monomorphisms in $\mathcal{K} =$ epimorphisms in \mathcal{K}^{op} and

6 PROPOSITION: $(\mathcal{M}, \mathcal{E})$ is an *image factorization system* in the opposite category \mathcal{K}^{op} .

Proposition 6 clearly plays a role in establishing the duality between results on reachability and corresponding results on observability -- e.g. the fact noted after Proposition 9.

7 PROPOSITION (DIAGONAL FILL-IN): Given a commutative square $ge = mf$



with $e \in \mathcal{E}$, $m \in \mathcal{M}$ there exists (necessarily unique) ψ with $\psi e = f$ and $m\psi = g$. [Hint for proof: take the images of f and g .]

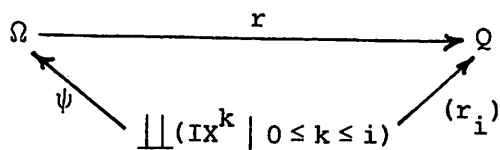
8 PROPOSITION (\mathcal{E} determines \mathcal{M}): The converse of diagonal fill-in holds.

That is, if m is an arbitrary morphism with the property that whenever $ge = mf$ with $e \in \mathcal{E}$ there exists ψ with $\psi e = f$ then necessarily $m \in \mathcal{M}$. [Hint for proof: factor $m = m' e$ and let $f = \text{id}_S$.] Dually, \mathcal{M} determines \mathcal{E} .

9 PROPOSITION: If $f \in \mathcal{E}$ and $f \in \mathcal{M}$ then f is an isomorphism.

10 PROPOSITION: If $f: Q \rightarrow R$ and $g: R \rightarrow S$ then $gf \in \mathcal{E}$ implies $g \in \mathcal{E}$ whereas $gf \in \mathcal{M}$ implies $f \in \mathcal{M}$. [Hint for proof: use 8.]

Reachability in bounded time implies reachable. To prove this, observe that

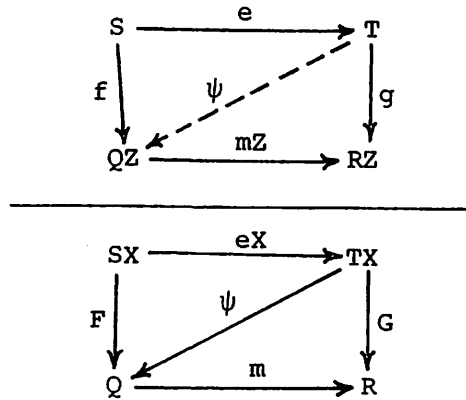


where ψ is defined by $\psi \text{in}_k = \text{in}_k$. Thus $(r_i) = r \cdot \psi$ in \mathcal{E} implies r in \mathcal{E} . Dually, observability in bounded time implies observable.

11 PROPOSITION: Given a family $f_i: Q_i \rightarrow R_i$ with each $f_i \in \mathcal{M}$ then the unique $f: \prod Q_i \rightarrow \prod R_i$ defined by $\text{pr}_i f = f_i \text{pr}_i$ is also in \mathcal{M} . Dually, given a family $f_i: Q_i \rightarrow R_i$ with each $f_i \in \mathcal{E}$, the unique $f: \coprod Q_i \rightarrow \coprod R_i$ with $f \text{in}_i = \text{in}_i f_i$ is again in \mathcal{E} . [Hint for proof: use 8.]

12 PROPOSITION. X preserves \mathcal{E} if and only if Z preserves \mathcal{M} .

Proof: Assuming X preserves \mathcal{E} we wish to show that $mZ : QZ \longrightarrow RZ \in \mathcal{M}$ given that $m : Q \longrightarrow R \in \mathcal{M}$. This is immediate from 8 and the adjoint correspondences



(where capital and lower case letters correspond). The converse result is dual. □

We can now state all necessary standing assumptions and summarize them here for convenience. For the balance of the paper we assume that:

\mathcal{K} is a category with products and coproducts of countable families.

(X, Z) is an adjoint process in \mathcal{K} .

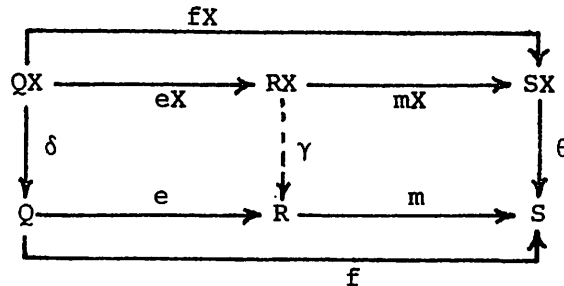
$(\mathcal{E}, \mathcal{M})$ is an image factorization system in \mathcal{K} .

X preserves \mathcal{E} (and hence Z preserves \mathcal{M}).

I is a fixed input object. Y is a fixed output object.

It is often the case that $\mathcal{E} =$ all epimorphisms or that $\mathcal{E} =$ all morphisms which are the coequalizer of some pair [3, section 1.3] [19, p. 64]. This is the case for the category of sets and for the category of modules over a ring, the unique \mathcal{E} being the class of epimorphisms = the class of all coequalizers. In these two cases, it is well known that X must preserve \mathcal{E} [19, v.5].

13 DYNAMORPHIC IMAGE LEMMA: Let $f : (Q, \delta) \longrightarrow (S, \theta)$ be a dynamorphism, that is, the perimeter of the diagram below commutes.



Let $f = m \circ e$ be an \mathcal{E} - \mathcal{M} factorization of f . Then there exists a unique dynamics $\gamma : RX \longrightarrow R$ rendering the above diagram commutative.

Proof: Since $eX \in \mathcal{E}$, this is immediate from 7. \square

14 DEFINITION: The canonical realization M_H of a Hankel matrix H_i^j is the system $(Q_H, \delta_H, \tau_H, \beta_H)$ defined as follows. Let $H : (\Omega, \mu_0) \longrightarrow (\Gamma, L)$ be the dynamorphism defined by $\text{pr}_j \cdot H \cdot \text{in}_i = H_i^j$. Let

$$\Omega \xrightarrow{r_H} Q_H \xrightarrow{\sigma_H} \Gamma$$

be an \mathcal{E} - \mathcal{M} factorization of H . By the dynamorphic image lemma, there exists a unique dynamics $\delta_H : Q_H X \longrightarrow Q_H$ rendering r_H and σ_H dynamorphisms.

Define $\tau_H = r_H \text{in}_0$ and $\beta_H = \text{pr}_0 \sigma_H$.

It is proved in [4, theorems 2.1, 3.15] that M_H is a realization of H_i^j , that the reachability and observability maps of M_H are r_H and σ_H (so that M_H is reachable and observable) and that any other reachable and observable realization is isomorphic to M_H .

The question of interest here, however, is under what conditions the canonical realization can be found from a finite fragment of the Hankel matrix $(H_i^j \mid 0 \leq i \leq k, 0 \leq j \leq n)$. We first present a partial realization which tells us when such a fragment lets us define an adjoint system whose

behavior is consistent with that portion of the Hankel matrix. Then, in the remaining sections, we present general conditions on m and n under which this partial realization will be isomorphic to the canonical realization.

Let us fix the following notations:

$$\bar{n} = \{0, 1, \dots, n\}$$

$$IX^{\bar{k}} = \coprod_{i \in \bar{k}} IX^i; \quad YX^{\bar{n}} = \prod_{j \in \bar{n}} YZ^j$$

while

$$H_{\bar{k}}^{\bar{n}} : IX^{\bar{k}} \longrightarrow YZ^{\bar{n}}$$

is defined by $\text{pr}_j \cdot H_{\bar{k}}^{\bar{n}} \cdot \text{in}_i = H_i^j$ for $i \in \bar{k}, j \in \bar{n}$.

Define $\tilde{\mu} : IX^{\bar{k}}X \longrightarrow IX^{\overline{k+1}}$ by $\tilde{\mu} \cdot \text{in}_i X = \text{in}_{i+1}$

$$\tilde{\epsilon} : YZ^{\overline{n+1}}X \longrightarrow YZ^{\bar{n}} \quad \text{by} \quad \text{pr}_j \cdot \tilde{\epsilon} = \epsilon_{YZ^j} \cdot \text{pr}_{j+1} X.$$

Then the Hankel crossover condition yields

15

$$\begin{array}{ccc} IX^{\bar{k}}X & \xrightarrow{\tilde{\mu}} & IX^{\overline{k+1}} \\ \downarrow H_{\bar{k}}^{\overline{n+1}}X & & \downarrow H_{\overline{k+1}}^{\bar{n}} \\ YZ^{\overline{n+1}}X & \xrightarrow{\tilde{\epsilon}} & YZ^{\bar{n}} \end{array}$$

(just precede the square by $\text{in}_i X$ and follow it by pr_j for $0 \leq i \leq k, 0 \leq j \leq n$ to recapture the square of 3(ii)).

Let $H_{\bar{k}}^{\overline{n+1}}$ have \mathcal{E} - \mathcal{M} factorization (\bar{e}, \bar{m}) with image \bar{Q} while $H_{\overline{k+1}}^{\bar{n}}$ factors as (\hat{e}, \hat{m}) with image \hat{Q} . Then, since X preserves \mathcal{E} , we may define

$\tilde{\delta} : \bar{Q}X \longrightarrow \hat{Q}$ by diagonal fill-in:

16

$$\begin{array}{ccc} IX^{\bar{k}}X & \xrightarrow{\tilde{\mu}} & IX^{\overline{k+1}} \\ \downarrow \bar{e}X & & \downarrow \hat{e} \\ \bar{Q}X & \xrightarrow{\tilde{\delta}} & \hat{Q} \\ \downarrow \bar{m}X & & \downarrow \hat{m} \\ YZ^{\overline{n+1}}X & \xrightarrow{\tilde{\epsilon}} & YZ^{\bar{n}} \end{array}$$

The important fact is that $\tilde{\delta}$ is completely determined by the H_i^j for $0 \leq i \leq k+1$, $0 \leq j \leq n+1$.

To obtain our partial realization theorem, we must establish conditions under which $\tilde{\delta}$ may be viewed as a dynamics. To this end, define

$$\begin{aligned} \text{in} : IX^{\bar{k}} &\longrightarrow IX^{\bar{k}+1} & \text{by} & \text{in} \cdot \text{in}_i = \text{in}_i \\ \text{pr} : YZ^{\bar{n}+1} &\longrightarrow YZ^{\bar{n}} & \text{by} & \text{pr}_j \cdot \text{pr} = \text{pr}_j . \end{aligned}$$

Then the bi-index principle, 1.10, yields

17

$$\begin{array}{ccc} IX^{\bar{k}} & \xrightarrow{H_k^{\bar{n}+1}} & YZ^{\bar{n}+1} \\ \text{in} \downarrow & \searrow & \downarrow \text{pr} \\ IX^{\bar{k}+1} & \xrightarrow{H_k^{\bar{n}}} & YZ^{\bar{n}} \\ & \xrightarrow{H_{k+1}^{\bar{n}}} & \end{array}$$

Forming the \mathcal{E}, \mathcal{M} factorization (e, m) of $H_k^{\bar{n}}$ with image R , we then obtain t and u by diagonal fill-in:

18

$$\begin{array}{ccccc} IX^{\bar{k}} & \xrightarrow{\bar{e}} & \bar{Q} & \xrightarrow{\bar{m}} & YZ^{\bar{n}+1} \\ \text{in} \downarrow & \searrow e & \downarrow t & & \downarrow \text{pr} \\ IX^{\bar{k}+1} & \xrightarrow{\hat{e}} & \hat{Q} & \xrightarrow{\hat{m}} & YZ^{\bar{n}} \\ & & \downarrow u & \nearrow m & \end{array}$$

19 PARTIAL REALIZATION THEOREM: If t, u are isomorphisms in 18, we may define the system $M = (\bar{Q}, \delta, \tau, \beta)$ by

20 $\delta = t^{-1} \cdot u^{-1} \cdot \tilde{\delta} : \bar{Q}X \longrightarrow \bar{Q}$ (using 16 and 18)

21 $\tau = \bar{e} \cdot \text{in}_0 : I \longrightarrow IX^{\bar{k}} \longrightarrow \bar{Q}$

22 $\beta = \text{pr}_0 \cdot \bar{m} : \bar{Q} \longrightarrow YZ^{\bar{n}+1} \longrightarrow Y$.

Then the Hankel matrix of M agrees with H_i^j for $0 \leq i \leq n$, $0 \leq j \leq k$.

Proof: Let r_i, σ_j be the i -step reachability and j -step observability maps, respectively, of M . We prove the theorem in two steps:

- (i) We show $r_i = IX^i \xrightarrow{in_i} IX^{\bar{k}} \xrightarrow{\bar{e}} \bar{Q}$ for $0 \leq i \leq k$.
(ii) We show $\sigma_j = \bar{Q} \xrightarrow{\bar{m}} YZ^{\bar{n}+1} \xrightarrow{pr_j} YZ^j$ for $0 \leq j \leq k$.

It is then immediate that $\sigma_j \cdot r_i = pr_j \cdot \bar{m} \cdot \bar{e} \cdot in_i = pr_j \cdot H_{\bar{k}}^{\bar{n}+1} \cdot in_i = H_i^j$.

Proof of (i): For $i = 0$, this is 21. Now, for $0 \leq i < k$, we have by induction

$$\begin{aligned}
r_{i+1} &= \delta \cdot r_i X \\
&= t^{-1} u^{-1} \cdot \tilde{\delta} \cdot r_i X \\
&= t^{-1} u^{-1} \cdot \tilde{\delta} \cdot \bar{e} X \cdot in_i X && \text{by induction hypothesis} \\
&= t^{-1} u^{-1} \cdot \hat{e} \cdot \tilde{\mu} \cdot in_i X && \text{by 16} \\
&= t^{-1} u^{-1} \cdot \hat{e} \cdot in_{i+1} && \text{by definition of } \tilde{\mu} \\
&= t^{-1} u^{-1} \cdot u \cdot t \cdot \bar{e} \cdot in_{i+1} && \text{by 18, and definition of } in \\
&= \bar{e} \cdot in_{i+1} && \text{as was to be shown.}
\end{aligned}$$

Proof of (ii): By definition $\sigma_0 = \beta = pr_0 \cdot \bar{m}$. But then, for $0 \leq j < k$,

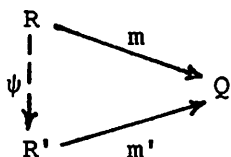
$$\begin{aligned}
\sigma_{j+1} &= \sigma_j Z \cdot \Delta \\
&\hline
&= \sigma_j \cdot \delta && \text{by 1.2} \\
&= pr_j \cdot \bar{m} \cdot \delta && \text{by induction hypothesis} \\
&= pr_j \cdot \hat{m} \cdot u \cdot t \cdot \delta && \text{by 18, and definition of } pr \\
&= pr_j \cdot \hat{m} \cdot \tilde{\delta} && \text{by definition of } \delta \\
&= pr_j \cdot \tilde{e} \cdot \bar{m} X && \text{by 16} \\
&= \epsilon_{YZ^j} \cdot pr_{j+1} X \cdot \bar{m} X && \text{by definition of } \tilde{e} \\
&\hline
&= pr_{j+1} \cdot \bar{m}
\end{aligned}$$

□

3. Generalizing the Notion of Finite Dimensionality

For a linear system M , the subspaces Q_i generated by the union of the images of $A^k B : I \rightarrow Q$, $0 \leq k \leq i$, constitute an ascending chain of subspaces of Q . If Q is finite-dimensional -- or more generally, for modules over a ring rather than vector spaces, if Q is Noetherian -- this chain is eventually stationary, $Q_m = Q_{m+1} = \dots$, and M is reachable in time m . In this section, we show how such dimensionality considerations may be extended to our category \mathcal{K} -- with dimension reducing, essentially, to cardinality in the case of Set. The notions of \mathcal{E} -height and \mathcal{M} -height were introduced in [1]. Further properties of Noetherian objects appear in [7].

1. DEFINITIONS: Let Q be an object of \mathcal{K} . The set of all pairs (R, m) with $m : R \rightarrow Q \in \mathcal{M}$ admits a reflexive and transitive order by defining $(R, m) \leq (R', m')$ if there exists ψ with $m'\psi = m$



(note that such ψ is necessarily unique and is itself in \mathcal{M}). Thus $(R, m) \sim (R', m')$ if $(R, m) \leq (R', m')$ and $(R', m') \leq (R, m)$ is an equivalence relation whose equivalence classes $[R, m]$ are called the *subobjects* of Q . $[R, m] \leq [R', m']$ if $(R, m) \leq (R', m')$ is a well-defined partial order on the subobjects of Q . It is easily seen that $[R, m] = [R', m']$ if and only if there exists an isomorphism ψ with $m'\psi = m$.

Q is *Noetherian* if every strictly ascending chain of subobjects of Q is finite. Let $h \geq 0$ be an integer. Q has \mathcal{M} -height h if Q admits a strict chain of proper subobjects of length h , but none of length $h+1$. Q has *finite \mathcal{M} -height* if Q has \mathcal{M} -height h for some h .

The dual concepts relative to \mathcal{K} are formulated by repeating the above definitions in \mathcal{K}^{op} (using 2.6). Thus, the ordering on *quotient objects* of Q is described by

$$\begin{array}{ccc}
 R & \xleftarrow{e} & Q \\
 \uparrow \psi & & \\
 R' & \xleftarrow{e'} & Q
 \end{array}
 \quad
 \begin{array}{l}
 [R, e] \leq [R', e'] \\
 e, e' \in \mathcal{E}
 \end{array}$$

(Note that we reverse the arrows, not the ordering.) We say Q is *Artinian* if Q is co-Noetherian, that is, if every strictly ascending chain of quotient objects of Q is finite. The definitions of ' \mathcal{E} -height' and '*finite* \mathcal{E} -height' are clear.

2 EXAMPLES: In the category of sets, subobjects may be identified with subsets of Q and quotient objects may be identified with the canonical quotient projections induced by equivalence relations on Q . A set with h elements has \mathcal{M} -height $h+1$ and \mathcal{E} -height h , except that the empty set has \mathcal{E} -height 1. For sets, Noetherian = Artinian = finite. Notice that for both subsets and quotient sets, ascending chains mean increasing cardinality.

In the category of modules over a ring, the passage from a submodule S to its cokernel Q/S establishes an anti-isomorphism of partially ordered sets between subobjects and quotient objects. For this reason, Artinian is equivalent to the descending chain condition on subobjects (the usual definition in module theory) and a module has finite height if and only if it is simultaneously Noetherian and Artinian. These two properties do not hold in a general category where descending chain conditions are not equivalent to the ascending chain conditions defined above, and do not seem to be well motivated in a system context.

In abelian groups, Noetherian = finitely-generated, whereas finite \mathcal{M} -height = finite. The group of additive integers is not Artinian. For vector spaces, on the other hand, Noetherian = Artinian = finite height and \mathcal{M} -height = \mathcal{E} -height = 1 + dimension.

It is unclear how to define 'proper subobject' so that the four finite-height conditions discussed above come out 'right'. One could exclude the proper subobject $[Q, \text{id}_Q]$ although from the system point of view this subobject is not the trivial one; it is the zero subobject that is trivial from the point of view of building increasing chains. Recall that an object 0 is *initial* if there is a unique morphism $0 \rightarrow Q$ to every Q and, dually, an object 1 is *terminal* if there is a unique morphism $Q \rightarrow 1$ for every Q . For sets, 0 is the empty set, 1 is a one-element set and for modules 0 is both initial and terminal. While the unique $0 \rightarrow Q$ is not always in \mathcal{M} , the image factorization of this map produces the least element of the partially ordered set of subobjects of Q . Dually, the image factorization of $Q \rightarrow 1$ produces the least quotient object of Q . It seems hard to posit a natural procedure to decide when to omit the zero subobject from chains which both preserves duality (i.e., the same procedure must be applied to quotient chains) and works right in the examples above.

Motivated by the sequence $r_i : IX^i \rightarrow Q$ induced by an adjoint system, we consider an arbitrary sequence of morphisms of form $f_i : P_i \rightarrow Q$, $i \in \mathcal{N} = \{0, 1, 2, \dots\}$ of natural numbers. The following four constructions are useful. We fix their notations for the remainder of the paper.

3 For each non-empty subset S of \mathcal{N} define

$$P_S = \coprod (P_i : i \in S)$$

$$f_S = P_S \rightarrow Q, \quad \text{where } f_S \text{ in}_i = f_i \quad (i \in S).$$

4 For $S \subset T \subset \eta$, $\text{in}_{ST} : P_S \longrightarrow P_T$ is defined by $\text{in}_{ST} \text{in}_i = \text{in}_i$ ($i \in S$).

5 Fix an \mathcal{E} - \mathcal{M} factorization of f_S :

$$P_S \xrightarrow{e_S} I_S \xrightarrow{m_S} Q .$$

6 For $S \subset T \subset \eta$, $m_{ST} : I_S \longrightarrow I_T$ is defined by diagonal fill-in (2.7):

$$\begin{array}{ccc}
 P_S & \xrightarrow{e_S} & I_S \\
 \text{in}_{ST} \downarrow & & \downarrow m_S \\
 P_T & & I_T \\
 e_T \downarrow & \swarrow m_{ST} & \downarrow m_T \\
 I_T & \xrightarrow{m_T} & Q
 \end{array}$$

(to prove that the square commutes observe that both paths are f_i when preceded by in_i).

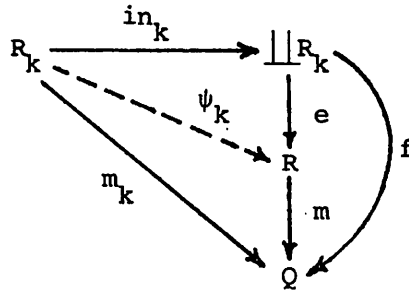
We observe at once, using the results of section 2, that $m_{ST} \in \mathcal{M}$, that $m_{TU} m_{ST} = m_{SU}$ for $S \subset T \subset U$, and that if $f_S \in \mathcal{E}$ then $f_T \in \mathcal{E}$ whenever $S \subset T$.

Motivated by system theory we should like to prove results such as 'if Q is Noetherian and M is reachable then M is reachable in bounded time' and 'if f_S is onto and if T is the subset of S obtained by deleting those k for which the union of the images of f_0, \dots, f_{k-1} is the same as the union of the images of f_0, \dots, f_k then f_T is still onto'. We observe that the reason these results are so easy to obtain in the category of sets is because the passage $S \mapsto I_S$ is union-preserving; I_S is, after all, just the union of the images of the $(f_s : s \in S)$. Our approach below is to show that, in general, this passage is sufficiently supremum-preserving to lift the theory to a category. We present general results about 'dimension in a category'

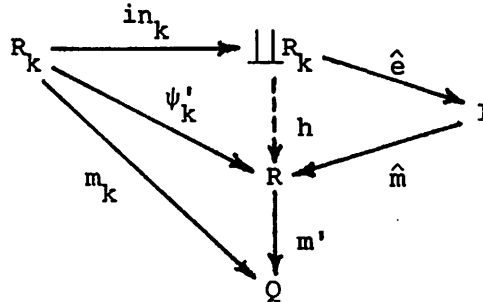
in this section; and turn to their system-theoretic application in section 4.

7 LEMMA: For any object Q , every non-empty countable family of subobjects of Q has a supremum.

Proof: Given $[R_k, m_k]$ define f by $f \text{ in}_k = m_k$, and consider



where (e, m) is an \mathcal{E} - \mathcal{M} factorization of f . Using diagonal fill-in on the 'square' $m(e \text{ in}_k) = m_k \text{ id}$ induces ψ_k , so that $[R_k, m_k] \leq [R, m]$ for all k . We will show that $[R, m]$ is the least upper bound. Suppose that $[R_k, m_k] \leq [R', m']$. Then there exist ψ'_k as shown

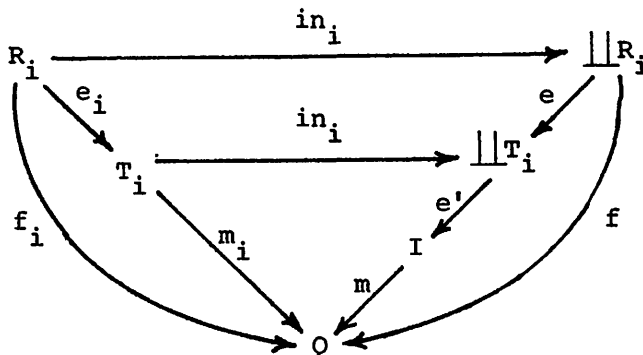


and hence a unique h with $h \text{ in}_k = \psi'_k$. Clearly $m' h = f$. Hence, if (\hat{e}, \hat{m}) is an \mathcal{E} - \mathcal{M} factorization of h , $(\hat{e}, m' \hat{m})$ is an \mathcal{E} - \mathcal{M} factorization of f so that $[R, m] = [I, m' \hat{m}]$. But then, via \hat{m} , $[R, m] \leq [R', m']$. \square

Given $f : R \rightarrow Q$, let $[f]$ denote the subobject of Q obtained by taking the image factorization of f .

8 LEMMA: Let $f_i : R_i \rightarrow Q$ be a non-empty countable family of morphisms and let $f : \coprod R_i \rightarrow Q$ be defined by $f \text{ in}_i = f_i$. Then $[f] = \sup(\{f_i\})$.

Proof: Consider the diagram shown in which $[f_i] = [T_i, m_i]$ and $[I, m] = \text{sup}([f_i])$ according to the construction of Lemma 7.

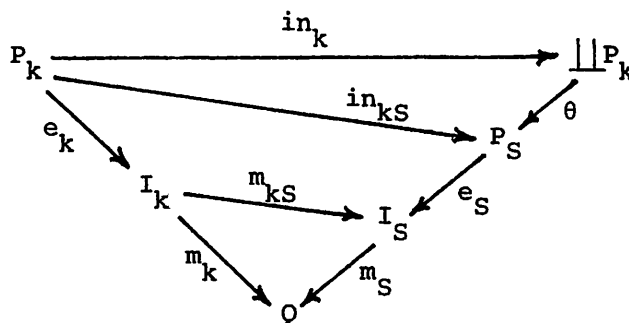


Define e by $e \text{ in}_i = \text{in}_i e$. Then the diagram commutes -- that is, $f = m e' e$ -- because both paths coincide with f_i when preceded by in_i . By 2.11, $e \in \mathcal{E}$ so that $[f] = [I, m]$ as desired. \square

Before continuing, we introduce the abbreviation I_S for the more cumbersome $[I_S, m_S]$, ' I_S is a subobject of Q '.

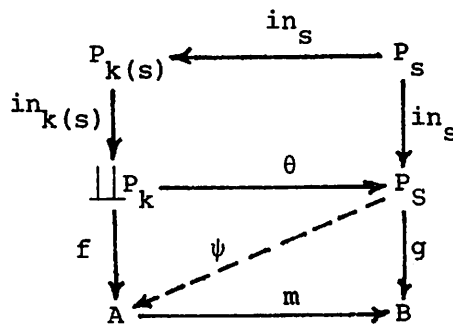
9 LEMMA: The passage $S \mapsto I_S$ preserves non-empty countable suprema.

Proof: Let $(S_k : k \in I)$ be a non-empty countable family of non-empty subsets of \mathcal{N} and set $S = \cup S_k$. We must show that I_S is the supremum of the I_k (where we use the subscript k for the more cumbersome S_k throughout). Consider the map $\theta : \coprod P_k \rightarrow P_S$ defined by $\theta \text{ in}_k = \text{in}_{kS}$ ($k \in I$). In view of the commutative diagram



it suffices to show that $\theta \in \mathcal{E}$, for then $(e_S \theta, m_S)$ is precisely the construction of the supremum in Lemma 7. To prove that $\theta \in \mathcal{E}$ we use the dual

of 2.8. For $s \in S$ choose $k(s)$ with $s \in S_{k(s)}$. Then consider the diagram



where g , f and m are only required to satisfy $g\theta = mf$ and $m \in \mathcal{M}$.

We must construct ψ with $m\psi = g$.

Define ψ by $\psi \text{ in}_s = f \text{ in}_{k(s)} \text{ in}_s$ as shown. Since $\theta \text{ in}_{k(s)} \text{ in}_s = \text{in}_{k(s)} P_s \text{ in}_s = \text{in}_s$ (see 4) we have $(m\psi) \text{ in}_s = m f \text{ in}_{k(s)} \text{ in}_s = g \theta \text{ in}_{k(s)} \text{ in}_s = g \text{ in}_s$ for all $s \in S$, so that $m\psi = g$. \square

For the next definition and two propositions we consider an arbitrary non-empty-countable-supremum-preserving map $I: R \rightarrow L$ where R is the partially ordered set of non-empty subsets of \mathcal{N} and L is an arbitrary partially ordered set. For the general I we write $I(S)$ instead of I_S . I_S is not the only application; see [8].

10 DEFINITION: Define $\bar{n} = \{0, \dots, n\} \in R$. I is stationary if $I(\bar{n}) = I(\bar{n}+1)$ for all n . $A \in R$ is adequate if $I(A) = I(S)$ whenever $A \subset S$. Equivalently, A is adequate if and only if $I(S) = I(T)$ whenever $A \subset S \subset T$.

11 PROPOSITION: If A is 'one-step adequate' in the sense that $I(A) = I(A \cup \{k\})$ for all k then A is adequate.

Proof: If $A \subset S$, $S = \cup \{A \cup \{k\} : k \in S\}$. \square

12 PROPOSITION: For each non-empty-countable-supremum-preserving map $I: R \rightarrow L$, the set

$$A = \{0\} \cup \{k \in \mathcal{N} \mid I(\bar{k-1}) < I(\bar{k})\}$$

is adequate.

Proof: Note that if I is stationary, then $A = \{0\}$, and certainly this A is adequate. Otherwise, it suffices to prove $I(A) = I(A \cup \{0, \dots, n\})$ for all n . Since $\{0\} \subset A$, this is certainly true for $n = 0$. Suppose now that $I(A) = I(A \cup \{0, \dots, n-1\})$. Then if $n \in A$, it is certainly true that $I(A) = I(A \cup \{0, \dots, n\})$. Otherwise, $I(\overline{n-1}) = I(\overline{n})$, and so

$$\begin{aligned} I(A \cup \{0, \dots, n\}) &= \sup(I(A), I(\overline{n})) \\ &= \sup(I(A), I(\overline{n-1})) \\ &= I(A \cup \overline{n-1}) = A \quad \square \end{aligned}$$

13 COROLLARY: If Q has \mathcal{M} -height h , $f_i : P_i \rightarrow Q$ has an adequate set with $h+1$ or fewer elements. □

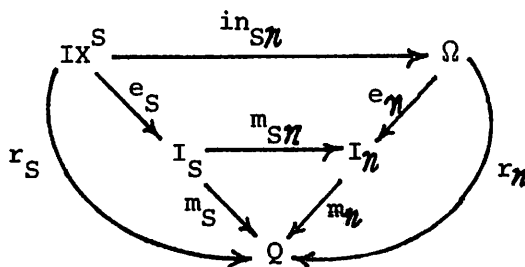
4. Adequacy for Systems and the Simple Recursion Principle

In this section, we study the implications of Section 3 for an adjoint system M . We introduce all of the notions of Section 3, with

$$f_i : P_i \longrightarrow Q = r_i : IX^i \longrightarrow Q. \text{ We write } IX^S \text{ instead of } P_S.$$

1 PROPOSITION: *If M is reachable and Q is Noetherian, M is reachable in bounded time. Dually, if M is observable and Q is Artinian, M is observable in bounded time.*

Proof: By definition, $r_{\mathcal{N}} = r$ is the reachability map of M and M is reachable in bounded time if and only if $r_S \in \mathcal{E}$ for some finite S . Since Q is Noetherian, there exists a finite one-step adequate set A . Then A is adequate by 3.11 and, in the diagram shown, $m_{S\mathcal{N}}$ is an isomorphism.



If M is reachable, $m_{\mathcal{N}}$ is an isomorphism (2.10 and 2.11) so that $r_S = m_{\mathcal{N}} m_{S\mathcal{N}} e_S \in \mathcal{E}$ (by IFS1 and IFS2). \square

2 DEFINITION: Let V be a set. A sequence v_n in V is defined by simple recursion if there exists a function $g : V \longrightarrow V$ such that $v_{n+1} = g(v_n)$, that is, $v_n = g^n(v_0)$.

3 EXAMPLE: The sequence $\text{span}(B, AB, \dots, A^n B)$ of subspaces of the state space of a linear system is defined by simple recursion. Define $g(S) = \text{span}(S \cup F(S))$ for each subspace S . The next result shows that this construction works for arbitrary adjoint systems.

Recall that $\bar{n} = \{0, \dots, n\}$.

4 SIMPLE RECURSION PRINCIPLE FOR ADJOINT SYSTEMS: Let $I_{\bar{n}}$ be the subobject $[r_{\bar{n}}]$ of the state object of an adjoint system 'reachable in time n '. Then the ascending sequence $I_{\bar{n}}$ is defined by simple recursion. Dually, the ascending sequence of observability quotient objects of the state object is also defined by simple recursion.

Proof: We define the endomorphism g on the subobjects of Q by

$g([R, m]) = \sup([R, m], [\delta \cdot mX])$. To verify that $g(I_{\bar{n}}) = I_{\overline{n+1}}$, recall that $r_{i+1} = \delta \cdot r_i X$, and that $r_i = r_{\bar{n}} \cdot in_i$ for $i \leq n$, and that we have the coproduct diagram

$$\begin{array}{ccccc}
 IX^i X & & & & \\
 \downarrow in_i X & \searrow r_i X & & \searrow r_{i+1} & \\
 IX^{\bar{n}} X & \xrightarrow{r_{\bar{n}} X} & QX & \xrightarrow{\delta} & Q
 \end{array}$$

Now, by lemma 3.8, $[\delta \cdot r_{\bar{n}} X] = \sup([r_1], \dots, [r_{n+1}]) = \sup(I_1, \dots, I_{n+1})$. But because X preserves \mathcal{E} ,

$$[\delta \cdot r_{\bar{n}} X] = [\delta \cdot m_{\bar{n}} X \cdot e_{\bar{n}} X] = [\delta \cdot m_{\bar{n}} X].$$

Moreover, $I_{\bar{n}} = \sup(I_0, \dots, I_n)$ by 3.9. Therefore

$$g(I_{\bar{n}}) = \sup(\sup(I_0, \dots, I_n), \sup(I_1, \dots, I_{n+1})) = \sup(I_0, \dots, I_{n+1}) = I_{\overline{n+1}}. \quad \square$$

An immediate consequence is a better proof of the general version [1, Theorem 4.6] of the 'if you stick you're stuck' result of [9]:

5 COROLLARY: If $I_{\bar{n}} = I_{\overline{n+1}}$, then \bar{n} is adequate.

Proof: $I_{\overline{n+k+1}} = g(I_{\overline{n+k}}) = g(I_{\overline{n+k-1}})$ (induction hypothesis) = $I_{\overline{n+k}}$. \square

While the proof of Corollary 5 bypasses Proposition 3.12, the latter is still a useful principle, as we shall see in [8].

6 COROLLARY: Let M be an adjoint system with state object Q . If Q has \mathcal{M} -height h then M is reachable in time h . Dually, if Q has \mathcal{E} -height h , M is observable in time h . \square

To tie this back to the realization theory of Section 2, and especially the Partial Realization theorem 2.19, we make the

7 OBSERVATION: It is clear from 2.18 and 2.10 that t is in \mathcal{E} and that u is in \mathcal{M} . It is then clear that if \mathcal{E} -height $(\bar{Q}) = \mathcal{E}$ -height (R) and both are finite, then t is an isomorphism; while if \mathcal{M} -height $(R) = \mathcal{M}$ -height (\hat{Q}) and both are finite, then u is an isomorphism. Thus the condition 't and u are isomorphisms' in 2.19 may be replaced by ' \mathcal{E} -height $(\bar{Q}) = \mathcal{E}$ -height (R) and \mathcal{M} -height $(R) = \mathcal{M}$ -height (\hat{Q}) and both are finite'.

8 COROLLARY: For adjoint processes in Vect, we may obtain a partial realization as soon as

$$\dim(Q) = \dim(R) = \dim(\hat{Q}) = \text{finite.}$$

This yields both Tether's [22] criterion for partial realization of linear systems, and Isidori's [16] criterion for partial realization of bilinear systems (internal sense).

Combining the argument for Corollary 6 with the Partial Realization Theorem 2.19 and observation 7 we have

9 THE HANKEL REALIZATION THEOREM: Let H_1^j be a Hankel matrix having a realization with state object Q having \mathcal{E} -height $\leq h$ and \mathcal{M} -height $\leq \ell$ with h and ℓ finite. Then the canonical realization of H_1^j may be constructed by applying the construction of 2.19 with $k = h$ and $n = \ell$.

Proof outline: The crucial point is that the finite height conditions imply that items (i) and (ii) of the proof of 2.19 -- $r_i = \bar{e} \cdot \text{in}_i$ and $\sigma_j = \text{pr}_j \cdot \bar{m}$ -- hold for all i and j respectively. But this not only shows that the M of 2.19 has Hankel matrix H_i^j , but also that M has reachability map \bar{e} in \mathcal{E} and observability map \bar{m} in \mathcal{M} -- so that M is canonical. \square

10 OBSERVATION: As in 8, we note that when $\mathcal{K} = \underline{\text{Vect}}$ the two height conditions in 9 collapse to the single condition ' Q having finite dimension h ', and we may then take $k = n = h$ in forming the realization.

For the biadequacy criterion for Hankel realization, see [8, Theorem 4.9].

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RECURRENCE CONDITIONS FOR GENERALIZED HANKEL MATRICES¹

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ABSTRACT

Recurrence and corecurrence of the Hankel matrix of an adjoint system is related to the existence of finite realizations. Adjoint systems include linear and bilinear systems, automata, and group systems in both the time-varying and time-invariant cases.

After a brief review of the setting of [4], we define recurrence of degree m for a Hankel matrix H_i^j . Such recurrence implies that H_i^m depends on H_i^0, \dots, H_i^{m-1} for all i . The dual notion is corecurrence. To relate these conditions to 'finiteness' we investigate ascending chain conditions for subobjects and quotient objects of an object in a category. Recurrence is related to Artinian realizations with injective Artinian input object whereas corecurrence is related to Noetherian realizations with projective Noetherian output object. The scarcity of Artinian injective modules in the context of linear systems over a ring puts the emphasis on corecurrence there. Our theorems unify the familiar linear results [9], [6, XVI.10] with the group machine result of [5, theorem 2].

The 'recurrence polynomials' in the context of modules over a noncommutative ring, as considered by [10], are algebraic operations in the sense of universal algebra [8, section 1.5] as opposed to morphisms. They cannot be subsumed in the discussion here and will be discussed elsewhere.

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1. Generalities

We quickly record some definitions and facts concerning the Hankel matrix of an adjoint system. For details, see [4] and the references cited there.

We work in an arbitrary category \mathcal{K} with countable products and co-products and provided with an image factorization system $(\mathcal{E}, \mathcal{M})$ (generalizing \mathcal{E} = surjections, \mathcal{M} = injections in familiar categories such as sets and functions, or vector spaces and linear maps).

We fix an adjoint process (X, Z) . This means that X and Z are functors from \mathcal{K} to itself with Z right adjoint to X . Thus there are bijective correspondences

$$\frac{RX \xrightarrow{g} S}{R \xrightarrow{\psi} SZ}$$

(one to each pair of \mathcal{K} -objects (R, S)) subject to the naturality condition

$$\frac{QX \xrightarrow{fX} RX \xrightarrow{g} S \xrightarrow{h} T}{Q \xrightarrow{f} R \xrightarrow{\psi} SZ \xrightarrow{hZ} TZ}$$

whenever $f: Q \rightarrow R$, $h: S \rightarrow T$. In this notation, ψ may alternatively appear on the top with g on the bottom. We also assume that X preserves \mathcal{E} (and hence [4, theorem 2.12] that Z preserves \mathcal{M}).

For example, let \mathcal{K} be the category of sets and functions. Let A be a fixed input alphabet. Define $QX = Q \times A$, $QZ = Q^A$, the set of functions from A to Q . For $f: Q \rightarrow R$, $fX: Q \times A \rightarrow R \times A$ is defined by $(q, a) \mapsto (f(q), a)$ whereas $fZ: Q^A \rightarrow R^A$ sends $g: A \rightarrow Q$ to $fg: A \rightarrow R$.

The adjointness correspondence

$$\frac{Q \times A \xrightarrow{g} R}{Q \xrightarrow{\psi} R^A}$$

is the familiar $(\psi q)(a) = g(q, a)$. Here, let \mathcal{E} = surjections and let \mathcal{M} = injections. It is indeed true that fX is surjective when X is and that gZ is injective when g is.

An *adjoint system* is $M = (Q, \delta, I, \tau, Y, \beta)$ where Q, I, Y are objects (the *state object, input object* and *output object* of M) and $\delta : QX \rightarrow Q$, $\tau : I \rightarrow Q$ and $\beta : Q \rightarrow Y$ are morphisms (the *dynamics, input map* and *output map* of M). (Note: 'map' is here a synonym for 'morphism'.) The *codynamics* of M is the map $\Delta : Q \rightarrow QZ$ which corresponds to δ under adjointness.

Given two dynamics $\delta : QX \rightarrow Q$ and $\theta : RX \rightarrow R$, a *dynamorphism* $h : (Q, \delta) \rightarrow (R, \theta)$ is a map $h : Q \rightarrow R$ which 'respects the dynamics':

$$\begin{array}{ccc} QX & \xrightarrow{hX} & RX \\ \delta \downarrow & & \downarrow \theta \\ Q & \xrightarrow{h} & R \end{array}$$

The *time- i reachability map* $r_i : IX^i \rightarrow Q$ and the *time- j observability map* $\sigma_j : Q \rightarrow YZ^j$ are defined by:

$$\begin{aligned} r_0 &= \tau \\ r_{i+1} &= IX^{i+1} \xrightarrow{r_i X} QX \xrightarrow{\delta} Q \\ \sigma_0 &= \beta \\ \sigma_{j+1} &= Q \xrightarrow{\Delta} QZ \xrightarrow{\sigma_j Z} YZ^{j+1}. \end{aligned}$$

The bisequence H_i^j , where $H_i^j : IX^i \rightarrow YZ^j$ is defined by $H_i^j = \sigma_j r_i$, is the *Hankel matrix* of M . The Hankel matrix for adjoint systems was introduced in [4], which also contains further references on related studies.

We continue our earlier example to describe 'automata theory'. (For the other examples mentioned in the abstract we refer the reader to [4].) Let I have a single element so that τ amounts to an element of Q , the initial state. The dynamics and output map have their usual forms $\delta : Q \times A \longrightarrow Q$, $\beta : Q \longrightarrow Y$. It is easily checked that $r_i : A^i \longrightarrow Q$ sends an i -tuple of input letters to the state reached from the initial state if the letters are inputted in sequence, whereas $\sigma_j : Q \longrightarrow Y^{(A^j)}$ sends q to that function $A^j \longrightarrow Y$ with β following the time- j reachability map if the initial state were q . Thus $H_i^j : A^i \longrightarrow Y^{(A^j)}$ is essentially a way of describing $\beta \circ r_{i+j}$ with emphasis on i as 'present time'.

As in [4] we define the object of inputs Ω to be the coproduct $\coprod (IX^i : i \geq 0)$ and we define the observability space Γ to be the product $\prod (YZ^j : j \geq 0)$. Then Ω and Γ carry canonical X -dynamical structure [4, 2.1 and 2.2]. The following realizability theorem is proved in [4, 2.3].

REALIZABILITY THEOREM: Let $H_j^i : IX^i \longrightarrow YZ^j$ be an arbitrary bisequence of morphisms and let $H : \Omega \longrightarrow \Gamma$ be the unique morphism with $\text{pr}_j H \text{ in}_i = H_i^j$. Then the following three conditions are equivalent (and we say H_i^j is a Hankel matrix if these conditions hold).

- (i) H_i^j is realizable, that is, is the Hankel matrix of some system.
- (ii) (The Hankel crossover condition). For all i, j :

$$\frac{IX^{i+1} \xrightarrow{H_{i+1}^j} YZ^j}{IX^i \xrightarrow{H_i^{j+1}} YZ^{j+1}}$$

- (iii) $H : \Omega \longrightarrow \Gamma$ is a dynamorphism.

Let M be an adjoint system. The *reachability map* $r : \Omega \rightarrow Q$ of M is defined by $r \text{ in}_i = r_i : IX^i \rightarrow Q$. The *observability map* $\sigma : Q \rightarrow \Gamma$ of M is defined by $\text{pr}_j \cdot \sigma = \sigma_j : Q \rightarrow YZ^j$. We say M is *reachable* if $r \in \mathcal{E}$, and *observable* if $\sigma \in \mathcal{M}$.

To conclude this section we recall that the *canonical realization* M_H of the Hankel matrix H_i^j is the system $(Q_H, \delta_H, \tau_H, \beta_H)$ defined as follows.

Let $H : (\Omega, \mu_0) \rightarrow (\Gamma, L)$ be the above dynamorphism. Let

$$\Omega \xrightarrow{r_H} Q_H \xrightarrow{\sigma_H} \Gamma$$

be an $\mathcal{E}\text{-}\mathcal{M}$ factorization of H . By the dynamorphic image lemma [4, 2.13], there exists a unique dynamics $\delta_H : Q_H X \rightarrow Q_H$ rendering r_H and σ_H dynamorphisms. Define $\tau_H = r_H \text{ in}_0$ and $\beta_H = \text{pr}_0 \sigma_H$.

It is proved in [2, theorems 2.1, 3.15] that M_H is a realization of H_i^j , that the reachability and observability maps of M_H are r_H and σ_H (so that M_H is reachable and observable) and that any other reachable and observable realization is isomorphic to M_H .

2. Recurrence for Matrices

For the duration of the paper fix an input object I and an output object Y .

1 DEFINITION: Given $H_i^j : IX^i \longrightarrow YX^j$ and an integer $m > 0$, a *recurrence morphism of degree m* is a morphism p such that

$$\begin{array}{ccc} IX^i & \xrightarrow{(H_i^j)} & \coprod (YZ^j : 0 \leq j < m) \\ & \searrow H_i^m & \swarrow p \\ & & YZ^m \end{array}$$

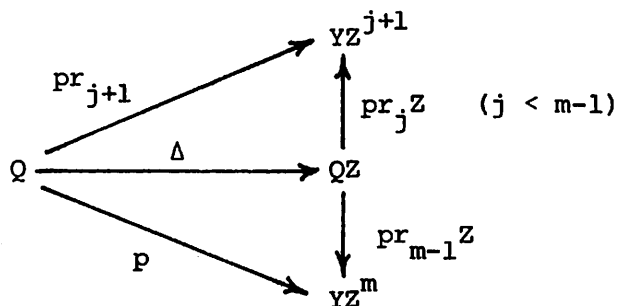
holds for all i . By (H_i^j) we mean the unique morphism f such that $\text{pr}_j f = H_i^j$ for all $0 \leq j < m$. If such p exists, H_i^j is said to be (*morphically*) *recurrent of degree m* . Dually, a *corecurrence morphism of degree m* is a morphism ρ such that

$$\begin{array}{ccc} \coprod (IX^i : 0 \leq i < m) & \xrightarrow{(H_i^j)} & YZ^j \\ & \swarrow \rho & \searrow H_m^j \\ & & IX^m \end{array}$$

holds for all j . If such ρ exists, H_i^j is *corecurrent of degree m* .

2 RECURRENCE THEOREM FOR MATRICES: Let H_i^j be a Hankel matrix which is recurrent of degree m . Then H_i^j has a realization with state object $Q = \coprod (YZ^j : 0 \leq j < m)$.

Proof: Define $\beta = \text{pr}_0 : Q \longrightarrow Y$, define $\tau : I \longrightarrow Q$ by $\text{pr}_j \tau = H_0^j : I \longrightarrow YZ^j$ and let $\delta : QX \longrightarrow Q$ correspond under adjointness to the Δ defined by



(where we make use of the fact [4, 1.9] that Z preserves products). We must show that $H_i^j = \sigma_j r_i$. We first show that $r_i = (H_i^j : 0 \leq j < m) : IX^i \rightarrow Q$ by induction on i . For $i = 0$ this is the definition of τ . For the inductive step, if $0 \leq j < m-1$ we have

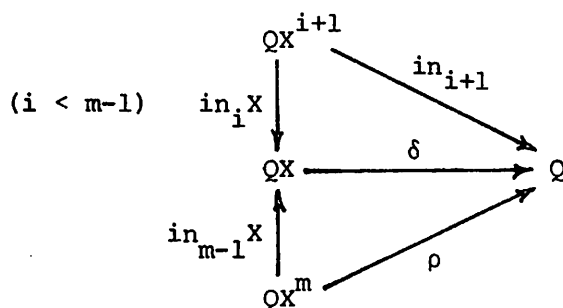
$$\begin{array}{c}
 IX^{i+1} \xrightarrow{r_i X} QX \xrightarrow{\delta} Q \xrightarrow{pr_j} YZ^j \\
 \hline
 IX^i \xrightarrow{r_i} Q \xrightarrow{\Delta} QZ \xrightarrow{pr_j Z} YZ^{j+1} \\
 = IX^i \xrightarrow{r_i} Q \xrightarrow{pr_{j+1}} YZ^{j+1} \\
 = H_i^{j+1} \\
 \hline
 H_{i+1}^j
 \end{array}$$

whereas, for $j = m-1$ we have the related argument that $pr_{m-1} Z \cdot \Delta \cdot r_i = p r_i = p (H_i^j : 0 \leq j < m)$ (by the induction hypothesis) $= H_i^m$ (by the definition of recurrence morphism) which corresponds under adjointness to H_{i+1}^{m-1} as desired. In particular, we have shown that $pr_0 r_i = H_0^i$ which is the case 'j = 0' in showing that $H_i^j = \sigma_j r_i$. But the inductive step on j is immediate from the fact that both H_i^j and $\sigma_j r_i$ satisfy the Hankel crossover condition. \square

3 CORECURRENCE THEOREM FOR MATRICES: Let H_i^j be a Hankel matrix which is corecurrent of degree m . Then H_i^j has a realization with state object $Q = \coprod (IX^i : 0 \leq i < m)$.

Proof: The proof is dual to that of 2 but we record the construction.

Define $\tau = in_0$, define β by $\beta in_i = H_i^0$ and define δ by



□

4 EXAMPLE: THE DECOMPOSABLE CASE. Here $X = Z$ is the identity functor of \mathcal{K} . The realization theory in this special case was studied in [1]. When \mathcal{K} is the category of vector spaces (or of modules over a ring) an adjoint system is just a linear system

$$I \xrightarrow{B} Q \quad Q \xrightarrow{A} Q \quad Q \xrightarrow{C} Y.$$

The same system description holds in any category. The adjointness correspondence is just

$$\frac{Q \xrightarrow{g} R}{Q \xrightarrow{g} R}$$

so that the codynamics is again A . We have $r_i = A^i B$ and $\sigma_j = CA^j$ so that $H_i^j = CA^{i+j} B$. A recurrence map takes the form $p : Y^m \rightarrow Y$ (where Y^m denotes the m -fold product of copies of Y). If \mathcal{K} is a category such as modules over a ring or groups in which objects are 'sets with structure', morphisms are determined as functions (though not all functions may be admissible as morphisms) and the underlying set of a finite product is the product of the underlying sets, the recurrence condition literally takes the form

$$p(H_1^0(u), \dots, H_1^{m-1}(u)) = H_1^m(u)$$

for all $i \geq 0$, $u \in I$. When \mathcal{K} is the category of groups, theorem 2 above is a direct generalization of the sufficiency proof of [5, theorem 2] (for the necessity part see 3.4 below). Owing to the fact that the coproduct is the free product in the category of groups (see [3, 3.2.3] for the details)

corecurrence for group systems may not be a very useful notion. When \mathcal{K} is the category of modules over a ring (or semiring), however, a corecurrence map has the form $\rho : I \longrightarrow I^m$ and the corecurrence condition is

$$H_m^j(u) = \sum_{k=0}^{m-1} H_k^j \rho_k(u)$$

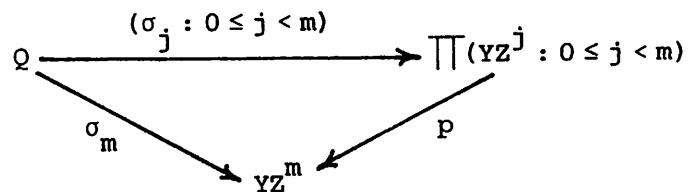
where $\rho(u) = (\rho_0(u), \dots, \rho_{m-1}(u))$ and $j \geq 0$. This essentially recaptures the recurrence polynomials familiar in linear system theory (see, e.g., [6, XVI.9.1] and [9, p. 3405]).

3. Recurrence for Systems

1 DEFINITIONS: Let M be an adjoint system. M is *reachable in time i* if $(r_k : 0 \leq k \leq i) : \coprod (IX^k : 0 \leq k \leq i) \longrightarrow Q \in \mathcal{E}$ and, dually, M is *observable in time j* if $(\sigma_k : 0 \leq k \leq j) \in \mathcal{M}$. M is *reachable in bounded time* if M is reachable in time i for some i , and M is *observable in bounded time* if M is observable in time j for some j .

Reachability in bounded time implies reachable [4, 2.10]. Dually, observability in bounded time implies observable.

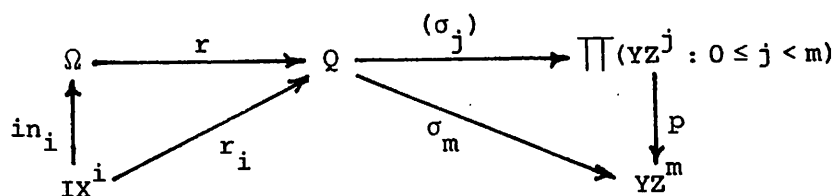
2 RECURRENCE THEOREM FOR SYSTEMS: Let M be an adjoint system with Hankel matrix $H_i^j = \sigma_j r_i$. Let $p : \prod (YZ^j : 0 \leq j < m) \longrightarrow YZ^m$ be a morphism. Consider the diagram



Then

- (i) If the diagram commutes, H_i^j is recurrent.
- (ii) If M is reachable and H_i^j is recurrent with recurrence map p , the diagram commutes.

Proof: Consider the diagram



In the notation below, $0 \leq j < m$. From the definition of a product, we have that $r_i(\sigma_j) = (r_i \sigma_j) = (H_i^j)$. Thus, if $p(\sigma_j) = \sigma_m$, $p(H_i^j) = p(\sigma_j r_i) = p(\sigma_j) r_i = \sigma_m r_i = H_i^m$ for all i , and thus p is a recurrence map. To prove (ii), if p is a recurrence map then for all i , $[p(\sigma_j) r] \text{in}_i = p(H_i^j) = H_i^m = [\sigma_m r] \text{in}_i$. By the definition of a coproduct, $p(\sigma_j) r = \sigma_m r$. As M is reachable, r is an epimorphism so that $p(\sigma_j) = \sigma_m$ as desired. \square

Theorem 2 reduces the problem of finding a recurrence morphism of degree m to the filling-in of a single diagram. Before exploiting the result, we pause to state the dual theorem:

3 CORECURRENCE THEOREM FOR SYSTEMS: Let M be an adjoint system with Hankel matrix H_i^j and let $\rho : IX^m \longrightarrow \coprod (IX^i : 0 \leq i < m)$ be a morphism. Then

- (i) If $(r_i : 0 \leq i < m) \rho = r_m$ H_i^j is corecurrent.
- (ii) If M is observable and H_i^j is corecurrent with corecurrence map ρ , then $(r_i : 0 \leq i < m) \rho = r_m$. \square

Our first application of 2 is to finite-state decomposable systems, fully recapturing theorem 2 of [5] by choosing \mathcal{K} to be the category of groups and \mathcal{F} to be the finite groups. We note that our proof specializes to theirs in all essentials. The generalization to any class of universal algebras is immediate.

4 RECURRENCE THEOREM FOR FINITE-STATE DECOMPOSABLE SYSTEMS: Let X and Z be the identity functor of \mathcal{K} as in example 2.4. Let \mathcal{F} be a class of 'finite' objects in the sense that the following two axioms hold:

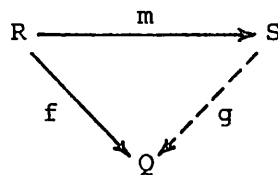
- (i) $Y^m \in \mathcal{F}$ for all $m > 0$.
- (ii) For all $Q \in \mathcal{F}$, the set $\mathcal{K}(Q, Y)$ of all morphisms from Q to Y is finite.

Let $H_i^j : I \longrightarrow Y$ be an arbitrary Hankel matrix. Then H_i^j has a realization with state object in \mathcal{F} if and only if H_i^j is recurrent.

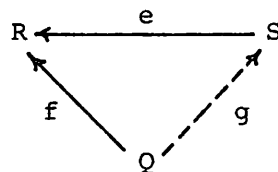
Proof: One direction is clear from 2.2. Conversely, Let M realize H_i^j with state object $Q \in \mathcal{F}$. Since $\sigma_j : Q \rightarrow Y$ and there are only finitely many \mathcal{K} -morphisms from Q to Y , there exists $0 \leq k < m$ with $\sigma_k = \sigma_m$. But then $\text{pr}_k(\sigma_j : 0 \leq j < m) = \sigma_k = \sigma_m$ so that $\text{pr}_k : Y^m \rightarrow Y$ is a recurrence map by 2.(i). \square

The reader may easily formulate the dual theorem to 4.

5 INJECTIVE AND PROJECTIVE OBJECTS: An object Q is *injective* if each diagram of form



may be completed (not necessarily uniquely) -- that is, $gm = f$ for some g -- whenever $m \in \mathcal{M}$. Dually, Q is *projective* if



whenever $e \in \mathcal{E}$.

The following result is immediate from 2 and 3. It is trivially checked that any product of injectives is injective and any coproduct of projectives is projective so that, e.g., the condition that YZ^m be injective for all m is often reducible to showing that Y is injective.

6 OBSERVATION: Let M be an adjoint system with Hankel matrix H_i^j . If YZ^m is injective and if M is observable in time m then H_i^j is recurrent of degree m . If IX^m is projective and if M is reachable in time m then H_i^j is corecurrent of degree m . \square

7 EXAMPLE: In the category of sets, all non-empty sets are projective and all sets with at least two elements are injective. It follows that, in the context of the example of section 1, even if I , Y or A are infinite, observability in bounded time implies recurrence, and reachability in bounded time implies corecurrence.

8 EXAMPLE: In the linear case of example 2.4 with \mathcal{K} the category of modules over a ring (or semiring), if I is a free (or projective) module then $IX^m = I^m$ is projective for all m . Using 6, we recapture the familiar result that if $(A^i_B : 0 \leq i < m) : I^m \longrightarrow Q$ has the same image as $(A^i_B : i \geq 0)$ then CA^{i+j}_B is recurrent of degree m . This result does not generalize to arbitrary universal algebras because -- unlike the module case where finite products are also finite coproducts -- there is no reason for I^m to be projective when I is free.

A converse result to 6 is a trivial extension of 2.2:

9 PROPOSITION: If H^j_i is recurrent of degree m , H^j_i has a realization which is reachable in time m . If H^j_i is corecurrent of degree m , H^j_i has a realization which is observable in time m .

Proof: Use the realizations of 2.2, 2.3. It is trivial to check that

$(\sigma_j : 0 \leq j < m)$ and $(r_i : 0 \leq i < m)$ are, respectively, the identity map. \square

4. Noetherian and Artinian Objects

Noetherian and Artinian objects were defined in [4, 3.1]. We briefly recall the essentials. An object Q of \mathcal{K} is *Noetherian* if every strictly ascending chain of subobjects of Q is finite. Here a subobject is an equivalence class $[R, m]$ of pairs (R, m) with $m: R \rightarrow Q$ in \mathcal{M} . The length of the longest strictly ascending chain of subobjects of Q is called the \mathcal{M} -height of Q . Thus finite \mathcal{M} -height implies Noetherian, but the converse is false. Dually Q is *Artinian* if every strictly ascending chain of quotient objects of Q is finite. The length of the longest strictly ascending chain of quotient objects of Q is called the \mathcal{E} -height of Q .

In the category of sets, subobjects and quotient objects take on their usual meanings. We note that the partial orderings have been chosen so that ascending chains -- of either subsets or quotient sets -- have increasing cardinality.

In the category of modules over a ring, the passage from a submodule S to its cokernel Q/S establishes an anti-isomorphism of partially ordered sets between subobjects and quotient objects. For this reason, Artinian is equivalent to the descending chain condition on subobjects (the usual definition in module theory) and a module has finite height if and only if it is simultaneously Noetherian and Artinian.

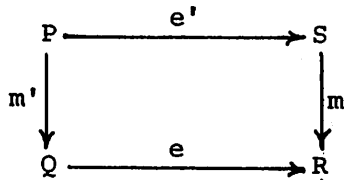
In abelian groups, Noetherian = finitely-generated whereas finite \mathcal{M} -height = finite. The additive integers is not Artinian. For vector spaces, on the other hand, Noetherian = Artinian = finite height and \mathcal{M} -height = \mathcal{E} -height = $1 + \text{dimension}$.

1 A class \mathcal{F} of objects is *closed under subobjects* if whenever $m: Q \rightarrow F \in \mathcal{M}$ with $F \in \mathcal{F}$, also $Q \in \mathcal{F}$. Similarly, \mathcal{F} is *closed under quotients* if $e: F \rightarrow Q \in \mathcal{E}$ and $F \in \mathcal{F}$ imply $Q \in \mathcal{F}$.

A class of 'finite' objects might well be expected to be closed under subobjects and quotients. It is obvious that 'Noetherian' is closed under

subobjects and that 'Artinian' is closed under quotients. We conclude this section with a proposition that concludes that 'Noetherian' is closed under quotients, leaving the statement of the dual theorem as an exercise for the reader.

2 LEMMA [8, 3.4.12]: *Given a pullback square*

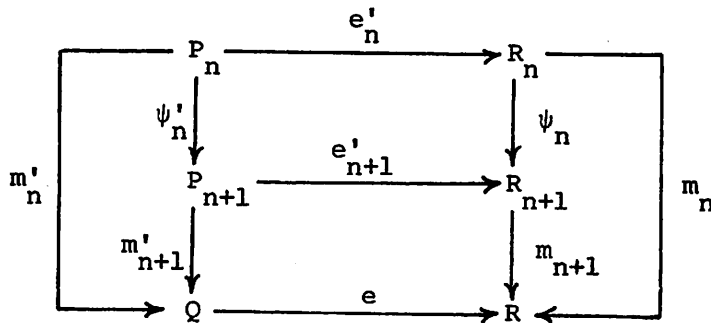


with $m \in \mathcal{M}$, also $m' \in \mathcal{M}$. It is easily verified that $[P, m']$ is uniquely determined by $[S, m]$; $[P, m']$ is the inverse image of $[S, m]$ under e .

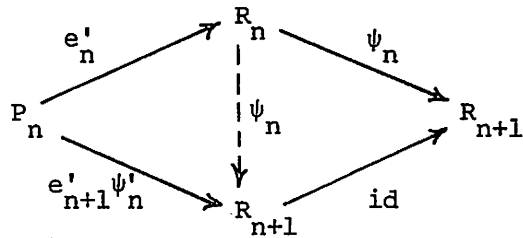
We say that \mathcal{E} is closed under inverse images if $e' \in \mathcal{E}$ whenever $e \in \mathcal{E}$ and $m \in \mathcal{M}$ in a pullback square as above.

3 PROPOSITION: *If \mathcal{E} is closed under inverse images then 'Noetherian' is closed under quotients.*

Proof: Let Q be Noetherian, let $e : Q \rightarrow R \in \mathcal{E}$ and let $[R_n, m_n]$ be an ascending chain of subobjects of R . Let (P_n, e'_n, m'_n) be pullbacks of



(e, m_n) as shown. If $m_{n+1} \psi_n = m_n$ defines ψ_n then, since $m_{n+1}(\psi_n e'_n) = m_n e'_n = e m'_n$, there exists unique ψ'_n as shown (by the definition of a pullback). Thus $[P_n, m'_n]$ is an ascending chain of subobjects of Q and there exists N such that ψ'_n is an isomorphism if $n \geq N$. But for $n \geq N$ we have two \mathcal{E} - \mathcal{M} factorizations of the same map:



This induces an isomorphism as shown which must be ψ_n . \square

In both the category of sets and the category of modules over a ring, both 'Noetherian' and 'Artinian' are closed under subobjects and under quotients. This may be shown to be a corollary of the proposition although, of course, these results are well-known.

5. Noetherian Corecurrence

1 LEMMA: Let M realize H_i^j with state object Q . Then there exists a diagram

$$Q_H \xleftarrow{e} R \xrightarrow{m} Q$$

with $m \in \mathcal{M}$, $e \in \mathcal{E}$. R is the 'reachable part' of Q .

Proof: Recall the canonical realization H_H of section 1 and consider the diagram

$$\begin{array}{ccccc}
 \Omega & \xrightarrow{e_1} & R & \xrightarrow{m} & Q \\
 \downarrow r_H & \nearrow e & & & \downarrow \sigma \\
 Q_H & & & \xrightarrow{\sigma_H} & \Gamma
 \end{array}$$

where (e_1, m) is an \mathcal{E} - \mathcal{M} factorization of r . The outside commutes by the bi-index principle [4, 1.10] inducing the diagonal fill-in e [4, 2.7] which is in \mathcal{E} by [4, 2.10]. □

2 THEOREM: Let \mathcal{F} be a class of Noetherian objects which is closed under \mathcal{E} , closed under \mathcal{M} , closed under the formation of finite coproducts and which contains IX^i for every $i \geq 0$. Assume that IX^i is projective for all i . Then the following three conditions on the Hankel matrix H_i^j are equivalent:

- (i) H_i^j is corecurrent.
- (ii) H_i^j has a realization with state object in \mathcal{F} .
- (iii) The canonical state object Q_H is in \mathcal{F} .

Proof: For (i) \implies (ii) use 2.3. For (ii) \implies (iii) use lemma 1. That (iii) \implies (ii) is obvious. To see that (iii) \implies (i) combine 3.6 with the fact [4, Proposition 4.1] that if M is reachable and Q is Noetherian then M is reachable in bounded time. □

3 COROLLARY: Let R be an arbitrary ring and let $H_i^j : I \longrightarrow Y$ be R -linear where I is a projective Noetherian R -module. Then H_i^j has a realization with a Noetherian state module if and only if H_i^j is corecurrent (see example 2.4). \square

4 COROLLARY: Let R be an arbitrary ring and let $H_i^j : I \longrightarrow Y$ be R -linear where I is a projective R -module. Then H_i^j has a realization whose state module has finite homological dimension if and only if H_i^j is corecurrent.
Proof: This is theorem 2 with \mathcal{F} the class of modules which have finite homological dimension. See [7, VII.1] noting exercise 3 on page 204. \square

Corollary 3 may also be stated when $\mathcal{F} =$ modules with finite \mathcal{M} -height and when $\mathcal{F} =$ finite modules (although the latter is also clear from 3.4). The duals of these theorems are also of some interest. We state the dual of corollary 3:

5 COROLLARY: Let R be an arbitrary ring and let $H_i^j : I \longrightarrow Y$ be R -linear where Y is an injective Artinian R -module. Then H_i^j has a realization with an Artinian state-module if and only if H_i^j is recurrent. \square

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ON THE EVOLUTION OF GENERALIZED HANKEL MATRICES¹

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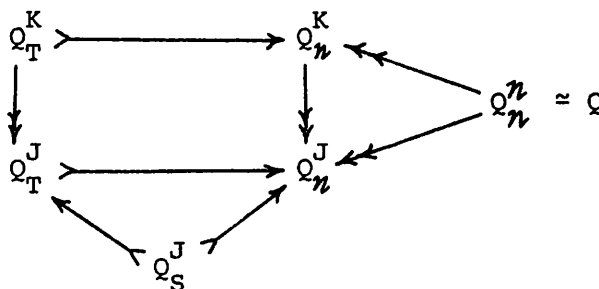
ABSTRACT

For the Hankel matrix of an adjoint system, the image of each submatrix is a subquotient of the minimal state space. While the simple recursion principle fails, the evolution of the Hankel matrix, as input length and observation time increase, follows a supremum-preserving principle. The theory applies to linear and bilinear systems, automata, and group systems in both the time-varying and time-invariant cases, and is new even for linear systems.

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1. Motivation

If $B : I \rightarrow Q$, $A : Q \rightarrow Q$, $C : Q \rightarrow Y$ is a reachable and observable linear system, let Q_S^J , for J, S non-empty subsets of the set \mathcal{N} of natural numbers, be the image of the partial Hankel matrix $(CA^{i+j}B : i \in S, j \in J)$ considered as a linear map from I^S to Y^J . Thus $Q_S^{\mathcal{N}}$ is the set of states reachable by inputs applied at times in S , and then Q_S^J is obtained by merging those states that are indistinguishable by observations made at times in J . Hence, given T, K finite subsets of \mathcal{N} with $S \subset T$ and $J \subset K$ there is an obvious commutative diagram



where \longrightarrow denotes a projection whereas \triangleright denotes an injection which fills in unused slots with zeroes. Thus each Q_S^J is a 'subquotient' of Q , that is, is a subobject of a quotient object of Q . The diagram above is used to motivate the definition of a partial order on the set of subquotients of an object in a category. The main abstract result is that, replacing the $CA^{i+j}B$ by a much more general bisequence of morphisms, the passage $(J, S) \mapsto Q_S^J$ preserves countable suprema. In particular, the theorem applies to the Hankel matrix of any adjoint system [2, 3] with projective input object.

Recall that a sequence v_n in V is defined by simple recursion if $v_n = g^n(v_0)$ for some $g : V \rightarrow V$. The ascending chain $\text{span}(B, AB, \dots, A^n B)$ of reachability subspaces of a linear system is defined by simple recursion:

let V be the set of subspaces of Q and set $g(S) = \text{span}(S \cup A(S))$. More generally, the ascending chain of reachability subobjects of the state object (and, dually, the ascending chain of observability quotient objects of the state object) of any adjoint system is defined by simple recursion [2, 4.4]. Examples abound to show that for the bisequence Q_S^J of a linear system, none of the sequences

$$Q_S^{\bar{n}} \quad (\text{fixed } S)$$

$$Q_{\bar{n}}^J \quad (\text{fixed } J)$$

$$Q_{\bar{n}}^{\bar{n}}$$

(where $\bar{n} = \{1, \dots, n\}$) need be defined by simple recursion. In particular, $Q_{\bar{n}}^{\bar{n}} = Q_{\bar{n}+1}^{\bar{n}+1}$ does not imply $Q_{\bar{n}+1}^{\bar{n}+1} = Q_{\bar{n}+2}^{\bar{n}+2}$ (as is well known in the study of partial realizations). This paper, then, was motivated by the desire to find a more positive result for Q_S^J .

2. Generalities

Let \mathcal{K} be a category and let $(\mathcal{E}, \mathcal{M})$ be an image factorization system in \mathcal{K} , that is, \mathcal{E}, \mathcal{M} are subclasses of morphisms satisfying the following four axioms:

IFS1. \mathcal{E} and \mathcal{M} are each closed under composition.

IFS2. Every isomorphism is both in \mathcal{E} and in \mathcal{M} .

IFS3. Every element of \mathcal{E} is an epimorphism and every element of \mathcal{M} is a monomorphism. (A map $f: R \rightarrow S$ is an *epimorphism* if whenever $g, h: S \rightarrow T$ satisfy $gf = hf$, then $g = h$; dually, f is a *monomorphism* if whenever $a, b: Q \rightarrow R$ satisfy $fa = fb$ then $a = b$.)

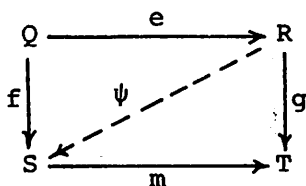
IFS4. Every morphism $f: Q \rightarrow R$ admits an \mathcal{E} - \mathcal{M} factorization (e, m) -- that is, $f = m e$ with $e \in \mathcal{E}$, $m \in \mathcal{M}$ -- and such factorizations are unique up to isomorphism in the sense that if (e', m') is another one then there exists a unique isomorphism ψ with $\psi e = e'$ and $m' \psi = m$.

The category of sets and the category of modules over a ring both have \mathcal{E} = surjections and \mathcal{M} = injections as unique image factorization system. For the theory of image factorization systems see [5, 3.4]. We record two general lemmas here. The first is standard. The second, pointed out to us by J. R. Isbell, improves [5, 3.4.14].

1 DIAGONAL FILL-IN LEMMA: *The following two statements about a morphism $e: Q \rightarrow R$ are equivalent.*

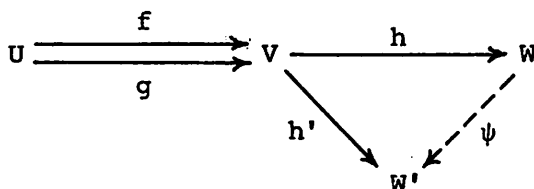
(i) $e \in \mathcal{E}$.

(ii) For every commutative square $ge = mf$ with $m \in \mathcal{M}$,



there exists unique ψ with $\psi e = f$ and $m\psi = g$. \square

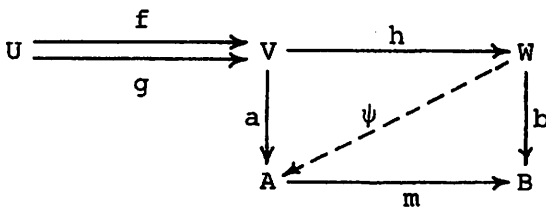
Recall that, given $f, g : U \rightarrow V$, their coequalizer is a morphism $h : V \rightarrow W$ such that $hf = hg$, and such that whenever $h'f = h'g$ there exists unique ψ with $\psi h = h'$.



The category \mathcal{K} has coequalizers if every pair f, g has a coequalizer. See [1, pp. 20, 34] and [4, pp. 64-65]. The category of modules over a ring has coequalizers -- let h be the canonical projection to the cokernel of $f - g$. The category of sets has coequalizers -- let h be the canonical projection to V/R where R is the smallest equivalence relation containing $\{(f(x), g(x)) \mid x \in U\}$.

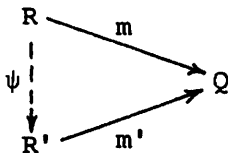
2 LEMMA: If h is the coequalizer of some f, g then $h \in \mathcal{E}$.

Proof: We use lemma 1. Consider the diagram shown where $ma = bh$ and



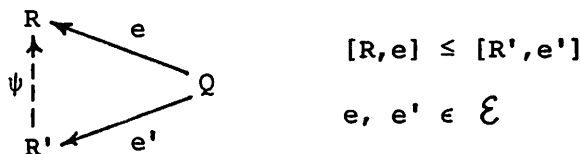
$m \in \mathcal{M}$. Since m is a monomorphism $af = ag$ and the desired ψ is induced by the coequalizer property. \square

Let Q be an object of \mathcal{K} . The set of all pairs (R, m) with $m : R \longrightarrow Q$ in \mathcal{M} admits a reflexive and transitive order by defining $(R, m) \leq (R', m')$ if there exists ψ with $m'\psi = m$



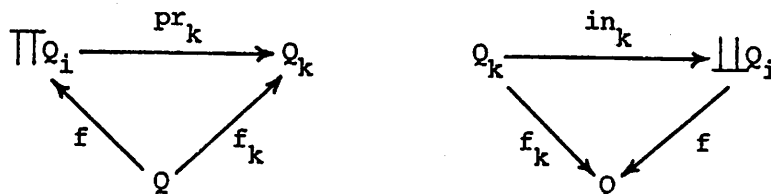
(note that such ψ is necessarily unique and is itself in \mathcal{M}). Thus $(R, m) \sim (R', m')$ if $(R, m) \leq (R', m')$ and $(R', m') \leq (R, m)$ is an equivalence relation whose equivalence classes $[R, m]$ are the *subobjects* of Q . $[R, m] \leq [R', m']$ if $(R, m) \leq (R', m')$ is a well-defined partial order on the subobjects of Q . It is easily seen that $[R, m] = [R', m']$ if and only if there exists an isomorphism ψ with $m'\psi = m$. Q is *Noetherian* if every strictly ascending chain of subobjects of Q is finite.

The dual concepts relative to \mathcal{K} are formulated by repeating the above definitions in the dual category \mathcal{K}^{op} . Thus, the ordering on *quotient objects* of Q is described by



(Note that we reverse the arrows, but not the ordering.) We say Q is *Artinian* if Q is co-Noetherian, that is, if every strictly ascending chain of quotient objects of Q is finite. Facts about Noetherian and Artinian objects appear in [2, 3].

We assume that every countable family Q_i of objects of \mathcal{K} has a product $pr_k : \prod Q_i \longrightarrow Q_k$ and a coproduct $in_k : Q_k \longrightarrow \coprod Q_i$ [1, 1.2], [4, III.3, III.4]. Hence there are bijective correspondences

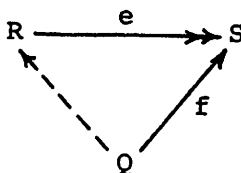


between morphisms f and families f_i as shown in the diagram. An immediate consequence is the

3 BI-INDEX PRINCIPLE: If $(Q_i : i \in I)$, $(R_j : j \in J)$ and $f_i^j : Q_i \longrightarrow R_j$ then, so long as the coproduct and product exist, there exists a unique morphism $f : \coprod Q_i \longrightarrow \prod R_j$ such that $\text{pr}_j f \text{in}_i = f_i^j$ for all i, j .

Proof: Define $f^j : \coprod Q_i \longrightarrow R_j$ by $f^j \text{in}_i = f_i^j$ and then define f by $\text{pr}_j f = f^j$. Uniqueness is left as an exercise. \square

An object Q is *projective* if each diagram of the form



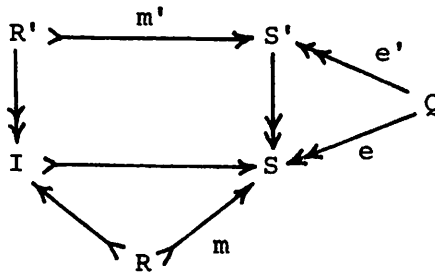
may be completed (not necessarily uniquely) as shown. Clearly, a coproduct of projective objects is projective. In the category of sets all objects are projective. In the category of modules over a ring, every free module is projective.

3. Subquotients

Fix an object Q . Consider (R, m, S, e) where

$$R \xrightarrow{m} S \xleftarrow{e} Q$$

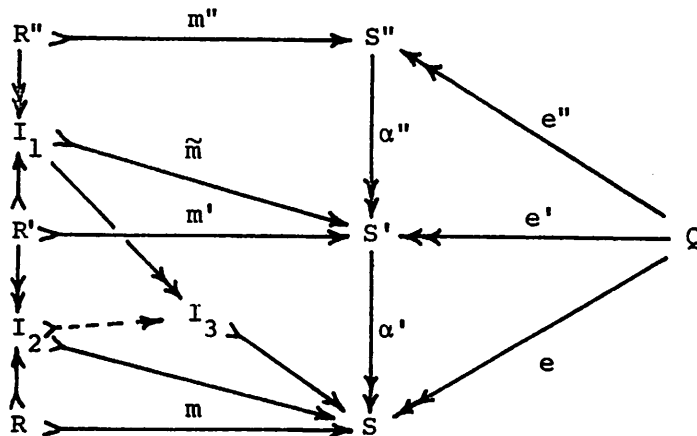
Motivated by the diagram of section 1, say that $(R, m, S, e) \leq (R', m', S', e')$ if there exists a commutative diagram



An equivalent way to say this is that $[S, e] \leq [S', e']$ as quotient objects of Q and that $[R, m]$ is contained in the image of $R' \rightarrow S' \rightarrow S$. It is clear that \leq is reflexive. We also have

1 LEMMA: \leq is transitive.

Proof: Suppose that $(R, m, S, e) \leq (R', m', S', e') \leq (R'', m'', S'', e'')$. This is expressed in the diagram

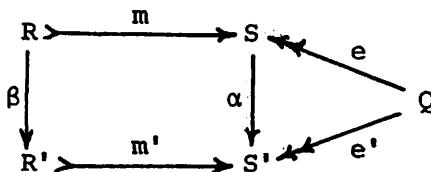


where $I_1 \twoheadrightarrow I_3 \twoheadrightarrow S$ is obtained by \mathcal{E} - \mathcal{M} factorization of $\alpha' \tilde{m}$, and where $I_2 \twoheadrightarrow I_3$ is then obtained by diagonal fill-in, Lemma 2.1. But

this shows that $(R, m, S, e) \leq (R'', m'', S'', e'')$. \square

2 DEFINITION: A *subquotient object* of Q is an antisymmetry class $[R, m, S, e]$ arising from the reflexive and transitive order above. The subquotients of Q are partially ordered by $[R, m, S, e] \leq [R', m', S', e']$ if $(R, m, S, e) \leq (R', m', S', e')$. The following result shows that the ordering relation is not "too abstract".

3 OBSERVATION: $[R, m, S, e] = [R', m', S', e']$ if and only if there exist isomorphisms α and β as shown.



Proof: If $[S, e] = [S', e']$ as quotient objects of Q then α is an isomorphism. But then $[R, m] = [R', \alpha^{-1} m']$ as subobjects of S so that β is an isomorphism. \square

4 PROPOSITION: Consider the two conditions

- (i) Every strictly ascending chain of subquotient objects of Q is finite.
- (ii) Q is Noetherian and Artinian.

Then (i) \implies (ii). If \mathcal{E} is closed under inverse images [3, 4.2] then

(ii) \implies (i).

Proof: If $[R_i, m_i]$ is a strictly ascending chain of subobjects then $[R_i, m_i, Q, id]$ is a strictly ascending chain of subquotient objects. If $[S_j, e_j]$ is a strictly ascending chain of quotient objects then $[S_j, id, S_j, e_j]$ is a strictly ascending chain of subquotient objects. Conversely, if $[R_n, m_n, S_n, e_n]$ is a strictly ascending chain of subquotients then, as Q is Artinian, we eventually have $[S_n, e_n] = [S_{n+1}, e_{n+1}] = \dots$ and $[R_{n+k}, m_{n+k}]$ is an ascending chain of subobjects of S_n . Now use [3, 4.3] which asserts that 'Noetherian' is closed under quotients providing \mathcal{E} is closed under inverse images. \square

5 COROLLARY: In the category of modules over a ring, a module is Noetherian and Artinian if and only if its every strictly ascending chain of its sub-quotients is finite. □

4. Biadequacy

We consider an arbitrary bisequence $f_i^j : P_i \longrightarrow R_j$, j being just a superscript.

1 For each pair (S, J) of non-empty subsets of \mathcal{N} define

$$P_S = \coprod (P_i : i \in S)$$

$$R^J = \prod (R_j : j \in J)$$

$f_S^J : P_S \longrightarrow R^J$ is defined, using the bi-index principle 2.3, by

$$\text{pr}_j f_S^J \text{in}_i = f_i^j \quad (i \in S, j \in J).$$

2 Pairs of non-empty subsets of \mathcal{N} constitute a partially ordered set with ordering $(S, J) \leq (T, K)$ if $S \subset T$ and $J \subset K$. If $(S, J) \leq (T, K)$ we have a commutative diagram

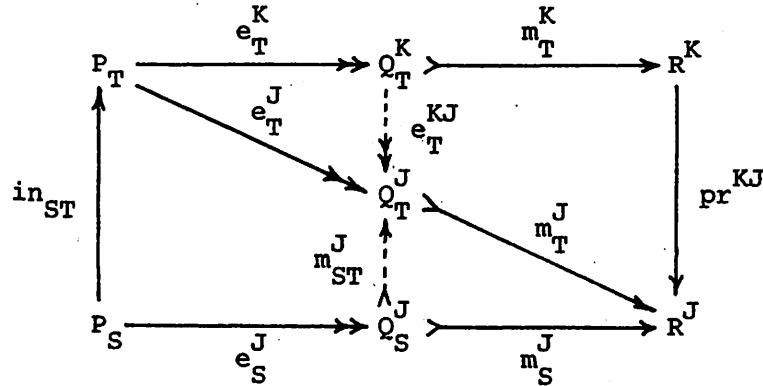
$$\begin{array}{ccc}
 P_T & \xrightarrow{f_T^K} & R^K \\
 \uparrow \text{in}_{ST} & \searrow f_T^J & \downarrow \text{pr}^{KJ} \\
 P_S & \xrightarrow{f_S^J} & R^J
 \end{array}$$

where $\text{in}_{ST} \text{in}_i = \text{in}_i$ ($i \in S$) and $\text{pr}_j \text{pr}^{KJ} = \text{pr}_j$ ($j \in J$). The diagram commutes by the bi-index principle.

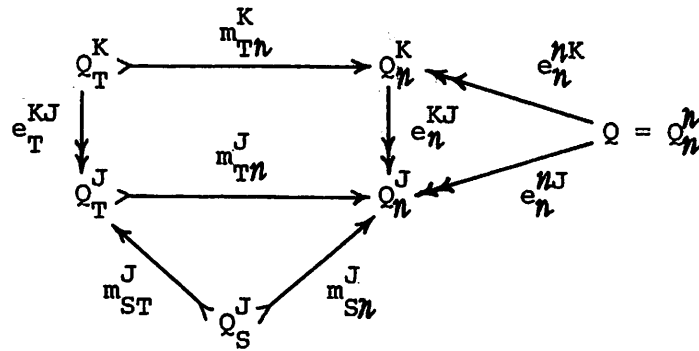
3 Fix an \mathcal{E} - \mathcal{M} factorization of f_S^J

$$P_S \xrightarrow{e_S^J} Q_S^J \xrightarrow{m_S^J} R^J .$$

4 Diagonal fill-in on the diagram of (2) yields



5 Whenever $(S, J) \leq (T, K)$ we have



This diagram recaptures that of section 1. Here Q abbreviates Q_n^J .

Thus Q_S is a subquotient object of Q (we use Q_S^J qua subquotient object for the more cumbersome $[Q_S^J, m_{S\mathcal{N}}^J, Q_n^J, e_n^{KJ}]$) and the passage from (S, J) to Q_S^J is order-preserving.

6 **DEFINITION:** Let A, B be non-empty subsets of \mathcal{N} . (A, B) is *biadequate* if $(A, B) \leq (S, J)$ implies $Q_A^B = Q_S^J$. By 3.3, $Q_A^B = Q_S^J$ if and only if e_n^{KJ} , e_T^{KJ} and m_{ST}^J are isomorphisms.

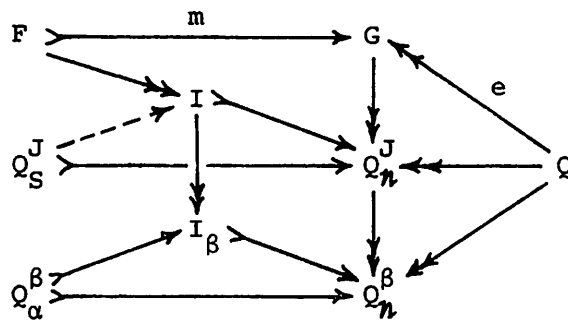
The following result is immediate from [2, 3.12] applied to the passage $s \mapsto Q_S^S$.

7 **THEOREM:** Let $\bar{i} = \{0, \dots, i\}$. Define $A = \{i \in \mathcal{N} \mid i = 0 \text{ or } Q_{i-1}^{\bar{i}-1} < Q_i^{\bar{i}}\}$. Then (A, A) is *biadequate*. □

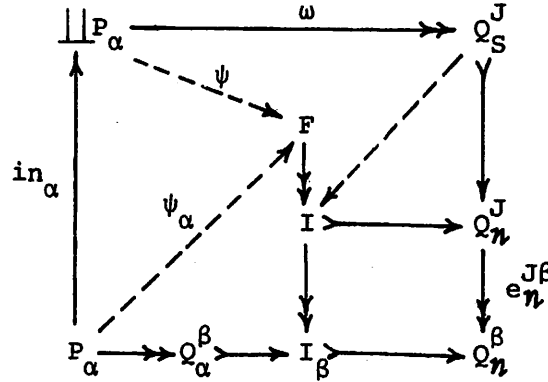
We now turn to the main result of this paper.

8 THEOREM: In the context of (1) - (5), assume that P_i is projective for all $i \in \mathcal{N}$ and assume that \mathcal{K} has coequalizers. Then the passage from (S, J) to the subquotient object Q_S^J of Q preserves suprema of non-empty countable families.

Proof: Let S_α, J_β be non-empty countable families of non-empty subsets of \mathcal{N} and set $S = \cup S_\alpha, J = \cup J_\beta$. We must show that Q_S^J is the supremum, in the partially ordered set of subquotient objects of Q , of the Q_α^β (we will use the subscript α for S_α and the superscript β for J_β). That Q_S^J is an upper bound follows from (5). Now let (F, m, G, e) be another upper bound, and consider the diagram shown. To explain the diagram, we first apply the dual



of [2, lemma 3.9] to the singly-indexed sequence $f_n^j : Q \rightarrow R^j$ to conclude that Q_n^J is the supremum of the Q_n^β as quotient objects of Q . This is why G factors through Q_n^J . Let I be the image of $F \rightarrow Q_n^J$ and then let I_β be the image of the various $I \rightarrow Q_n^\beta$. Then I_β is the image of $F \rightarrow Q_n^\beta$ so that, by hypothesis, Q_α^β factors through I_β for all α and β . We must show that Q_S^J factors through I . The argument is outlined in the second diagram. The desired factorization $Q_S^J \rightarrow I$ is constructed by diagonal fill-in.



This breaks down into three steps: (i) defining $\omega \in \mathcal{E}$; (ii) defining ψ ; (iii) proving that the two maps $t = \coprod P_\alpha \longrightarrow F \longrightarrow Q_n^J$ and $u = \coprod P_\alpha \longrightarrow Q_S^J \longrightarrow Q_n^J$ are equal.

Step (i). Define $\theta : \coprod P_\alpha \longrightarrow P_S$ by $\theta in_\alpha = in_{\alpha S}$. Then $\theta \in \mathcal{E}$. To prove it use 2.1 and the fact that S is the union of the S_α ; proof details (with the same notations) appear in [2, proof of lemma 3.9]. Then $\omega = e_S^J \theta : \coprod P_\alpha \longrightarrow P_S \longrightarrow Q_S^J \in \mathcal{E}$.

Step (ii). Any coproduct of projectives is again projective, so each P_α is projective. Thus there exist ψ_α and hence ψ as shown.

Step (iii). It is routinely checked that $e_n^{J\beta} t in_\alpha = m_{\alpha n}^\beta e_\alpha^\beta = e_n^{J\beta} t in_\alpha$ for all α and β . By the coproduct property it follows that $e_n^{J\beta} t = e_n^{J\beta} u$ for all β . From this we conclude that $t = u$ as follows. Let $h : Q_n^J \longrightarrow W$ be the coequalizer of t and u . Then $h \in \mathcal{E}$ by 2.2. Thus, as a quotient object of Q , $Q_n^J \geq W$. On the other hand, $e_n^{J\beta}$ factors through h because $e_n^{J\beta} t = e_n^{J\beta} u$ and h is the coequalizer of t and u , so that $W \geq Q_n^\beta$ for each β . But as Q_n^J is the supremum of the Q_n^β , $Q_n^J = W$ and h is an isomorphism. Since $th = uh$ we have $t = thh^{-1} = uhh^{-1} = u$. \square

As a corollary, we obtain:

9 THEOREM: If (A,B) is 'one-step biadequate' in the sense that
 $Q_{AU\{i\}}^B = Q_A^B = Q_A^{BU\{j\}}$ for all $i, j \in \mathcal{I}$ then (A,B) is biadequate.

Proof: If $(A,B) \leq (S,J)$ then $(S,J) = \sup\{AU\{i\}, BU\{j\} \mid i \in S, j \in J\}$. \square

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