PARTIALLY-ADDITIVE SEMANTICS: A PROGRESS REPORT[1]

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In this short report, I wish to briefly and informally discuss the scope of joint work with Michael Arbib on 'partially-additive semantics'. The time of this writing is May 1979.

1. Why Order Semantics?

The Knaster-Tarski theorem was introduced by Kleene to define a recursive function as the least fixpoint of the continuous functional induced by its recursive definition. Owing to the work of Scott and many of his colleagues, this approach provides a wide-spread foundation for the semantics of recursive programs. Indeed, in the textbook [Manna, 1974] order semantics is introduced as if it were the only conceivable foundation.

My interest in 'fixpoint semantics' is recent. In a first pass at the literature I was unable to isolate compelling 'computer science' motivations for the proliferation of papers and books which explore the effects of introducing ordered sets ([Bloom, 1976] [Kamin, 1979], [Goguen et al., 1976, 1977], [Lehmann, 1978], [Lehmann & Smyth, 1977], [Meseguer, 1978], [Milne-Stratchey, 1976], Scott, 1970, 1971], [Stoy, 1977], [Tennent, 1976], [Tiuryn, 1977], [Wagner et al., 1979], [Wand, 1977], to name some). I feel that 'why order semantics?' is a challenge that needs to be met.

Michael Arbib and I introduce partially-additive semantics as a viable alternative, worthy of further study. The spirit of our investigation also provides guidelines for other, as yet uninvented, semantics.

2. Canonical Fixpoints

I have heard many people say that "one needs ordered sets to be able to talk about least fixpoints." But this is misleading, for we shall now see that what makes least fixpoints special is not that they are least.

To begin, what is the context of the Knaster-Tarski theorem? As it is usually stated, one is given a single object $A = (A,\leq,h)$ where $(A,\leq)$ is a partially-ordered set with least element $\perp$ and in which every countable ascending chain has a supremum, and $h: A \rightarrow A$ preserves such suprema; the theorem asserts that

$$\mathrm{Sup}(h^n(\perp) : n \geq 0) \tag{1}$$

exists and provides the least fixpoint of h. What is striking about (1) is that its form is independent of A. To a category-theorist, this immediately suggests making A an object in a category in the hope of discovering 'functoriality' or 'naturality' axioms characterizing (1) as an A-indexed construction.

I shall immediately generalize. Let <u>End</u> be the category whose objects are pairs (A,h) where A is a set and $h: A \rightarrow A$ is a function and whose morphisms $\phi: (A,h) \rightarrow (A',h')$ are functions $\phi: A \rightarrow A'$ satisfying $h'\phi = \phi h$. Let $\mathcal{A}$ be an arbitrary category and let $\Gamma: \mathcal{A} \longrightarrow$ <u>End</u> be an arbitrary functor. The motivating example is: $\mathcal{A}$-object $= (A,\leq,h)$ as above; a morphism $\phi: (A,\leq,h) \longrightarrow (A',\leq',h')$ preserves $\perp$ and suprema of countable chains, and satisfies $h'\phi = \phi h$; $\Gamma(A,\leq,h) = (A,h)$.

<u>Definition</u>: Given $\Gamma: \mathcal{A} \rightarrow$ <u>End</u>, for $A \in \mathcal{A}$ write $\Gamma A = (A_0,h)$. A <u>fixpoint of</u> A is $a \in A_0$ with $h(a) = a$. A <u>canonical fixpoint of</u> $(\mathcal{A},\Gamma)$ is an assignment α of a fixpoint α_A of A to each A subject to the requirement that for all $\phi: A \rightarrow B$ in $\mathcal{A}$, $(\Gamma\phi)\alpha_A = \alpha_B$.

<u>Canonical fixpoint theorem</u>: Let $\Gamma: \mathcal{A} \rightarrow$ <u>End</u>, and let $\mathcal{A}$ have an initial object I. Then there is a bijective correspondence between fixpoints i_0 of I and canonical fixpoints of $(\mathcal{A},\Gamma)$ given by $\alpha_A = \Gamma(I \rightarrow A)(i_0)$. □

<u>Corollary</u>: For $\Gamma: \mathcal{A} \rightarrow$ <u>End</u> as in the motivating example, the Knaster-Tarski formula (1) provides the unique canonical fixpoint.

<u>Proof</u>: $I = (\mathbb{N} \cup \{\infty\}, \leq, s)$ with $s(n) = n+1$, $s(\infty) = \infty$ is an initial object of $\mathcal{A}$, the unique map $\psi: I \rightarrow A = (A,\leq,h)$ being $\psi(n) = h^n(\perp)$, $\psi(\infty) = \mathrm{Sup}(h^n(\perp))$. Since ∞ is the only fixpoint of I, $\alpha_A = \psi(\infty) = \mathrm{Sup}(h^n(\perp))$ is the unique canonical fixpoint. □

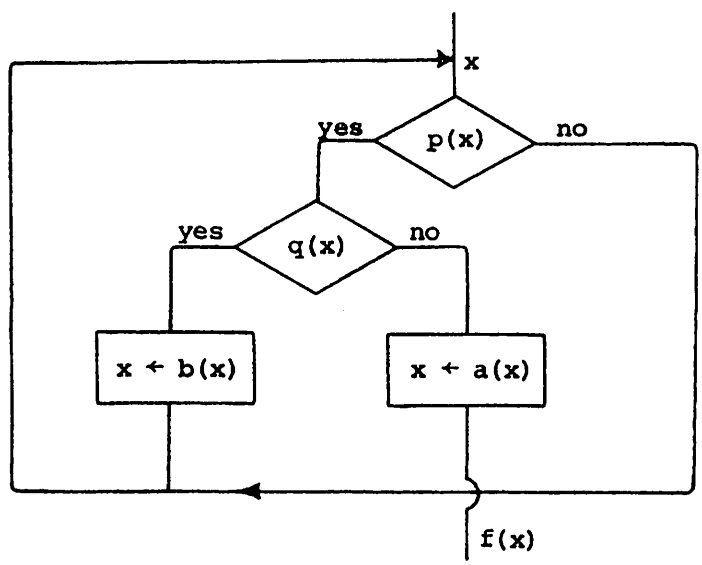
The whole point of the above corollary is that it allows (1) without mentioning the word 'least'. (1) is the unique canonical fixpoint for a certain $(\mathcal{A},\Gamma)$. Having begun with the somewhat negative question "why order semantics", I should now like to rephrase it in a more positive form: Why not a better $(\mathcal{A},\Gamma)$? Indeed, a new $(\mathcal{A},\Gamma)$ is offered in Section 6.

3. Metric Spaces, a Diversion

A number of workers [Bloom, 1977] [Schützenberger, 1962] have used the Banach contraction theorem instead of the Knaster-Tarski theorem for problems of 'fixpoint semantics'. This theorem also provides an instance of the canonical fixpoint theorem. Let $\mathcal{A}$ have objects (A,d,h) where (A,d) is a non-empty metric space and $h: (A,d) \longrightarrow (A,d)$ satisfies $d(hx,hy) \leq Kd(x,y)$ for all x, y, for some $K < 1$. Morphisms ϕ can belong to any subcategory of the category of all functions $A \longrightarrow A'$ satisfying $h'\phi = \phi h$. The one element set with its unique metric and endomorphism is the initial object of $\mathcal{A}$ and possesses its only element as unique fixpoint. (The proof here requires the well-known fact that each object of $\mathcal{A}$ has a unique fixpoint.)

4. Partially-additive Analysis of a Simple Iterative Program

Consider the following program scheme computing a partial function $f: \underline{\underline{N}} \longrightarrow \underline{\underline{N}}$:



(2)

Here $a,b: \underline{\underline{N}} \longrightarrow \underline{\underline{N}}$ are partial functions and $p,q: \underline{\underline{N}} \longrightarrow \{true,false\}$ are total functions called 'predicates'. By inspection, it is clear that

If $\bar{p}(x)$ (meaning 'not p(x)')
then f(x) is not defined. (3)

In a larger program, the deduction of such facts is more complicated and it would be nice to have a systematic method to isolate (3) from irrelevant aspects of (2). I do not think that the Kleene sequence $(h^n(\bot) : n \geq 0)$ is optimal here, because it does not 'deal with pieces any smaller' than the original program.

On the other hand, in the partially-additive analysis of (2) we shall see that each path of the flowchart becomes a 'term' in an infinite series expansion. There is one 'straight-through term' $a\bar{q}p$ and there are two 'return terms' bqp and $\bar{p}$. This suggests that the semantics of (2) is

$$f = \sum_{n=0}^{\infty} (a\bar{q}p)(bqp + \bar{p})^n. \qquad (4)$$

The precise definition of (4) is as follows. Each predicate $r: A \to \{true, false\}$ induces two partial functions $r_{true}, r_{false}: A \to A$ where

$$r_{true}(a) = \begin{cases} a & \text{if } r(a) = true \\ \text{undefined else} \end{cases} \qquad r_{false} = \begin{cases} a & \text{if } r(a) = false \\ \text{undefined else} \end{cases} \tag{5}$$

These notations are cumbersome and henceforth I shall write r for r_{true} and $\bar{r}$ for r_{false}. Then $a\bar{q}p$, bqp and p in (4) are partial functions $\underline{\underline{N}} \to \underline{\underline{N}}$, using the operation of composition of partial functions. The <u>sum</u> of any family $(f_i : i \in I)$ of partial functions $f_i: A \to B$ is defined only when the domain $dom(f_i)$ is disjoint from $dom(f_j)$ when $i \neq j$; and then

$$(\Sigma f_i)(a) = \begin{cases} f_j(a) & \text{for the unique } j \text{ with } a \in dom(f_j) \\ \text{undefined} & a \notin \cup dom(f_i) \end{cases} \tag{6}$$

But it is now clear that (4), with its two uses of sum (finitary Σ is written with infix +) is the correct semantics of (2).

A number of authors such as [Zeiger, 1969] [de Roever, 1976] have discussed sums of multi-functions and relations which can be applied to partial functions but, as far as we are aware, the emphasis on a partially-defined addition yielding an axiomatic theory such as that discussed in the next section is new.

Summation of partial functions has a number of pleasant properties:

If $f_i + f_j$ is defined $i \neq j$, Σf_i is defined. (7)

If $f: A \to B$, $g_i: B \to C$, $h: C \to D$ with Σg_i defined then $\Sigma h g_i f$ is defined and is $h(\Sigma g_i)f$. (8)

If I is partitioned into $(I_j : j \in J)$ then $\Sigma(f_i : i \in I) = \Sigma(\Sigma(f_i : i \in I_j) : j \in J)$ in the sense that if either side is defined then so is the other, and they are equal. (9)

For any two sets A, B there is a zero for the addition, the partial function $\perp: A \to B$ which has empty domain. Any composition of partial functions is $\perp$ if one of the factors is.

If $p, \bar{p}: A \to A$ is a predicate, the following equations hold:

$$\begin{aligned} p\bar{p} &= \perp = \bar{p}p \\ p^2 &= p, \quad \bar{p}^2 = \bar{p} \\ p + \bar{p} &= id. \end{aligned} \tag{10}$$

Here is the analysis of (4). Expanding $(a\bar{q}p)(bqp + \bar{p})^n$ for $n = 1,2$ gives

$$(a\bar{q}p)(bqp + \bar{p}) = a\bar{q}pbqp + a\bar{q}p\bar{p} = a\bar{q}pbqp + a\bar{q}\perp = a\bar{q}pbqp$$
$$(a\bar{q}p)(bqp + \bar{p})^2 = a\bar{q}pbqpbqp + a\bar{q}pbqp\bar{p} = a\bar{q}p(bqp)^2$$

which leads one to conjecture that $(a\bar{q}p)(bqp + \bar{p})^n = (a\bar{q}p)(bqp)^n$; this is easily verified by induction. Then (3) is obvious since every term ends with p and $p\bar{p} = \perp$.

We may further deduce that if $f(x)$ is defined, $(a\bar{q}p)(bqp)^n(x)$ is defined for a unique n. Since $qp(x) = x$ when defined, the final value of $f(x)$ in this case is $ab^n(x)$.

5. Partially-additive Categories: Axioms and Diatribe

The summation operation on the set of partial functions from A to B suggests a number of abstract definitions. Let us get them out of the way.

Definition: A partially-additive monoid (M,Σ) is a set M equipped with a partially defined operation Σ on finite or denumerable sequences of M subject to

Partition associativity axiom: The same as (9), but I is countable.

Limit axiom: Given $(m_i : i \in I)$, if $\Sigma(m_i : i \in F)$ is defined for each finite subset F of I, $\Sigma(m_i : i \in I)$ is defined. This weakens (7).

Unary sum axiom: (If I has one element) Σm is defined and is m.

The empty sum provides an additive zero, denoted $\perp$.

Definition: A morphism $f: (M,\Sigma) \longrightarrow (M',\Sigma')$ of partially-additive monoids satisfies: if Σm_i is defined then $\Sigma' fm_i$ is defined and equals $f(\Sigma m_i)$.

Definition: A partially-additive semiring is $(R,\Sigma,\cdot,1)$ where (R,Σ) is a partially-additive monoid, $(R,\cdot,1)$ is a monoid and the left and right multiplications $m \mapsto ma$, $m \mapsto am$ are morphisms $(R,\Sigma) \longrightarrow (R,\Sigma)$.

Definition: A partially-additive category is a category $\mathcal{A}$ with countable coproducts in which for each pair of objects (A,B) the set $\mathcal{A}(A,B)$ of $\mathcal{A}$-morphisms from A to B has the structure of a partially-additive monoid in such a way that the following three axioms hold:

Distributivity axiom: If $f: A' \to A$, $g: B \to B'$ then $g\circ-\circ f: \mathcal{A}(A,B) \to \mathcal{A}(A',B')$ is a morphism of partially-additive monoids. (In particular, endomorphism monoids are partially-additive semirings.)

For A in $\mathcal{A}$, I a countable set, write copower injections as
$in_j : B \to I\cdot B$ and define 'quasiprojections' $pr_i : I\cdot B \to B$
by $pr_i in_j = \delta_i^j$ (Kronecker delta). (11)

Compatible sum axiom: If $f: A \to I\cdot B$, $\Sigma(pr_i f : i \in I)$ exists in $\mathcal{A}(A,B)$.

Untying axiom: If $f_i : A \to B$ and Σf_i exists then $\Sigma in_i f_i : A \to I\cdot B$ exists.

In theoretical physics, aesthetically-appealing mathematical models must also be validated by experimental evidence, but it may be detrimental to progress to fail to publish results which 'almost work' (cf. the Klein-Gordon equation which would have been Schrödinger's if he had not been too scared to publish it [Dirac, 1971, pp. 37-40]).

I suggest similar criteria for programming semantics. We should publish all appealing ideas, but should measure them for 'reality'. For example

Appealing idea	Reality
Data types are ordered sets.	It's not always true. Besides, should any formulation restrict, a priori, the internal structure of data types?
Iterative algebraic theories motivated by 'timed' partial functions. [Elgot, 1975]	'Time' is implementation-dependent.
Relations, not just partial functions.	For ultimate progress in 'parallel processing', very important. For the standard case, it's just not true.

Some of my colleagues have insisted that it is unnecessary to deal with partially-additive monoids since the union operation for relations deals correctly with the issues involved and allows Σ to be always-defined; 'at the end', if necessary, one can verify that 'partial functions lead to partial functions'.

Primary rebuttal: Relations are there, just as complex eigenvalues and eigenvectors are there in the process of finding real solutions to linear differential equations with constant real coefficients. But in any foundations -- which, for me, should be robust enough to specialize in several ways -- one specialization should be, exactly, partial functions since this is the major concrete case in programming.

Secondary rebuttal: After all, the set of relations from A to B with $\Sigma = \cup$ is a partially-additive monoid. The primary objection to the general Σ seems to be psychological: partially-defined algebraic operations are 'unpleasant'. But even matrix notation struck engineers as 'unintuitive' when it was first introduced, but is now accepted as an aid to intuition. Arbib and I experienced the same transition with growing familiarity with partial-addition.

6. Partially-additive Monoids and Recursive Calls

Definition: Let (M,Σ) be a partially-additive monoid. Say that $f\colon M^n \longrightarrow M$ is n-additive if when all but one of the n arguments are fixed, the result is a morphism $(M,\Sigma) \longrightarrow (M,\Sigma)$.

Definition: An abstract recursion scheme is (M,Σ,H) with (M,Σ) a partially-additive monoid and $H = (H_n : n = 0,1,2,\ldots)$ with $H_n\colon M^n \longrightarrow M$ n-additive (0-additive = constant) subject to the requirement that

$$h(m) = \sum_{n\geq 0} H_n(m,\ldots,m) \qquad (12)$$

is defined for all m in M.

By a scheme morphism $\phi\colon (M,\Sigma,H) \longrightarrow (M',\Sigma',H')$ we mean a morphism of partially-additive monoids such that $H'_n\phi = \phi H_n$ for all n.

It follows at once that $h'\phi = \phi h$ and hence that $\Gamma(M,\Sigma,H) = (M,h)$ is a functor Γ from the category AbsRS of abstract recursion schemes to End.

> Our thesis: a recursive definition induces an abstract recursion scheme whose canonical fixpoint (discussed below) provides semantic meaning to the definition. (13)

Let's test this thesis. Here is an APL program which evaluates the determinant of a square matrix by cofactor expansion along the first row.

```
∇Z←DET MAT;I;N
  N←(⍴MAT)[1]
  →(N=1)/END
  Z←I←0
 LOOP:→(N<I←I+1)/0
  Z←Z+((¯1)*1+I)×MAT[I;1]×DET MAT[(I≠⍳N)/⍳N;1+⍳(N-1)]
  →LOOP
 END:Z←MAT[1;1]
∇
```

The desired function is an element of the partially-additive monoid (M,Σ) with M the set of all partial functions from the set of all square matrices with real entries to the set of reals (with Σ as in (6)). Then define

$H_0 \in M$ by $H_0 =$ if MAT $= [a_{11}]$ is 1-by-1 then a_{11} else undefined

$H_1: M \to M = \bot$ always undefined

$$H_2: M^2 \to M \text{ by } H_2(m_1,m_2) = \underline{\text{if}} \text{ MAT} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{ is 2-by-2}$$
$$\underline{\text{then}}\ a_{11}m_1([a_{22}]) - a_{12}m_2([a_{11}]) \quad \underline{\text{else}} \text{ undefined}$$

$$H_3: M^3 \to M \text{ by } H_3(m_1,m_2,m_3) = \underline{\text{if}} \text{ MAT} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ is 3-by-3}$$
$$\underline{\text{then}}\ a_{11}m_1\left(\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}\right) - a_{12}m_2\left(\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}\right) + a_{13}m_3\left(\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}\right) \quad \underline{\text{else}} \text{ undefined}$$

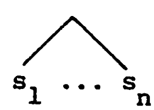
and so on. H_n is non-trivial for all $n \neq 1$. Clearly

$$\text{DET} \leftarrow h(\text{DET}) = \sum_{n \geq 0} H_n(\text{DET},\ldots,\text{DET})$$

is the intended recursive call. Other examples, including vector equations and Ackermann's function, appear in [Arbib & Manes, POC].

<u>Theorem</u> [Arbib & Manes, POC]: <u>AbsRC</u> $\xrightarrow{\Gamma}$ <u>End</u> has a unique canonical fixpoint.

<u>Proof Outline</u>: Let <u>Tree</u> be the set of all finite-depth finitely-branching trees with one root. Let I be the set of all subsets of <u>Tree</u>. Then $I = (I,\hat{\Sigma},\hat{H})$ is an abstract recursion scheme if $\hat{\Sigma}$ is defined only for pairwise disjoint families to be union, and $\hat{H}_n(s_1,\ldots,s_n)$ is the set of all



with $s_i \in S_i$. In fact, I is the initial object, and the unique $\psi_A: (I,\hat{\Sigma},\hat{H}) \to (M,\Sigma,H) = M$ to the arbitrary M is $\psi_M(S) = \Sigma(s^{\#} : s \in S)$ where $s^{\#}$ is that element of M obtained by labeling each n-ary branch of s with H_n and each terminal node of s by H_0, and evaluating. Moreover, I has unique fixpoint, namely all of <u>Tree</u>. The stated result then follows from the canonical fixpoint theorem. □

Indeed the proof above gives a specific semantics which should be regarded as the partially-additive version of the Knaster-Tarski formula (1):

The <u>partially-additive semantics</u> of the abstract recursion scheme (M,Σ,H) is

$$\Sigma(s^{\#} : s \in \underline{\text{Tree}}) \tag{14}$$

(the theory guarantees that this sum is always defined).

<u>Notation</u>: Let <u>Pfn</u> denote the category of sets and partial functions. The set <u>Pfn</u>(A,B) of partial functions from A to B is both a partially-additive monoid (6) and a partially-ordered set with least element and suprema of countable-ascending chains.

Theorem: Given an abstract recursion scheme of form ($\underline{Pfn}(A,B)$,H) (Σ as in (6)), then $h(m) = \Sigma H_n(m,\ldots,m)$ preserves suprema of countable ascending chains. Moreover, the partially-additive and least fixpoint semantics coincide:

$$\Sigma(s^{\#} : s \in \underline{Tree}) = \text{Sup}(h^n(\bot) : n \geq 0). \tag{15}$$

Proof Outline: We showed in [Arbib & Manes, 1978] that the partially-additive semantics of a recursive definition coincides with its interpretive semantics, which is known to equal the least fixpoint semantics. The present more general theorem is proved with the canonical fixpoint theorem.

Let $\mathcal{A}$ be the category of (A,B,H) such that ($\underline{Pfn}(A,B)$,H) is an abstract recursion scheme (with Σ as in (6)) and with morphisms ψ: (A,B,H) $\longrightarrow$ (A',B',H') being ψ: ($\underline{Pfn}(A,B)$,H) $\longrightarrow$ ($\underline{Pfn}(A',B')$,H') in $\underline{AbsRS}$, and define $\Gamma: \mathcal{A} \to \underline{End}$ by $\Gamma(A,B,H) =$ ($\underline{Pfn}(A,B)$,h). Recalling the proof outline of the previous theorem and observing that subsets of $\underline{Tree}$ under disjoint union correspond to elements of $\underline{Pfn}(\underline{Tree},1)$ (where 1 is a one-element set) in a Σ-preserving way, ($\underline{Tree}$,1,$\hat{H}$) -- with the $\hat{H}$ of the preceding theorem -- is an initial object of ($\mathcal{A}$,Γ) with a unique fixpoint, and $\Sigma(s^{\#} : s \in \underline{Tree})$ is then the unique canonical fixpoint.

We must show that $\text{Sup}(h^n(\bot) : n \geq 0)$ is also a canonical fixpoint. To begin, in any partially-additive monoid (M,Σ) say that '$m \in \lim m_k$' if there exist x_k with

$$\begin{aligned} m_k &= m_{k-1} + x_k \qquad (k \geq 1) \\ m &= m_0 + \sum_{k=1}^{\infty} x_k \end{aligned} \tag{16}$$

and say that a function c between partially-additive monoids is <u>continuous</u> if $c(m) \in \lim c(m_k)$ whenever $m \in \lim m_k$. It is easily shown that if $F: (M,\Sigma)^n \longrightarrow (M',\Sigma')$ is n-additive then $m \mapsto F(m,\ldots,m)$ is continuous and that any pointwise sum (if defined) of continuous functions is continuous. In particular, the h of any abstract recursion scheme is continuous and all morphisms in $\underline{AbsRS}$ are continuous.

To conclude, note that when our partially-additive monoid is $\underline{Pfn}(A,B)$ under disjoint union, then the x_k of (16) are unique, and m must be the supremum. Hence observe that a function $\underline{Pfn}(A,B) \longrightarrow \underline{Pfn}(A',B')$ preserves suprema of countable chains if and only if it is continuous and hence, by the corollary of Section 2, $\text{Sup}(h^n(\bot) : n \geq 0)$ is canonical for ($\mathcal{A}$,Γ).

□

Let me discuss further the relationship between order and sum on $\underline{Pfn}(A,B)$. The order is easily recovered from the addition since $f \leq g$ if and only if $g = f + h$ for some h. There is, however, no equally direct way to recover Σ (or even +) from $\leq$. in Pfn(A,B) (although one can easily express disjointness in a lattice and then $\Sigma = \text{Sup}$ for disjoint families).

As is well-known, the least fixed point of an arbitrary continuous h: $\underline{Pfn}(A,B) \to \underline{Pfn}(A,B)$ is not necessarily recursive. The 'polynomials' $h(f) = \Sigma H^n(f,\ldots,f)$: $\underline{Pfn}(A,B) \longrightarrow \underline{Pfn}(A,B)$ are a smaller subclass of continuous maps that appear adequate for all recursive definitions.

When $H_n = \bot$ for $n > 1$, the semantics (14) of (M,Σ,H) specializes to

$$\sum_{n=0}^{\infty} H_1^n(H_0). \tag{17}$$

There are many examples of this:

(i) (20) below in any partially-additive category;

(ii) The Kleene iteration operation for subsets of X^*, $(A,B) \mapsto A^*B$, (and I point out that Kleene [1956] used this form, not $A \mapsto A^*$) arises as in (i) in the partially-additive category $\underline{\mathrm{Mat}}_R$ (see below) for R the complete semiring of subsets of X^*;

(iii) The behavior of an automaton (convert to recursive definition first as in [Eilenberg, 1974, Theorem 6.4];

(iv) The semantics of an iterative program (convert to recursive definition using the McCarthy transform [Manna, 1974, Theorem 4.5].

Let $f: A \rightarrow B$ be the semantics of $(\underline{\mathrm{Pfn}}(A,B),H)$. There are two useful proof rules [Arbib & Manes, POC], the first of which uses both H and ≤:

<u>Tree induction rule for partial correctness</u>: For any $g \in \underline{\mathrm{Pfn}}(A,B)$, $f \leq g$ if and only if

<u>Basis step</u>: $H_0 \leq g$

<u>Inductive step</u>: If $H_n \neq \bot$ and $s_1,\ldots,s_n \in$ <u>Tree</u> are such that $s_i^\# \leq g$ for all i then $H_n(s_1^\#,\ldots,s_n^\#) \leq g$.

<u>Termination rule</u>: For any $a \in A$, $f(a)$ is defined if and only if there exists $s \in$ <u>Tree</u> with $s^\#(a)$ defined.

7. Partially-additive Categories: Mathematical Results

<u>Theorem</u>: A category is partially-additive in at most one way; that is, if Σ exists it is unique.

<u>Proof Outline</u>: We say $f_i: A \rightarrow B$ are <u>compatible</u> if $f_i = pr_i f$ $(i \in I)$ for some $f: A \rightarrow I \cdot B$. But compatible is equivalent to summable since if Σf_i exists so does $f = \Sigma in_i f_i$ by the untying axiom and $f_i = pr_i f$. Moreover, if $\sigma: I \cdot B \rightarrow B$ is defined by $\sigma in_i = id$, $\Sigma f_i = \sigma f$.

□

It follows that <u>Pfn</u> with Σf_i extended to include all families which agree on their overlaps is <u>not</u> partially-additive. (In fact the untying axiom fails.)

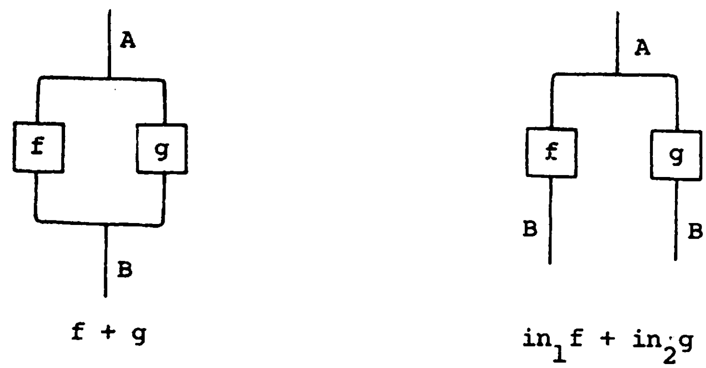
As explained in [Arbib & Manes, PaCat], ~~the~~ our theory extends that of semi-additive categories [Mitchell, 1965, 27-31][Arbib & Manes, 1975, Sec. 5.2]. Here, <u>Pfn</u> provides the central example of a partially-additive category whereas the category of abelian monoids is the prototype semi-additive category.

An abstract characterization of partially-additive categories in terms of four axioms which assert that 'countable coproducts are almost products' is given in [Arbib & Manes, PaCat, Section 5]. This suggests the 'half direct sum' symbol ⊕ for coproducts. To say a little more, if 𝒜 is partially-additive the coproduct B ⊕ C is 'almost a product' in the sense that there exists a bijection between maps $f: A \longrightarrow B \oplus C$ and pairs $(f_B: A \rightarrow B,\ f_C: A \rightarrow C)$ defined by

$$f_B = pr_B f, \quad f_C = pr_C f$$
$$f = in_B f_B + in_C f_C. \tag{18}$$

Further understanding comes from considering Pfn. Exercise: check directly that (f_i) is compatible if and only if $\mathrm{dom}(f_i) \cap \mathrm{dom}(f_j)$ when $i \neq j$. Coproducts are disjoint unions with the usual injections. Thus $\{true,false\} = \{true\} + \{false\}$ and the decomposition of $r: A \to \{true,false\}$ into $r_{true}: A \to \{true\}$, $r_{false}: A \to \{false\}$ is closely related to that of (5).

Still in Pfn, let $f,g: A \to B$ and consider



(19)

We are extending usual flowchart syntax; the diagrams are semantically meaningful only when $\mathrm{dom}(f) \cap \mathrm{dom}(g) = \emptyset$. The flowchart on the left represents $f + g$. The flowchart on the right then shows that the untying axiom asserts that the 'codomain join lines may be untied'.

Proposition [Arbib & Manes, PaCat, 3.10]: If $\mathcal{A}$ is partially-additive, $f_i: A \to B$, $g_i: B \to C$ $(i \in I)$ then if Σf_i exists, $\Sigma g_i f_i$ exists. □

This motivates

Definition: Let R be a partially-additive semiring. A family $(r_j : j \in J)$ is abstractly compatible if for every J-tuple $(s_j : j \in J)$ in R, $\Sigma(s_j r_j : j \in J)$ exists.

The matrix category Mat_R of R is defined as follows. Object = countable set. A morphism $I \to J$ is an I-by-J matrix (I indexing rows, J columns) of scalars in R whose rows are abstractly compatible. Composition is matrix multiplication and identities are the usual identity matrices.

Proposition [Arbib & Manes, PaCat, 7.8]: Mat_R is partially-additive.

If R is the 2-element semiring [Eilenberg, 1974, p. 123] Mat_R is the category of sets and relations. If R is the two-element partially-additive semiring with 1+1 undefined, Mat_R is Pfn.

Comparisons between Mat_R and the matricial theories of [Elgot, 1976] as well as a partially-additive version of free iterative theories [Elgot et al., 1978] appear in [Arbib & Manes, PaCat, Section 7].

C. C. Elgot was the first to point out the advantages of regarding 'the conditional' as a map $f: A \to B \dotplus C$ in a category (I will use '$\dotplus$' for coproducts in an arbitrary category), and was also the first to regard iteration as a passage $f: A \to A \dotplus B \mapsto f^{\dagger}: A \to B$.

In an arbitrary partially-additive category, we decompose f into $f_A: A \to A$, $f_B: A \to B$ by (17) and define

$$\text{for}\quad f\colon A \longrightarrow A + B,\quad f^{\dagger} = \sum_{n\geq 0} f_B f_A^n. \tag{20}$$

Theorem: The sum in (20) is always defined.

Proof outline: Much as in LISP, express $f^{\dagger}$ in recursive form:

$$f^{\dagger} \longleftarrow f_A f^{\dagger} + f_B \tag{21}$$

thereby inducing the abstract recursion scheme ($\underline{Pfn}(A,B)$,H) with $H_0 = f_B$, $H_1(g) = f_A g$ and $H_n = \bot$ for $n > 1$). Then (20) is just the semantics of (14). □

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