VARIETORS AND MACHINES[1]

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INTRODUCTION

Some landmark dates in universal algebra:

1935 G. Birkhoff's introduction of (finitary and infinitary) Ω-algebras and their free algebras.

1950's Rigorous attention to the construction of infinitary free algebras by (at least) Diener, Felscher, Harzheim, Henkin, Kerkhoff, Lowig, Monk, Słomiński and Tarski.

1963 F. W. Lawvere's thesis, "Functorial semantics of algebraic theories."

1965 S. Eilenberg and J. C. Moore's introduction of algebras over a 'triple'.

1966 F. E. J. Linton's varietal functors.

1974 The Trnková Seminar in Prague: varietors.

The last item has yet to receive expository treatment. I have asked my friends, Jiří Adámek and Věra Trnková to write one and they have kindly complied.

Varietors (a term first introduced here) generalize operator domains Ω. They provide an alternative approach to as basic a construction as the syntax of terms in first order logic. Their systematic use (for example in the framework of decision problems in universal algebra) remains unexplored.

In the language of 'triples', every varietor is a functor which generates a free triple (and the converse is true in the category of sets and in many other categories). Each operator domain Ω induces a corresponding varietor F_Ω (see A,2) whose generated free triple is that for Ω-algebras.

Now the way I recall the atmosphere in, say, 1972 is as follows. The problem of generating free triples is a special case of the problem of generating free monoids in a monoidal category. The construction should be a transfinite generalization of the construction of ordinary free monoids. The conjectured result is that an endofunctor of sets generates a free triple if and only if it is a quotient functor of an F_Ω where Ω has rank. At the very least, a free triple has rank. Concrete evidence that someone (besides me) believed any of this is found in the introduction of [D]. (Dubuc's construction has been 'localized' in E,4 below.)

But the truth is quite different. An endofunctor F of sets generates a free triple if and only if Fn has cardinal at most n for arbitrarily large cardinals n. Thus more functors than I would have originally suspected generate free triples and not all free triples have rank.

The most important contribution in my opinion, however, is the free algebra construction of A,II which is interesting even for F_Ω with Ω finitary. This algorithm may ultimately come to be regarded as a generalized 'Knaster-Tarski least fixed-point construction'. For example, an absolutely free Ω-algebra is constructed as the smallest set of trees closed under the operation of adjoining a new root. Notice, however, that for an arbitrary varietor, the ordinal at which the construction stops may increase with the cardinal of the generating set. This is counterintuitive to the experience derived from operator domains.

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PART A: FREE ALGEBRAS

A,I: Algebras of type F

A groupoid is (Q,d) with $d : Q \times Q \longrightarrow Q$ arbitrary. In universal algebra one regards d as a basic operation and imposes equations. From our point of view 'basic operations' can be more general. Thus a commutative groupoid is any function $P_2Q \longrightarrow Q$ if P_2Q denotes the set of doubletons of Q (see A,2 example 2). But no such construction captures the associative law. For future reference, we define a functor $F_{(2)} : \underline{\underline{Set}} \longrightarrow \underline{\underline{Set}}$ by $F_{(2)}X = X \times X$ and $F_{(2)}f = f \times f$, so that a groupoid is essentially a map $F_{(2)}Q \longrightarrow Q$.

A,1 Definition Let $\mathcal{K}$ be a category and let $F : \mathcal{K} \longrightarrow \mathcal{K}$ be a functor. An algebra of type F is a pair (Q,d) with Q an object, $d : FQ \longrightarrow Q$ a morphism. A homomorphism from (Q,d) to (Q',d') is a morphism $f : Q \longrightarrow Q'$ for which $f \cdot d = d' \cdot Ff$.

The resulting category of algebras is denoted by $\mathcal{K}(F)$.

Categories $\mathcal{K}(F)$ were introduced by Barr in [Bl], who denoted them by $\mathcal{K} : F$ (see E,II below). Arbib and Manes denote them by Dyn(F).

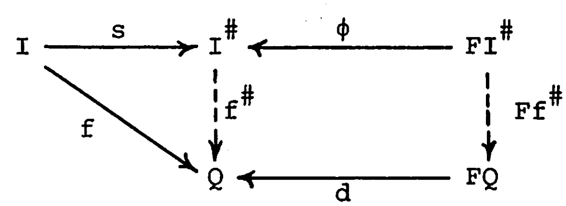
A,2 Examples 1) Generalizing $F_{(2)}$, let Ω be a type (i.e., a set with an arity function $ar : \Omega \longrightarrow$ cardinals). A universal algebra of type Ω consists of a set Q and an Ω-labelled system of operations d_ω, $\omega \in \Omega$, $d_\omega : Q^{ar(\omega)} \longrightarrow Q$. This can be viewed as a pair (Q,d), where $d : \coprod_{\omega \in \Omega} Q^{ar(\omega)} \longrightarrow Q$ is a map, defined as d_ω on $Q^{ar(\omega)}$. Thus, a universal algebra of type Ω is just an algebra of type $F_\Omega : \underline{\underline{Set}} \longrightarrow \underline{\underline{Set}}$, where

$$F_\Omega X = \coprod_{\omega\in\Omega} X^{ar(\omega)} \qquad \text{and} \qquad F_\Omega f = \coprod_{\omega\in\Omega} f^{ar(\omega)}.$$

Homomorphisms, as defined above, coincide with those in universal algebra.

2) Commutative groupoids are just algebras of type P_2 where P_2X is the set of all subsets $T \subset X$ with at most two elements and P_2f sends $\{x_1,x_2\}$ to $\{f(x_1),f(x_2)\}$.

A.3 Definition (See [AM 1].) Let I be an object (of "generators") in $\mathcal{K}$. A free algebra of type F, generated by I, is an algebra $(I^\#,\phi)$ together with a morphism $s : I \longrightarrow I^\#$, universal in the following sense. For every algebra (Q,d) of type F and for every morphism $f : I \longrightarrow Q$ there exists a unique homomorphism $f^\#: (I^\#,\phi) \longrightarrow (Q,d)$ with $f = f^\# \cdot s$



If every object I generates a free algebra then F is called a varietor. (Arbib and Manes use the terms input process and recursion process [AM 3] depending on context.)

A.4 Examples The free algebra of type $F_{(2)}$ is the free groupoid $I^\#$ (of all binary trees with leaves labelled in I). The free algebra of type P_2 is the free commutative groupoid.

As we shall see, there exists no free algebra of type P, where P is the covariant power-set functor $(PX = \exp X;\ PF : A \mapsto f(A))$.

A,II: The Free-algebra construction

Free universal algebras of a (possibly infinitary) type were formulated in the founding 1935 paper of Birkhoff, but were first treated carefully by Słomiński [S]. He defines sets $W_i, i = 0,1,2,\ldots$ of "terms of complexity i" by transfinite induction:

$$W_0 = I,\ W_{i+1} = I + F_\Omega I,\ W_\gamma = \bigcup_{i<\gamma} W_i \qquad \text{for } \gamma \text{ a limit ordinal}$$

and then he shows that $W_{i_0} = I^{\#}$ for a sufficiently big ordinal i_0.

This construction was naturally generalized to all type functors $F : \underline{Set} \longrightarrow \underline{Set}$ in [KPK]. A further generalization from $\underline{Set}$ to any category $\mathcal{K}$ was done in [A 1]. $\mathcal{K}$ is assumed to have finite coproducts and colimits of chains (= well-ordered diagrams).

A,5 Construction. Given an object I we shall define a chain W (a functor from the ordinals to $\mathcal{K}$) by transfinite induction.

a) $W_0 = I$, $W_1 = I + FI$ and $w_{0,1} : W_0 \longrightarrow W_1$ is the coproduct injection.

b) $W_{j+1} = I + FW_j$ and, given $w_{j,k} : W_j \longrightarrow W_k$ then

$$w_{j+1,k+1} = id_I + Fw_{j,k} : I + FW_j \longrightarrow I + FW_k.$$

c) For a limit ordinal γ, W_γ is the colimit of the preceding chain (and $w_{j,\gamma} : W_j \longrightarrow W_\gamma$ are colimit injections):

$$W_\gamma = \operatorname*{colim}_{j<\gamma} W_j.$$

Finally, $w_{\gamma,\gamma+1} : W_\gamma \longrightarrow I + FW_\gamma$ is the unique morphism such that $w_{\gamma,\gamma+1} \cdot w_{0,\gamma} : I \longrightarrow I + FW_\gamma$ is the injection and, for each $j < \gamma$, $w_{\gamma,\gamma+1} \cdot w_{j+1,\gamma} = id_I + Fw_{j,\gamma} : I + FW_j \longrightarrow I + FW_\gamma$.

The free-algebra construction is said to stop after α steps if each $w_{\alpha,i}$ $(i > \alpha)$ is an isomorphism -- equivalently, $w_{\alpha,\alpha+1}$ is an isomorphism.

A.6 Note. For $\alpha = \omega$ the construction has a much simpler form: denote, for short, by $t : I \longrightarrow I + FI$ the injection, then we have:

$$I \xrightarrow{t} I + FI \xrightarrow{id+Ft} I + F(I + FI) \xrightarrow{id+F(id+Ft)} I + F(I + F(I + FI)) \longrightarrow \dots W_\omega .$$

A.7 Example. Let $\mathcal{K} = \underline{Set}$ and $F = F_{(2)}$. Then $W_0 = I$; $W_1 = I + (I \times I)$ = binary I-trees of length ≤ 1 (leaves are labelled in I); $W_2 = I + (W_1 \times W_1)$ = binary I-trees of length ≤ 2, etc. Then W_ω is the set of all binary I-trees. The map $W_{\omega,\omega+1} : W_\omega \longrightarrow I + (W_\omega \times W_\omega)$ sends each singleton tree $i \in I$ to itself, and each tree $\tau = <\tau_1, \tau_2>$ (where τ_1, τ_2 are the two maximal proper subtrees of $\tau \in W_\omega - I$) it sends to the pair $(\tau_1, \tau_2) \in W_\omega \times W_\omega$. Clearly, $W_{\omega,\omega+1}$ is an isomorphism. Hence, the free-algebra construction stops after ω steps.

A.8 Theorem [A 1]. Let the free-algebra construction stop after α steps. Denote by $\phi : FW_\alpha \longrightarrow W_\alpha$ the composition of the injection $FW_\alpha \longrightarrow W_{\alpha+1} = I + FW_\alpha$ and $W^{-1}_{\alpha,\alpha+1} : W_{\alpha+1} \longrightarrow W_\alpha$. Then (W_α, ϕ) is the free algebra, generated by I.

We call F a <u>constructive varietor</u> if the free-algebra construction stops for every object I. And F is an <u>α-nary varietor</u> if it stops after α steps, no matter what I is. Example: $F_{(2)}$ is a <u>finitary varietor</u>, i.e. $\alpha = \omega$. (More generally, F_Ω is an α-nary varietor with α the least infinite regular cardinal bigger than all arities.) Clearly, each functor preserving colimits of α-chains, is an α-nary varietor. In part B a better criterion will be given.

A.9. Varietors preserving monos are constructive and have an external characterization. More precisely:

Let $\mathcal{M}$ be a class of monomorphisms in $\mathcal{K}$ with the following properties:

(a) Each coproduct injection $A \longrightarrow A + B$ is in $\mathcal{M}$;

(b) $\mathcal{M}$ is closed under finite coproducts and colimits of chains (i.e., if W is a chain with $W_{i,j} \in \mathcal{M}$ then also the injections $W_i \longrightarrow \operatorname{colim} W$ are in $\mathcal{M}$ and, given a compatible family $u_i : W_i \longrightarrow U$ in $\mathcal{M}$ then the unique $u : \operatorname{colim} W \longrightarrow U$ is in $\mathcal{M}$, too);

(c) $\mathcal{K}$ is $\mathcal{M}$-well powered.

Then the following is proved in [TAKR]:

A.10 Theorem. For any functor $F : \mathcal{K} \longrightarrow \mathcal{K}$, preserving $\mathcal{M}$ (i.e., such that $m \in \mathcal{M}$ implies $Fm \in \mathcal{M}$) and any object I the following conditions are equivalent:

(i) $I^\#$ exists;

(ii) the free-algebra construction for I stops;

(iii) there exists an object J, isomorphic to $I + FJ$.

A.11 Corollary [KPK]. A functor $F : \underline{Set} \longrightarrow \underline{Set}$ is a (constructive) varietor iff for each set I there exists a set J with $I \subset J$ and $\operatorname{card} FJ \leq \operatorname{card} J$.

A functor $F : R\text{-}\underline{Vect} \longrightarrow R\text{-}\underline{Vect}$ (= the category of real vector spaces) is a varietor iff for each space I there exists a space J with $I \subset J$ and $\dim FJ \leq \dim J$. For other concrete base-categories see [KR 3], [R 4], where other sufficient conditions for F to be a varietor are studied.

A.12 Example. For a class $C \subset \text{Card}$ of cardinals denote by $P_C : \underline{Set} \longrightarrow \underline{Set}$ the following functor, introduced in [K 1]:

$$P_C X = \{T \subset X; \ \operatorname{card} T \in C\} \cup \{\emptyset\} ;$$

$P_C f : T \mapsto f(T)$ if f is one-to-one on T; $T \mapsto \emptyset$ else.

Whether or not P_C is a varietor depends on C: e.g., if $C = \text{Card}$ then always $\text{card}\ P_C J = 2^{\text{card} J}$, hence by A,11 the free algebra $I^{\#}$ exists for no set I. On the other hand, classes C_1, C_2 can be constructed such that

P_{C_1} and P_{C_2} are varietors but

neither $P_{C_1} \times P_{C_2}$ nor $P_{C_1} + P_{C_2}$ are varietors.

A simple example, under the generalized continuum hypothesis: let

$$\text{Ev} = \{\aleph_j \in \text{Card};\ j \text{ is an even ordinal}\} \text{ and } \text{Od} = \text{Card} - \text{Ev}.$$

Then for every set Y with $\text{card}\ Y \in \text{Od}$ we have $\text{card}\ P_{\text{Ev}} Y = \text{card}\ Y$. Thus, again by A,11, the functor P_{Ev} is a constructive varietor -- but <u>not</u> α-nary for any α! Analogously, P_{Od} is a constructive varietor. On the other hand, for every set Y

$$\text{card}\ (P_{\text{Ev}} Y \times P_{\text{od}} Y) = \text{card}\ (P_{\text{Ev}} Y + P_{\text{Od}} Y) = 2^{\text{card}\ Y}.$$

Hence, neither $P_{\text{Ev}} \times P_{\text{Od}}$ nor $P_{\text{Ev}} + P_{\text{Od}}$ is a varietor.

Subfunctors and factor-functors of varietors in <u>Set</u> are varietors, too. This is true in a general category $\mathcal{K}$ under very mild assumptions, see [AK 3; R 4].

<u>A,13 Note</u>. The free-algebra construction is a natural generalization of the construction of absolutely free universal algebras (the case $F = F_\Omega$) as algebras of terms. The basic common feature is the recursive formula $W_{j+1} = I + FW_j$. Thus, free commutative groupoids (the case $F = P_2$, see A,2) are much closer to free groupoids than are, say, free semigroups.

Some varieties of universal algebras appear to be "more free" than others: those of the form <u>Set</u>(F). They are precisely those varieties which have an equational presentation in which all equations contain on both sides only basic operations (as opposed to derived operations). On the other hand, there are (big) varietors F such that <u>Set</u>(F) fails to be a variety in the classical sense of universal algebra, e.g., P_E from A,9.

An interesting special case of free algebras is when $I = 0$ is the initial object in $\mathcal{K}$. Then $(I^{\#}, \phi)$ (if it exists) is just the initial algebra of type F, i.e., initial object in $\mathcal{K}(F)$. Since always $0 + X \simeq X$, the free-algebra construction obtains a simpler form, studied in [AK 2,4]:

A,14. Least-fixed-point construction. Define a chain W by $W_0 = 0$, $W_{i+1} = FW_i$ and $W_\gamma = \operatorname{colim}_{i<\gamma} W_i$ for a limit ordinal γ:

$$0 \xrightarrow{c} F0 \xrightarrow{Fc} F^2 0 \xrightarrow{F^2 c} \cdots \longrightarrow W_\omega = \operatorname{colim} F^n 0 \longrightarrow FW_\omega \longrightarrow F^2 W_\omega \cdots$$

where $c : 0 \longrightarrow F0$ is the (unique) morphism.

If this construction stops, it yields the least fixed point of F (an object X is a fixed point if $FX \simeq X$). Conversely, under the hypothesis of A,9, whenever F has any fixed point, the construction stops. Properties of fixed points are investigated in [Ar, AK 2,4].

A,III: Cofree algebras and free co-algebras

Co-free algebras, defined in [AM 2], serve as a tool for study of observability of machines (cf. D,I below). Let I be an object of $\mathcal{K}$. A co-free algebra of type F, generated by I, is an algebra $(I_{\#}, \psi)$ together with a morphism $t : I_{\#} \longrightarrow I$, co-universal in the following sense. For each algebra (Q,d) of type F and each morphism $f : Q \longrightarrow I$ there exists a unique homomorphism $f_{\#} : (Q,d) \longrightarrow (I_{\#}, \psi)$ with $f = t \cdot f_{\#}$. If $I_{\#}$ always exists, F is called a co-varietor.

A,15 Example. Denote by V_Σ : Set $\longrightarrow$ Set (Σ a set) the functor $V_\Sigma X = X \times \Sigma$ and $V_\Sigma f = f \times id_\Sigma$. Algebras of type V_Σ are just unary algebras with Σ-labelled operations. Denote by Σ^* the free monoid of strings in Σ. Then for each set I the cofree algebra is $I_\# = I^{\Sigma^*}$ with $\psi : I^{\Sigma^*} \times \Sigma \longrightarrow I$ sending (p,σ) to $\hat{p}$ where $\hat{p} : \Sigma^* \longrightarrow I$ is defined by $p(\sigma_1 \ldots\ldots \sigma_n) = p(\sigma\sigma_1 \ldots \sigma_n)$. Hence, V_Σ is a co-varietor.

A,16 Theorem [T 5]. V_Σ is the only co-varietor in $\mathcal{K}$ = Set.

Each left adjoint functor $F : \mathcal{K} \longrightarrow \mathcal{K}$ is a co-varietor. A certain converse is proved in [AK 3]: for reasonable categories (like sets, vector spaces, topological spaces, posets) every co-varietor is a left adjoint. The only co-varietor in R-Vect is V_Σ, defined by $V_\Sigma X = X \otimes \Sigma$ and $V_\Sigma f = f \otimes id_\Sigma$ ($\otimes$ is the tensor product).

A,17 Note. We always consider functors only up to natural equivalence. Thus, A,16 actually states that all co-varietors are naturally equivalent to some V_Σ.

While algebras of type F are morphisms $d : FQ \longrightarrow Q$, the dual notion are co-algebras of type $G(: \mathcal{K} \longrightarrow \mathcal{K})$ which are morphisms $d : Q \longrightarrow GQ$. Their common generalization are bialgebras of type F-G, where $F,G : \mathcal{K} \longrightarrow \mathcal{K}$ are functors. There are pairs (Q,d) where $d : FQ \longrightarrow GQ$ is a morphism. Homomorphisms $f : (Q,d) \longrightarrow (Q',d')$ are such morphisms $f : Q \longrightarrow Q'$ for which $Gf \cdot d = d' \cdot Ff$. The resulting category is denoted by $\mathcal{K}(F,G)$.

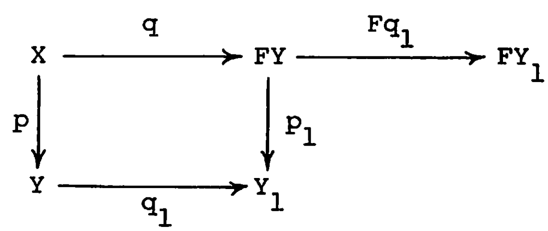
For $\mathcal{K}$ = Set, categories of bialgebras are investigated in a number of papers under the name "generalized algebraic categories". The pairs of functors F, G for which these categories have certain limits are characterized [A 3, KP, Pt, TG]; analogously for colimits [A 3, AK 1, AKP, P], free algebras

[KPK], etc. Example: free coalgebras exist (i.e., the forgetful functor Set(I,G) $\longrightarrow$ Set is a right adjoint) iff G is a hom-functor, i.e., a right adjoint [KPK]. Cf. A,16: in Set, V_{Σ} are the only left adjoints.

PART B: THE PUSHOUT CONSTRUCTION WITH APPLICATIONS

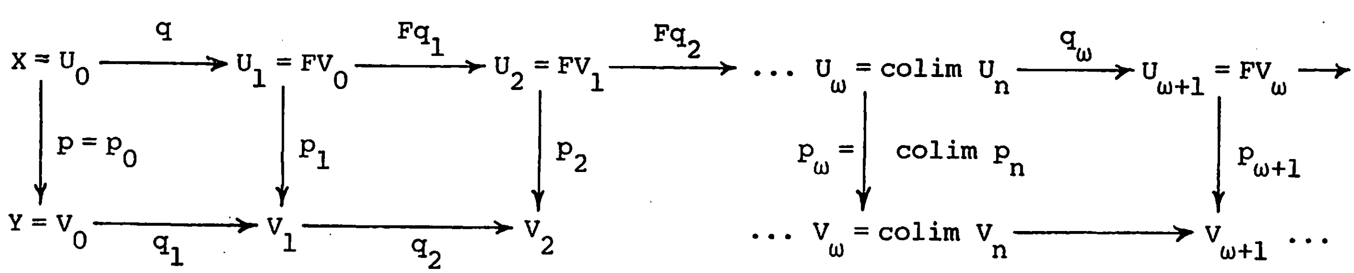
B,I: The Pushout construction

V. Koubek noticed that the essence of the above free-algebra construction lies in a simple idea. Given a pair of morphisms with a common domain and with ranges Y, FY, their pushout $p_1 \cdot q = q_1 \cdot p$ leads to another pair of this kind, viz., p_1 and Fq_1:



We can construct a pushout of p_1 and Fq_1 and iterate this procedure. On limit steps we form the colimits of the upper and lower chains.

More precisely, starting from p and q we define chains U, V and a transformation $U \longrightarrow V$ as follows. First, $U_{01} = q$ and $V_{01} = q_1$ while $p : U_0 \longrightarrow V_0$ starts the transformation.

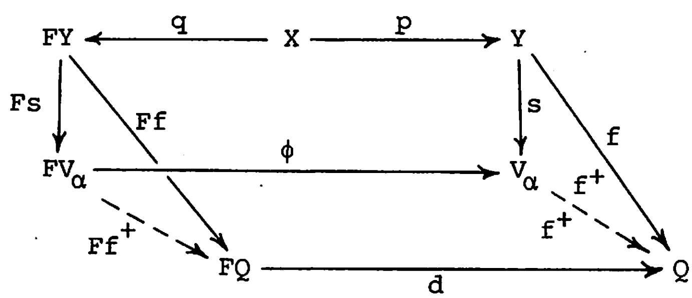


Given $p_j : U_j \longrightarrow V_j$ and $U_{j,j+1} : U_j \longrightarrow U_{j+1} = FV_j$, their pushout defines V_{j+1}, p_{j+1} and $V_{j,j+1}$, and we put $U_{j+1,j+2} = FV_{j,j+1}$. Finally, for a limit ordinal γ, $V_\gamma = \operatorname{colim}_{j<\gamma} V_j$ and $U_\gamma = \operatorname{colim}_{j<\gamma} U_j$ as well as $p_\gamma = \operatorname{colim}_{j<\gamma} p_j :$ $U_j \longrightarrow V_j$. Since always $U_{j+1} = FV_j$, we have a canonical $U_{\gamma,\gamma+1} : U_\gamma \longrightarrow FV_\gamma = U_{\gamma+1}$.

This transfinite diagram is called the pushout construction. If each $V_{\alpha,j}$ $(j > \alpha)$ is an isomorphism (equivalently, if $V_{\alpha,\alpha+1}$ is) then the construction is said to stop after α steps. In this case, the pushout construction converges to an algebra of type F, viz., (V_α, ϕ) with $\phi = V_{\alpha,\alpha+1}^{-1} \cdot p_\alpha$, together with a morphism $s = V_{0,\alpha} : Y \longrightarrow V_\alpha$, universal in the following sense.

(i) $s \cdot p = \phi \cdot Fs \cdot q$

(ii) for each algebra (Q,d) and each morphism $f : Y \longrightarrow Q$ with $f \cdot p = d \cdot Ff \cdot q$ there exists a unique homomorphism $f^+ : (V_\alpha, \phi) \longrightarrow (Q,d)$ for which $f = f^+ \cdot s$.



B.1 Example. Let $X = 0$ be the initial object (p, q the unique maps). Then the pushout construction is just the free-algebra construction over Y. Thus, if the pushout construction always stops then F is a constructive varietor. Conversely:

B.2 Theorem [KR 5]. Let $\mathcal{K}$ be cocomplete and have a factorization system $(\mathcal{E}, \mathcal{M})$ such that $\mathcal{K}$ is $\mathcal{E}$-cowell powered. Let F be a constructive varietor, preserving $\mathcal{E}$ (i.e., $e \in \mathcal{E}$ implies $Fe \in \mathcal{E}$). Then the pushout construction stops for each p and q.

An important result of Koubek and Reiterman concerns functors with rank α i.e. such that they preserve unions of α-chains of subobjects (= monos in $\mathcal{M}$). Generalizing a theorem of Barr [B 1] they prove:

B,3 Theorem [KR 5]. Let $\mathcal{K}$, $(\mathcal{E},\mathcal{M})$ be as in B,2. Then every functor with rank is a constructive varietor; in fact, the pushout construction stops for each p and q.

(This theorem is trivial if F preserves colimits of all α-chains but non-trivial if only chains of $\mathcal{M}$-morphisms are considered.)

B,II: Colimits

If $\mathcal{K}$ is a cocomplete category then the pushout construction allows a computation of colimits in the category $\mathcal{K}(F)$ of algebras. Let $D : \mathcal{D} \longrightarrow \mathcal{K}(F)$ be a diagram with, say, $Dz = (Q_z, d_z)$ for $z \in \mathcal{D}$. Let $\hat{D}$ denote the underlying diagram in $\mathcal{K}$ ($\hat{D}z = Q_z$). Consider colimits in $\mathcal{K}$:

$Y = \operatorname{colim} \hat{D}$, with injections $j_z : Q_z \longrightarrow Y$

$X = \operatorname{colim} F\hat{D}$, with injections $i_z : FQ_z \longrightarrow X$.

We have natural morphisms $p : X \longrightarrow Y$, $q : X \longrightarrow FY$:

$p = \operatorname{colim} d_z : \operatorname{colim} FQ_z \longrightarrow \operatorname{colim} Q_z$

$q \cdot i_z = Fj_z$.

B,4 Theorem [AK 3]. If the pushout construction stops for the above p and q then the algebra (V_α, ϕ), to which it converges, is the colimit of D in $\mathcal{K}(F)$.

B,5 Corollary [KR 5]. Let $\mathcal{K}$, $(\mathcal{E},\mathcal{M})$ be as in B,2. If F has rank or if it is a constructive varietor preserving $\mathcal{E}$, then $\mathcal{K}(F)$ is cocomplete.

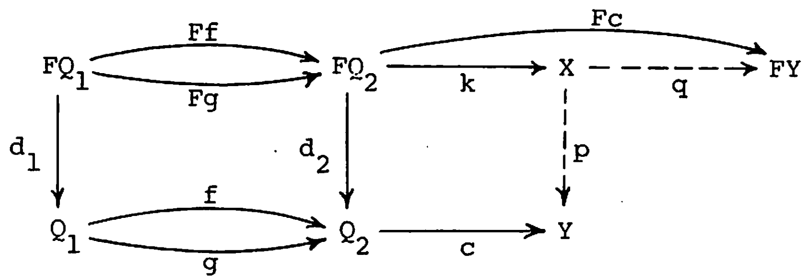
The following example is important for algebraic theories (see E,1 below) and it illustrates the general procedure.

B,6 Example [AK 3]. If $\mathcal{K}$, $(\mathcal{E},\mathcal{M})$ are as in B,2 and if F preserves $\mathcal{E}$ then $\mathcal{K}(F)$ has coequalizers (even if F is not a varietor!) obtained as follows. For each pair of homomorphisms $f,g : (Q_1,d_1) \longrightarrow (Q_2,d_2)$ denote by

$c : Q_2 \longrightarrow Y$ the coequalizer of f, g in $\mathcal{K}$

$k : FQ_2 \longrightarrow X$ the coequalizer of Ff, Fg in $\mathcal{K}$.

We get natural morphisms $p : X \longrightarrow Y$ with $p\cdot k = c\cdot d_2$



(because $c\cdot d_2\cdot Ff = c\cdot d_2\cdot Fg$) and $q : X \longrightarrow FY$ with $q\cdot k = Fc$. Now we start the pushout construction noting that $q \in \mathcal{E}$ (proof: since c is a co-equalizer, $c \in \mathcal{E}$ and so $Fc = q\cdot k \in \mathcal{E}$, hence $q \in \mathcal{E}$). By transfinite induction, each $U_{j,k}$ is in $\mathcal{E}$. Since $\mathcal{K}$ is $\mathcal{E}$-cowell powered, the pushout construction for p and q must stop.

B,7 Example [A 7]. If a varietor fails to preserve $\mathcal{E}$ (or have rank) then $\mathcal{K}(F)$ need not be cocomplete. E.g., let $\mathcal{K}$ be the category of graphs $\langle X,R\rangle$ (X a set, $R \subset X \times X$) and compatible maps. Let $F : \mathcal{K} \longrightarrow \mathcal{K}$ be defined by

$$F\langle X,R\rangle = \langle \exp H, S\rangle \quad \text{where}$$

$$H = \{(x,y,z) \in X^3;\ (x,y) \in R \text{ and } (y,z) \in R\}$$

$$S = \{(A,B) \in \exp H \times \exp H;\ A = \emptyset,\ B \neq \emptyset\}$$

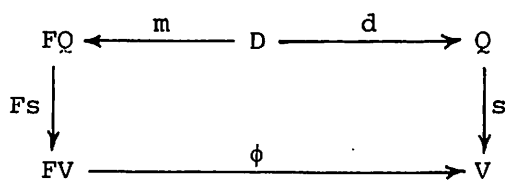
and, for a compatible map $f : \langle X,R\rangle \longrightarrow \langle X',R'\rangle$, by

$$Ff(A) = f^3(A) = \{(fx,fy,fz);\ (x,y,z) \in A\} \qquad (A \subset H).$$

For each graph $I = \langle X,R\rangle$ put $I^{\#} = I + FI$. Clearly, $FI = FI^{\#}$ and so $(I^{\#},\phi)$ with $\phi : FI \longrightarrow I + FI$ the injection, is a free algebra of type F, generated by I. Hence, F is a varietor. Yet $\mathcal{K}(F)$ fails to have coequalizers.

B,III: Partial Algebras and Algebraized Chains

Partial algebras have a natural model in a category equipped with a factorization system $(\mathcal{E},\mathcal{M})$: operations are defined not on all of FQ but on a subobject. Thus, a partial algebra of type F is a quadruple (Q,d,D,m) consisting of an object Q, a subobject $m : D \longrightarrow FQ$ (in $\mathcal{M}$) and the operation morphism $d : D \longrightarrow Q$.



We can ask about the free completion of a partial algebra. This is naturally defined as a (total) algebra (V,ϕ) and a morphism $s : Q \longrightarrow V$, universal in the following sense:

(i) $s\cdot d = \phi\cdot Fs\cdot m$

(ii) Given an algebra (Q',d') and a morphism $f : Q \longrightarrow Q'$ with $f\cdot d = d'\cdot Ff\cdot m$, there exists a unique morphism $f^{+} : (V,\phi) \longrightarrow (Q',d')$ such that $f = f^{+}\cdot s$.

This is just the universal property of the algebra, obtained by the pushout construction for d and m. Hence, for $\mathcal{K}$ cocomplete and $\mathcal{E}$-cowell powered, all partial algebras have free completion provided that

F has rank or

F is a varietor, preserving $\mathcal{E}$.

We can also consider more general relational algebras $(d : FQ \longrightarrow Q$ a relation) and the result would be the same. These, and related, problems are studied in [KR 5].

As a generalization of various iterative constructions, Reiterman develops a theory of 'algebraized chains' [R 1, R 2, KR 5]. These are pairs (W,ω) where W is a chain (with objects W_i and morphisms $W_{ij} : W_i \longrightarrow W_j$, $i \leq j$ running through all ordinals) and $\omega_i : FW_i \longrightarrow W_{i+1}$ is a transformation, i.e., $\omega_j \cdot FW_{i,j} = W_{i+1,j+1} \cdot \omega_i$ for all $i \leq j$. A morphism from one algebraized chain to another is a transformation $h : W \longrightarrow W'$ compatible with ω, ω'. The resulting category $\mathcal{K}^*(F)$ of algebraized chains allows an easy technical manipulation with iterative constructions even if they fail to stop.

Reiterman investigates varieties in $\mathcal{K}(F)$, equational and Birkhoff (= classes closed under products, subalgebras and quotients) by means of $\mathcal{K}^*(F)$. It turns out that, for varietors F in a suitable $\mathcal{K}$, equational and Birkhoff varieties coincide (the Birkhoff theorem). (The power-set functor P : Set $\longrightarrow$ Set provides an example that the Birkhoff theorem can fail in general.) Moreover, varieties in $\mathcal{K}(F)$ are coextensive with categories of Eilenberg-Moore algebras over $\mathcal{K}$, provided that $\mathcal{K}$ is cocomplete and cowell-powered.

Reiterman also uses algebraized chains to study functors with rank, not necessarily preserving monos or epis, see B,I above. In [R 4] he exhibits iterative constructions of transformations between functors.

PART C: COGENERATION

C,I: Cogeneration in Sets

Given a factorization system $(\mathcal{E},\mathcal{M})$ in $\mathcal{K}$, subalgebras and quotient algebras are defined in a natural way: a subalgebra of an algebra (Q,d) of type F is any $\mathcal{M}$-subobject in $\mathcal{K}(F)$, i.e., a homomorphism $m : (Q' d') \longrightarrow (Q,d)$ with $m \in \mathcal{M}$; analogously for quotients. For a fixed algebra (Q,d), a subobject $m_0 : Q_0 \longrightarrow Q$ (in $\mathcal{K}$) is said to generate the least subalgebra of (Q,d) containing m_0. If $\mathcal{K}$ has big intersections then each subobject m_0 generates a subalgebra: the intersection of all subalgebras containing m_0. (This does not depend on the type F.)

Inspired by reductions of automata (see D,I) we introduce the dual notion: cogeneration of quotient algebras. First, we recall that quotients of an object are naturally ordered: given $e_1 : X \longrightarrow Y_1$ and $e_2 : X \longrightarrow Y_2$ both in $\mathcal{E}$ then $e_1 \leq e_2$ means that there exists $k : Y_1 \longrightarrow Y_2$ with $e_2 = k \cdot e_1$.

C,1 Definition. Let (Q,d) be an algebra of type F and let $e_0 : Q \longrightarrow Q_0$ be a quotient $(e_0 \in \mathcal{E})$. Then e_0 is said to cogenerate a quotient algebra $e : (Q,d) \longrightarrow (Q',d')$ if e is the biggest quotient algebra with $e \leq e_0$.

E.g., in universal algebra an equivalence $\sim$ on Q will be said to cogenerate the biggest (= coarsest) congruence $\equiv$ smaller than $\sim$.

Example: For a monoid $(Q,\cdot,1)$ this congruence is defined by $x \equiv y$ iff $u\cdot x\cdot v \sim u\cdot y\cdot v$ for all pairs $u, v \in Q$.

More generally, equivalences on finitary algebras always cogenerate a congruence, while this is not so for infinitary algebras. (E.g., the equivalence on the complete lattice [0,1] with classes [0,1), {1} does not cogenerate any congruence.)

C,2 Theorem [T 4]. A functor $F : \underline{\underline{Set}} \longrightarrow \underline{\underline{Set}}$ admits cogeneration (in the sense that any equivalence on any algebra cogenerates a congruence) iff F is a quotient functor of F_Ω with a finitary Ω (see A,2).

For functors which are not quotients of a finitary F_Ω, a further discussion of noncogeneration appears in [T 5]. Let us mention a special case:

C,3 Theorem. Let Ω be an infinitary type. Then there exists an equivalence of order 2 (with two classes) on the free algebra $1^\#$ of type F_Ω on one generator, which cogenerates no congruence.

On the other hand, cogeneration in $\underline{\underline{Set}}$ causes no difficulties for regular equivalences. An equivalence ~ on an algebra (Q,d) is regular if there exists a congruence of finite order on (Q,d), smaller than ~.

C,4 Theorem [A 5]. For every functor $F : \underline{\underline{Set}} \longrightarrow \underline{\underline{Set}}$, each regular equivalence on each algebra of type F cogenerates a congruence.

C,II: Cogeneration in General Categories

The basic operation for generation is intersection; dually, the basic operation for cogeneration is cointersection. The cointersection of a collection of quotients $e_i : X \longrightarrow Y_i$ is their supremum in the above mentioned order, i.e., the least quotient $e : X \longrightarrow Y$ bigger or equal to all e_i.

For the next theorem we assume that:

(a) $\mathcal{K}$ is cocomplete and $\mathcal{E}$-cowell powered;

(b) sum injections $A \longrightarrow A + B$ are in $\mathcal{M}$;

(c) $\mathcal{K}$ is connected, i.e., $\hom(X,Y) \neq \emptyset$ whenever Y is not initial;

(d) $\mathcal{K}$ has a terminal object without non-trivial quotients.

C,5 Theorem [A 5]. Let $\mathcal{K}$ fulfill a-d above and let $F : \mathcal{K} \longrightarrow \mathcal{K}$ be a functor preserving $\mathcal{E}$. Then F admits cogeneration iff F preserves co-intersections.

If, moreover, pullbacks in $\mathcal{K}$ commute with colimits of sequences, then each F, which admits cogeneration, is a finitary varietor (cf. A,8).

C,6 Corollary. A functor $F : R\text{-}\underline{\underline{\text{Vect}}} \longrightarrow R\text{-}\underline{\underline{\text{Vect}}}$ admits cogeneration iff it is finitary, i.e. preserves filtered colimits (equivalently: preserves well-ordered unions).

C,7 Open problem. For each finitary type Ω we can define $F_\Omega : R\text{-}\underline{\underline{\text{Vect}}} \longrightarrow R\text{-}\underline{\underline{\text{Vect}}}$ by $F_\Omega X = \coprod_{\omega\in\Omega} X^{ar(\omega)}$ where $X^n = X \otimes X \otimes \dots \otimes X$ is the n-fold tensor product. Clearly, F_Ω is a finitary functor. Is it true that each finitary functor in R-$\underline{\underline{\text{Vect}}}$ is a quotient of some F_Ω?

It is in general hard to decide whether a given functor F preserves co-intersections. Fortunately, an easy sufficient condition exists: F preserves well-ordered unions (under mild side conditions). This is proved in [A 4] as a generalization of Barr's result in [B 2].

PART D: MACHINES

D,I: Machines in the Category of Sets

A theory of machines in a category is developed by Arbib and Manes in a number of papers, see [AM 1, 2] and references there. Here we shall briefly recall their basic definitions (D,II) and some natural modifications (D,I; D,III) and show an interconnection between algebraic results from the previous parts and automata-theoretic results. For expository reasons, we restrict ourselves to the category of sets. As a motivation we start by describing sequential machines (the theory of Arbib and Manes includes a number of other kinds of machines and systems, see [AM 1]).

A sequential Σ-machine (Σ = set of inputs) consists of a set Q of states, a next-state function $d : Q \times \Sigma \longrightarrow Q$ (if q is the present state and σ is the present input then $d(q,\sigma)$ is the next state) and an output function $y : Q \longrightarrow Y$ (Y = set of outputs). If the machine in a state $q_0 \in Q$ receives a sequence of inputs $\sigma_1\sigma_2 \ldots \sigma_n \in \Sigma^*$ then it changes its states successively to $q_1 = d(q_0,\sigma_1)$, ..., $q_n = d(q_{n-1},\sigma_n)$ and emits the output $y(q_n)$. This defines a map $b(q_0) : \Sigma^* \longrightarrow Y$. The set $\{b(q_0);\ q_0 \in Q\} \subset Y^{\Sigma^*}$ is called the <u>total behavior</u> of the machine; the map $b : Q \longrightarrow Y^{\Sigma^*}$ is the <u>observability map</u>.

Now, the pair (Q,d) is just an algebra of type V_Σ (see A,15). Denoting $1 = \{0\}$, the reader will easily verify that

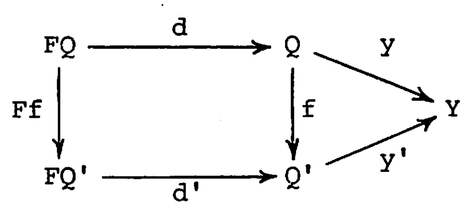
(i) Σ^* is the free algebra of type V_Σ on one generator, i.e., $\Sigma^* = 1^\#$ (with $\phi : \Sigma^* \times \Sigma \longrightarrow \Sigma^*$ the concatenation $\phi(\sigma_1 \ldots \sigma_n, \sigma) = \sigma\sigma_1 \ldots \sigma_n$);

(ii) $b(q_0) = y \cdot i^\#_{q_0} : \Sigma^* \longrightarrow Q \longrightarrow Y$ where $i_{q_0} : 1 \longrightarrow Q$ is defined by $i_{q_0}(0) = q_0$ and $i^\#_{q_0} : (1^\#,\phi) \longrightarrow (Q,d)$ is the free extension of i_{q_0}.

D.1 Definitions. Let $F : \underline{Set} \longrightarrow \underline{Set}$ be a functor which has a free algebra on one generator, $(1^{\#},\phi)$.

A machine of type F is a quadruple $M = (Q,d,Y,y)$ consisting of a set Q of states, a next-state map $d : FQ \longrightarrow Q$ and an output map $y : Q \longrightarrow Y$. For each state $q \in Q$ the map $b(q) = y \cdot i_q^{\#} : 1^{\#} \longrightarrow Y$ is the behavior of M at state q (where $i_q : 1 \longrightarrow Q$ sends 0 to q). The total behavior of M is the set $\{b(q);\ q \in Q\} \subset Y^{1^{\#}}$ and the map $b : Q \longrightarrow Y^{1^{\#}}$ is the observability map of M.

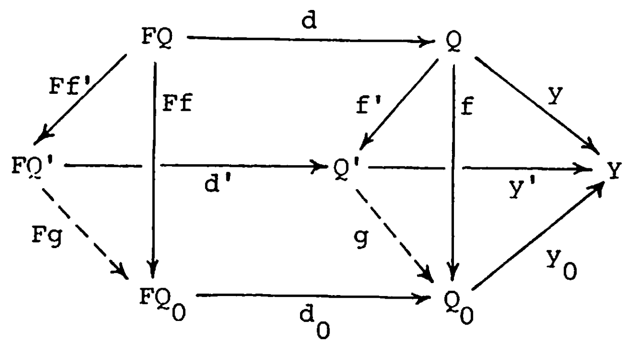
Given machines $M = (Q,d,Y,y)$ and $M' = (Q',d',Y',y')$ with a common output set Y, a homomorphism $f : (Q,d) \longrightarrow (Q',d')$ which commutes with output maps $(y = y' \cdot f)$ is called a simulation $f : M \longrightarrow M'$.



An important field of problems is the minimization of a machine M, i.e., finding the 'smallest' machine, equivalent (in a sense) to M. This can be categorically expressed in several ways (studied, in a different setting, by Ehrig et al. [E]).

Reduction. A reduction of a machine M is a name for a quotient machine, i.e., a simulation $f : M \longrightarrow M'$ with f surjective. If M' is a reduction of M then they have the same total behaviors (and M' has "fewer states" than M). A machine is reduced if it has no proper reduction (no reduction other than an isomorphism).

Given a machine M, its minimal reduction is a reduction M_0 such that for each other reduction M' of M, M' can be reduced to M_0.



More precisely, a reduction $f : M \longrightarrow M_0$ is minimal if for each reduction $f' : M \longrightarrow M'$ there exists a reduction $g : M' \longrightarrow M_0$ with $g \cdot f' = f$.

Minimal reductions are closely related to cogeneration: given a machine $M = (Q,d,Y,y)$ the quotient algebra of (Q,d), cogenerated by ker y, is the minimal reduction of M (for a suitable y_0). Conversely, given an algebra (Q,d) and an equivalence $\sim$, let $y : Q \longrightarrow Q/\!\sim\, = Y$ be the canonical map, then the minimal reduction of (Q,d,Y,y) yields the cogenerated quotient algebra.

Hence, results of Part C can be applied. E.g., by C,2 the only machines with minimal reduction are ("varieties" of) Ω-tree machines of Thatcher and Wright [TW]. And by C,4 each machine with a finite reduction has a minimal one.

Minimization.

A machine M is observable if distinct states q, q' have distinct behaviors b(q), b(q'), i.e., if the observability map is one-to-one. Observable machines are always minimal machines with the given total behavior (i.e., in being equal to their minimal reduction). A functor F is said to admit natural minimization if every machine has an observable reduction. This turns out to be a very special property.

D,2 Theorem [T 5]. The only functors $F : \underline{Set} \longrightarrow \underline{Set}$, which admit natural minimization, are V_{Σ_1,Σ_0}, defined by

$$V_{\Sigma_1,\Sigma_0} X = (X \times \Sigma_1) + \Sigma_0$$

and

$$V_{\Sigma_1,\Sigma_0} f = (f \times id_{\Sigma_1}) + id_{\Sigma_0} .$$

D,3 Remark. Put $\Sigma = \Sigma_1 + \Sigma_0$; then machines of type V_{Σ_1,Σ_0} are just sequential Σ-machines in which the transformations $d(\sigma,-) : Q \longrightarrow Q$ are constant maps for all $\sigma \in \Sigma_0$. Such inputs σ are called resets. Therefore, the above theorem states that sequential machines with resets are the only machines in Set with natural minimization.

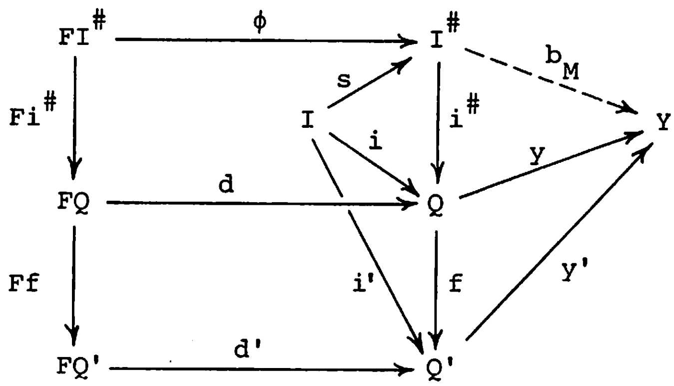
A weaker property than natural minimization is a minimization: for each machine there exists an observable machine with the same behavior. A set functor, other than V_{Σ_1,Σ_0}, with both minimal reduction and minimization, is presented in [T 5].

D,II: Initial machines

Next we consider machines with a set of initial states. An initial machine (which is the original definition of Arbib and Manes) is a 6-tuple $M = (Q,d,I,i,Y,y)$ where (Q,d,Y,y) is a machine as above and $i : I \longrightarrow Q$ is the initialization map (for $I = 0$, $i(0)$ is simply an initial state of M; for bigger I there are more initial states).

Let F be a varietor. Then the (external) behavior of the machine M is characterized by the map $b_M = y \cdot i^{\#}$, where $i^{\#} : (I^{\#},\phi) \longrightarrow (Q,d)$ is the free extension of i (called the reachability map of M). If $i^{\#}$ is surjective, M is said to be reachable. Simulations $f : M \longrightarrow M'$ are defined as homomorphisms $f : (Q,d) \longrightarrow (Q',d')$ commuting with initialization $(f \cdot i = i')$ and outputs

$(y = y' \cdot f)$.



Whenever a simulation $f : M \longrightarrow M'$ exists then $b_M = b_{M'}$.

The problem of synthesis is to find, for a given behavior (i.e., map $b : I^{\#} \longrightarrow Y$) its realization, i.e., a machine M with $b_M = b$. Every behavior has a free (maximal) reachable realization, viz., $M = (I^{\#}, \phi, I, s, Y, b)$. A non-trivial problem is the existence of a minimal realization of a behavior b, which is a machine M_0 such that

(i) M_0 is a reachable realization of b;

(ii) for every reachable realization M of b, M_0 is a reduction of M, i.e., there exists a simulation $f : M \longrightarrow M_0$ which is surjective.

D,4 Observation. A varietor admits minimal realization (in the sense that every behavior has a minimal realization) iff it admits minimal reduction, i.e., iff it admits cogeneration.

The notion of initial machine and its behavior can be formulated for an arbitrary varietor $F : \mathcal{K} \longrightarrow \mathcal{K}$, of course. If $\mathcal{K}$ is equipped with a factorization system $(\mathcal{E}, \mathcal{M})$ then a machine will be reachable iff $i^{\#} \in \mathcal{E}$. Hence, minimal realizations can also be formulated. Again, to admit minimal realization is equivalent to cogeneration. Thus, e.g., under the hypothesis of C,5 we have

<u>D,5 Theorem</u>. A functor $F : \mathcal{K} \longrightarrow \mathcal{K}$, preserving $\mathcal{E}$, is a varietor which admits minimal realizations iff F preserves cointersections.

Also large parts of Section D,I can be formulated in the general language. E.g., the observability maps $b : Q \longrightarrow Y^{I^{\#}}$ are modelled in [T 5] as colimit injections of the colimit of the category of machines (considered as a large diagram in the underlying category $\mathcal{K}$).

D,III: Universal realization

"Realization is universal", states Goguen in [G], meaning that for sequential machines minimal realization (as a functor from the category <u>Beh</u> of behaviors into the category <u>Mach</u> of reachable machines) is right adjoint to behavior (as a functor from <u>Mach</u> to <u>Beh</u>). This turns out to be a very special property of sequential machines. We shall formulate the problem of universality for machines of type F (in a category $\mathcal{K}$ with a factorization system $(\mathcal{E},\mathcal{M})$). Then we shall show that realization is seldom universal. Results in this section are from [TA 2].

Denote by <u>Mach</u> the category of reachable (initial) machines of type F and machine functions. A <u>machine function</u> is a triple (f,f^1,f^2) : $(Q,d,Y,y,I,i) \longrightarrow (Q',d',Y',y',I',i')$ consisting of a homomorphism $f : (Q,d) \longrightarrow (Q',d')$ and morphisms $f^1 : I \longrightarrow I'$ with $i' \cdot f^1 = f \cdot i$ and $f^2 : Y \longrightarrow Y'$ with $f^2 \cdot y = y' \cdot f$. (Thus, a simulation is a machine function $(f,1,1)$ if $I = I'$ and $Y = Y'$.) Denote by <u>Beh</u> the category of behaviors (I,b,Y) where $b : I^{\#} \longrightarrow Y$ is a morphism in $\mathcal{K}$; the morphisms in <u>Beh</u> are pairs $(f^1,f^2) : (I,b,Y) \longrightarrow (I',b',Y')$ of $\mathcal{K}$-morphisms $f^1 : I \longrightarrow I'$, $f^2 : Y \longrightarrow Y'$ such that $f^2 \cdot b = b' \cdot (f^1)^{\#}$.

Assigning to each reachable machine M its behavior $\mathcal{B}(M) = (I, b_M, Y)$ we obtain a functor $\mathcal{B}$: <u>Mach</u> $\longrightarrow$ <u>Beh</u>, defined on morphisms by $\mathcal{B}(f, f^1, f^2) = (f^1, f^2)$. There is no obvious way how to define minimal realization functor <u>Beh</u> $\longrightarrow$ <u>Mach</u> on morphisms, however.

<u>D,6 Definition</u>. A varietor F is said to <u>admit universal realization</u> if it admits minimal realization and if there exists a functor $\mathcal{R}$: <u>Beh</u> $\longrightarrow$ <u>Mach</u> right adjoint, right inverse to $\mathcal{B}$ (then, necessarily, $\mathcal{R}(I, b, Y)$ is always a minimal realization of $b : I^{\#} \longrightarrow Y$).

<u>D,7 Theorem</u> [T 5]. In sets, the only machines with universal realization are sequential machines with resets (see D,3). More precisely: F : <u>Set</u> $\longrightarrow$ <u>Set</u> admits universal realizations iff $F = V_{\Sigma_1, \Sigma_0}$ for some sets Σ_1, Σ_0.

A more general result can be stated: under side conditions, which we shall mention in some detail, universality of realizations implies that F <u>preserves unions</u>. This (very strong) condition means that given $\mathcal{M}$-monos $m = \bigcup_{i \in I} m_i$ then $\operatorname{im}(Fm) = \bigcup_{i \in I} \operatorname{im}(Fm)$, where im denotes the image in the factorization system $(\mathcal{E}, \mathcal{M})$. In the following definition, S_I denotes the coproduct of I copies of S (e.g. $S + S = S_I$ if card I = 2).

<u>D,8 Definition</u>. An object S is a <u>stone</u> if

(i) S is an $\mathcal{E}$-projective generator, i.e., a morphism $f : X \longrightarrow Y$ is in $\mathcal{E}$ iff for each morphism $h : S \longrightarrow Y$ there exists $k : S \longrightarrow X$ with $h = f \cdot k$;

(ii) For every functor $F : \mathcal{K} \longrightarrow \mathcal{K}$ such that the canonical map $FS + FS \longrightarrow F(S + S)$ is not in $\mathcal{E}$ there exist non-disjoint sets I, J such that also the canonical map $FS_I + FS_J \longrightarrow FS_{I \cup J}$ is not in $\mathcal{E}$.

D,9 Stone criterion. Let $\mathcal{K}$ be a category, concrete over Set or over R-Mod (= the category of left R-modules) with a representable forgetful functor $\mathcal{U} = \hom(S,-) : \mathcal{K} \longrightarrow$ Set (or, $\mathcal{K} \longrightarrow$ R-Mod). Let $\mathcal{U}$ preserve finite coproducts and epis from $\mathcal{E}$. Then S is a stone in $\mathcal{K}$.

D,10 Examples. Every (non-void) set, discrete poset, discrete topological space is a stone. Free Abelian groups, free discrete topological abelian groups, vector spaces are stones.

D,11 Open problem. Do the categories of lattices, groups and Boolean algebras have a stone?

D,12 Theorem. Let $\mathcal{K}$ be a cocomplete category with a stone S, in which

a) pushouts of a morphism and an $\mathcal{E}$-epi are pullbacks and

b) coproduct injections $A \longrightarrow A + B$ are in $\mathcal{M}$. Let F be a varietor, preserving $\mathcal{E}$-epis and with $\hom(FS,S) \neq \emptyset$.

If F admits universal realization then F preserves unions.

D,13 Corollary. The only endofunctors of R-Vect (R a field) with universal realizations are V_{Σ_1,Σ_0}, defined by $V_{\Sigma_1,\Sigma_0} X = (X \otimes \Sigma_1) \times \Sigma_0$.

A certain converse to Theorem D,12 (unions imply universality) will be presented in D,IV. Another necessary and sufficient condition is exhibited in E,8 below.

D,IV: Nerode equivalences

For sequential machines the minimal realization of a behavior $b : \Sigma^* \longrightarrow Y$ is obtained as a quotient Σ^*/E where the equivalence E, called the Nerode equivalence, is defined by

$$v \; E \; w \quad \text{iff for each} \quad t \in \Sigma^*, \quad b(tv) = b(tw).$$

In other words, E is the biggest relation on Σ^* such that $v \; E \; w$ implies

(0) $b(v) = b(w)$

(1) $b(\sigma v) = b(\sigma w)$ for each $\sigma \in \Sigma$

(2) $b(\sigma_1 \sigma_2 v) = b(\sigma_1 \sigma_2 w)$ for each $\sigma_1, \sigma_2 \in \Sigma$, etc.

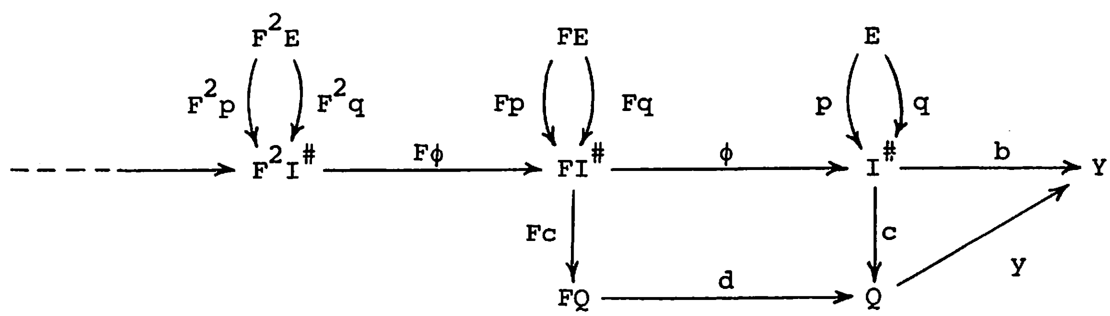
Motivated by this, Arbib and Manes introduce in [AM 1] Nerode equivalences for a behavior $b : I^\# \longrightarrow Y$ in the context of a varietor $F : \mathcal{K} \longrightarrow \mathcal{K}$. A relation on $I^\#$ is categorically expressed by a subobject of $I^\# \times I^\#$ or, equivalently, by a pair $p,q : E \longrightarrow I^\#$ of morphisms such that the induced morphism $E \longrightarrow I^\# \times I^\#$ is mono (in sets, E will be the set of pairs in the relation and p, q will be projections). These pairs are naturally ordered (as subobjects of $I^\# \times I^\#$).

<u>D,14 Definition</u> [AM 1]. The <u>external Nerode equivalence</u> of a behavior $b : I^\# \longrightarrow Y$ is the biggest relation $p,q : E \longrightarrow I^\#$ such that

(0) $b \cdot p = b \cdot p$;

(1) $(b \cdot \phi) \cdot Fp = (b \cdot \phi) \cdot Fq$;

(2) $(b \cdot \phi \cdot F\phi) \cdot F^2 p = (b \cdot \phi \cdot F\phi) \cdot F^2 q$, etc.



Let $\mathcal{K}$ have coequalizers and regular factorizations, i.e., a factorization system $(\mathcal{E},\mathcal{M})$ with $\mathcal{M}$ = monos, $\mathcal{E}$ = coequalizers. We can form the coequalizer $c : I^{\#} \longrightarrow Q$ of p and q and we can ask whether (as for sequential machines) Q is the state object of a minimal realization of b. Notice that since $bp = bq$, we have a unique $y : Q \longrightarrow Y$ with $b = c \cdot y$. Thus, it suffices to find $d : FQ \longrightarrow Q$ such that c becomes a morphism (in other words, it suffices that c be a congruence on $(I^{\#},\phi)$) to obtain a machine $M^{*} = (Q,d,I,c\cdot s,Y,y)$.

<u>D,15 Definition</u>. A varietor F <u>admits external Nerode realization</u> if every behavior $b : I^{\#} \longrightarrow Y$ has an external Nerode equivalence p, q whose coequalizer c is a congruence on $(I^{\#},\phi)$ such that Fc is epi.

<u>D,16 Theorem</u> [AM 1]. If F admits external Nerode realization then the above machine M^{*} is a minimal realization of b.

In view of D,12, external Nerode realization is very special:

<u>D,17 Theorem</u> [A 6]. Every varietor which admits external Nerode realization also admits universal realization.

Now we want to show that preservation of unions (as in D,12) often implies external Nerode realizations. There is a catch in it, however: since in D,12 S was an $\mathcal{E}$-projective generator, it follows that the epis are "weak" (e.g., in the category of topological spaces, $\mathcal{E}$ will be the class of all continuous onto maps, not only quotient maps). But in the present context, $\mathcal{E}$ is the class of all coequalizers, i.e., "strong" epis.

<u>D,18 Theorem</u> [A 6]. Let $\mathcal{K}$ be a cocomplete and finitely complete, well-powered category with regular factorizations. Let F be a varietor which admits minimal realization and preserves unions (of monos) and such that Fe is epi for each coequalizer e. Then F admits external Nerode realization.

D,19 Corollary. Let $\mathcal{K}$ be a balanced category which satisfies the hypothesis of both D,12 and D,18. Then the following conditions are equivalent for a functor preserving epis and fulfilling $\hom(FS,S) \neq \emptyset$, S a stone:

(i) F admits universal realization

(ii) F admits external Nerode realization

(iii) F preserves unions.

D,20 Example. Both $\mathcal{K}$ = Set and $\mathcal{K}$ = R-Vect fulfill the hypothesis of D,12 and D,18. The only functors with external Nerode realization are V_{Σ_1,Σ_0} (see D,2 and D,13).

D,21 Open problem. Can D,19 be proved for non-balanced categories?

A minor change in the above conditions (0), (1), (2), ... for Nerode equivalence yields a much more general model. For sequential machines E is also the biggest relation such that v E w implies

(0') $b(xv) = b(xw)$ for $x = \Lambda$ (the void string);

(1') $b(xv) = b(xw)$ for $x \in \Sigma + \{\Lambda\}$;

(2') $b(xv) = b(xw)$ for $x \in (\Sigma \times \Sigma) + \Sigma + \{\Lambda\}$ etc.

To express these conditions categorically, use the free-algebra construction (A,5) denoting $W_0^E = E$, $W_{n+1}^E = E + FW_n^E$; analogously $W_0^{I^\#} = I^\#$, $W_{n+1}^{I^\#} = I^\# + FW_n^{I^\#}$. Further, given morphisms $p,q : E \longrightarrow I^\#$ we get natural morphisms $p^n, q^n : W_n^E \longrightarrow W_n^{I^\#}$ with $p^{n+1} = p + Fp^n \cdot q^{n+1} = q + Fq^n$. Finally $\phi : FI^\# \longrightarrow I^\#$ can be iterated to $\phi_{(n)} : W_{n+1}^{I^\#} \longrightarrow W_n^{I^\#}$: $\phi_{(1)} = 1 \vee \phi$: $I^\# + FI^\# \longrightarrow I^\#$ and $\phi_{(n+1)} = 1 + F\phi_{(n)} : I + FW_{n+1}^{I^\#} \longrightarrow I + FW_n^\#$. The conditions on p, q are:

(0') $b \cdot p = b \cdot q$ (i.e., $b \cdot p^0 = b \cdot q^0 : E \longrightarrow Y$)

(1') $b \cdot \phi_{(1)} \cdot p^1 = b \cdot \phi_{(1)} \cdot q^1 : W_1^E \longrightarrow Y$;

(2') $b \cdot \phi_{(1)} \cdot \phi_{(2)} \cdot p^2 = b \cdot \phi_{(1)} \cdot \phi_{(2)} \cdot q^2 : W_2^E \longrightarrow Y$, etc.

Furthermore, we shall restrict ourselves to reflexive pairs $p,q : E \longrightarrow I^{\#}$, i.e., such that there exists $\delta : I^{\#} \longrightarrow E$ with $p \cdot \delta = q \cdot \delta = 1$.

D,22 Definition. The inner Nerode equivalence of the behavior $b : I^{\#} \longrightarrow Y$ is the biggest reflexive relation $p,q : E \longrightarrow I^{\#}$, satisfying (0'), (1'), (2'), ... A varietor admits inner Nerode realization if every behavior has an inner Nerode equivalence, the coequalizer c of which is a congruence with Fc epi.

For the next theorem, $\mathcal{K}$ is assumed to fulfill the hypothesis of C,5 and have finite limits.

D,23 Theorem [A 6]. Let F be a varietor which preserves regular epis and which admits minimal realizations. Then F admits inner Nerode realization.

D,24 Remark. Other approaches to Nerode equivalences are studied in [AAM].

D,V: Acceptors and non-determinism

An interesting problem of automata theory is the characterization of behaviors of finite machines. This is the well-known Kleene theorem for sequential machines, naturally generalized by Thatcher and Wright [TW] to tree machines. Generally, acceptors and finite acceptors in a category $\mathcal{K}$ can be investigated, see [AT]. Here we shall consider only $\mathcal{K}$ = Set.

An acceptor of type $F : \underline{Set} \longrightarrow \underline{Set}$ is a quadruple $M = (Q,d,I,T)$

where (Q,d) is an algebra and I, T are subsets of Q (of initial and terminal states, respectively). Thus, it is an initial machine with i mono, $Y = \{0,1\}$ and y the characteristic function of T. The <u>language</u> L_M accepted by M is defined by the reachability map $i^{\#} : (I^{\#},\phi) \longrightarrow (Q,d)$:

$$L_M = (i^{\#})^{-1}(T) = \{x \in I^{\#};\ i^{\#}(x) \in T\}.$$

A language, i.e., a subset $L \subseteq I^{\#}$, is said to be <u>recognizable</u> if there exists a finite acceptor M (i.e., Q is a finite set) with $L = L_M$.

It turns out that an abstract characterization of recognizable languages is a hard problem even for finitary set functors. Some results can be found in [TA 2].

An important concept in this field is that of a non-deterministic acceptor, i.e., $M = (Q,d,I,T)$ where $d : FQ \longrightarrow Q$ is a relation ($d \subset FQ \times Q$). It is possible to define the language of a non-deterministic acceptor in a natural (though not obvious) way. And languages of non-deterministic acceptors are easier to study than in the deterministic case. But it is non-trivial to determine for which types F the languages of deterministic and non-deterministic acceptors coincide. These problems are studied in [T 7, 8].

PART E: ALGEBRAIC THEORIES

E,1: Construction of colimits

Let $\mathbb{T} = (T,\mu,\eta)$ be an algebraic theory (triple) in a cocomplete category $\mathcal{K}$. If $\mathcal{K}$ has a factorization system $(\mathcal{E},\mathcal{M})$ such that $\mathbb{T}$ preserves $\mathcal{E}$ (i.e., $e \in \mathcal{E}$ implies $Te \in \mathcal{E}$) and $\mathcal{K}$ is $\mathcal{E}$-cowell powered, then an explicit construction of colimits in $\mathcal{K}^{\mathbb{T}}$ can be exhibited as an application of the pushout construction (B,II).

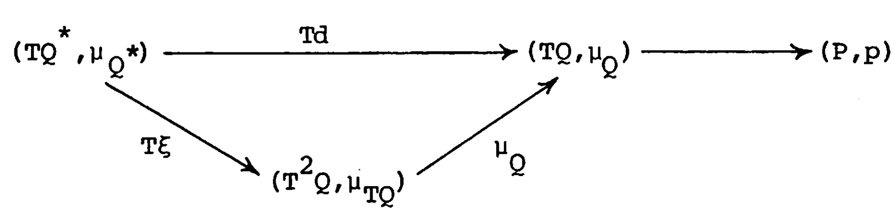
It was observed by Linton [L] that coproducts in $\mathcal{K}^{\mathbb{T}}$ (= the category of $\mathbb{T}$-algebras) can be constructed from coequalizers in the following way. Given $\mathbb{T}$-algebras (Q_i,d_i), $i \in I$, put

$$Q = \coprod_{i\in I} Q_i \quad \text{and} \quad Q^* = \coprod_{i\in I} TQ_i$$

(coproducts in $\mathcal{K}$) and consider the canonical maps

$$\xi : Q^* \longrightarrow TQ \quad \text{and} \quad d = \coprod_{i\in I} d_i : Q^* \longrightarrow Q.$$

__E,1 Theorem__ [L]. Let the following be a coequalizer in $\mathcal{K}^{\mathbb{T}}$:



Then (P,p) is the coproduct of $\mathbb{T}$-algebras (Q_i,d_i) in $\mathcal{K}^{\mathbb{T}}$.

Now, the category $\mathcal{K}^{\mathbb{T}}$ is a full subcategory of $\mathcal{K}(T)$, the category of algebras of type T. Since $\mathbb{T}$ preserves $\mathcal{E}$, it is easy to see that $\mathcal{K}^{\mathbb{T}}$ is closed under coequalizers in $\mathcal{K}(T)$ (see [A 7]). Thus, it suffices to find the coequalizer of Td and $\mu_Q \cdot T\xi$ in $\mathcal{K}(T)$ -- and this is done in B,6. A completely analogous situation applies to other colimits in $\mathcal{K}^{\mathbb{T}}$.

(Given a diagram D in $\mathcal{K}^{\mathbb{T}}$ with the underlying diagram $\hat{D}$ in $\mathcal{K}$, put $Q = \operatorname{colim} \hat{D}$ and $Q^* = \operatorname{colim} T\hat{D}$; then we have canonical maps $\xi : Q^* \longrightarrow TQ$ and $d : Q^* \longrightarrow Q$. The coequalizer of Td and $\mu \cdot T\xi$ yields the colimit of D.) Hence, we obtain a constructive proof of the following

<u>E,2 Theorem</u>. Let $\mathcal{K}$ be a cocomplete, $\mathcal{E}$-cowell powered category with a factorization system $(\mathcal{E}, \mathcal{M})$. Then for each algebraic theory $\mathbb{T}$ in $\mathcal{K}$, preserving $\mathcal{E}$, the category $\mathcal{K}^{\mathbb{T}}$ is cocomplete, too.

This theorem, improving results of Linton [L] and Barr [B 1] was first formulated by J. Reiterman; see [KR 5] or, for a different proof, [A 7].

An important concept are <u>simple colimits</u>, i.e., colimits for which the coequalizer of Td and $\mu \cdot T\xi$ can be computed in $\mathcal{K}$ (more precisely, the coequalizer in $\mathcal{K}^{\mathbb{T}}$ exists and is preserved by the forgetful functor $\mathcal{K}^{\mathbb{T}} \longrightarrow \mathcal{K}$). E.g., under mild additional assumption, <u>every finitary theory $\mathbb{T}$ has simple colimits</u>. See [AK 5] for details about simple and (non-simple) constructive colimits. Another construction of colimits (for theories with rank) is exhibited in [Sch] and yet another in [KR 5].

<u>E,II: Free theories, free monoids</u>

Barr [B 1] defines an algebraic theory <u>freely generated</u> by a functor $F : \mathcal{K} \longrightarrow \mathcal{K}$. This is a theory $\mathbb{T} = (T, \mu, \eta)$ with a transformation $\sigma : F \longrightarrow T$ universal in the sense that for each theory $\mathbb{T}' = (T', \mu', \eta')$ and each transformation $\tau : F \longrightarrow T'$ there exists a unique theory map (triple transformation) $\tau^* : \mathbb{T} \longrightarrow \mathbb{T}'$ with $\tau = \tau^* \cdot \sigma$.

<u>Example</u>. Let F be a varietor. Denote by $\mathbb{F}^{\#}$ the theory (T, μ, η) with $TX = X^{\#}$, $\eta_X = s$ (the universal morphism $X \longrightarrow X^{\#}$) and $\mu_X = (\mathrm{id}_{X^{\#}})^{\#} : X^{\#\#} \longrightarrow X^{\#}$. Then $\mathbb{F}^{\#}$ is freely generated by F with respect to

$\sigma_X : FX \longrightarrow X^\#$ defined by $\sigma_X = \phi \cdot Fs$.

Conversely.

E,3 Theorem [B 1]. The only free algebraic theories in a complete, well-powered category are $\mathbb{F}^\#$. More precisely, a functor F freely generates a theory iff F is a varietor.

Barr also introduced the notion of rank (B,I) and he exhibited a non-constructive proof of Theorem B,3 in case $\mathcal{K}$ is complete (and cocomplete). The first constructive approach to free theories appears in a paper of Dubuc [D]. He observes that free theories are exactly free monoids in the monoidal (pseudo-)category $\mathcal{K}^{\mathcal{K}}$ of endofunctors (where $\otimes$ is composition) and he presents an iterative construction of free monoids. In a closed monoidal category, the free monoid over M is $M^* = \coprod_{n=0} M^n$ with $M^n = M \otimes M \otimes \ldots \otimes M$, of course. In a general monoidal category the situation becomes more complicated. Translating Dubuc's result to the language of free algebras, we get:

E,4 Dubuc's construction of a free algebra $I^\#$ of type F (in a cocomplete category $\mathcal{K}$). We define a diagram R in $\mathcal{K}$ with objects $R_0, R_1, \ldots, R_j, \ldots$, j an ordinal, by transfinite induction. The diagram R is not an ordered one; nevertheless, the existence of a morphism from R_j to R_k implies $j \le k$.

a) $R_0 = I$; $R_{j+1} = R_j + FR_j$.

b) For a limit ordinal γ, R_γ is the colimit of the preceding part and hom $(R_j, R_\gamma) = \{\xi_j\}$ where ξ_j is the colimit injection $(j < \gamma)$.

c) hom $(R_0, R_1) = \{s_I\}$ where $s_X : X \longrightarrow X + FX$ denotes the sum injection;

hom $(R_{j+1}, R_{j+2}) = \{f + Ff;\ f \in \text{hom}\ (R_j, R_{j+1})\} + \{s_{R_{j+1}}\}$

hom $(R_\gamma, R_{\gamma+1}) = \{s_{R_\gamma}\}$ for a limit ordinal γ.

$$X \xrightarrow{s_X} X + FX \underset{s_{X+FX}}{\overset{s_X + Fs_X}{\rightrightarrows}} \underbrace{X + FX + F(X + FX)}_{R_2} \begin{array}{c} \xrightarrow{s_X + Fs_X + F(s_X + Fs_X)} \\ \xrightarrow{s_{X+FX} + Fs_{X+FX}} \\ \xrightarrow[s_{X+FX+F(X+FX)}]{} \end{array} R_2 + FR_2 \begin{array}{c} \longrightarrow \\ \longrightarrow \\ \longrightarrow \\ \longrightarrow \end{array}$$

$$\begin{array}{c} \longrightarrow \\ \longrightarrow \\ \longrightarrow \\ \longrightarrow \end{array} \cdots\ R_\omega \xrightarrow{s_{R_\omega}} R_\omega + FR_\omega \underset{s_{R_\omega + FR_\omega}}{\overset{s_{R_\omega} + Fs_{R_\omega}}{\rightrightarrows}} \cdots$$

Dubuc proves that for any uncountable regular cardinal α the following holds: if F preserves α-directed colimits than $R_\alpha = I^\#$. It is not clear, however, whether for a finitary functor, $R_\omega = I^\#$ (because the part of R up-to ω is not directed). Comparing the free-algebra construction (A,5) with the Dubuc construction, it is easy to find a transformation $\varepsilon_j : R_j \longrightarrow W_j$ with each ε_j a split epi. Hence, the free-algebra construction is at least as quick as Dubuc's construction.

<u>E,5 Open problem</u>. Is there an example of $\mathcal{K}$, F, I for which Dubuc's construction stops after a bigger number of steps than the free-algebra construction (or, does not stop, though the free-algebra construction does)?

E,III: Machines -- minimal and universal realization

The theory of machines, based on free algebraic theories (see D,II) can be as well formulated for an arbitrary algebraic theory $\mathbb{T}$ in a category $\mathcal{K}$ with a factorization system $(\mathcal{E},\mathcal{M})$. A machine is a 6-tuple $M = (Q,d,I,i,Y,y)$ where (Q,d) is a $\mathbb{T}$-algebra and $i : I \longrightarrow Q$; $y : Q \longrightarrow Y$ are morphisms. The reachability map (previously $i^{\#}$), is now the homomorphism $d \cdot Ti : (TI,\mu_I) \longrightarrow (Q,d)$. Thus, M is reachable if $d \cdot Ti \in \mathcal{E}$; and the behavior of the machine M is $b_M = y \cdot (d \cdot T_i) : TI \longrightarrow Y$.

Conversely, given a behavior, i.e., a morphism $b : TI \longrightarrow Y$, its minimal realization is a reachable machine M_0 such that

(i) M_0 is a realization of b, i.e., $b_{M_0} = b$;

(ii) For each reachable realization M of b there exists a unique simulation (= homomorphism, commuting with i, y) from M to M_0. An analogous theorem to C,5 holds, yet without the heavy side conditions!

E,6 Theorem [A 8]. Let $\mathcal{K}$ be a cocomplete, $\mathcal{E}$-cowell powered category with a factorization system $(\mathcal{E},\mathcal{M})$. Let $\mathbb{T}$ be an algebraic theory, preserving $\mathcal{E}$. Then $\mathbb{T}$ admits minimal realization iff $\mathbb{T}$ preserves cointersections.

E,7 Example. For $\mathcal{K}$ = Set, $\mathbb{T}$ admits minimal realization iff it is the theory of a variety of finitary universal algebras.

This gives a more general theorem for cogeneration in $\mathcal{K}(F)$ with F a varietor preserving $\mathcal{E}$. F admits cogeneration iff $\mathbb{F}^{\#}$ preserves cointersections (without the side conditions of C,5). Analogously, the following theorem has an interpretation for universal realizations in $\mathcal{K}(F)$, originally exhibited in [TA 1].

Universality can be defined quite analogously to D,III: realization is universal if minimal realizations exist and constitute a right adjoint, right inverse functor to behavior. We shall characterize theories with universal realization in terms of co-preimages = pushouts of a morphism and an $\mathcal{E}$-epi (as preimages are pullbacks of a morphism and an $\mathcal{M}$-mono).

E,8 Theorem [A 8]. Let $\mathcal{K}$, $(\mathcal{E},\mathcal{M})$ and $\mathbb{T}$ be as in E,6. Then $\mathbb{T}$ admits universal realization iff $\mathbb{T}$ preserves cointersections and co-preimages.

E,9 Example. For $\mathcal{K}$ = Set, $\mathbb{T}$ admits universal realization iff it is the theory of a variety of unary algebras.

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