PARTIALLY-ADDITIVE CATEGORIES AND COMPUTER PROGRAM SEMANTICS[1]

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ABSTRACT

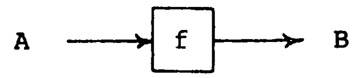
We develop a model in which computer programs find semantic interpretation as morphisms in a category. In the first section we use a basic example to motivate the primacy of a partially-defined sometimes infinitary addition in interpreting iteration. The remainder of the paper is devoted to axiomatic models with interrelationships and examples. The resulting class of categories generalizes semiadditive categories, and we see that coproducts $\oplus A_i$ in a partially-additive category are equipped with morphisms $(pr_j : \oplus A_i \longrightarrow A_j)$ which enjoy many of the properties of products.

Such programming topics as 'procedure calls' and 'declarations' may be defined in a partially-additive category, as we shall show in a sequel to this paper.

1. Motivation

The syntax of a programming language tells us what structures to regard as legal programs, while the semantics tells us how to build from the interpretation of complete programs [8]. In the present section, we introduce three ways of building new programs from old -- sequential composition, the conditional, and iteration -- and how in each case how the interpretation of the constituents in the category $\underline{\underline{Pfn}}_D$ (to be defined shortly) yields an interpretation of the new program in $\underline{\underline{Pfn}}_D$. This will provide the motivation for the introduction of partially-additive categories in Section 2. We show that a rich range of alternative semantics -- including the semantics of nondeterministic programs -- falls within our general theory. We show in Section 3 that any such category is rich enough to support the semantics of sequential composition, the conditional and iteration, while a later paper will show how procedure calls, and declarations in block structure languages, may be handled in such categories. Section 4 provides a new axiomatization of iteration which allows us to prove a fixpoint property rather than assume it and which includes iteration in a partially-additive category as a special case. In Section 5, we characterize partially-additive categories using an axiomatization which spells out a sense in which coproducts are 'almost' products. Finally, in Section 6, we introduce ω-complete partial semirings, and show that appropriately defined matrix categories over such semirings are equivalent to 'declaration-free' semantics in a partially-additive category.

Let us return, then, to our motivation. Given two sets A and B, we may consider a flow diagram with a set A of entry lines and a set B of exit lines to be a morphism $f\colon A \longrightarrow B$



in a syntactic category. To see one way of assigning an interpretation to such an f, imagine that some set D of 'data values' is fixed. Computation starts on some entry line a in A with some initial data value d. We then work our way through the flow diagram, operating on the data and branching on the result of tests on the data until (if ever) we exit on some line b in B with some value d' in D. In this way, the interpretation of f will be a <u>partial function</u> (partial because computation may not terminate for certain entry data)

$$\hat{f}: A \times D \longrightarrow B \times D$$

where $\hat{f}(a,d) = (b,d')$ as above. The ubiquity of this example is well-known -- see, e.g., [6]. Such interpretations lie in the category $\underline{\underline{\mathrm{Pfn}}}_D$ whose objects are countable sets A and whose morphisms $A \longrightarrow B$ are partial functions $A \times D \longrightarrow B \times D$ (for the fixed choice of the set D). Composition and identity are as for partial functions.

Countable coproducts play an important role in our theory. In $\underline{\underline{\mathrm{Pfn}}}_D$, the coproduct of an arbitrary collection (A_i) of sets is just the disjoint union A_A of the A_i and

$$in_i: A_i \longrightarrow A, \; (a,d) \mapsto ((a,i),d).$$

In general, we employ the notation $\overset{\frown}{+} A_i$ for coproducts because the axioms of Section 2 will give coproducts some product-like properties so that $\overset{\frown}{+}$ is 'between $+$ and $\oplus$'.

We now study three ways in which we may build new flow diagrams from old ones, and in each case show how the interpretation of the constituents in $\underline{\underline{\mathrm{Pfn}}}_D$ yields an interpretation of the new flow diagram in $\underline{\underline{\mathrm{Pfn}}}_D$.

1. Sequential Composition

Syntax: Given flow diagrams $A \longrightarrow \boxed{f} \longrightarrow B$ and $B \longrightarrow \boxed{g} \longrightarrow C$ we define their sequential composition

$$A \longrightarrow \boxed{f;g} \longrightarrow C \;=\; A \longrightarrow \boxed{f} \longrightarrow \boxed{g} \longrightarrow C$$

by merging the b-exit line of f with the b-entry line of g.

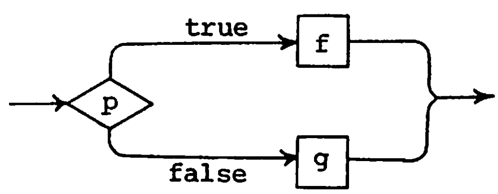
Semantics: If f and g are interpreted as $\underline{\underline{Pfn}}_D$-morphisms $\hat{f}: A \longrightarrow B$, $\hat{g}: B \longrightarrow C$, respectively, then

$$\widehat{f;g} = \hat{g} \cdot \hat{f}: A \longrightarrow C .$$

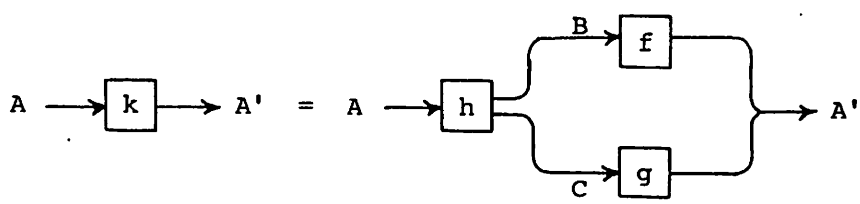
2. Conditional

We generalize the familiar conditional

if p then f else g



Syntax: Given a flow diagram $h: A \longrightarrow B + C$ (i.e. the set of exit lines is partitioned into two disjoint sets labelled B and C respectively) which we may represent as $A \longrightarrow \boxed{h} \rightrightarrows {B \atop C}$, together with $B \longrightarrow \boxed{f} \longrightarrow A'$ and $C \longrightarrow \boxed{g} \longrightarrow A'$, we may define a new flow diagram



by identifying the b-exit line of h with the b-entry line of f for each b in B; by identifying the c-exit line of h with the c-entry line of g for each c in C; and by merging the a'-exit lines of f and g for each a' in A'.

Semantics: Let f, g and h have interpretations $\hat{f}$, $\hat{g}$, $\hat{h}$, respectively, in $\underline{\underline{Pfn}}_D$. Then the interpretation of k is just

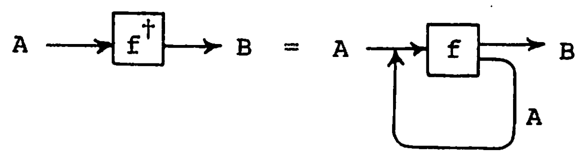
$$\hat{k}: A \xrightarrow{h} B \dotplus C \xrightarrow{\binom{f}{g}} A'$$

in $\underline{\underline{Pfn}}_D$, which unpacks to

$$\hat{k}(a,d) = \begin{cases} \hat{f}(b,d') & \text{if } \hat{h}(a,d) = (b,d') \text{ with } b \in B \\ \hat{g}(c,d') & \text{if } \hat{h}(a,d) = (c,d') \text{ with } c \in C \\ \text{undefined} & \text{otherwise} \end{cases}$$

3 Iteration

Syntax: Given $f: A \longrightarrow A \dotplus B$ we form its iterate $f^{\dagger}: A \longrightarrow B$ as "repeat f until B":



by linking each a-exit line of f to the a-entry line of f for each a in A.

Semantics: Informally, we think of $\hat{f}^{\dagger}: A \times D \longrightarrow B \times D$ as being obtained by 'going round the loop' with f until a result in $B \times D$ is obtained. We may split $\hat{f}$ into two parts

$$\hat{f}_A: A \longrightarrow A, \quad (a,d) \mapsto \begin{cases} \hat{f}(a,d) & \text{if } \hat{f}(a,d) \text{ is in } A \times D \\ \text{undefined} & \text{if not} \end{cases}$$

$$\hat{f}_B: A \longrightarrow B, \quad (a,d) \mapsto \begin{cases} \hat{f}(a,d) & \text{if } \hat{f}(a,d) \text{ is in } B \times D \\ \text{undefined} & \text{if not} \end{cases}$$

corresponding to 'loop again' and 'exit' respectively. The idea is to repeat $\hat{f}_A$ as many times as it is defined, and then apply $\hat{f}_B$:

$$\hat{f}^{\dagger}(a,d) = \begin{cases} \hat{f}_B\hat{f}_A^n(a,d) & \text{if } n \geq 0 \text{ is such that } f_A^n(a,d) \text{ is defined} \\ & \quad\text{but } f_A^{n+1}(a,d) \text{ is not defined} \\ \text{undefined} & \text{if no such } n \text{ exists.} \end{cases} \tag{4}$$

Noting that the choice of n is unique (if it exists) we may rewrite (4) as $\hat{f}^{\dagger} = \sum_{n\geq 0} \hat{f}_B\hat{f}_A^n$ where for any set of partial functions $f_i: A \longrightarrow B$ <u>with disjoint domains of definition</u> we define their <u>sum</u> $\sum_i f_i$ by

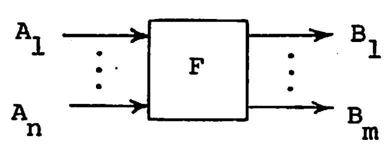
$$(\sum_i f_i)(a,d) = \begin{cases} f_j(a,d) & \text{if } (a,d) \text{ is in } \mathrm{dom}(f_j) \\ \text{undefined} & \text{if no such } j \text{ exists.} \end{cases} \tag{5}$$

Note that with this definition of sum we have

$$f = in_A \cdot f_A + in_B \cdot f_B : A \longrightarrow A \overset{\frown}{+} B.$$

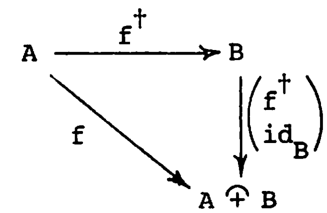
(Finite Σ is written + which should not be confused with $\overset{\frown}{+}$.)

In a more general setting, we allow declaration and expunging of variables. Thus it is not enough to give a set of entry and exit lines -- we must specify the 'data space' holding on each line. How this is done in a way consonant with an actual programming language like Pascal, say, must be left for onother paper. The general setting of this paper does, however, embrace this 'variable data space' semantics. We interpret a flow diagram of the form



as a morphism $F: A_1 \overset{\frown}{+} \ldots \overset{\frown}{+} A_n \longrightarrow B_1 \overset{\frown}{+} \ldots \overset{\frown}{+} B_m$.

Elgot has made the diagram



the starting point for his axiomatic study of iteration [3, 4, 6, 7], and we thus call it the <u>Elgot iteration equation</u>.

It is well-known that, in $\underline{\underline{Pfn}}_D$, for any $f: A \longrightarrow A + B$ the $f^\dagger$ defined as $\sum_{n\geq 0} f_B f_A^n : A \longrightarrow B$ is the least solution (in the inclusion ordering on subsets of $(A \times D) \times (B \times D)$) of the Elgot iteration equation. But is is not the unique solution. This obstruction has led many workers to view the general setting for program semantics as one in which a suitable partial order exists -- such as complete lattices [13]. On the other hand, Elgot's approach was to abandon $\underline{\underline{Pfn}}_D$ for categories in which unique solutions exist. Our approach, rather, is to observe, returning to (5) above, that a key ingredient of this approach to iteration is the use of a partially-defined sometimes infinitary addition: we decompose $f: A \longrightarrow A + B$ into f_A and f_B, and then define $f^\dagger$ as $\sum_{n\geq 0} f_B f_A^n$ -- showing that this sum is indeed defined. We shall see in Section 3 that this approach to iteration does hold in any category which is partially-additive in the sense we introduce in Section 2. Then in Section 4 we shall explore the notation of a "functorial dagger" (a notion we introduced in [2]), which is a passage $(f: A \longrightarrow A + B) \mapsto (f^\dagger: A \longrightarrow B)$ which satisfies a basic "functoriality condition" together with $(in_2: A \longrightarrow A + A)^\dagger = id_A$.

We prove that any functorial dagger satisfies the Elgot iteration equation. In particular, this holds for the iterate $f^\dagger = \sum_{n\geq 0} f_B f_A^n$ in a partially-additive category, since it is a functorial dagger, as we see in Section 3. This supports the claim that Elgot's use of algebraic theories as the setting for his algebraic semantics adds an unnecessary complication. However, we do wish to acknowledge that Elgot's observations provided the major motivation for the present work.

2. Partially-Additive Categories

The crucial property of $\underline{\underline{Pfn}}_D$ that we used in the motivating discussion of program semantics in Section 1 was that for any sets A and B, the hom-set $\underline{\underline{Pfn}}_D(A,B)$ admits a partial-additive structure, i.e. if $(f_i\colon A \longrightarrow B \mid i \in I)$ have disjoint domains (so that $i \neq j \Longrightarrow \mathrm{dom}(f_i) \cap \mathrm{dom}(f_j) = \emptyset$) then we can define their sum $f = \sum_{i \in I} f_i : A \longrightarrow B$ by $f(a) = f_j(a)$ if $a \in \mathrm{dom}(f_j)$, $f(a)$ undefined otherwise.

In definition 1, we introduce the notion of an ω-complete partial abelian monoid to axiomatize key properties of this partially-additive structure; while definition 4 defines a partially-additive category as one whose hom-sets have this partially-additive structure and fit together in an appropriate way. Throughout this paper, <u>countable</u> means 'finite or denumerable'.

1 <u>Definition</u> An ω-<u>complete partial abelian monoid</u> is a pair (M,Σ) where M is a set and Σ is a partial operation on countable sequences in M subject to the following axioms:

<u>Partition-associativity axiom</u>: If the countable set I is partitioned into $(I_j : j \in J)$ then for each family $(x_i : i \in I)$ in M, $\Sigma(x_i : i \in I) = \Sigma(\Sigma(x_i : i \in I_j) : j \in J)$ in the sense that the left side is defined if and only if the right side is defined and then the values are equal.

Limit axiom: If $(x_i : i \in I)$ is a countable family in M and if $\Sigma(x_i : i \in F)$ is defined for every finite subset F of I, then $\Sigma(x_i : i \in I)$ is defined.

Unary-sum axiom: For one-element families, Σx is defined and $\Sigma x = x$.

In the partition-associativity axiom I_j may be empty. Since the unary-sum axiom ensures that some sums exist, it follows that the empty sum is defined and provides an additive zero. Notice, also, that for any permutation $\sigma: I \longrightarrow I$, $\Sigma(x_i : i \in I) = \Sigma(x_{\sigma i} : i \in I)$.

Notice that (in the presence of the other axioms) the limit axiom is equivalent to the following: given $(x_i : i = 0,1,2,...)$ if $\Sigma(x_i : 0 \le i \le k)$ is defined for every finite k, then the infinite sum $(x_i : 0 \le i)$ is also defined.

We recapture the following concept (Eilenberg [5, pp. 124-125]):

2. Definition A countably complete abelian monoid is an ω-complete partial abelian monoid for which Σ is total. (If Σ_2 is the restriction of Σ to $M \times M \longrightarrow M$, it is readily checked that $(M,\Sigma_2,0)$ is then an abelian monoid, and for finite tuples (x_i), the sum $\Sigma(x_i)$ has its usual meaning.)

3. Definition A morphism $f: (M,\Sigma) \longrightarrow (M',\Sigma')$ of ω-complete partial abelian monoids is a function $f: M \longrightarrow M'$ such that for all $(x_i : i \in I)$ in M, if Σx_i is defined then $\Sigma' f(x_i)$ is defined and $f(\Sigma x_i) = \Sigma' f(x_i)$.

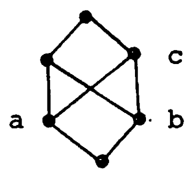
4 Examples and non-examples of ω-complete partial abelian monoids

(i) M = the set of partial functions from A to B, with Σf_i defined as their union when the domains are disjoint.

(ii) M = the set of relations from A to B, with Σ totally defined by taking ΣR_i as the union of the R_i.

(iii) A countably-complete poset with Σ = supremum. These are characterized by requiring Σ to be totally defined and idempotent in the sense that if $x_i = x$ then Σx_i is defined and is x. Example (vi) below shows that $x + x = x$ alone will not do.

(iv) The poset with Hasse diagram



is not an example because sup(a,b,c) is defined whereas sup(a,b) in sup(sup(a,b),sup(c)) is not.

(v) $M = \mathbb{N} \cup \{\infty\}$, with Σ the usual addition.

(vi) $M = \mathbb{N} \cup \{\infty\}$, with $\Sigma x_i = 0$ or ∞ if the usual sum is 0 or ∞, respectively, but $\Sigma x_i = 1$ otherwise. The subset $S = \{0,1,\infty\}$ with the same Σ is also an example. In S, Σ is totally defined and $x + x = x$ for all x. Moreover, $1 + \ldots + 1 = 1$ for each finite sequence of 1's, but the sum of the infinite sequence $1 + 1 + \ldots = \infty$.

(vii) While an ω-complete partial abelian monoid is often an abelian monoid, no nonzero x can have an inverse. To see this, observe that if $x + y = 0$ then

$$0 = (x + y) + (x + y) + \ldots$$
$$= x + (y + x) + (y + x) + \ldots = x.$$

<u>5 Notations and conventions</u> Our index sets I, J are always countable (i.e. finite or denumerable). Our categories shall always have countable coproducts $in_j : A_j \longrightarrow \oplus (A_i : i \in I)$ including an initial object 0. The copower of I copies of A we denote $I \times A$. Two special maps are $\sigma : I \times A \longrightarrow A$ defined by $\sigma in_i = id_A$ and the 'diagonal' $\Delta : \oplus (A_i : i \in I) \longrightarrow I \times \oplus (A_i : i \in I)$ defined by

$$\begin{array}{ccc} \oplus A_i & \xrightarrow{\ \Delta\ } & I \times \oplus A_i \\ {\scriptstyle in_j}\uparrow & & \uparrow{\scriptstyle in_j} \\ A_j & \xrightarrow[\ in_j\ ]{} & \oplus A_i \end{array}$$

since the initial object exists (as the empty coproduct) the category has zero maps if and only if the initial object is terminal. We may note that the empty set $\emptyset$ is the zero object of <u>Pfn</u>, and that $0 : A \longrightarrow B$ is then the nowhere-defined partial function from A to B. Let us further note that although $A_1 \oplus A_2$ is not a product in <u>Pfn</u> we do have 'quasi-projections'

$$pr_j : A_1 \oplus A_2 \longrightarrow A_j, \quad a \mapsto \begin{cases} a & \text{if } a \in A_j \\ \text{undefined} & \text{if not} \end{cases}$$

These quasi-projections allow us to decompose any partial function $f : A \longrightarrow A_1 \oplus A_2$ into the family $(pr_j \cdot f)$.

<u>6 Observation</u> Let $\mathcal{K}$ be a category with countable coproducts which has zero maps. Then for $J \subset I$, we may define the <u>quasi-projection</u> $pr_J : \oplus (A_i : i \in I) \longrightarrow \oplus (A_i : i \in J)$ by

$$pr_J \, in_i = \begin{cases} in_i & \text{if } i \in J \\ 0 & \text{if } i \notin J. \end{cases}$$

We write pr_j if $J = \{j\}$, and δ_{ij} for $pr_j in_i$.

With this definition, it is readily checked that

$$\begin{array}{ccc} \oplus A_i & \xrightarrow{pr_j} & A_j \\ \Delta \downarrow & & \downarrow in_j \\ I \times \oplus A_i & \xrightarrow[pr_j]{} & \oplus A_i \end{array}$$

commutes for each j -- simply check that the two paths are equal $(in_j \cdot \delta_{ij} = \delta_{ij} \cdot in_i)$ when preceded by in_i for each i in I.

7 Definition A partially-additive category $\mathcal{K}$ is a category with countable coproducts which is based on the category of ω-complete partial abelian monoids (i.e., hom-sets are ω-complete partial abelian monoids and composing on either side is a morphism) in such a way that the following two axioms hold:

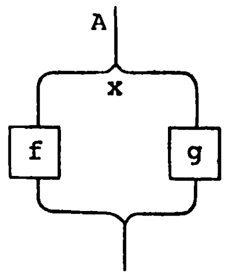
Disjoint sum axiom: If a family $(f_i : i \in I)$ of morphisms from A to B is disjoint in the sense that there exists $f: A \longrightarrow I \times B$ with $pr_i f = f_i$ for all i, then Σf_i is defined.

Untying axiom: Given a family $f_i : A \longrightarrow B$ with Σf_i defined, $\Sigma in_i f_i : A \longrightarrow I \times B$ is defined.

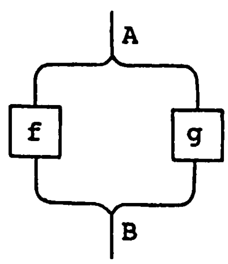
Note that since composing on either side is a morphism of ω-complete partial abelian monoids, the zero element of $\mathcal{K}(A,B)$ provides zero maps.

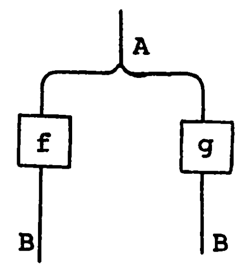
8 Discussion of the Motivating Example In $\underline{Pfn}_D$, define Σf_i if and only if the $(f_i : A \longrightarrow B)$ are disjoint to be their union. (The untying axiom fails if this construction is extended to families which agree on their overlaps.)

When f + g is defined, we may view it as interpreting the flowchart



the point being that a 'fanout' such as that at x is an extension of conventional program syntax whose semantic interpretation includes the assertion that f and g are disjoint. The name "untying axiom" then comes from its flowchart interpretation:

If 

exists, then so does 

$f + g\colon A \longrightarrow B$ $\qquad$ $in_1 f + in_2 g\colon A \longrightarrow B \oplus B$

We now begin to examine the extent to which the pr_j enjoy certain of the properties of projections in a category which is only partially-additive.

<u>9 Proposition</u> Given $f\colon A \longrightarrow \oplus (B_i : i \in I)$ there exist unique $f_i\colon A \longrightarrow B_i$ with $f = \Sigma in_i f_i$, namely $f_i = pr_i f$.

<u>Proof</u>: The family $(in_i f_i)$ is disjoint because $\Delta \cdot f\colon A \longrightarrow I \times \oplus B_i$ satisfies the disjointness condition $pr_j (\Delta \cdot f) = in_j \cdot f_j$, using (6). Thus $\Sigma in_j f_j$ is defined, by the disjoint sum axiom.

Noting that $pr_j \cdot id = pr_j$, we conclude, by the disjoint sum axiom again, that $pr_j\colon I \times \oplus B_i \longrightarrow \oplus B_i$ is summable so that, by the untying axiom, $\Sigma in_j pr_j\colon I \times \oplus B_i \longrightarrow I \times \oplus B_i$ exists. But $(\Sigma in_j pr_j) in_i = \Sigma in_j \delta_{ij} = in_i$ for each i, and so $\Sigma in_j pr_j = id$. But then, noting that $\sigma\Delta = id$, we have that $f = \sigma(\Sigma in_j pr_j)\Delta f = \Sigma(\sigma \cdot in_j)(pr_j \cdot \Delta) f = \Sigma in_j (pr_j f)$. If the g_i satisfy

$f = \Sigma in_j \cdot g_j$, then $pr_i f = \Sigma pr_i in_j g_j = \Sigma \delta_{ji} g_j = g_i$. □

<u>10 Corollary</u> In a partially additive category, $\sum_i in_i pr_i = id: \oplus A_i \longrightarrow \oplus A_i$. □

<u>11 Theorem</u> The additive structure of a partially-additive category is unique as follows. If $\mathcal{K}$ is partially-additive then a family $f_i: A \longrightarrow B$ is summable if and only if it is disjoint; in that case, the $f: A \longrightarrow I \times B$ with $pr_i f = f_i$ is unique and

$$\Sigma f_i = A \xrightarrow{f} I \times B \xrightarrow{\sigma} B.$$

<u>Proof</u>: If Σf_i exists then $f = \Sigma in_i f_i$ exists and $pr_j f = \Sigma(pr_j in_i f_i : i \in I) = \Sigma(\delta_{ij} f_i : i \in I) = f_j$. This shows summable families are disjoint.

That $pr_i f = pr_i g$ implies $f = g$ is immediate from (9). Moreover, if $pr_i f = f_i$, $\sigma f = \sigma \Sigma in_i f_i = \Sigma \sigma in_i f_i = \Sigma f_i$. □

<u>12 Proposition</u> Given $f_i: A \longrightarrow B$, $g_i: B \longrightarrow C$ in a partially-additive category, if Σf_i exists then $\Sigma g_i f_i$ exists.

<u>Proof</u>: By (9) and (11) if Σf_i exists there exists $f: A \longrightarrow I \times B$ with $f = \Sigma in_i f_i$. Consider $g = (g_i): I \times B \longrightarrow C$. Then $gf = \Sigma g in_i f_i = \Sigma g_i f_i$. □

We now see what happens when the Σ on each $\mathcal{K}(A,B)$ is totally defined.

<u>13 Definition</u> An <u>ω-additive category</u> is a category with countable coproducts which is based on the category of countably complete abelian monoids.

Totality of Σ means that the disjoint sum axiom and untying axiom are trivially satisfied, and so every ω-additive category is a partially-additive category. Equivalently, then, an ω-additive category is a partially-additive category in which every countable family $(f_i: A \longrightarrow B)$ is summable.

We thus immediately have:

14 Theorem Let $\mathcal{K}$ be a category with countable coproducts and zero maps. Then $\mathcal{K}$ is ω-additive if and only if for every set I and object A $pr_i: I \times A \longrightarrow A$ is a product. □

The preceding theorem may be proved without essential change if 'I countable' is replaced by 'card I < α' or 'card I ≤ α' for any cardinal α, or by 'I arbitrary'. The case 'card I < ω' is the familiar characterization theorem for semiadditive categories [12, pp. 27-31], [2, Sec. 5.2].

15 Example For a fixed set D, $\underline{\underline{Rel}}_D$ is the category with all countable sets as objects and in which a morphism from A to B is a relation from $A \times D$ to $B \times D$. $\underline{\underline{Rel}}_D$ is ω-additive. A disjoint union $A = \cup A_i$ is a coproduct in the obvious way and is also a product in that a subset of A is tantamount to the tuple obtained by intersecting with the A_i. This category provides the setting for nondeterministic program semantics -- initial data do not determine a unique outcome but rather a set of possible outcomes.

$\underline{\underline{Pfn}}_D$ and $\underline{\underline{Rel}}_D$ motivate the following:

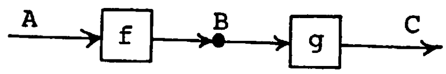
16 Definition Let $\mathcal{K}$ be a category and let D be an object whose countable copowers exist. The localization $\mathcal{K}_D$ of $\mathcal{K}$ to D is the category whose objects are countable sets and in which a morphism from A to B is a $\mathcal{K}$-morphism $A \times D \longrightarrow B \times D$ with composition and identities as in $\mathcal{K}$.

It is easy to check that disjoint unions provide coproducts in $\mathcal{K}_D$ so that $\mathcal{K}_D$ is essentially an algebraic theory [11, Chap. 1] in the category of sets. Moreover $\mathcal{K}_D$ is partially additive when $\mathcal{K}$ is (an independent proof is given in our characterization theorem 6.6 of $\mathcal{K}_D$).

3. Basic Structuring Concepts

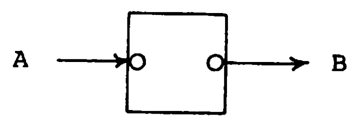
The main aim of this section is to show that the interpretation of sequential composition, the conditional and iteration all carry over from the motivating setting of Section 1 to the setting of a general partially-additive category $\mathcal{K}$. Along the way, we shall also write down the interpretation of a number of other flowchart constructs.

1 Sequential composition The flowchart



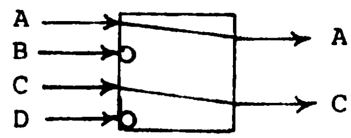
is simply interpreted by the composition of $f: A \longrightarrow B$ and $g: B \longrightarrow C$ in $\mathcal{K}$. (In this section, we use the same notation for the syntactic construct, the flowchart, and the semantic construct, the morphism in $\mathcal{K}$. This should make for ease of exposition with no loss of clarity.)

2 Zero Maps We view the zero map $A \longrightarrow B$ as interpreting the 'no through road' flow chart



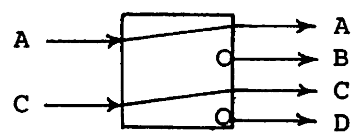
Since (following the influence of Scott [13]) the symbol $\perp$ has achieved wide use among theoretical computer scientists to denote 'the completely undefined', we shall henceforth use $\perp$ to denote the zero map $A \longrightarrow B$ in a partially-additive category $\mathcal{K}$ -- without, of course, imputing any partial ordering to $\mathcal{K}(A,B)$.

3 Projection Maps We view a quasi-projection such as pr_{AC}: $A \oplus B \oplus C \oplus D \longrightarrow A \oplus C$ as interpreting the corresponding 'straight-through connection', in this case



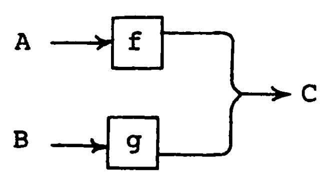
Zero maps are included as a special case.

<u>4 Injection Maps</u> We view an injection such as $in_{AC}: A \dotplus C \longrightarrow A \dotplus B \dotplus C \dotplus D$ as interpreting the corresponding 'straight-through connection', in this case



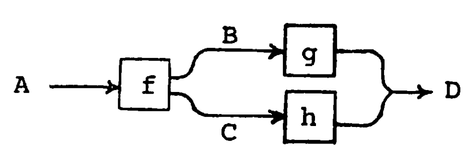
Identity maps are included as a special case.

<u>5 Common-Exit Joins</u> Given $f: A \longrightarrow C$ and $g: B \longrightarrow C$, the flowchart



is interpreted by the induced $\binom{f}{g}: A \dotplus B \longrightarrow C$, with $\binom{f}{g} \cdot in_A = f$, $\binom{f}{g} \cdot in_B = g$.

<u>6 Conditional Composition</u> By 2.9, any $f: A \longrightarrow B \dotplus C$ may be written as $in_B \cdot f_B + in_C \cdot f_C$ where f_B and f_C are uniquely characterized as $pr_B \cdot f$ and $pr_C \cdot f$, respectively. Thus, combining (1) and (5) above, we see that, given $g: B \longrightarrow D$ and $h: B \longrightarrow D$, the conditional composition flowchart



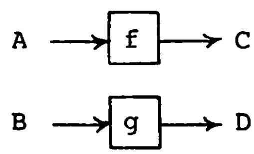
is interpreted by the map

$$\begin{pmatrix} g \\ h \end{pmatrix}\cdot f = \begin{pmatrix} g \\ h \end{pmatrix}\cdot (in_B\cdot f_B + in_C\cdot f_C)$$

$$= \begin{pmatrix} g \\ h \end{pmatrix}\cdot in_B\cdot f_B + \begin{pmatrix} g \\ h \end{pmatrix}\cdot in_C\cdot f_C$$

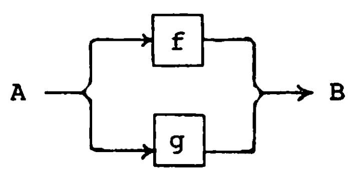
$$= g\cdot f_B + h\cdot f_C, \quad \text{so that this sum is well-defined.}$$

7 Coproduct Given $f\colon A \longrightarrow C$, $g\colon B \longrightarrow D$, the flowchart

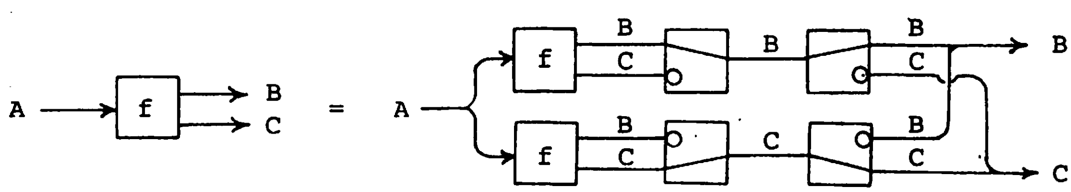


is interpreted as the map $f \overset{\frown}{+} g\colon A \overset{\frown}{+} B \longrightarrow C \overset{\frown}{+} D$, i.e. $(f + g)\cdot in_A = in_C\cdot f$ and $(f \overset{\frown}{+} g)\cdot in_B = in_D\cdot g$.

8 Disjoint Sums If a family $(f_i\colon A \longrightarrow B)$ is disjoint, then that sum exists by the disjoint sum axiom. In such a case, we may abuse the usual flowchart notation by representing this sum by joining the entries and exits of the f_i, so that $f + g$ is the interpretation of



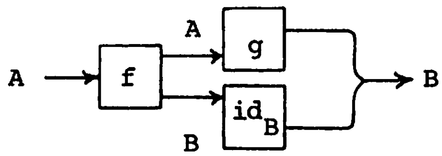
9 The Exclusive-Or Principle In 2.9 and 2.11, we saw that every $f\colon A \longrightarrow \overset{\frown}{+} B_i$ equals the disjoint sum $\Sigma in_i pr_i f$. In flowchart notation, this corresponds to such equalities as



We now show that the formula $\sum_{n\geq 0} f_B f_A^n$ for the iterate $f^\dagger$ introduced in Section 1 yields a well-defined sum in any partially-additive category.

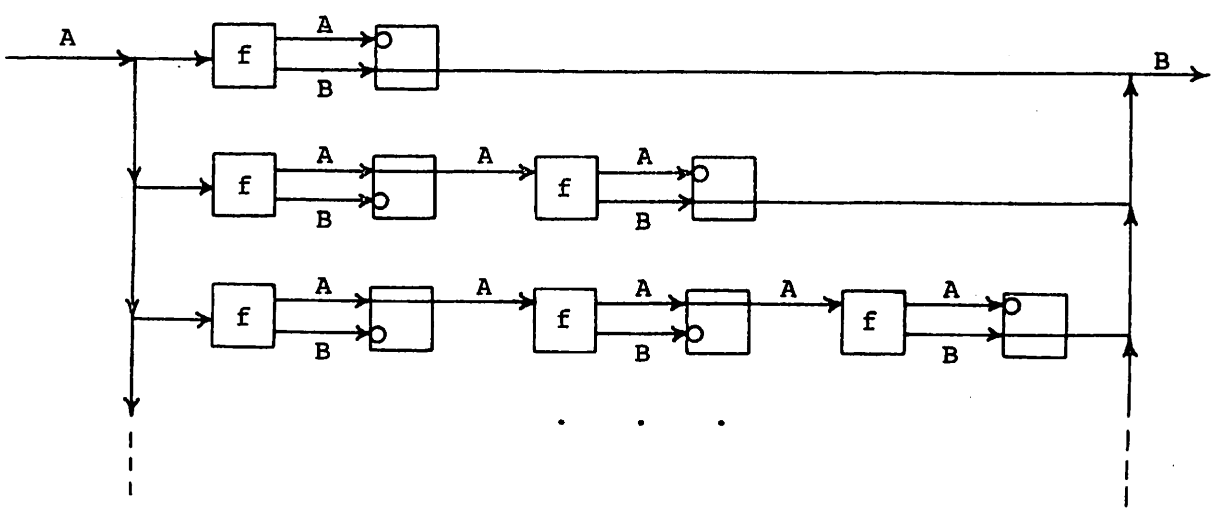
<u>10 Proposition</u> Given $f: A \longrightarrow A + B$, take $f_A = pr_A \cdot f: A \longrightarrow A$, $f_B = pr_B \cdot f: A \longrightarrow B$. Then the sum $f^\dagger = \sum_{n \geq 0} f_B f_A^n$ is defined.

<u>Proof</u>: By the remark following Definition 2.1, it suffices to prove that $\sum_{n=0}^{k} f_B f_A^n$ is defined for each finite k. But for any $g: A \longrightarrow B$ we may form the conditional



whose interpretation is the sum $g \cdot f_A + f_B$, so that this sum is well-defined. Now $\sum_{n=0}^{k} f_B f_A^n$ is certainly defined for k = 0, and taking $g = \sum_{n=0}^{k} f_B f_A^n$ and using partition-associativity we have the induction step that $\sum_{n=0}^{k+1} f_B f_A^n$ is defined. □

Recalling that all sums are disjoint, we can represent $f^\dagger$ as the infinite disjoint sum flowchart

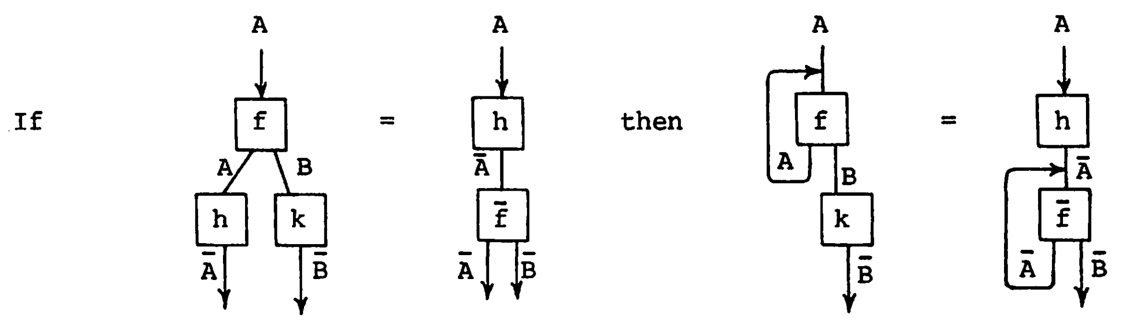


11 Proposition The passage $f \mapsto f^\dagger$ is functorial, that is,

$$\text{if}\quad \begin{array}{ccc} A & \xrightarrow{f} & A \oplus B \\ h\downarrow & & \downarrow h \oplus k \\ \bar{A} & \xrightarrow[\bar{f}]{} & \bar{A} \oplus \bar{B} \end{array} \qquad \text{then} \qquad \begin{array}{ccc} A & \xrightarrow{f^\dagger} & B \\ h\downarrow & & \downarrow k \\ \bar{A} & \xrightarrow[\bar{f}^\dagger]{} & \bar{B} \end{array}$$

Proof: Writing $f = in_A f_A + in_B f_B$ and $\bar{f}$ similarly, it is easily verified that $(h \oplus k)f = in_{\bar{A}} h f_A + in_{\bar{B}} k f_B$ whereas $\bar{f}h = in_{\bar{A}} \bar{f}_A h + in_{\bar{B}} \bar{f}_B h$ so that the leftmost square is equivalent to $hf_A = \bar{f}_A h$ and $kf_B = \bar{f}_B h$. It follows that $kf^\dagger = \Sigma\, kf_B f_A^n = \Sigma\, \bar{f}_B h f_A^n = \Sigma\, \bar{f}_B \bar{f}_A^n h = \bar{f}^\dagger h$. □

In flowchart pictures, functoriality takes the form



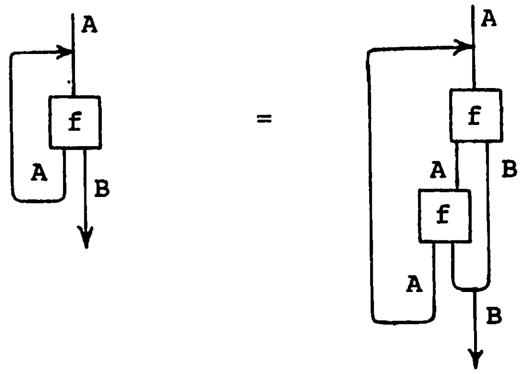
We announced this principle in [2]. The previous proposition proves it in $\underline{Pfn}_D$. Upon reflection, the result is seen directly as a truism of flowchart semantics, a new one so far as we know. Some of its consequences are further explored in the next section.

The following two results are immediate from the definition.

12 Proposition For any A, $(in_2 : A \longrightarrow A \oplus A)^\dagger = id_A$.

13 Proposition For any A, B $(in_A : A \longrightarrow A \oplus B)^\dagger = \bot$.

In conversation with J. D. Wright we learned that the following truism of flowchart semantics had not been deduced from other axioms studied.



We now show that the identity holds in any partially-additive category.

<u>14 Proposition</u> Given $f: A \longrightarrow A \oplus B$,

$$f^{\dagger} = \left(A \xrightarrow{f} A \oplus B \xrightarrow{\binom{f}{in_B}} A \oplus B\right)^{\dagger}$$

<u>Proof</u>: $\binom{f}{in_B} f = \binom{f}{in_B}(in_A f_A + in_B f_B) = ff_A + in_B f_B$

$$= in_A f_A^2 + in_B f_B f_A + in_B f_A$$

$$= in_A f_A^2 + in_B (f_B f_A + f_A).$$

For the last step, we must note that the existence of $f_B f_A + f_A$ was seen in the proof of proposition 10 with $g = f_B$. Thus, using partition-associativity,

$$\left(\binom{f}{in_B} f\right)^{\dagger} = \Sigma\, (f_B f_A + f_A)(f_A^2)^n = \sum_{n\geq 0} f_B (f_A^{2n+1} + f_A^{2n})$$

$$= \sum_{m\geq 0} f_B f_A^m$$

$$= f^{\dagger}. \qquad \square$$

4. Axiomatic Iteration

In this section we explore the properties of a category which satisfies:

1 Definition A category $\mathcal{K}$ admits functorial dagger if $\mathcal{K}$ has countable coproducts, and is equipped with an operation $\dagger$ (called a functorial dagger)

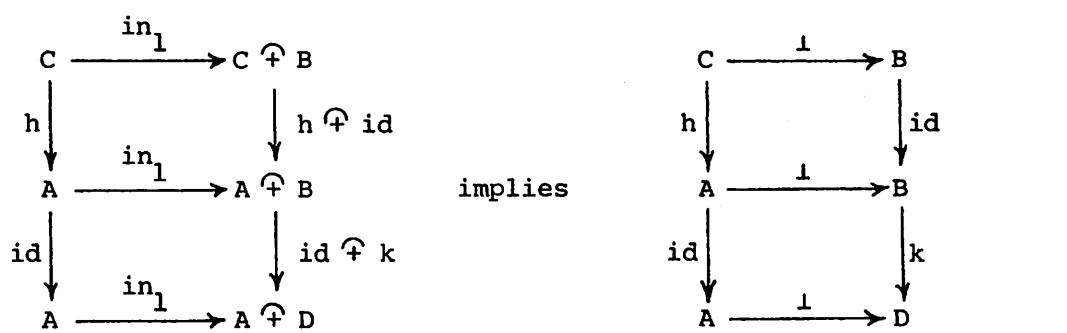
$$f: A \longrightarrow A + B \;\mapsto\; f^{\dagger}: A \longrightarrow B$$

which satisfies the functorial axiom of 3.11 and the axiom $(in_2: A \longrightarrow A + A)^{\dagger} = id_A$ of 3.12.

Note that functoriality alone is insufficient since we wish to exclude such degenerate solutions as $f^{\dagger} \equiv \bot$, and $f^{\dagger} = f_B$.

2 Proposition A category $\mathcal{K}$ with functorial dagger has zero maps.

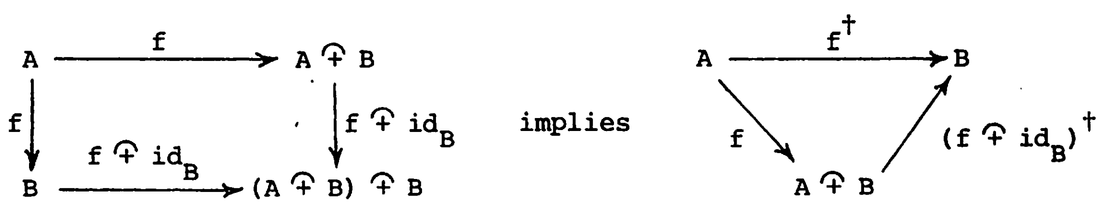
Proof: Define $\bot: A \longrightarrow B = (in_1: A \longrightarrow A + B)^{\dagger}$. Then for all $h: C \longrightarrow A$, $k: B \longrightarrow D$ we have



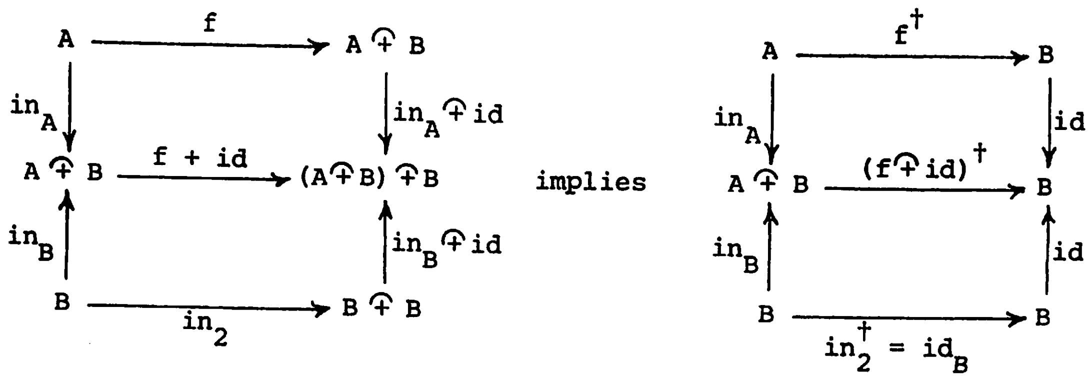
□

3 Proposition If $\mathcal{K}$ admits functorial dagger, then the Elgot iteration equation holds for every $f: A \longrightarrow A + B$.

Proof:



We must show that $(f \oplus id_B)^{\dagger} = \begin{pmatrix} f^{\dagger} \\ id_B \end{pmatrix}$. Indeed,

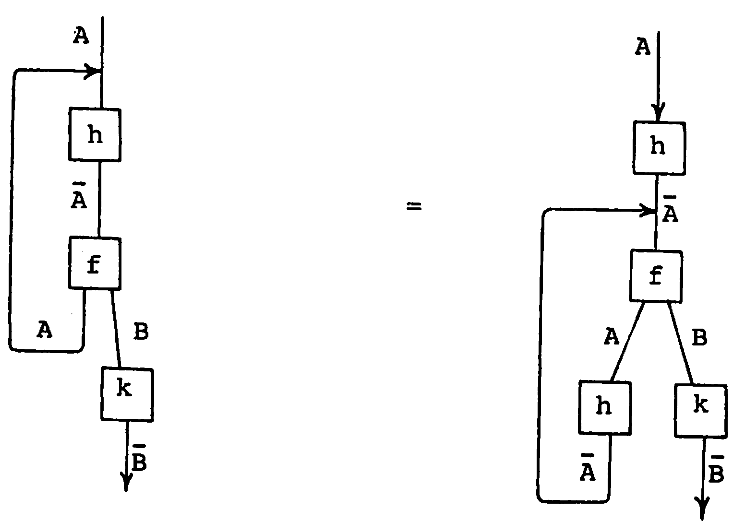


□

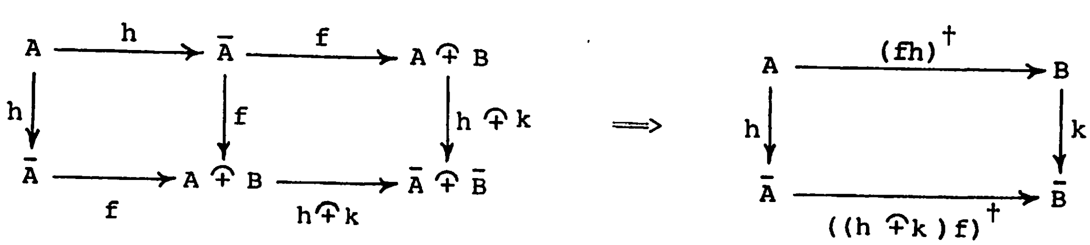
We do not know, however, whether the result of 3.14 must hold in this context.

For a general discussion of dinaturality see [10, IX.4]; we have

<u>4 Proposition</u> The following dinaturality condition holds:



<u>Proof</u>:



□

Our 'dinaturality' corresponds to propositions 5.6.2 and 5.6.3 of [6]. The context there requires (in our notation) that $f\colon A \longrightarrow A \mathbin{\widehat{+}} B$ be 'ideal' in an iterative algebraic theory. We also observe that Elgot's proof of [6, 5.6.5] works for any functorial dagger. Moreover, Elgot's 'anti-iterate' [6, p. 225], a construction $f \mapsto f^{\ddagger}$ with $f^{\ddagger\dagger} = f$ is much simpler in our formulation because we can iterate coproduct injections (which Elgot cannot). Thus if $f\colon A \longrightarrow B$, set $f^{\ddagger} = in_B f\colon A \longrightarrow A \mathbin{\widehat{+}} B$ so that $f^{\ddagger}_A = \perp$, $f^{\ddagger}_B = f$ and hence $(f^{\ddagger})^{\dagger} = f$.

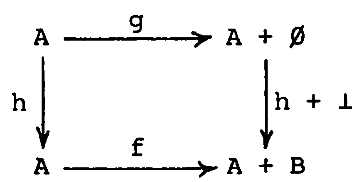
As discussed at the end of Section 1, the dilemma posed by the Elgot iteration equation -- should one use least solutions or abandon $\underline{\underline{Pfn}}_D$ for unique solutions -- now admits a third approach. The next theorem provides the desired uniqueness statement.

<u>5 Theorem</u> The passage $(f\colon A \longrightarrow A + B) \mapsto (\sum_{n\geq 0} f_B f_A^n\colon A \longrightarrow B)$ is the only functorial dagger in $\underline{\underline{Pfn}}_D$.

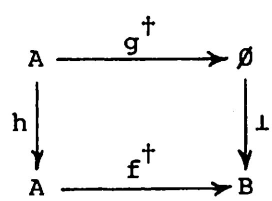
<u>Proof</u>: Let $(\ )^{\dagger}$ be a functorial dagger. By (3), $\sum_{n\geq 0} f_B f_A^n \subset f^{\dagger}$ in the usual sense of inclusion for partial functions. We must show that the inclusion is in fact equality. Let P be the complement of $\mathrm{dom}(\sum_{n\geq 0} f_B f_A^n)$ in $A \times D$, and let $h\colon A \longrightarrow A$ in $\underline{\underline{Pfn}}_D$ be the partial function

$$h(a,d) = \begin{cases} (a,d) & \text{if } (a,d) \in P \\ \text{undefined} & \text{else} \end{cases}$$

Now $P = \{(a,d) \mid f^n(a,d) \in A \times D \text{ for all } n \geq 0\}$ and so for $g = in_A pr_A f$



which implies



Hence $f^{\dagger} h = \perp$, i.e. $\mathrm{dom}(f^{\dagger}) \subset \mathrm{dom}(\sum_{n\geq 0} f_B f_A^n)$ as was to be shown. □

6 Open Question Does every partially-additive category have a unique functorial dagger?

7 Open Question What conditions on a topos $\mathcal{E}$ guarantee that a suitably internalized form of Theorem 5 holds for the category of partial $\mathcal{E}$-morphisms?

5. A Characterization Theorem

In this section we seek criteria that help to decide if a given category $\mathcal{K}$ is partially-additive. It follows from 3.11, 4.2 and 2.11 that the following two axioms must hold (in addition to our standing requirement that $\mathcal{K}$ have countable coproducts).

Zero axiom $\mathcal{K}$ has zero maps.

Covering axiom For any countable set I and object A the family $pr_i : I \times A \longrightarrow A$ is jointly monic.

In this context, if $f_i : A \longrightarrow B$ is disjoint (so that there exists unique $f: A \longrightarrow I \times B$ with $pr_i f = f_i$) Σf_i is defined as σf as in 2.11.

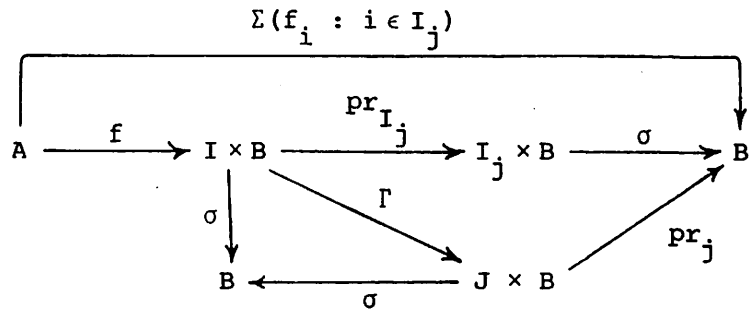
1 Theorem For $\mathcal{K}$ as above, $\mathcal{K}$ is partially-additive if and only if

(i) Given $(f_i : i \in I)$ and a partition $(I_j : j \in J)$ of I, if $\Sigma((\Sigma f_i : i \in I_j) : j \in J)$ is defined then $\Sigma(f_i : i \in I)$ is defined, and

(ii) A family $(f_i : i \in I)$ is disjoint providing each of its finite subfamilies is disjoint.

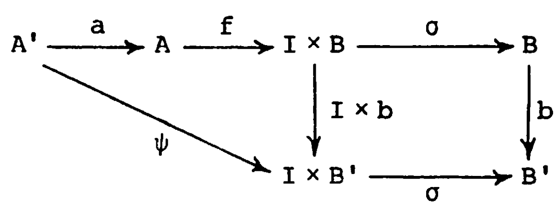
Proof: We have only to show that if $\mathcal{K}$ satisfies the zero and covering axioms then $(\mathcal{K},\Sigma)$ satisfies all of 2.7 except (i) and (ii) above.

Suppose $\Sigma(f_i : i \in I)$ is defined and $(I_j : j \in J)$ partitions I. Let $f\colon A \longrightarrow I \times B$ with $pr_i f = f_i$. For $i \in I$ let $j(i)$ be the unique $j \in J$ with $i \in I_j$. (The fact that some I_j may be empty does not obstruct the proof.) Define $\Gamma\colon I \times B \longrightarrow J \times B$ by $\Gamma in_i = in_{j(i)}$. That $\Sigma(\Sigma(f_i : i \in I_j) : j \in J)$ exists and coincides with $\Sigma(f_i : i \in I)$ is immediate from the diagram



The unary-sum axiom of 2.1 is trivial and the limit axiom of 2.1 is immediate from (ii), so $\mathcal{K}(A,B)$ is an ω-complete partial abelian monoid.

To see that composition preserves Σ let $f\colon A \longrightarrow I \times B$, $f_i = pr_i f$, $a\colon A' \longrightarrow A$, $b\colon B \longrightarrow B'$. Then in the diagram



$pr_i\psi = bf_i a$ so that $\Sigma bf_i a = \sigma\psi$ exists and equals $b(\Sigma f_i)a$.

Of the remaining axioms in 2.7, the disjoint-sum axiom is trivial so we need only check that if $f\colon A \longrightarrow I \times B$ then $(in_i pr_i f : i \in I)$ is disjoint. Indeed, using the diagonal map of 2.5, 2.6 $\Delta f\colon A \longrightarrow I \times (I \times B)$ satisfies $pr_i(\Delta f) = in_i pr_i f$. □

It is also true that the proofs of 2.9, 2.10 go through in the context of the above proof. However,

<u>2 Example</u> If $\mathcal{K}$ is the category of real vector spaces and linear maps, $\mathcal{K}$ satisfies the zero and covering axioms but satisfies neither (i) nor (ii) of Theorem 1. Here $f_i : A \longrightarrow B$ is disjoint if and only if $\{i \in I : f_i(a) \neq 0\}$ is finite for all $a \in A$. $(\Sigma f_i)(a) = \Sigma f_i(a)$. A sequence such as $f - f + f - f + \ldots$ shows why (i) fails. The failure of (ii) is obvious.

We do not know whether (i) and (ii) of Theorem 1 are independent if the zero and covering axioms hold.

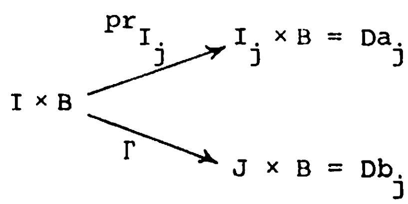
For practical purposes Theorem 1 is the most useful we can offer. More abstract versions of (i) and (ii) in that theorem are discussed below primarily because they make explicit that all depends on the structure of the quasi-projections.

<u>Abstract limit axiom</u> For each countable set I and object A the family $pr_F : I \times A \longrightarrow F \times A$ indexed by the finite non-empty subsets of I is the inverse limit of the $F \times A$ (where, if $G \subset F$, $F \times A \longrightarrow G \times A$ is itself a quasi-projection; that pr_F is a cone is always true).

<u>Partition axiom</u> For each countable set I, partition $(I_j : j \in J)$ of I and object B consider the diagram scheme

$$a_j \xrightarrow{\ \alpha_j\ } b_j \xleftarrow{\ \beta_j\ } c \qquad (j \in J)$$

over which we define the diagram D by $Da_j = I_j \times B$, $Db_j = B$, $Dc = J \times B$, $D\alpha_j = \sigma : I_j \times B \longrightarrow B$, $D\beta_j = pr_j : J \times B \longrightarrow B$. Define $\Gamma : I \times B \longrightarrow J \times B$ by $\Gamma in_i = in_{j(i)}$ as in the proof of Theorem 1. Then



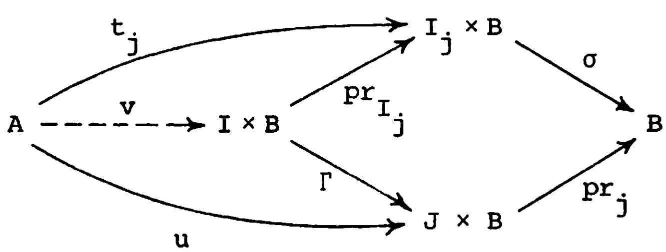
(which is always a cone over D) is the limit of D.

Notice that all of the axioms on projections -- the covering, abstract limit and partition axioms -- are true about projections from a product B^I, and amount to a weakening of the context of Theorem 2.14.

3 Characterization Theorem Let $\mathcal{K}$ be a category with countable coproducts. Then $\mathcal{K}$ is partially-additive if and only if $\mathcal{K}$ satisfies the zero, covering, abstract limit and partition axioms.

Proof: We use Theorem 1. Given the zero and covering axioms, (ii) is clearly equivalent to the abstract limit axiom and we have only to show that (i) is equivalent to the partition axiom.

First assume (i) and assume given t_j and u with $\sigma t_j = pr_j u$ so that we seek unique v, all as shown.



Define $f_i = pr_i t_{j(i)} : A \longrightarrow B \quad (i \in I)$. Then $\Sigma(f_i : i \in I_j) = \sigma t_j$ exists and hence $\Sigma(\Sigma(f_i : i \in I_j) : j \in J) = \sigma u$ exists so that, using (i) and Theorem 1, $v = \Sigma(in_i f_i : i \in I)$ exists. We have

$$pr_{I_j} v = \Sigma(pr_{I_j} in_i f_i : i \in I) = \Sigma(in_i f_i : i \in I_j) = t_j,$$

the last equality by 2.9 and the remark following Theorem 1. Using the covering axiom, the pr_{I_j} are easily seen to be jointly monic so v is unique.

As $pr_j : J \times B \longrightarrow B$ is jointly monic, $\Gamma v = u$.

That the partition axiom implies (i) is obvious from the above diagram. □

Any flowchart may be 'unfolded' to yield a sequence of tree-like flowcharts which provide non-decreasing loop-free approximations to the given flowchart, and this sequence may be viewed as approximating a (possibly) infinite tree which can be interpreted in a natural way to yield the semantics of the original flowchart. In the remainder of this section we show how a suitable collection of infinite trees may be properly viewed as a partially-additive category.

4 Definition A tree shape is a subset σ of $\mathbb{N}^*$ satisfying

(i) $\Lambda \in \sigma$

(ii) If $wk \in \sigma$ with $w \in \mathbb{N}^*$ and $k \in \mathbb{N}$, then $w, w0, \ldots, w(k-1)$ are in σ.

We say w in σ has degree n, $\partial w = n$, if $wk \in \sigma$ iff $k < n$. We say w is a leaf of σ if $\partial w = 0$. Notice that any non-empty union of tree shapes is again a tree shape.

Let Ω be a finitary operator domain such that Ω_0 has a single element $\bot$. An Ω-tree in A is a pair (σ,t) where σ is a tree shape and $t: \sigma \longrightarrow \Omega \cup A$ is such that $t(w) \in \Omega_n$ if $\partial w = n > 0$, while $t(w) \in A \cup \{\bot\}$ if w is a leaf.

Imagine Ω as labelling non-exit nodes of a one-entry flowchart, while the exits are labelled with elements of A or with $\bot$ (indicating that a computation reaching that point is henceforth totally undefined). The Elgot school emphasizes the 'free iterative theory' generated by Ω. This is the 'algebraic theory' $\underline{S}$ (using the notation of [11]) for which AS is just the set of Ω-trees in A with only a finite number of nonisomorphic subtrees -- i.e. the trees obtained by 'unfolding' finite flowcharts. [Note: Their formulation always allows Ω_0 to be an arbitrary nonempty set.]

The Kleisli category [11, Section 1.3] of the theory $\underset{\sim}{S}$ does not have zero maps, and so cannot be partially-additive. However we show that a modification, appropriate to our program semantics, does yeild a partially-additive category. Recall that we view an Ω-tree in A as a loop-free flowchart with entry at the root and with leaves labelled a in A corresponding to 'exit a' and leaves labelled $\perp$ corresponding to undefinedness. We thus introduce the equivalence relation ~ on each AS generated by identifying each subtree in $\emptyset$S with $\perp$ (since on entering such a subtree, computation cannot reach a defined exit). It is a straightforward exercise to show that the resulting equivalence classes AT = AS/~ correspond to an algebraic theory $\underset{\sim}{T}$, and we now concentrate attention on the full subcategory $\mathcal{A}$ of the Kleisli category of $\underset{\sim}{T}$ whose objects are the countable sets. For convenience, we restate the definition of $\mathcal{A}$ explicitly within the next theorem.

<u>5 Theorem</u> Let $\mathcal{A}$ be the category defined as follows:

(i) Objects: Countable sets.

(ii) Morphisms: A morphism $A \longrightarrow B$ in $\mathcal{A}$ is a map $A \longrightarrow BS$ in <u>Set</u> (where each element in BS may be viewed as an Ω-tree in B with the property that every subtree which is an Ω-tree in $\emptyset$ comprises a single node labelled $\perp$).

(iii) Composition: Clone composition (Given $\alpha: A \longrightarrow B$ and $\beta: B \longrightarrow C$, the tree $\beta \circ \alpha(a)$ in CS is that obtained from the tree $\alpha(a)$ by inserting the tree $\beta(b)$ from CS in place of each leaf labelled b and then reducing each subtree all of whose leaves are $\perp$ to $\perp$. This corresponds to the program interpretation: "On reaching exit b of the loop-free flowchart $\alpha(a)$, transfer control to the entry of the loop-free flowchart $\beta(b)$.")

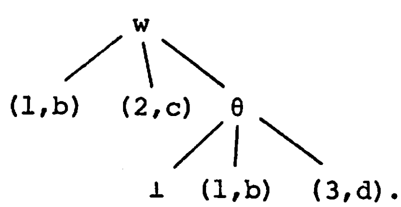
Then $\mathscr{A}$ is a partially-additive category.

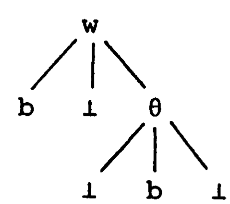
Proof: It is clear that $\emptyset$ is a zero object in $\mathscr{A}$ ($\emptyset S$ contains the single element $\bot$) and that countable products = disjoint union (since disjoint union = coproduct in any Kleisli category). Recall that in any Kleisli category, a morphism $\alpha: A \longrightarrow B$ induces a map $\alpha^{\#}: AT \longrightarrow BT$ (for t in AT, $\alpha^{\#}(t)$ is obtained by replacing each a-node by $\alpha(a)$ from BT).

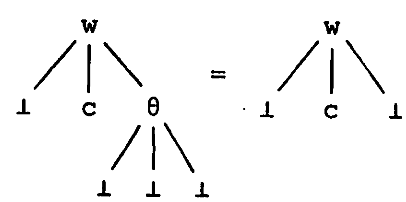
Now consider $I \times B$. A morphism $A \xrightarrow{f} I \times B$ is a function $A \xrightarrow{f} (I \times B)T$. The composition of f with $pr_i: I \times B \longrightarrow B$ is $pr_i^{\#} f$. Thus to prove the covering axiom, we must prove that the functions $pr_i^{\#}$: $(I \times B)T \longrightarrow BT$ are jointly monic. Now

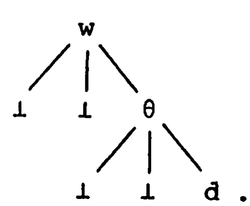
$$pr_i(j,b) = \begin{cases} \bot \in BT & \text{if } i \neq j \\ b & \text{if } i = j. \end{cases}$$

Thus $pr_i^{\#}$ substitutes b for each leaf of form (i,b), $\bot$ for each leaf of form (j,b) $j \neq i$, and then replaces each $\emptyset S$ subtree by $\bot$. For example:

Let t = 

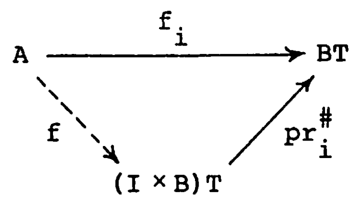
Then $pr_1^{\#}(t)$ = 

$pr_2^{\#}(t)$ = 

$pr_3^{\#}(t)$ = 

That the $pr_i^{\#}$ are jointly-monic in general is clear. To recover the original tree (σ,t) from $(\sigma_i,t_i) = pr_i^{\#}(\sigma,t)$, set $\sigma = \cup \sigma_i$. When $w \in \sigma$ is non-terminal, $\{i : w \text{ is non-terminal in } \sigma_i\}$ is non-empty; and if i, j are in this set, $t_i(w) = t_j(w)$ recaptures the label t(w). When $w \in \sigma$ is terminal, then either all $t_i(w) = \bot$ so that $t(w) = \bot$, or there is a unique (i,b) with $t_i(w) = b$ so that $t(w) = (i,b)$.

We then see that a family $A \xrightarrow{f_i} BT$ is disjoint in the sense of 2.7 that there exists f with



iff for all $a \in A$ $(f_i(a) : i \in I)$ is disjoint as a family of Ω-trees, where we say that a <u>family</u> $((\sigma_i, t_i) : i \in I)$ <u>of</u> <u>Ω-trees</u> <u>in</u> B <u>is</u> <u>disjoint</u> if and only if there exists an Ω-tree (σ,t) in $I \times B$ with $pr_i^{\#}(\sigma,t) = (\sigma_i, t_i)$. But this says that for each $w \in \sigma_i \cap \sigma_j$ either $t_i(w) = t_j(w)$ or one of them equals $\bot$. We may thus define the partial-addition $\Sigma(\sigma_i, t) = (\cup\sigma_i, \max t_i)$ where max refers to the ordering "not defined" $< \bot <$ B.

By Theorem 1, it only remains to show

(i) If $(I_j : j \in J)$ partitions I and $\Sigma((\Sigma(\sigma_i, t_i) : i \in I_j) : j \in J)$ is defined, then $\Sigma((\sigma_i, t_i) : i \in I)$ is defined.

(ii) $((\sigma_i, t_i) : i \in I)$ is disjoint if each of its finite subfamilies is disjoint.

To prove (i): Let $w \in \sigma_m \cap \sigma_n$. If $m, n \in I_j$ for some $j \in J$ then $t_m(w) = t_n(w)$ or one equals $\bot$, and we are done. Else $m \in I_j$, $n \in I_k$, $j \neq k$. Then $w \in (\cup(\sigma_i : i \in I_j)) \cap (\cup(\sigma_i : i \in I_k))$ so $\max(t_i w : i \in I_j) = \max(t_i w : i \in I_k)$ or one of these equals $\bot$. If $t_m(w)$ or $t_n(w)$ equals $\bot$ we are done. Otherwise, $t_m(w) = \max(t_i w : i \in I_j) = \max(t_i w : i \in I_k) = t_n w$.

(ii) is immediate. □

Note: The fact that $\cup\sigma_i$ is admissible for all countable {i} is where 'infinite trees' are needed.

Note: Let $\Omega_n = \emptyset$ if $n > 0$, $\Omega_0 = \{\bot\}$. If (σ,t) is a tree in B and if $w \in \sigma$ then $\{k \in P : wk \in \sigma\} = \emptyset$ since $t(w) \in \Omega_n$ for $n > 0$ is impossible. Thus {1} is the only tree shape. $AT = A + \{\bot\}$ and $\mathcal{A}$ = countable sets and partial functions.

6. Matrix Categories over ω-Complete Partial Semirings

A morphism $f : A \longrightarrow B$ in $\underline{\underline{Pfn}}_D$ may be regarded as an $A \times B$-matrix of $\underline{\underline{Pfn}}_D$-morphisms $f_{ab} : 1 \longrightarrow 1$ where

$$f_{ab}(d) = \begin{cases} d' & \text{if } f(a,d) = (b,d') \\ \text{undefined} & \text{if no such } d' \text{ exists} \end{cases}$$

and we see that the matrix (f_{ab}) is 'row-disjoint', i.e. for each a in A the $(f_{ab} : b \in B)$ are disjoint. In this section, we introduce the notion of an ω-complete partial semiring R to axiomatize the necessary properties of $\underline{\underline{Pfn}}_D(1,1)$, and then introduce the notion of the matrix category $\underline{\underline{Mat}}(R)$ of such an R to capture the way in which $\underline{\underline{Pfn}}_D$ is built up from $\underline{\underline{Pfn}}_D(1,1)$. Then in Theorem 6 we see that a category $\mathcal{K}$ is a $\underline{\underline{Mat}}(R)$ iff it is isomorphic to the localization $\mathcal{A}_D$ for some partially-additive category $\mathcal{A}$ and object D of $\mathcal{A}$.

1 Definition An ω-complete partial semiring is $R = (R,\Sigma,\cdot,1)$ where (R,Σ) is an ω-complete abelian monoid, $(R,\cdot,1)$ is a monoid and for all r the maps $s \mapsto sr$, $s \mapsto rs$ are endomorphisms of (R,Σ).

When Σ is totally defined, these are the complete semirings of [5].

2 Example If $\mathcal{K}$ is partially-additive and if $D \in \mathcal{K}$, $\mathcal{K}(D,D)$ is an ω-complete partial semiring under Σ, composition, and id_D.

An important instance is the set of partial functions from a set to itself.

3 Example If M is a monoid then $R = 2^M$ is a complete semiring under ∪, $A \cdot B = \{ab : a \in A, b \in B\}$ and $\{\Lambda\}$.

Important cases are $M = 1$ (the Boolean semiring), and M = the free monoid A^* for some finite alphabet A.

Motivated by 2.12, we have the following notion of disjointness:

4 Definition Let R be an ω-complete partial semiring. Say that a (countable) family (r_i) in R is abstractly disjoint if for arbitrary families s_i, $\Sigma s_i r_i$ exists in R.

With this we may define the category of 'row-disjoint matrices over R':

5 Definition Let R be an ω-complete partial semiring. The matrix category of R Mat(R) is defined as follows:

Object: Any countable set

Morphism: $(r_{ij}): I \longrightarrow J$ with $r_{ij} \in R$ $(i \in I,\ j \in J)$ such that for all $i \in I$, $(r_{ij} : j \in J)$ is abstractly disjoint.

Identity: $\delta_{ij}: I \longrightarrow I$. Indeed, for fixed i and arbitrary (s_j), $\Sigma s_j \delta_{ij} = s_i$ is defined.

Composition: Usual matrix multiplication. For $I \xrightarrow{r} J \xrightarrow{s} K$ $(s \cdot r)_{ik} = \sum_j s_{jk} r_{ij}$, which exists as $(r_{ij} : j \in J)$ is abstractly disjoint.

6 Theorem For an arbitrary category $\mathcal{K}$, the following are equivalent

(i) $\mathcal{K}$ is isomorphic to the localization $\mathcal{A}_D$ for some partially-additive $\mathcal{A}$ and object $D \in \mathcal{A}$.

(ii) $\mathcal{K}$ is isomorphic to Mat(R) for some ω-complete partial semiring R.

Proof: (i) implies (ii): Set $R = \mathcal{A}(D,D)$. Given $f: I \times D \longrightarrow J \times D$ in $\mathcal{A}_D$ define $(f_{ij} : i \in I,\ j \in J)$ by

$$f_{ij} = D \xrightarrow{in_i} I \times D \xrightarrow{f} J \times D \xrightarrow{pr_j} D$$

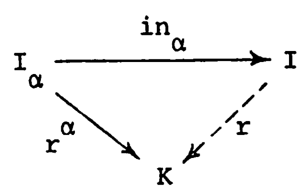
For fixed i, (f_{ij}) is disjoint so is abstractly disjoint by 2.12. If also $g: J \times D \longrightarrow K \times D$, $(g \cdot f)_{ik} = pr_k (g \cdot \sum_j in_j pr_j \cdot f) in_i = \sum_j g_{kj} \cdot f_{ij}$. If $f = id$, $f_{ij} = \delta_{ij}$. This constructs a functor $\mathcal{A}_D \longrightarrow$ Mat(R) which is the identity on objects. If $f \neq g : I \times D \longrightarrow J \times D$, then $fin_i \neq gin_i$ for some i.

By the covering property, $f_{ij} \neq g_{ij}$ for some i, j. Finally, given (f_{ij}): $I \longrightarrow J$ in $\underline{\underline{Mat}}(R)$ we must find f with $(f)_{ij} = f_{ij}$. For fixed i, f_{ij} is abstractly disjoint and hence (taking $s_i = id$ in 4) disjoint. Thus there exists g_i: $D \longrightarrow J \times D$ with $pr_j g_i = f_{ij}$. Define f by $fin_i = g_i$.

<u>(ii) implies (i)</u>: It is clear that $\underline{\underline{Mat}}(R) \cong (\underline{\underline{Mat}}(R))_D$ if D has one element, so it suffices to show that $\underline{\underline{Mat}}(R)$ is partially-additive. We use Theorem 5.1.

The empty set $\emptyset$ provides a zero object, $I \longrightarrow \emptyset$, $\emptyset \longrightarrow J$ being the empty matrices (and their composition $I \longrightarrow J$ being given by r_{ij} = empty sum = $\perp$).

Non-empty coproducts are disjoint unions. For let (I_α) partition I and, for i in I, write $\alpha(i)$ for the unique α with $i \in I_\alpha$. Define injections in_α: $I_\alpha \longrightarrow I$ by $(in_\alpha)_{ij} = \delta_{ij}$. Given $(r^\alpha$: $I_\alpha \longrightarrow K)$



set $r_{ik} = r_{ik}^{\alpha(i)}$. For fixed i, $r_{ik} = r_{ik}^{\alpha(i)}$ is abstractly disjoint. $(rin_\alpha)_{ik} = \Sigma r_{jk}(in_\alpha)_{ij} = r_{ik} = r_{ik}^\alpha$ since $i \in I_\alpha$. If $sin_\alpha = r^\alpha$ then $s_{ik} = \Sigma s_{jk}(in_\alpha)_{ij} = r_{ik}^\alpha = r_{ik}$. This verifies the coproduct property.

For the covering property, observe that pr_m: $K \times I \longrightarrow I$ $(m \in K)$ is defined by $pr_m in_k = \delta_{mk}$ so that $pr_m = (\rho^m_{ki_1,i})$ where

$$\rho^m_{ki_1,i} = \begin{cases} 1 & \text{if } k = m,\ i_1 = i \\ 0 & \text{else.} \end{cases}$$

Thus if t,u: $J \longrightarrow K \times I$ satisfy $pr_m t = pr_m u$ (all m) then for all j, m, i we have

$$t_{j,mi} = \sum_{k,i_1} \rho^m_{ki_1,i}\, t_{j,ki_1} = \sum_{k,i_1} \rho_{ki_1,i}\, u_{j,ki_1} = u_{j,mi} .$$

A family $r^k: I \longrightarrow J$ is disjoint if and only if there exists $r: I \longrightarrow K \times J$ with $r_{i,kj} = r^k_{i,j}$. Equivalently, for all i and arbitrary $(s_{k,j} : k \in K,\ j \in J)$, $\sum_{k,j} s_{k,j} r^k_{i,j}$ exists in R. In that case the sum given by

$$\Sigma r^k = I \xrightarrow{\ r\ } K \times J \xrightarrow{\ \sigma\ } J$$

is seen to be $(\Sigma r^k)_{ij} = \sum_k r^k_{ij}$. In view of this pointwise addition formula, (i) and (ii) of 5.1 are clear. □

7 Example Let R be the unit interval with the following complete semiring structure. Let $(x_i : i \in I)$ with $0 \leq x_i \leq 1$. If I is finite define Σx_i to be the minimum of 1 and the usual sum in $\mathbb{R}$. If I is infinite, define Σx_i to be the limit of the net of finite partial sums. The multiplicative monoid structure is the usual one. Iteration in this Mat(R) generalizes the behavior of Markov chains with absorbing states [9, p. 52] when we view a substochastic $f: A \longrightarrow A + B$ as representing a Markov chain on A + B with set B of absorbing states.

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