Sum-Ordered Partial Semirings

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COINS Technical Report #85-01

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A Dissertation Presented

By

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Submitted to the Graduate School of the
University of Massachusetts in partial fulfillment
of the requirements for the degree of

DOCTOR OF PHILOSOPHY

February 1985

Department of Computer and Information Science

This research was supported in part by National Science Foundation grants MCS-8205168 and MCS-8003433.

For P. and D.

Acknowledgements

Ernie Manes and Michael Arbib first introduced me to partially-additive structures and for the past five years have provided me with much needed guidance, criticism, and support. To them I owe special thanks.

I thank readers of the thesis – Michael Arbib, David Benson, Stephen Bloom, Murray Eisenberg, Ernie Manes, and Rich Sutton – for making many helpful suggestions for its improvement.

I thank the COINS staff members, who provide logistic support, and the RCF staff members, who maintain the computing facilities, for making the department a very pleasant environment in which to work. Additional thanks are due Andy Cromarty for EMACS and Lenny Wesley for TEX.

Above all, I thank my advisor Ernie Manes for stimulating discussions (and tea ceremonies), for snaring many erroneous and poorly written sections of the thesis and rendering them correct and readable, and for patience and friendship.

ABSTRACT

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February 1985

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Directed by: Professor Ernest G. Manes

If we endow the set of partial functions from a data set to itself with an addition (disjoint-domain sums) and a multiplication (functional composition), then any iterative algorithm may be described formally as the solution to a matrix equation, where the matrix entries are partial functions which describe the parts of the algorithm. This suggests that algorithms may be transformed by manipulation of matrices of partial functions. Hence, it becomes necessary to understand how such matrices behave. The partial functions under disjoint-domain sums and functional composition do not form a field, and thus conventional linear algebra is not applicable. However, they can be regarded as a sum-ordered partial semiring or "so-ring", an algebraic structure possessing a natural partial ordering, an infinitary partial addition, and a binary multiplication, subject to a set of axioms. The majority of this dissertation is devoted to a detailed study of the properties and interesting substructures of so-rings themselves; preliminary results illustrating the behavior of matrices over so-rings are also presented. We hope that this study in part provides a basis for a matrix theory of algorithm transformation.

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CHAPTER I

INTRODUCTION

A partial semiring is an algebraic structure consisting of a set together with an additive operation and a multiplicative operation. The additive structure is that of a partial monoid, meaning that the additive operation is an infinitary partial addition subject to a set of axioms. These axioms imply, among other things, that the addition is associative and commutative, that an additive zero exists, but that additive inverses cannot exist. The multiplicative structure is that of a monoid, and multiplication distributes over addition on both sides. The sum-ordering is a binary relation defined by $x \leq y$ if there exists an h with $x + h = y$. If this binary relation is a partial order, then the partial semiring is called a sum-ordered partial semiring or so-ring, for short.

Why study such structures? Our answer is two-fold. Our motivation comes in part from the work done in partially-additive semantics by Arbib and Manes [1980, 1982] and [Manes and Arbib, 1985]. They note that the set of partial functions from a set to itself is a so-ring, with the addition of two partial functions f and g defined if their domains are disjoint, in which case

$$x(f+g) = \begin{cases} xf, & \text{if } x \in dom(f); \\ xg, & \text{if } x \in dom(g); \\ \text{undefined}, & \text{otherwise}; \end{cases}$$

and with multiplication of two partial functions defined as the usual functional composition.[1]

Matrices over the so-ring of partial functions have arisen in connection with the semantics of iterative algorithms. McCarthy [1960] demonstrated that any iterative flowscheme can be transformed into a set of recursive equations, one for each cutpoint in the flowscheme. Manes and Arbib recast these equations in terms of the so-ring of partial functions and consolidated them into one matrix equation

$$\bar{x} = A\bar{x} + \bar{b}.$$

Here $\bar{x}$ is the vector containing the cutpoints, each of which is associated with a set of iterative loops in the flowscheme; A is a square matrix, each row of which corresponds to a cutpoint and contains a set of coefficients, one for each loop path passing through a cutpoint in the flowssheme (recursive portion); $\bar{b}$ is the vector containing the coefficients for each loop path passing through no cutpoints in the flowscheme (non-recursive portion). The least solution of this recursive matrix equation as dictated by operational semantics is the vector

$$\bar{x} = \sum_{n \geq 0} A^n \bar{b}$$

whose first component is the partial function describing the iterative algorithm in question.

This suggests that program transformations could be accomplished by manipulation of matrices over partial functions, thus requiring the development of a matrix algebra for matrices over so-rings. Matrices over semirings have been previously introduced as algebraic structures applied to problems of interest to computer scientists. In particular, they have been used to determine the relationships between formal power series and automata and formal language theory. Among the people who have studied these applications of formal power series are Schützenberger

[1] We have chosen to apply functions on the right.

[1960,1961,1962], Nivat [1968], Fliess [1972], Eilenberg [1974], and Salomaa and Soittola [1978]. Matrices over semirings have also been used in solving combinatorial optimization problems, such as finding the shortest path through a graph [Gondran, 1975], [Gondran and Minoux, 1979], [Cuninghame-Green, 1979], [Zimmerman, 1981]. Hence, there is a small but expanding literature concerning the algebra of matrices over semirings. However, this literature does not address the more general case of matrices over partial semirings or over so-rings.

If we are going to develop a matrix theory of so-rings, we must begin by investigating so-rings themselves. This brings us to the second reason for studying so-rings. From a mathematical standpoint, we want to discover what properties so-rings possess. Of particular interest is the effect of an infinite partial addition on the nature of these properties. In chapter II, we introduce partial monoids and partial semirings and give many examples of each. Then, in chapter III, we present the basic properties and substructures of so-rings, and we describe their interrelationships.

The following three concepts, introduced in the third chapter, are fundamental to our understanding of the structure of so-rings. The first is the center, a Boolean substructure which is a subset of all so-ring elements from 0 to 1. The next are domain and range which are generalizations of these concepts as they apply to functions. Most but not all so-rings are such that each element has a domain and a range. The last is the property of "inversibility" which is a generalization of the usual notion of invertibility with respect to multiplication. An arbitrary element of a so-ring need not have an "inverse".

For motivation, we here note what these three concepts mean in the context of the so-ring of partial functions from a set D to D. The center corresponds to the set of guards, where a guard of a subset D' of D is a partial function which is the identity on D' and undefined elsewhere. The domain of a partial function is the least guard g such that $gf = f$; the range is the least guard g such that $fg = f$.

A partial function is "inversible" if and only if it is injective.

Chapter IV is devoted to matrices over so-rings. Several of the results from classical linear algebra carry over to matrices over so-rings. This is somewhat surprising, because in the classical case the matrix entries are over a field which implies that all non-zero elements have both an additive and a multiplicative inverse, whereas in the context of so-rings the entries are not guaranteed to have a multiplicative inverse and there are no additive inverses. Specifically, we define what we mean by a matrix over a so-ring, and we give a characterisation of matrix invertibility. We then show that for a large class of so-rings, an $n \times n$ matrix X over a so-ring R is invertible if and only if the columns of X form a basis for the space of n-vectors over R. In addition, we demonstrate that the cardinality of bases for the space of n-vectors over a so-ring depends upon the so-ring, and we show that any eigenvalue for an invertible matrix must be inversible.

In the fifth chapter, we extend some results in semigroup theory to so-rings. The first subject that we explore is so-ring representation, in which we generalise results on the representation of semigroups to so-rings. We show that any so-ring R may be embedded in the so-ring of additive maps from R to R, and we give a partial characterisation of the so-rings which may be embedded in a so-ring of partial functions. The remainder of the chapter is concerned with alternative partial orderings on so-rings, and their relationship to the sum-ordering. Again, we generalise results from semigroups to so-rings.

Chapter VI contains an investigation of the relationships of partial monoids to other algebras with infinitary partial additions. Upon searching the literature, we discovered that few people have looked at such algebras; the earliest reference appears to be the work of Tarski on cardinal algebras [1949] which we studied at the suggestion of D. S. Scott.

The last chapter is a summary of the main results and a discussion of their relevance to issues in theoretical computer science.

In this dissertation, we explore in detail the structure of so-rings, and we provide examples from the domain of computer science in order to illustrate the properties that we have discovered. It is our hope that this mathematical treatment in part provides the foundations on which a matrix theory of program transformation can be built, although such a theory is beyond the scope of this thesis.

CHAPTER II

PARTIAL MONOIDS AND SO-RINGS

Partially-defined infinitary operations appear in contexts ranging from integration theory to programming language semantics. In this chapter, we describe two related algebraic structures – partial monoids and partial semirings – each of which possesses a partially-defined infinitary additive operation. First, however, we make some definitions which will provide us with a framework in which to discuss these structures.

Let M be a non-empty set, and let I be a (possibly empty, possibly nondenumerable) set. An *I-indexed family* in M is a function $x: I \to M$. Such a family is denoted by $(x_i : i \in I)$, where $x_i = ix$ for each i in I. The *cardinality* of the family $(x_i : i \in I)$ is the cardinality of its index set I. Two families $(x_i : i \in I)$ and $(y_k : k \in K)$ in M are *isomorphic* if there is a bijection $\sigma: I \to K$ with $y_{i\sigma} = x_i$ for each i in I. A *subfamily* of $(x_i : i \in I)$ is a family $(x_j : j \in J)$ such that $J \subseteq I$. The *empty family* in M is the unique such family indexed by $\emptyset$.

Now let us consider an infinitary operation $\sum$ which takes families in M to elements of M, but which may not be defined for all families in M. By "infinitary", we mean that $\sum$ may be applied to a family $(x_i : i \in I)$ in M, for which the cardinality of the index set I is infinite. This does not preclude applying $\sum$ to finite families in M. Since $\sum(x_i : i \in I)$ need not be defined for an arbitrary family $(x_i : i \in I)$ in M, $\sum$ is said to be *partially-defined.* We wish to view the operation $\sum$ as a generalised addition (hence, the choice of symbol) and will often refer to it as a *partial addition.* A family $(x_i : i \in I)$ in M is said to be *summable* if $\sum(x_i: i \in I)$ is defined and is in M.[1]

We have investigated four different partially-defined infinitary operations and their associated algebraic structures described in the existing literature, devoting most of our attention to partial monoids (and partial semirings). The other three structures – generalised cardinal algebras, $\sum$-structures, and infinite sums in commutative topological groups – are described in chapter VI and are used as points of comparison with partial monoids.

Partial Monoids

Arbib and Manes [1980] introduced partial monoids as an algebraic tool in connection with describing the semantics of sequential programming languages. A motivating example of the partial monoid structure is the set of partial functions from a set A to a set B, equipped with a suitable partial addition.[2]

2.1 DEFINITION. A *partial monoid* is a pair $(M, \sum)$ where M is a non-empty set and $\sum$ is a partial addition defined on some, but not necessarily all families $(x_i: i \in I)$ in M subject to the following two axioms:

(1) *Unary Sum Axiom.* If $(x_i: i \in I)$ is a one-element family in M and $I = \{j\}$, then $\sum(x_i: i \in I)$ is defined and equals x_j.

(2) *Partition-Associativity Axiom.* If $(x_i: i \in I)$ is a family in M and $(I_j: j \in J)$ is a *partition* of I (by which we mean that $\bigcup_{j \in J} I_j = I$, that $I_j \cap I_k = \emptyset$ for j, k in J, $j \neq k$, and that $I_j = \emptyset$ is allowed for any number of j), then $(x_i: i \in I)$ is summable if and only if $(x_i: i \in I_j)$ is summable for every j in J

[1] We use the notations $\sum(x_i: i \in I)$, $\sum_{i \in I} x_i$, and $\sum_i x_i$ interchangeably.

[2] Many of the basic definitions and theorems pertaining to partial monoids and partial semirings can be found in the forthcoming Manes and Arbib book *Algebraic Approaches to Program Semantics* and in the Manes and Benson [1985] paper "The inverse semigroup of a sum-ordered semiring."

and $\left(\sum(x_i: i \in I_j) : j \in J\right)$ is summable, and then

$$\sum(x_i: i \in I) = \sum\left(\sum(x_i: i \in I_j) : j \in J\right).$$

Before we proceed with a description of examples, we note several immediate consequences of the partition-associativity axiom.

(1) $\sum$ is an associative and a commutative operation.

(2) Any two isomorphic families have the same sum.

(3) Every subfamily of a summable family is itself summable.

(4) In particular, the empty family is summable, and its sum acts as an additive zero.

(5) There do not exist any nontrivial additive inverses.

We prove (2), (4), and (5).

2.2 OBSERVATION. If $(x_i: i \in I)$, $(y_k: k \in K)$ are two summable isomorphic families in a partial monoid $(M, \sum)$, then $\sum(x_i: i \in I) = \sum(y_k: k \in K)$.

PROOF. Let $\sigma: I \to K$ be a bijection such that $y_{i\sigma} = x_i$ for all i. Let $J = I$, and let $I_j = \{j\sigma\}$ for each j in J. Then

$$\begin{aligned}\sum(y_k: k \in K) &= \sum\left(\sum(y_m: m \in I_j) : j \in J\right)\\ &= \sum(y_{i\sigma}: i \in I)\\ &= \sum(x_i: i \in I).\end{aligned}$$

2.3 OBSERVATION. In a partial monoid $(M, \sum)$, the empty family is summable. Its sum, denoted by 0, is such that the sum of an arbitrary number of 0s is itself equal to 0. Furthermore, 0 acts as an additive zero in M.

PROOF. Let $x_n \in M$, $I = \{n\}$, $J = \{1,2\}$, $I_1 = I$, and $I_2 = \emptyset$. Then

$$\begin{aligned} x_n &= \sum (x_i : i \in I) \\ &= \sum \left(\sum (x_i : i \in I_j) : j \in J \right) \\ &= \sum (x_i : i \in I_1) + \sum (x_i : i \in I_2) \\ &= x_n + \sum (x_i : i \in \emptyset) \end{aligned}$$

which implies both that $\sum (x_i : i \in \emptyset)$ exists and that it acts as an additive zero for binary sums.

Now, let $I = \emptyset$. Then $\sum (x_i : i \in I) = \sum (x_i : i \in \emptyset) = 0$. Let J be any set and let $I_j = \emptyset$ for each j in J. Then $(I_j : j \in J)$ is a partition of I. Hence,

$$0 = \sum (x_i : i \in I) = \sum \left(\sum (x_i : i \in I_j) : j \in J \right) = \sum (0 : j \in J).$$

Thus, the sum of an arbitrary number of 0s equals 0.

To show that 0 is the additive zero for arbitrary sums, let $(x_i : i \in I)$ be a summable family, and let K be any set disjoint from I. For j in $I \cup K$, define

$$I_j = \begin{cases} \{j\}, & \text{if } j \in I; \\ \emptyset, & \text{if } j \in K. \end{cases}$$

Thus, $(I_j : j \in I \cup K)$ is a partition of I. Hence, $\sum (\sum (x_i : i \in I_j) : j \in I \cup K)$ exists and equals $\sum (x_i : i \in I)$, and for each j in $I \cup K$,

$$\sum (x_i : i \in I_j) = \begin{cases} x_j, & \text{if } j \in I; \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, 0 acts as an additive zero for arbitrary sums.

We define the *support* of a family $(x_i : i \in I)$ in M to be the subfamily $(x_i : i \in J)$ where $J = \{i \in I : x_i \neq 0\}$.

2.4 OBSERVATION. For a partial monoid $(M, \sum)$, if $\sum (x_i : i \in I)$ is defined and equals 0, then $x_i = 0$ for all i in I.

PROOF. Let $(x_i : i \in I)$ be a family in M such that $\sum(x_i : i \in I) = 0$. Let j be an element of I and let $y = \sum(x_i : i \neq j)$. Then

$$\begin{aligned} 0 &= x_j + y \\ &= (x_j + y) + (x_j + y) + \dots \\ &= x_j + y + x_j + y + \dots \\ &= x_j + (y + x_j) + (y + x_j) + \dots \\ &= x_j. \end{aligned}$$

This implies that additive inverses must be absent from partial monoids. Therefore, any ring fails to be a partial monoid for any $\sum$ extending the usual addition. Nonetheless, there are many examples of partial monoids, some of which we describe below.

2.5 EXAMPLE. The set of non-negative real numbers is a partial monoid with summability characterised as series convergence. That is, a family is summable if it has countable support and if the sequence of finite sums over the support of the family converges to a non-negative real number, in which case the sum is that number.

2.6 EXAMPLE. Any bounded upper (lower) semilattice is a partial monoid with $\sum$ defined as supremum (infimum) over families of finite support. Note that for an arbitrary family, the supremum (infimum) may not be defined.

Throughout this and subsequent chapters we will make repeated reference to three different examples of partial monoids useful in theoretical computer science – partial functions, multifunctions, and multisets – each of which is described below in detail.

2.7 EXAMPLES. Let D, E be sets and let the set of partial functions from D to E be denoted by $Pfn(D, E)$. Then $(Pfn(D, E), \sum)$ is a partial monoid if $\sum$ is defined such that a family $(x_i : i \in I)$ is summable if and only if for i, j in I and

$j \neq i$, $dom(x_i) \cap dom(x_j) = \emptyset$, where $dom(f)$ is the domain of definition of the partial function f. If $(x_i : i \in I)$ is summable then for any d in D

$$d\left(\sum_i x_i\right) = \begin{cases} dx_i, & \text{if } d \in dom(x_i) \text{ for some} \\ & \text{(necessarily unique) } i \in I; \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

We may extend this definition to include all families whose members agree on all domain overlaps. If we let $\widehat{\sum}$ denote the extension of $\sum$ to overlapping families, then $(Pfn(D,E), \widehat{\sum})$ is also a partial monoid. Arbib and Manes, working in a category-theoretic setting, have chosen the disjoint domain sum as the partial addition over the partial functions because this sum is the only sum for which the category of partial functions is a *partially-additive category*. The reader is referred to Manes and Arbib [1985, 3.2] for the details. For our purposes, both $\sum$ and $\widehat{\sum}$ are acceptable, but for definiteness we have selected $\sum$ as the partial addition for $Pfn(D,E)$.

2.8 EXAMPLE. Again, let D, E be sets. A *multifunction* $x: D \to E$ maps each element in D to an arbitrary subset of E. Such multifunctions correspond bijectively to relations $r \subseteq D \times E$, where $(d,e) \in r$ if and only if $e \in dx$. The set of multifunctions from D to E, denoted by $Mfn(D,E)$, together with $\sum$ defined such that for d in D, $d(\sum_i x_i) = \bigcup_i (dx_i)$, is a partial monoid in which every family is summable.

2.9 EXAMPLE. Let E be a set. A *finite multiset* on E is a function $m: E \to \mathbf{N}$ (where $\mathbf{N}$ represents the natural numbers) such that $\{e \in E : em \neq 0\}$ is finite. In other words, a finite multiset on E is a finite-support family in $\mathbf{N}$ indexed by E. Intuitively, a finite multiset is a set in which a member may occur with arbitrary finite multiplicity. For instance, consider the multiset $\{a, b, c, c, a, a\}$. Here $am = 3$, $bm = 1$, $cm = 2$, while $zm = 0$ for z different from a, b, and c. Let D be a set. Let $Mset(D,E)$ represent the set of total functions from the set D to the set of finite multisets on E. A family $(x_i : i \in I)$ in $Mset(D,E)$ is summable if for each d in D, $\{i: e(dx_i) \neq 0$ for at least one $e \in E\}$ is finite. In this case, for d in D and e in E, $e(d(\sum_i x_i)) = \sum_i e(dx_i)$, where the second sum is the usual addition over $\mathbf{N}$. For this choice of $\sum$, $(Mset(D,E), \sum)$ is a partial monoid. Zeiger [1969] appears to be the first to use multisets, which he refers to as rings of transformations, as a way to describe programs.

Other Partial Monoids

By appending other axioms to the basic axioms for a partial monoid, we obtain various special forms of partial monoid. A partial monoid in which a family is summable only if it has countable support is called an *ω-monoid.* An ω-monoid which satisfies the following axiom is called a *partially-additive monoid.*

Countable Limit Axiom. If $(x_i : i \in I)$ is a family in M of countable support and if for every finite $F \subseteq I$ the subfamily $(x_i : i \in F)$ is summable, then $(x_i : i \in I)$ is summable.

Our definition of partially-additive monoid differs from that of Manes and Arbib [1985, 3.1.2] in that we do not require summable families to be countable. Instead, we only require summable families to have countable support. In a partial monoid, noting that the sum of an arbitrary number of 0s is equal to 0 (observation 2.3), we know that the summability of the family $(x_i : i \in I)$, where I is countable, implies the summability of the family $(x_i : i \in I \cup J)$, where J has arbitrary cardinality and $x_j = 0$ for all j in J. Thus, our definition of partially-additive monoid is equivalent to that of Arbib and Manes in the following sense.

2.10 DEFINITION. Two partial monoids $(M, \sum)$ and $(M', \sum')$ are said to be *equivalent* if

(1) $M = M'$.

(2) For each summable family $(x_i : i \in I)$ in $(M, \sum)$, its support $(x_i : i \in J)$ is a summable family in $(M', \sum')$ and $\sum'(x_i : i \in J) = \sum(x_i : i \in I)$.

(3) For each summable family $(x_i : i \in I)$ in $(M', \sum')$, its support $(x_i : i \in J)$ is a summable family in $(M, \sum)$ and $\sum(x_i : i \in J) = \sum'(x_i : i \in I)$.

A partial monoid satisfying the following axiom is called a *generalised partially-additive monoid.*

Limit Axiom. If $(x_i : i \in I)$ is an arbitrary family in M and if for every finite $F \subseteq I$ the subfamily $(x_i : i \in F)$ is summable, then $(x_i : i \in I)$ is summable.

2.11 EXAMPLE. The set of natural numbers, **N**, is an ω-monoid if $\sum$ is defined only for families $(x_i : i \in I)$ of finite support, in which case $\sum$ is the usual addition.

2.12 EXAMPLE. By extending the previous example to $\hat{\mathbf{N}} = \mathbf{N} \cup \{\infty\}$, we obtain a partially-additive monoid, provided we extend the definition of $\sum$ to families of countable support as follows:

$$\sum(x_i : i \in I) = \begin{cases} \text{usual sum,} & \text{if } \{i : x_i \neq 0\} \text{ is finite and no } x_i = \infty; \\ \infty, & \text{otherwise.} \end{cases}$$

2.13 EXAMPLE. Any complete upper (lower) semilattice with $\sum$ defined as supremum (infimum) over arbitrary families is a generalised partially-additive monoid.

There is one more variant of the partial monoid, the sum-ordered monoid, which we will discuss in some detail, since most of the material in subsequent chapters is based on this particular structure.

Sum-ordered Monoids. There is a natural relation $\leq$ on the elements of a partial monoid.

2.14 DEFINITION. Let $(M, \sum)$ be a partial monoid. Then the *sum-ordering* on $(M, \sum)$ is the binary relation $\leq$ such that if x, y are in M, then $x \leq y$ if and only if there exists h in M such that $y = x + h$.

2.15 OBSERVATION. For any partial monoid M, the sum-ordering is a quasi-order.

PROOF. For any $x \in M$, $x = x + 0$, and so $x \leq x$, implying that $\leq$ is reflexive. Let x, y, z be in M such that $x \leq y$ and $y \leq z$. Then there exist h, k in M such that $y = x + h$ and $z = y + k$. Hence, $z = (x + h) + k = x + (h + k)$, and so $x \leq z$, implying that $\leq$ is transitive. Therefore, $\leq$ is a quasi-order.

Manes and Benson [1985] have investigated partial monoids (and partial semirings) for which the sum-ordering is a partial order.

In all of the examples of partial monoids thus far introduced, the sum-ordering is a partial order. Such partial monoids are called *sum-ordered partial monoids* or *so-monoids*, for short. However, in general, the sum-ordering need not be antisymmetric.

2.16 EXAMPLE. (Manes and Arbib, 1985, 8.3.2) Let $X = \{0,1,2,\infty\}$ and define $\sum$ as follows:

$$\sum(x_i: i \in I) = \begin{cases} \infty, & \text{if some } x_i = \infty \text{ or } x_i \neq 0 \text{ infinitely often;} \\ 0, & \text{if } x_i = 0 \text{ for all } i \in I; \\ 1, & \text{if no } x_i = \infty,\ \{i: x_i \neq 0\} \text{ is finite and nonempty, and} \\ & \text{the number of } i \text{ with } x_i = 2 \text{ is even;} \\ 2, & \text{if no } x_i = \infty,\ \{i: x_i \neq 0\} \text{ is finite and nonempty, and} \\ & \text{the number of } i \text{ with } x_i = 2 \text{ is odd.} \end{cases}$$

Then $(X,\sum)$ is a partial monoid. Now $1+2=2$ and $2+2=1$ so that $1 \leq 2$ and $2 \leq 1$, but $1 \neq 2$. Hence, we see that $\leq$ fails to be antisymmetric in this instance.

The following observation is true in any partial monoid.

2.17 OBSERVATION. If $(x_i: i \in I)$ and $(y_i: i \in I)$ are two families in a partial monoid M such that $x_i \leq y_i$ for all i in I and if $\sum(y_i: i \in I)$ is defined, then $\sum(x_i: i \in I)$ is defined and $\sum(x_i: i \in I) \leq \sum(y_i: i \in I)$.

PROOF. For all i in I, since $x_i \leq y_i$, there exists h_i in M such that $y_i = x_i + h_i$. Then, by the partition-associativity axiom, we have that $\sum(y_i: i \in I) = \sum(x_i + h_i: i \in I) = \sum(x_i: i \in I) + \sum(h_i: i \in I)$.

Partial Sub-monoids and Additive Maps

2.18 DEFINITION. Let $(M,\sum)$ be a partial monoid. Then $(M',\sum')$ is a *partial sub-monoid* of $(M,\sum)$ if

(1) M' is a subset of M;

(2) $(M', \sum')$ is a partial monoid; and

(3) $(x_i : i \in I)$ is a summable family in M' implies that $(x_i : i \in I)$ is a summable family in M whose sum is in M', in which case $\sum'_i x_i = \sum_i x_i$.

2.19 EXAMPLE. Let $(M, \sum)$ be a partial monoid, and let M' be a subset of M and let $\sum'$ be the restriction of $\sum$ to M'. If M' is closed under $\sum'$, then it is easily verified that $(M', \sum')$ is a partial monoid, and furthermore, that it satisfies the conditions of definition 2.18. Therefore, $(M', \sum')$ is a partial sub-monoid of $(M, \sum)$.

2.20 OBSERVATION. Any partial sub-monoid M' of a so-monoid M is itself a so-monoid.

PROOF. Let $\leq'$ denote the sum-ordering on M'. Suppose that x, y are two elements of M' such that $x \leq' y$ and $y \leq' x$. Then there exist h, k in M' such that $y = x +' h = x + h$ and $x = y +' k = y + k$. Thus, $x \leq y$ and $y \leq x$, which by the antisymmetry of $\leq$ implies that $x = y$. Therefore, $\leq'$ is antisymmetric and so M' is a so-monoid.

In this case, M' is said to be a *sub-so-monoid* of M.

2.21 DEFINITION. Let $(M, \sum)$, $(\widehat{M}, \widehat{\sum})$ be partial monoids. An *additive map* $\theta : (M, \sum) \to (\widehat{M}, \widehat{\sum})$ is a function $\theta : M \to \widehat{M}$ such that whenever $(x_i : i \in I)$ is a summable family in M, then $(x_i\theta : i \in I)$ is a summable family in $\widehat{M}$ and $(\sum_i x_i)\theta = \widehat{\sum}_i x_i\theta$.

Clearly, the identity map on a partial monoid is an additive map. We also note that the composition of additive maps is again an additive map.

2.22 OBSERVATION. If M_1 and M_2 are partial monoids and if $\theta : M_1 \to M_2$ is an additive map, then $0\theta = 0$.

PROOF. Recall that in a partial monoid, $0 = \sum (x_i : i \in \emptyset)$. Using the additivity of θ, we have that $0\theta = (\sum (x_i : i \in \emptyset))\theta = \sum (x_i\theta : i \in \emptyset) = 0$.

If M_1 and M_2 are partial monoids and if $\theta: M_1 \to M_2$ is an additive map, then it need not be the case that $M_1\theta$ is closed under the additive operation of M_2. Hence, $M_1\theta$ equipped with the additive operation of M_2 is not necessarily a partial sub-monoid of M_2.

2.23 COUNTEREXAMPLE. Let M_1 be the partial monoid $\{0,1\}$ with

$$\sum(x_i : i \in I) = \begin{cases} 0, & \text{if } x_i = 0 \text{ for all } i \in I; \\ 1, & \text{if } \exists j \in I \text{ such that } x_j = 1 \text{ and } x_i = 0 \text{ for } i \neq j; \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

Let M_2 be the so-ring of natural numbers with the usual addition over families of finite support. Define an additive map $\theta: M_1 \to M_2$. By observation 2.22, it must be the case that $0\theta = 0$. Let $1\theta = 1$. In M_1, the sum $1+1$ is not defined, and so in $M_1\theta$, the image $(1+1)\theta$ does not exist. However, in $M_1\theta$, the sum $1\theta + 1\theta$ does exist, but $1\theta + 1\theta = 1 + 1 = 2$ is not an element of $M_1\theta$. Thus, $M_1\theta$ is not closed under the additive operation of M_2.

However, if in $M_1\theta$ we restrict the additive operation on M_2 in a particular way, then $M_1\theta$ becomes a partial sub-monoid of M_2.

2.24 OBSERVATION. Let M_1 and M_2 be partial monoids, and let $\theta: M_1 \to M_2$ be an additive map. Then an additive operation can be defined on $M_1\theta$ such that $M_1\theta$ is a partial sub-monoid of M_2.

PROOF. If $(x_i : i \in I)$ is a summable family in M_1, then since θ is an additive map, $(\sum_i x_i)\theta = \sum_i x_i\theta$ is an element of $M_1\theta$. However, as was demonstrated in counterexample 2.23, $M_1\theta$ may not be closed under the additive operation of M_2. Nevertheless, $M_1\theta$ becomes a partial sub-monoid of M_2, if we define the additive operation $\sum'$ on $M_1\theta$ such that $\sum'_i x_i\theta$ is defined only if $(x_i : i \in I)$ is a summable family in M_1, in which case $\sum'_i x_i\theta = \sum_i x_i\theta = (\sum_i x_i)\theta$. Hence, $M_1\theta$ inherits its partial monoid structure directly from M_1. Since $M_1\theta$ satisfies the conditions of definition 2.18, $(M_1\theta, \sum')$ is a partial sub-monoid of M_2.

We say that $(M_1\theta, \sum')$ is a *relative partial sub-monoid* of M_2, following the terminology of universal algebra [Grätzer, 1968, 2.13].

2.25 OBSERVATION. Let M_1 and M_2 be partial monoids, and let $\theta: M_1 \to M_2$ be an additive map. If x, y are elements of M_1 such that $x \leq y$, then $x\theta \leq y\theta$.

PROOF. Let x, y be elements of M_1 such that $x \leq y$. Then there exists h in M_1 such that $x + h = y$. Since θ is additive, $y\theta = (x+h)\theta = x\theta + h\theta$. Thus, $x\theta \leq y\theta$.

Hence, the sum-ordering is preserved by additive maps.

There are two different types of embeddings of partial monoids. One is the embedding of a partial monoid in a partial monoid, and the other is the embedding of a partial monoid in an abelian monoid. For the most part, we are concerned with the former type of embedding, but an important example of the latter type arises in connection with the partial functions, as we will see in example 2.46.

2.26 DEFINITIONS. An *embedding* of partial monoids is an injective additive map. An *embedding* of a partial monoid $(M, \sum)$ in an abelian monoid $(A, +)$ is an injective function $\theta: M \to A$ such that whenever the finite family $(x_i : 1 \leq i \leq n)$ in M is summable, $(\sum_i x_i)\theta = x_1\theta + \cdots + x_n\theta$.

2.27 OBSERVATION. Let M_1 be a partial monoid on which is defined a quasi-order $\preceq$, and let M_2 be a so-monoid. If $\theta: M_1 \to M_2$ is an additive map such that $x\theta \leq y\theta$ if and only if $x \preceq y$, then θ is an embedding if and only if $\preceq$ is a partial order.

PROOF. Suppose that θ is an embedding. Let x, y be elements of M_1 such that $x \preceq y$ and $y \preceq x$. Then $x \preceq y$ implies that $x\theta \leq y\theta$, and $y \preceq x$ implies that $y\theta \leq x\theta$. By the antisymmetry of $\leq$, $x\theta = y\theta$. This in turn implies that $x = y$, since θ is injective. Thus, $\preceq$ is antisymmetric, and therefore a partial order.

Now, suppose that $\preceq$ is a partial order. Let x, y be elements of M_1 such that $x\theta = y\theta$. Then $x\theta \leq y\theta$ implies that $x \preceq y$, and $y\theta \leq x\theta$ implies that $y \preceq x$. Hence, $x = y$, since $\preceq$ is antisymmetric. Therefore, θ is injective, and so θ is an embedding.

Constructions of Partial Monoids

Thus far, we have supplied a variety of specific examples of partial monoids. We now show how to construct new partial monoids from given ones. The objects constructed are the usual ones from category theory; the category under consideration is *Pmon* with partial monoids as objects and with additive maps as morphisms. In each case, we give the construction of the new partial monoid, but we leave it to the reader as a routine exercise to verify that the constructed object possesses the given universal property for that construction in *Pmon*.

Products. Let $((M^i, \sum^i): i \in I)$ be a family of partial monoids. Their product is the partial monoid $(\prod_i M^i, \sum)$ together with the projection maps $pr_i: (\prod_i M^i, \sum) \to (M^i, \sum^i)$, defined as follows. The set $\prod_i M^i$ is the cartesian product of the M^is. Let $(x_j: j \in J)$ be a family in $\prod_i M^i$. Then each $x_j = (x_j^i: i \in I)$, where x_j^i is in M^i. The family $(x_j: j \in J)$ is summable in $\prod_i M^i$ if for each i in I, $(x_j^i: j \in J)$ is summable in M^i, in which case

$$\sum (x_j: j \in J) = \sum \Big((x_j^i: i \in I): j \in J\Big) = \Big(\sum^i (x_j^i: j \in J): i \in I\Big).$$

Each of the projections maps

$$pr_i: (\prod_i M^i, \sum) \to (M^i, \sum^i): x \mapsto x^i$$

is easily seen to be additive.

Coproducts. Let $((M^i, \sum^i): i \in I)$ be a family of partial monoids. Their coproduct is the partial monoid $(\coprod_i M^i, \sum)$ together with the injections maps $in_i: (M^i, \sum^i) \to (\coprod_i M^i, \sum)$, defined as follows. The set

$$\coprod_i M^i = \{0\} \cup \bigcup_{i \in I} \{(x, i): x \in M^i - \{0^i\}\}.$$

Let $(x_j: j \in J)$ be a family in $\coprod_i M^i$. Then for each j in J, $x_j = 0$ or $x_j = (y, i)$ where y is in M^i. Let J' be the support index set of $(x_j: j \in J)$. If $J' = \emptyset$, then $(x_j: j \in J)$ is summable and

$$\sum (x_j: j \in J) = 0.$$

If $J' \neq \emptyset$, then $(x_j: j \in J)$ is summable if $(x_j: j \in J') = ((y_j, i): j \in J')$ for some i in I and if $(y_j: j \in J')$ is summable in M^i, in which case

$$\sum (x_j: j \in J) = \sum ((y_j, i): j \in J') = \left(\sum\nolimits^i (y_j: j \in J'), i\right).$$

Each of the injection maps

$$in_i: (M^i, \sum\nolimits^i) \to (\coprod_i M^i, \sum): 0^i \mapsto 0, y \mapsto (y, i)$$

is easily seen to be additive.

The category *Pmon* has a zero object, namely $M = \{0\}$.

Quotients. We define quotients in *Pmon* modulo a congruence.

2.28 DEFINITION. Let M be a partial monoid, and let E be an equivalence relation on the elements of M. Then E is a *partial monoid congruence* on M if E is closed under the additive operation of the product partial monoid $M \times M$.

Let $(M, \sum)$ be a partial monoid, and let E be a partial monoid congruence on M. Their quotient is the partial monoid $(M/E, \sum')$ together with the canonical surjection map $p: (M, \sum) \to (M/E, \sum')$, defined as follows. The set $M/E = \{\bar{x}: x \in M\}$, where for each x in M, $\bar{x}$ is the equivalence class of x modulo E. The family $(\bar{x}_j: j \in J)$ is summable in M/E if $(x_j: j \in J)$ is summable in M, in which case

$$\sum\nolimits' (\bar{x}_j: j \in J) = \overline{\sum (x_j: j \in J)}.$$

The reader may easily verify that this sum is well-defined, since E is closed under the additive operation of $M \times M$. The surjection map

$$p: (M, \sum) \to (M/E, \sum'): x \mapsto \bar{x}$$

is readily seen to be additive.

Limits and Colimits. Let M_1, M_2 be two partial monoids, and let $f, g: M_1 \to M_2$ be two additive maps. Then the equaliser of f and g is the subset $M = \{x \in M_1 : xf = xg\}$ of M_1 together with the set inclusion map $h: M \to M_1$. The reader may easily show that M is closed under the additive operation of M_1 and is thus a partial sub-monoid of M_1, implying that h is an additive map. We observe that *Pmon* has all limits, since it has all products and all equalisers.

Let M_1, M_2 be partial monoids, and let $f, g: M_1 \to M_2$ be two additive maps. Then the coequalizer of f and g is the partial monoid M_2/E', where E' is the intersection of all partial monoid congruences on M_2 which contain $\{(xf, xg): x \in M_1)\}$, together with the canonical surjection $h: M_2 \to M_2/E'$. We observe that *Pmon* has all colimits, since it has all coproducts and all coequalisers.

Free Partial Monoids. Let X be any set. Then the free partial monoid generated by X is the set $M = X \cup \{0\}$ with the additive operation defined as follows. Let $(x_j : j \in J)$ be a family in M with support index set J'. Then,

$$\sum (x_j : j \in J) = \begin{cases} 0, & \text{if } J' = \emptyset; \\ x_i, & \text{if } \exists i \in J \text{ such that } J' = \{i\}; \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

The additive operation thus defined is called the *trivial addition.*

Partial Semirings

By marrying a multiplicative structure with the existing additive structure of a partial monoid, we form a partial semiring. More formally,

2.29 DEFINITION. A *partial semiring* is a quadruple $(R, \sum, \circ, 1)$, where $(R, \sum)$ is a partial monoid, $(R, \circ, 1)$ is an ordinary monoid with multiplicative operation $\circ$ and unit 1, and the additive and multiplicative structures obey the following distributive laws.[3] If $\sum(x_i : i \in I)$ is defined in R, then for all y in R, $\sum_i y \circ x_i$ and $\sum_i x_i \circ y$ are defined and

$$y \circ \left(\sum_i x_i\right) = \sum_i y \circ x_i;$$

$$\left(\sum_i x_i\right) \circ y = \sum_i x_i \circ y.$$

2.30 EXAMPLE. The set of nonnegative real numbers with $\sum$ defined as in example 2.5 and with $\circ$ defined as the usual product is a partial semiring.

2.31 EXAMPLE. Any bounded distributive lattice L is a partial semiring, with $\sum$ defined as supremum over families of finite support and with $\circ$ defined as the meet of two elements. (Such a lattice is also a partial semiring with $\sum$ defined as infimum over families of finite support, and with $\circ$ defined as the join of two elements.) The distributive laws in the sense of lattice theory guarantee that join distributes over meet (and that meet distributes over join), and hence that $\circ$ distributes over $\sum$.

2.32 DEFINITION. The *opposite* or *dual* of a partial semiring $(R, \sum, \circ, 1)$ is a partial semiring $(R, \sum, \bullet, 1)$, in which for x, y in R, $x \bullet y = y \circ x$. Henceforth, we denote the opposite of a partial semiring R by R^{op}.

Just as there are variant partial monoids, there are variant partial semirings. In each case, the name describing the type of partial semiring refers to the type of partial monoid which forms its additive structure.

[3] We use the notations $x \circ y$ and xy interchangeably, often dropping the $\circ$ and denoting multiplication by adjacency only.

In the examples below, each of the partial monoid variants given in examples 2.11 - 2.13 is supplied with a multiplicative structure which turns it into a partial semiring.

2.33 EXAMPLE. We can easily extend the ω-monoid $(\mathbf{N}, \sum)$ of example 2.11 to an ω-semiring by defining $\circ$ as the usual product with unit 1.

2.34 EXAMPLE. To extend the partially-additive monoid $(\hat{\mathbf{N}}, \sum)$ of example 2.12 to a partially-additive semiring, we define $\circ$ as follows. For $x, y \in \hat{\mathbf{N}}$,

$$x \circ y = \begin{cases} \text{usual product}, & \text{if } x \neq \infty \text{ and } y \neq \infty; \\ 0, & \text{if } x = 0 \text{ or if } y = 0; \\ \infty, & \text{if } x = \infty,\ y \neq 0 \text{ or if } x \neq 0,\ y = \infty. \end{cases}$$

2.35 EXAMPLES.

(1) If we parallel example 2.13, we would expect that any complete distributive lattice is a generalised partially-additive semiring, with $\sum$ defined as supremum (infimum) over arbitrary families and with $\circ$ defined as the meet (join) of two elements. In order to be a generalised partially-additive semiring, such a lattice must satisfy the following infinite distibutive laws

$$y \wedge \left(\bigvee_i x_i\right) = \bigvee_i y \wedge x_i$$

$$y \vee \left(\bigwedge_i x_i\right) = \bigwedge_i y \vee x_i,$$

but as is demonstrated by Birkhoff [1967, V.5], not all complete distributive lattices satisfy these laws. However, any complete Boolean lattice satisfies both infinite distributive laws, and thus is a generalized partially-additive semiring with $\sum$ equal to supremum (or infimum) and with $\circ$ equal to meet (or join).

(2) By contrast, a complete lattice is Brouwerian if and only if

$$y \wedge \left(\bigvee_i x_i\right) = \bigvee_i y \wedge x_i.$$

Thus, the Brouwerian condition guarantees infinite distributivity of meet over join, and so any complete Brouwerian lattice is a generalised partially-additive semiring with $\sum$ equal to supremum and with $\circ$ equal to meet.

In a partial semiring, we refer to the lack of nontrivial additive inverses as *positivity* because the most familiar examples are the positive integers and the positive real numbers. We say that a partial semiring is *complete* if all families are summable. Note that the previous three examples depict complete partial semirings. Eilenberg [1974, pp.122-126] also introduces the terms *positive* and *complete* in relation to semirings. His complete semirings are the same as our complete partial semirings, but his use of "positive" also requires that if $xy = 0$, then $x = 0$ or $y = 0$. We do not impose this restriction, both because we want the term positive to refer only to the additive structure of a partial semiring and because the fundamental example 2.38 (below) as well as many other examples do not satisfy this property.

We now come to sum-ordered partial semirings, the objects on which we shall concentrate in subsequent chapters.

2.36 DEFINITION. A *sum-ordered partial semiring* or *so-ring* for short, is a partial semiring in which the additive structure is a so-monoid.

2.37 OBSERVATION. The sum-ordering on a so-ring R is a compatible partial order.
PROOF. (Recall that a relation $\preceq$ is compatible if $x \preceq y$ implies that $zx \preceq zy$ and $xz \preceq yz$ for any z.) Compatibility is a direct consequence of distributivity. Let x, y be elements of R such that $x \leq y$. Then there exists h in R such that $y = x + h$. Let z be any element of R. Then $zy = z(x + h) = zx + zh$, which implies that $zx \leq zy$, and hence that $\leq$ is left compatible. Right compatibility can be proved similarly. Therefore, $\leq$ is a compatible partial order on any so-ring.

Each of the computer science examples discussed earlier (examples 2.7 - 2.9) can be extended to a so-ring. Both partial functions and multifunctions admit natural extensions to so-rings. This is also true of multisets, although here it is convenient to choose an alternative representation in order to adjoin a multiplicative structure.

2.38 EXAMPLE. The so-monoid $(Pfn(D,D),\sum)$ becomes a so-ring, with $\circ$ defined as the usual functional composition and with unit defined as the identity function on D. From the rules of functional composition, it is immediate that $(Pfn(D,D),\circ,1)$ is a monoid. However, we need to show that the distributive laws for a partial semiring are satisfied by $(Pfn(D,D),\sum,\circ,1)$.

We demonstrate distributivity on the left; distributivity on the right is left to the reader. Let $(x_i: i \in I)$ be a summable family in $Pfn(D,D)$. Then for $j \neq i$, $dom(x_i) \cap dom(x_j) = \emptyset$. Let y be any partial function. First, we show that $(x_i y: i \in I)$ is a summable family. For each i in I, the set $dom(x_i y) = \{d \in D: d \in dom(x_i) \text{ and } dx_i \in dom(y)\} \subseteq dom(x_i)$. Thus, for $j \neq i$, $dom(x_i y) \cap dom(x_j y) = \emptyset$, and so $(x_i y: i \in I)$ is a summable family. Now,

$$\begin{aligned} dom\left((\sum_i x_i)y\right) &= \{d \in D: d \in dom(\sum_i x_i) \text{ and } d\sum_i x_i \in dom(y)\} \\ &= \{d \in D: d \in dom(x_j) \text{ for some } j \in I \text{ and } dx_j \in dom(y)\} \\ &= dom\left(\sum_i x_i y\right). \end{aligned}$$

Hence, for d in D, if $d \notin dom((\sum_i x_i)y)$, then $d \notin dom(\sum_i x_i y)$ and so neither $d(\sum_i x_i)y$ nor $d(\sum_i x_i y)$ are defined. However, if $d \in dom((\sum_i x_i)y)$, then $d \in dom(\sum_i x_i)$ which implies that there exists a single j in I such that $d \in dom(x_j)$. In this case, $d(\sum_i x_i)y = dx_j y = d(\sum_i x_i y)$. Thus, distributivity on the left holds.

2.39 EXAMPLE. To convert the so-monoid $(Mfn(D,D),\sum)$ to a so-ring, we define $\circ$ as the usual relational composition. That is, for each d in D and for x, y in $Mfn(D,D)$, $d(x\circ y) = \bigcup(ey: e \in dx)$, and $d1 = \{d\}$. From the study of relations, it is immediate that $(Mfn(D,D),\circ,1)$ is a monoid. What remains to be shown is that the distributive laws for a partial semiring hold.

We leave distributivity on the right to the reader, but demonstrate distributivity on the left. Let $(x_i : i \in I)$ be any family in $Mfn(D,D)$, and let y be any element of $Mfn(D,D)$. For any d in D,

$$\begin{aligned} d(\sum_i x_i)y &= \bigcup(ey: e \in d\sum_i x_i) \\ &= \bigcup(ey: e \in \bigcup_i dx_i) \\ &= \bigcup_i(ey: e \in dx_i) \\ &= \bigcup_i dx_i y \\ &= d\sum_i x_i y. \end{aligned}$$

Hence, the left distributive law holds.

2.40 EXAMPLE. Extending $(Mset(D,D),\sum)$ to a so-ring requires a little more work. First, we must represent each x in $Mset(D,D)$ as a matrix with each matrix entry x_{ij} set equal to $j(ix)$. This new characterization of $Mset(D,D)$ yields a bijection between the members of $Mset(D,D)$ and the nonnegative integer matrices with at most finitely many nonzero entries in each row. We then define $\sum$ as matrix addition and $\circ$ as matrix multiplication with unit equal to the identity matrix. In this representation of $Mset(D,D)$, a family $(x_k : k \in K)$ is summable if for each i in D, $\{k$: the ij^{th} entry of x_k is nonzero for at least one $j \in D\}$ is finite.

We now show that the multiplicative structure is indeed a monoid. Let A, B be two matrices in $Mset(D,D)$. Certainly the product of A and B is a nonnegative integer matrix. Column j of row i of AB is the result of multiplying row i of A by column j of B. Suppose that row i of AB contains infinitely many nonzero entries. Then it must be the case that there are an infinite number of j such that the inner product of row i of A with column j of B is nonzero. Since A is a member of $Mset(D,D)$, row i of A must contain at most finitely many nonzero entries. Thus, for at least one of the nonzero entries in row i of A, say the entry in column k, it must be the case that its product with an entry in row k of B must be nonzero infinitely often. However, this implies that row k of B has infinitely many nonzero entries and thus that B is not in $Mset(D,D)$, a contradiction. Therefore, AB is a member of $Mset(D,D)$. Associativity under $\circ$ and proper behavior of the unit are a consequence of the laws of matrix multiplication. Hence, $(Mset(D,D),\circ,1)$ is a monoid. The partial semiring distributive laws hold as an immediate consequence of the distributive laws for matrix addition and multiplication.

2.41 DEFINITION. A family $(x_i: i \in I)$ in a partial semiring R is said to be *supersummable* if for any family $(y_i: i \in I)$ in R, $\sum(x_i y_i: i \in I)$ is defined.

Hence, all families in a complete partial semiring are supersummable. Clearly, any supersummable family is summable. Although the converse is not true in every partial semiring, we do have that

2.42 OBSERVATION. In $Pfn(D,D)$, $Mfn(D,D)$, and $Mset(D,D)$ all summable families are also supersummable.

PROOF. This is immediate in $Mfn(D,D)$, since $Mfn(D,D)$ is a complete so-ring. We supply the proof for both $Pfn(D,D)$ and $Mset(D,D)$. Let $(x_i: i \in I)$ be a summable family in $Pfn(D,D)$. Then for $j \neq i$, $dom(x_i) \cap dom(x_j) = \emptyset$. Let $(y_i: i \in I)$ be any family in $Pfn(D,D)$. For each i in I,

$$\begin{aligned} dom(x_i y_i) &= \{d \in D: d \in dom(x_i) \text{ and } dx_i \in dom(y_i)\} \\ &\subseteq \{d \in D: d \in dom(x_i)\} \\ &= dom(x_i). \end{aligned}$$

Hence, for $j \neq i$, $dom(x_i y_i) \cap dom(x_j y_j) = \emptyset$. Therefore, $(x_i y_i: i \in I)$ is a summable family in $Pfn(D,D)$.

Now, let $(x_i: i \in I)$ be a summable family in $Mset(D,D)$. Then for each j, the set $S_j = \{i\text{: the } jk^{\text{th}} \text{ entry of } x_i \text{ is nonzero for at least one } k\}$ is finite. Hence, for all i not in S_j, row j of x_i consists entirely of zeroes. Let $(y_i: i \in I)$ be any family in $Mset(D,D)$. Then each $x_i y_i$ is in $Mset(D,D)$, and so each row of $x_i y_i$ contains finitely many nonzero entries. Note that if row j of x_i contains all zeroes, then so does row j of $x_i y_i$. Therefore, for each j, $\{i\text{: the } jk^{\text{th}}$ entry of x_iy_i is nonzero for at least one $k\} \subseteq S_j$, and so is finite. Hence, $(x_iy_i : i \in I)$ is a summable family in $Mset(D, D)$.

Partial Sub-semirings and Homomorphisms

2.43 DEFINITIONS. Let $(R, \sum, \circ, 1)$ be a partial semiring. Then $(R', \sum', \circ, 1)$ is a *partial sub-semiring* of R if

(1) R' is a subset of R;

(2) $(R', \sum')$ is a partial sub-monoid of $(R, \sum)$;

(3) $(R', \circ, 1)$ is a sub-monoid of $(R, \circ, 1)$; and

(4) the left and right distributive laws hold in $(R', \sum', \circ, 1)$.

If $(R', \sum')$ is a relative partial sub-monoid of $(R, \sum)$, then $(R', \sum', \circ, 1)$ is a *relative partial sub-semiring* of R. If $R' = R$, then $(R', \sum', \circ, 1)$ is called a *full partial sub-semiring* of R.

By observation 2.20, we immediately have that a partial sub-semiring of a so-ring is itself a so-ring.

2.44 DEFINITION. A *homomorphism* of partial semirings is an additive map which is also a monoid homomorphism of the multiplicative structures.

2.45 DEFINITIONS. An *embedding* of partial semirings is a partial semiring homomorphism which is an embedding of the partial monoid structures. An *embedding* of a partial semiring in a ring is an injective monoid homomorphism of the multiplicative structures which is an embedding of the additive structures in the sense of embedding a partial monoid in an abelian monoid.

2.46 EXAMPLES. The so-ring $Pfn(D, D)$ embeds in both the so-ring $Mfn(D, D)$ and the so-ring $Mset(D, D)$. In the first case, the embedding is the set inclusion map of the partial functions in the multifunctions, whereas in the second case the embedding $\theta : Pfn(D,D) \to Mset(D,D)$ is defined by

$$e(d(f\theta)) = \begin{cases} 1, & \text{if } df \text{ is defined and } df = e; \\ 0, & \text{otherwise.} \end{cases}$$

In turn, we may embed $Mset(D,D)$ in the ring $Mat(D,D)$ of $D \times D$ complex matrices. The embedding of $Mset(D,D)$ in $Mat(D,D)$ is defined by $(f\theta)_{ij} = j(if)$. By composing the embedding of $Pfn(D,D)$ in $Mset(D,D)$ with the embedding of $Mset(D,D)$ in $Mat(D,D)$, we obtain an embedding of $Pfn(D,D)$ in $Mat(D,D)$.

This particular embedding of $Pfn(D,D)$ in the ring $Mat(D,D)$ gives a characterisation of partial functions as $D \times D$ 0-1 matrices with at most one 1 per row. We make use of this representation of $Pfn(D,D)$ in later chapters.

It is not possible, however, to embed $Mfn(D,D)$ in the ring $Mat(D,D)$ because a partial monoid which is embeddable in an abelian group must satisfy the following cancellability property: if $k+f$ and $k+g$ are defined and are equal then $f = g$. This criterion is not satisfied by $Mfn(D,D)$ because $f+f = f+0$ even when $f \neq 0$.

Constructions of Partial Semirings

We can extend each of the constructions in the category *Pmon* to the category *Prng* of partial semirings and their homomorphisms. We note that each of the constructions in *Prng* also applies directly to the category *Srng* of so-rings and their homomorphisms.

Products. Let $(R^i, \sum^i, \circ^i, 1^i)$ be a family of partial semirings. Their product is the partial semiring $(\prod_i R^i, \sum, \circ, 1)$ together with the projection maps $pr_i : (\prod_i R^i, \sum, \circ, 1) \to (R^i, \sum^i, \circ^i, 1^i)$ defined as follows. The product $(\prod_i R^i, \sum)$ is the partial monoid product of the partial monoids $(R^i, \sum^i)$. The product of two elements x, y in $\prod_i R^i$ is defined as

$$x \circ y = (x^i \circ^i y^i : i \in I),$$

and

$$1 = (1^i : i \in I).$$

Each of the projection maps

$$pr_i : (\prod_i R^i, \sum, \circ, 1) \to R^i, \sum^i, \circ^i, 1^i) : x \mapsto x^i$$

is easily seen to be a monoid homomorphism, and thus a partial semiring homomorphism.

Coproducts. At this writing, we do not have a general construction for the coproduct of an arbitrary family of partial semirings. Besides the usual coproduct of the empty family and of a one-element family, we do have the following construction. Let $(R^i, \sum^i, \circ^i, 1^i)$ be a family of partial semirings such that either $1^i + 1^i$ is undefined or $1^i + 1^i = 1^i$. Their coproduct is the partial semiring $(\coprod_i R^i, \sum, \circ, 1)$ together with the injection maps $in_i : (R^i, \sum^i, \circ^i, 1^i) \to (\coprod_i R^i, \sum, \circ, 1)$ defined as follows. First, define the set

$$X = \bigcup_{i \in I} \{(x, i) : x \in R^i - \{0^i, 1^i\}.$$

Then form the set $\overline{X}$, the set of reduced strings in X^*.[4] By reduced strings we mean that if $(x, i)(y, i)$ is a string in X^*, then the one-element string $(w \circ^i x, i)$ replaces the two-element string $(x, i)(y, i)$. We are now ready to define the coproduct set as

$$\coprod_i R^i = \{0\} \cup \overline{X}.$$

Let $(x_j : j \in J)$ be a family in $\coprod_i R^i$. Then for each j in J, $x_j = 0$, $x_j = 1$, or $x_j = (y, i)$ where y is in R^i. If the support of $(x_j : j \in J)$ is empty, then $(x_j : j \in J)$ is summable and

$$\sum (x_j : j \in J) = 0.$$

[4] We use the notation X^* to refer to the set of all finite strings of elements of X together with the empty string 1.

If the family contains one non-zero member, then the family is summable and the sum is that member. If the support contains more than one member, then let K be the index set of the elements that are different from 1 and let L be the index set of all 1s. The family $(x_j: j \in J)$ is summable if $(x_j: j \in K) = ((y_j, i): j \in K)$ for some i in I and if $(y_j: j \in K)$, $(1^i: j \in L)$, and $\sum^i (y_j: j \in K) + \sum^i (1^i: j \in L)$ are summable in R^i, in which case

$$\begin{aligned}\sum (x_j: j \in J) &= \sum((y_j, i): j \in K) + \sum(1: j \in L) \\ &= \left(\sum^i (y_j: j \in K) + \sum^i (1^i: j \in L), i\right).\end{aligned}$$

Otherwise, the family is not summable. The product of two elements x, y in $\coprod_i R^i$ is defined as

$$x \circ y = \begin{cases} x, & \text{if } y = 1; \\ y, & \text{if } x = 1; \\ 0, & \text{if } x = 0 \text{ or if } y = 0; \\ (w \circ^i z, i), & \text{if } x = (w, i) \text{ and } y = (z, i); \\ xy, & \text{otherwise.} \end{cases}$$

Hence, the product of x and y is string concatenation in most cases. Each of the injection maps

$$in_i: (R^i, \sum^i, \circ^i, 1^i) \rightarrow (\coprod_i R^i, \sum, \circ, 1): 0^i \mapsto 0, 1^i \mapsto 1, y \mapsto (y, i)$$

is easily seen to be a monoid homomorphism which is additive, and therefore, a partial semiring homomorphism.

The previous construction also works in the case in which there is one j such that $1^j + 1^j$ is defined and not equal to 1^j, and such that $1^i + 1^i$ is undefined for $i \neq j$. However, in the cases where there exists at least one such j and at least one k such that $1^k + 1^k$ is defined, the coproduct construction is unknown at this time.

The category *Prng* has an initial object, namely the partial semiring $R = \{0, 1\}$ with $1+1$ undefined. The terminal object of *Prng* is the partial semiring $R = \{0\}$.

Quotients.

2.47 DEFINITION. Let R be a partial monoid, and let E be an equivalence relation on the elements of R. Then E is a *partial semiring congruence* on R if E is a partial monoid congruence on R, E is closed under the multiplicative operation of the product partial semiring $R \times R$, and E contains the multiplicative identity of $R \times R$.

Let $(R, \sum, \circ, 1)$ be a partial semiring, and let E be a partial semiring congruence on R. Their quotient is the partial semiring $(R/E, \sum', \circ', 1')$ together with the canonical surjection $p: (R, \sum, \circ, 1) \to (R/E, \sum', \circ', 1')$ defined as follows. The quotient $(R/E, \sum')$ is the partial monoid quotient. The product of two elements $\bar{x}$, $\bar{y}$ in R/E is defined as

$$\bar{x} \circ' \bar{y} = \overline{x \circ y},$$

and

$$1' = \bar{1}.$$

The reader may easily verify that $\circ'$ is well-defined, since E is closed under the multiplicative operation of $R \times R$. The surjection map

$$p: (R, \sum, \circ, 1) \to (R/E, \sum', \circ', 1'): x \mapsto \bar{x}$$

is readily seen to be a monoid homomorphism, and thus a partial semiring homomorphism.

Limits and Colimits. Let R_1, R_2 be two partial semirings, and let f, g: $R_1 \to R_2$ be two partial semiring homomorphisms. Then the equaliser of f and g is the partial monoid equaliser R together with the inclusion map $h: R \to R_1$. The reader may also easily show that R is closed under the multiplicative operation of R_1 and that 1 is in R, since f, g are monoid homomorphisms. Hence, R is a partial sub-semiring of R_1, implying that h is a partial semiring homomorphism. We observe that $Prng$ has all limits, since it has all products and all equalizers.

Let R_1, R_2 be two partial semirings, and let $f, g: R_1 \to R_2$ be two partial semiring homomorphisms. Then the coequaliser of f and g is the partial semiring R_2/E', where E' is the intersection of all partial semiring congruences on R_2 which contain $\{(xf, xg): x \in R_1\}$, together with the canonical surjection $h: R_2 \to R_2/E'$. Although $Prng$ has coequalisers of all pairs of morphisms, it is unclear at this point whether $Prng$ has all colimits, since the general coproduct construction is unknown.

Free Partial Semirings. Let X be any set. Then the free partial semiring generated by X is the set $R = X^* \cup \{0\}$ with the trivial addition and with the multiplicative operation defined as follows. Let x, y be two elements in R. Then,

$$x \circ y = \begin{cases} x, & \text{if } y = 1; \\ y, & \text{if } x = 1; \\ 0, & \text{if } x = 0 \text{ or if } y = 0; \\ xy, & \text{otherwise.} \end{cases}$$

Hence, the multiplication is string concatenation. Thus, we see that the additive structure is that of a free partial monoid on the set X^* and that the multiplicative structure is that of a free monoid on the set X.

Now, let $(W, \circ', 1)$ be any monoid. Then the free partial semiring over the monoid W is the set $R = W \cup \{0\}$, with the trivial addition and with the multiplicative operation defined as follows. Let x, y be elements of R. Then,

$$x \circ y = \begin{cases} 0, & \text{if } x = 0 \text{ or if } y = 0; \\ x \circ' y, & \text{otherwise.} \end{cases}$$

Hence, any monoid can be extended to a partial semiring by appending one element. Similarly, any partial monoid can be extended to a partial semiring by appending one element, as we now demonstrate.

2.48 OBSERVATION. Any partial monoid $(R', \sum')$ can be extended to a partial semiring $(R, \sum, \circ, 1)$.

PROOF. Let $R = R' \cup \{1\}$, and let $(x_j : j \in J)$ be a family in R. Then,

$$\sum (x_j : j \in J) = \begin{cases} \sum' (x_j : j \in J), & \text{if } x_i \in R' \text{ for all } i \in I; \\ 1, & \text{if } \exists i \in J \text{ such that } x_i = 1 \text{ and } x_j = 0 \text{ for } j \neq i; \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

Let x, y be in R. Then,

$$x \circ y = \begin{cases} x, & \text{if } y = 1; \\ y, & \text{if } x = 1; \\ 0, & \text{otherwise.} \end{cases}$$

The reader may easily verify that $(R, \sum, \circ, 1)$ is a partial semiring. The multiplicative operation thus defined is called the *trivial multiplication*.

We leave the discussion of the more technical results relating to so-rings until the following chapter.

CHAPTER III

PROPERTIES OF SO-RINGS

In this chapter, we present the basic properties and substructures of so-rings. These results provide the background for the subsequent two chapters. For a more detailed study of some of the various properties of so-rings, consult Manes and Benson [1985].

First, we describe the "center"; it is a generalization of the notion of center in a distributive lattice, which is the Boolean algebra of its complemented elements. Next, we define "domain" and "range" which are generalizations of the concepts of domain and range as they apply to partial functions. Finally, we define the concept of "inversibility", that is, whether or not an element of a so-ring possesses a multiplicative "inverse". This notion of inverse is closely related to the notion of inverse in semigroup theory. We conclude with a discussion of an alternative ordering on the elements of a so-ring.

The Center

The concept of center occurs in the literature in reference to different algebraic structures. Perhaps the two types of center most generally known are the center of a semigroup and the center of a poset. The center of a semigroup S consists of all x in S such that $xy = yx$ for each y in S. The center of a poset P with least element 0 and greatest element 1 [Birkhoff, 1967, III.8] is the set of all x in P for which there exist posets $(T, 0_T, 1_T)$, $(U, 0_U, 1_U)$ and an order isomorphism

$\psi: P \to T \times U$ such that $x\psi = (1_T, 0_U)$. We define yet another notion of center which is a Boolean algebra formed by the complemented elements of a so-ring, and we show how the three different types of center are related.[1]

3.1 DEFINITION. A *complement* of an element r in a so-ring R is an element r' of R such that $r + r' = 1$ and $rr' = 0 = r'r$.

3.2 OBSERVATION. Complements, when they exist, are unique.
PROOF. The well-known argument for distributive lattices works equally well for so-rings. For suppose r' exists and suppose there also exists s such that $r + s = 1$ and $rs = 0 = sr$. Then $r' = r'(r+s) = r'r + r's = r's = r's + rs = (r' + r)s = s$.

We hence write r' for the complement of r, when it exists. The uniqueness of complements implies that when r' exists, r'' exists and $r'' = r$.

3.3 DEFINITION. The *center* C of a so-ring R is $\{r \in R : r' \text{ exists}\}$.

3.4 EXAMPLES. The centers of $Pfn(D,D)$, $Mfn(D,D)$, and $Mset(D,D)$ are each isomorphic to the Boolean algebra of subsets of D. The isomorphism is such that each subset $\hat{D}$ of D corresponds to

(1) f in $Pfn(D,D)$ such that

$$\hat{d}f = \begin{cases} \hat{d}, & \text{if } \hat{d} \in \hat{D}; \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

(2) g in $Mfn(D,D)$ such that

$$\hat{d}g = \begin{cases} \{\hat{d}\}, & \text{if } \hat{d} \in \hat{D}; \\ \emptyset, & \text{otherwise.} \end{cases}$$

(3) h in $Mset(D,D)$ such that

$$\hat{e}(\hat{d}h) = \begin{cases} 1, & \text{if } \hat{d}, \hat{e} \in \hat{D} \text{ and } \hat{d} = \hat{e}; \\ 0, & \text{otherwise.} \end{cases}$$

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[1] See the "Boolean semirings" of Elgot [1979] and the "guard modules" of Manes [1985] for applications to the flow of control in programs.

In the following pages, we demonstrate the extent to which the centers of so-rings, posets, and semigroups are related.

3.5 DEFINITION. Let $(P, \leq)$ be a poset, and let $a \leq b$. The *interval with endpoints* a, b is

$$[a,b] = \{x \in P : a \leq x \text{ and } x \leq b\}.$$

In any poset with 0 and 1, $[0,1]$ is called the *unit interval*.

The center of a so-ring is contained within its unit interval. In $Pfn(D,D)$, $Mfn(D,D)$, and $Mset(D,D)$, the center coincides with the unit interval. However, this is not always the case.

3.6 COUNTEREXAMPLE. Let R be the complete distributive lattice consisting of all real numbers from 0 to 1, with supremum as the additive operation and with meet as the multiplicative operation. Then R is a so-ring which is its own unit interval. The center of R, however, consists only of 0 and 1.

3.7 LEMMA. Let R be a so-ring. For $n \geq 2$, let $r_1, \ldots, r_n$ be in $[0,1]$, and suppose $r_i r_j = 0$ for $j \neq i$. Then $r_1 + \cdots + r_n$ exists and is in $[0,1]$.

PROOF. We show the result by induction. Let $k = 2$. Since $r_1 \leq 1$ and $r_2 \leq 1$, then there exist h_1, h_2 such that $r_1 + h_1 = 1 = r_2 + h_2$. Hence, $r_1 = r_1(r_2 + h_2) = r_1r_2 + r_1h_2 = r_1h_2$, and $h_2 = (r_1 + h_1)h_2 = r_1h_2 + h_1h_2 = r_1 + h_1h_2$. By substitution, $1 = r_2 + h_2 = r_2 + (r_1 + h_1h_2) = (r_1 + r_2) + h_1h_2$. Thus, $r_1 + r_2$ exists and is ≤ 1.

Assume that for all $k \leq n$, $r_1 + \cdots + r_k$ exists and is in $[0,1]$. Consider the two summands $r_1 + \cdots + r_n$ and r_{n+1}. Since $(r_1 + \cdots + r_n)r_{n+1} = r_1r_{n+1} + \cdots + r_nr_{n+1} = 0$, $(r_1 + \cdots + r_n) + r_{n+1} = r_1 + \cdots + r_{n+1}$ exists and is in $[0,1]$.

The following theorem implies that if we consider the multiplicative structure of a so-ring as a semigroup, then the center of this semigroup contains the center of the so-ring.

3.8 THEOREM. Let R be a so-ring with center C, and let a be in C. Then the following hold:

(1) If $x \leq a$, then $ax = x = xa$ and $a'x = 0 = xa'$.

(2) a is idempotent.

(3) For all x in $[0,1]$, $ax = xa$.

PROOF. (1): Observe that $x = x(a + a') = xa + xa'$. Since $x \leq a$, there exists h such that $a = x + h$. Hence, $0 = aa' = (x + h)a' = xa' + ha'$, which implies that $xa' = 0$. Therefore, $x = xa$. Similarly, $a'x = 0$ and $x = ax$.

(2): If $x = a$, then by (1), $a = a^2$ and thus is idempotent.

(3): Let $x \leq 1$. Since $\leq$ is compatible, $xa \leq a$ and $ax \leq a$. Hence, by (1), $xa = a(xa) = (ax)a = ax$.

We now turn our attention to the relationship between the poset center and the so-ring center.

3.9 THEOREM. (Manes and Benson, 1985, 3.7) Let R be a so-ring. Then the center of R coincides with the poset center of $[0,1]$.

3.10 THEOREM. (Birkhoff, 1967, III.8.10) Let P be a poset with 0 and 1. Then the center of P is a Boolean lattice in which join and meet represent respectively least upper bound and greatest lower bound in P.

We thus observe that the center of a so-ring is a Boolean lattice, since it is identical to the center of the poset $([0,1], \leq)$ and since the center of any poset is a Boolean lattice. Furthermore, the center of a so-ring is a sublattice of $([0,1), \leq)$. Next, we show how the lattice operations relate to the so-ring operations in the center of a so-ring.

3.11 THEOREM. For a so-ring R with center C, the Boolean operations on C are:

$$r \wedge s = rs$$
$$r \vee s = rs + r's + rs' = r + r's.$$

PROOF. Let r, s be elements of C. As $r \wedge s \leq r$ and $r \wedge s \leq s$, we use theorem 3.8(1) to get $s(r \wedge s) = r \wedge s = r(r \wedge s) = r(s(r \wedge s)) = rs(r \wedge s)$. Hence,

$rs = rs((r \wedge s) + (r \wedge s)') = rs(r \wedge s) + rs(r \wedge s)' = (r \wedge s) + rs(r \wedge s)'$, which implies that $r \wedge s \leq rs$. Now, $r \leq 1$ and $s \leq 1$ imply that $rs \leq r$ and $rs \leq s$, since $\leq$ is compatible, and so $rs \leq r \wedge s$. Therefore, $rs = r \wedge s$, since $\leq$ is antisymmetric.

Define an operator $\sqcup$ such that $r \sqcup s = rs + r's + rs$. First, we show that $\sqcup$ is indeed well-defined and yields a result in C. We then show that $\sqcup = \vee$. Now, $1 = (r+r')(s+s') = rs+r's+rs'+r's' = (rs+r's+rs')+r's' = r\sqcup s+r's'$, so that $r\sqcup s$ is defined and is ≤ 1. In addition, $rs+r's+rs' = (rs+rs')+r's = r(s+s')+r's = r+r's$, so that $r+r's = r\sqcup s$. Since $r+r's \leq 1$ and $r's' \in C$, theorem 3.8(3) gives $(r+r's)r's' = r's'(r+r's) = r's'r + r's'r's = s'(r'r) + r'r'(s's) = 0$. Hence, $r's' = (r+r's)'$ and thus $r+r's$ is in C. Note that for u, v in C, $u \leq v$ if and only if $u \wedge v = u$, which implies by the fact $u \wedge v = uv$, that $u \leq v$ if and only if $uv = u$. Now, $r(r \sqcup s) = r(r+r's) = r^2 + (rr')s = r$, and so $r \leq r \sqcup s$. Similarly, $s \leq r \sqcup s$. Therefore, $r \vee s \leq r \sqcup s$. Let t be any element of C such that $r \leq t$ and $s \leq t$. Then $rt = r$ and $st = s$. Hence, $(r \sqcup s)t = (r+r's)t = rt + r'(st) = r + r's = r \sqcup s$. Thus, $r \sqcup s \leq t$ and in particular, $r \sqcup s \leq r \vee s$. Therefore, by the antisymmetry of $\leq$, $r \sqcup s = r \vee s$.

3.12 OBSERVATION. Let R be a so-ring with center C. Let r, s be elements of C such that $r \leq s$. Then there exists h in C such that $s = r + h$ and $rh = 0 = hr$.

PROOF. Since $r \leq s$, there is a k in R such that $s = r + k$. Thus, $0 = ss' = (r+k)s' = rs' + ks'$, and so $rs' = 0 = ks'$. Now, $s = (r+r')s = rs + r's = rs + r's + rs' = r \vee s = r + r's$. Clearly, $r's$ is in C, since r' and s are. Moreover, $r(r's) = 0$ and $(r's)r = s(r'r) = 0$. Therefore, define h to equal $r's$.

3.13 THEOREM. Let R be a so-ring with center C. For $n \geq 2$, let $r_1, \ldots, r_n$ be in R and suppose that $r_i r_j = 0$ for $j \neq i$. Then the following two statements are equivalent:

(1) r_i is in C for each i;

(2) $r_1 + \cdots + r_n$ exists and is in C.

When either of these conditions hold, $r_1 + \cdots + r_n = r_1 \vee \cdots \vee r_n$.

PROOF. (1) implies (2): r_i is in C implies that r_i is in $[0,1]$ and that r_i' exists.

By lemma 3.7, $r_1 + \cdots + r_n$ exists and is in $[0,1]$ for $n \geq 2$. A modification of the proof of lemma 3.7 can be used to show that $r_1 + \cdots + r_n$ is in C as follows. For $k = 2$, obtain $1 = (r_1+r_2)+h_1h_2$ as in lemma 3.7. In particular, we can let $h_1 = r_1'$ and $h_2 = r_2'$. Then, $r_1'r_2'(r_1 + r_2) = r_1'r_2'r_1 + r_1'r_2'r_2 = (r_1r_1')r_2' + r_1'(r_2'r_2) = 0$. Therefore, $(r_1 + r_2)'$ exists and equals $r_1'r_2'$, and so $r_1 + r_2$ is in C. By De Morgan's law, $r_1 \vee r_2 = (r_1' \wedge r_2')' = (r_1'r_2')' = r_1 + r_2$. Proceeding inductively, we have that $r_1 + \cdots + r_n = r_1 \vee \cdots \vee r_n$. Then $r_1 + \cdots + r_{n+1} = (r_1 + \cdots + r_n) + r_{n+1} = (r_1 \vee \cdots \vee r_n) \vee r_{n+1} = r_1 \vee \cdots \vee r_{n+1}$.

(2) implies (1): Since $r_1 + \cdots + r_n$ is in C, $s = (r_1 + \cdots + r_n)'$ exists. Thus, $(r_1 + \cdots + r_n) + s = 1$, and $0 = (r_1 + \cdots + r_n)s = r_1s + \cdots + r_ns$, which implies that $r_is = 0$ for all i. Similarly, $sr_i = 0$ for all i. Let $t_i = \sum(r_j : j \neq i)$. Then for each i, $r_i + (t_i + s) = 1$ and $r_i(t_i + s) = r_it_i + r_is = 0 = t_ir_i + sr_i = (t_i + s)r_i$. Hence, $t_i + s = r_i'$ so that r_i must be in C.

In the center of a so-ring, an element and its complement form a 2-partition in the sense of the following:

3.14 DEFINITIONS. An n-*partition* of a so-ring R is an n-tuple $(r_1, \ldots, r_n)$ of elements in R such that $r_ir_j = 0$ for $j \neq i$ and such that $\sum_{i=1}^n r_i$ exists and equals 1. Such a set of elements $r_1, \ldots, r_n$ is said to *partition* R. A set $r_1, \ldots, r_n$ in R is said to *cover* R, if $\sum_{i=1}^n r_i = 1$, whether or not $r_ir_j = 0$ for $j \neq i$.

If $(r_1, \ldots, r_n)$ is an n-partition in R, then since $r_1 + \cdots + r_n = 1$ is in C, each r_i is in C by theorem 3.13. This in turn implies that $\sum_{i=1}^n r_i = \bigvee_{i=1}^n r_i = 1$ and that $r_ir_j = r_i \wedge r_j = 0$ for $j \neq i$.

We will return to n-partitions when we discuss matrices over so-rings.

Domains and Ranges

Although the concepts of domain and range are usually associated with functions, they can be generalized to be applied to elements of any so-ring. The motivation stems from $Pfn(D,D)$, where, for a partial function f, the domain is $\{d \in D: df \text{ is defined}\}$, and the range is $\{df: d \in D \text{ and } df \text{ is defined}\}$. In example 3.4, it was shown that each subset of D corresponds to an element of the center C of $Pfn(D,D)$. Hence, the domain and the range of a partial function can each be identified with an element of C. For the domain, the element is the least r in C such that $rf = f$, while for the range, it is the least r in C such that $fr = f$. Alternatively, the domain is the least r in C with $r'f = 0$, and the range is the least r in C with $fr' = 0$. These formulations are equivalent, as we now show. If r is in C with $rf = f$, then $r'f = r'(rf) = 0$. If on the other hand, r is in C with $r'f = 0$, then $f = (r + r')f = rf + r'f = rf$. We arbitrarily opt for the first formulation in our formal definitions, but make use of both formulations in this chapter and in those that follow.

3.15 DEFINITIONS. In a so-ring R with center C, the *domain* of any x in R, denoted by $\overleftarrow{x}$, is the least element (if it exists) of $\{r \in C: rx = x\}$. The *range* of x, denoted by $\overrightarrow{x}$, is the least element (if it exists) of $\{r \in C: xr = x\}$.

Although in most so-rings each element has a domain and a range, it is possible to construct a so-ring in which not every element has a domain and a range. For instance,

3.16 COUNTEREXAMPLE. Let B be a non-atomic Boolean algebra. Let P be a non-principal prime filter in B. Let $B' = B \cup \{\infty\}$. Let $(x_i : i \in I)$ be any family in B', and define $\sum$ as follows:

$$\sum_{i\in I} x_i = \begin{cases} \bigvee_{i\in I} x_i, & \text{if } \forall i \in I,\ x_i \in B; \\ \infty, & \text{if } \exists j \in I \text{ such that } x_j = \infty \text{ and} \\ & \text{if } \forall i \neq j \text{ either } x_i = \infty \text{ or } x_i = 0; \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

Let x, y be elements of B', and define $\circ$ as follows:

$$x \circ y = \begin{cases} x \wedge y, & \text{if } x,\, y \in B; \\ \infty, & \text{if } y = \infty \text{ and either } x \in P \text{ or } x = \infty; \\ 0, & \text{otherwise.} \end{cases}$$

With these definitions of $\sum$ and $\circ$, $(B', \sum, \circ, 1)$ is a partial semiring. The verification of this fact is straightforward but somewhat lengthy, and so is left to the reader. We show that ∞ does not have a domain, proceeding as follows. First, we note that for r in B, $r \in P$ if and only if $r' \in B - P$. We also note that $B - P = \{r' \in B : r'\infty = 0\}$. Hence, $P = \{r \in B : r'\infty = 0\}$. Thus, $\overleftarrow{\infty}$ is the least element of P, if it exists. But P is not principal and so has no least element. Therefore, ∞ has no domain.

3.17 DEFINITION. A so-ring R has *domains and ranges* if both $\overleftarrow{x}$ and $\overrightarrow{x}$ exist for each x in R.

The following definitions apply to those so-rings which do possess domains and ranges for each of their elements.

3.18 DEFINITION. In a so-ring R with domains and ranges *sums* are *disjoint* if the summability of a family $(x_i : i \in I)$ in R implies that $\overleftarrow{x_i}\,\overleftarrow{x_j} = 0$ for $j \neq i$.

3.19 DEFINITION. A so-ring R with domains and ranges is *adequate* if $xy = 0$ implies that $\overrightarrow{x}\,\overleftarrow{y} = 0$ for x, y in R.

3.20 EXAMPLE. The so-ring $Pfn(D, D)$ possesses the three properties described in definitions 3.17, 3.18, and 3.19. At the beginning of this section, we established the fact that $Pfn(D, D)$ has domains and ranges. Hence, using this fact and example 2.7, we can deduce that in $Pfn(D, D)$ sums are disjoint. Now, let x, y be elements of $Pfn(D, D)$ such that $xy = 0$. If $\overrightarrow{x}\,\overleftarrow{y} \neq 0$, then there exists d in D such that $d \in \overrightarrow{x}$ and $d \in \overleftarrow{y}$. This in turn implies that there exists d_1 in D such that $d_1 x = d$ and d_2 in D such that $dy = d_2$. Thus, $d_1 xy = dy = d_2$, and so $xy \neq 0$, which is a contradiction. Therefore, $\overrightarrow{x}\,\overleftarrow{y} = 0$, and so $Pfn(D, D)$ is adequate.

We now prove some basic properties of domains and ranges. In many instances, we supply only the proof for the domain, since the proof for the range is identical to the proof for the domain in the opposite so-ring (definition 2.32). The following observations generalize some familiar properties of partial functions.

3.21 OBSERVATION. Let R be a so-ring, and let x be an element of R. Then
(1) if $\overleftarrow{x}$ exists, then $\overleftarrow{x}\,x = x$ and $\overleftarrow{x}'x = 0$;
(2) if $\overrightarrow{x}$ exists, then $x\overrightarrow{x} = x$ and $x\overrightarrow{x}' = 0$.
PROOF. (1): Let C be the center of R. Since $\overleftarrow{x}$ is in C, $\overleftarrow{x}'$ exists. Thus, $x = (\overleftarrow{x} + \overleftarrow{x}')x = \overleftarrow{x}\,x + \overleftarrow{x}'x$. Since $\overleftarrow{x}$ is the least element of $\{r \in C\colon r'x = 0\}$, $\overleftarrow{x}'x = 0$. Hence, $\overleftarrow{x}\,x = x$.

3.22 OBSERVATION. Let R be a so-ring. Let x, y be in R, and let $x + y$ be defined. Then
(1) if $\overleftarrow{x}$, $\overleftarrow{y}$ exist, then $\overleftarrow{x+y}$ exists and equals $\overleftarrow{x} \vee \overleftarrow{y}$;
(2) if $\overrightarrow{x}$, $\overrightarrow{y}$ exist, then $\overrightarrow{x+y}$ exists and equals $\overrightarrow{x} \vee \overrightarrow{y}$.
PROOF. (1): Let C be the center of R. Let $X = \{r \in C\colon r'x = 0\}$, let $Y = \{r \in C\colon r'y = 0\}$, and let $Z = \{r \in C\colon r'(x+y) = 0\}$. Since $0 = r'(x+y) = r'x + r'y$ implies that $r'x = 0 = r'y$, we can rewrite Z as $\{r \in C\colon r'x = 0 = r'y\} = X \cap Y$. By assumption, the least element of X is $\overleftarrow{x}$, and the least element of Y is $\overleftarrow{y}$. Since both $\overleftarrow{x}$ and $\overleftarrow{y}$ are in C, $\overleftarrow{x} \vee \overleftarrow{y}$ is the least element t of C such that both $\overleftarrow{x} \le t$ and $\overleftarrow{y} \le t$. To see that $\overleftarrow{x} \vee \overleftarrow{y}$ is in Z, observe that $(\overleftarrow{x} \vee \overleftarrow{y})' = \overleftarrow{x}' \wedge \overleftarrow{y}' = x'y'$. Now, $x'y'(x+y) = x'y'x + x'y'y = (x'x)y' + x'(y'y) = 0$. Hence, $\overleftarrow{x} \vee \overleftarrow{y}$ must be in Z, and so $\overleftarrow{x+y} = \overleftarrow{x} \vee \overleftarrow{y}$.

3.23 OBSERVATION. Let R be a so-ring, and let $(x_i\colon i \in I)$ be a summable family in R.
(1) If $\overleftarrow{x_i}$ exists for all i in I, then $\overleftarrow{\sum_i x_i}$ exists if and only if $\bigvee_i \overleftarrow{x_i}$ exists, and in either case $\overleftarrow{\sum_i x_i} = \bigvee_i \overleftarrow{x_i}$.
(2) If $\overrightarrow{x_i}$ exists for all i in I, then $\overrightarrow{\sum_i x_i}$ exists if and only if $\bigvee_i \overrightarrow{x_i}$ exists, and

in either case $\overrightarrow{\sum_i x_i} = \bigvee_i \overrightarrow{x_i}$.

PROOF. (1): Let C be the center of R. Suppose that $\overleftarrow{x_i}$ exists for all i in I. The least element t of C such that $\overleftarrow{x_i} \leq t$ is $\bigvee_i \overleftarrow{x_i}$, if it exists. The least element u of the following set

$$\begin{aligned}
X &= \{r \in C : r' \sum_i x_i = 0\} \\
&= \{r \in C : \sum_i r' x_i = 0\} \\
&= \{r \in C : r' x_i = 0 \text{ for all } i \in I\} \\
&= \bigcap_i \{r \in C : r' x_i = 0\} \\
&= \bigcap_i \{r \in C : r x_i = x_i\}
\end{aligned}$$

is $\overleftarrow{\sum_i x_i}$, if it exists. Now, $\overleftarrow{x_i}$ is the least element r of C such that $rx_i = x_i$. Hence, for each i in I, $\overleftarrow{x_i} \leq u$, if u exists. Thus, u is the least element of C which is greater than or equal to $\overleftarrow{x_i}$ for each i in I. But this means that $u = t$. Therefore, if either $\bigvee_i \overleftarrow{x_i}$ or $\overleftarrow{\sum_i x_i}$ exists, then so does the other, in which case $\overleftarrow{\sum_i x_i} = \bigvee_i \overleftarrow{x_i}$.

3.24 OBSERVATION. Let R be a so-ring in which sums are disjoint. If $x_1, \ldots, x_n$ are elements of R such that $\sum_{i=1}^n x_i$ exists, then $\overleftarrow{\sum_{i=1}^n x_i} = \sum_{i=1}^n \overleftarrow{x_i}$.

PROOF. Let C be the center of R. Since sums are disjoint in R, $\overleftarrow{x_i}\,\overleftarrow{x_j} = 0$ for $j \neq i$. Each $\overleftarrow{x_i}$ is in C, and so by theorem 3.13, $\sum_{i=1}^n \overleftarrow{x_i}$ exists. Furthermore,

$$\begin{aligned}
\sum_{i=1}^n \overleftarrow{x_i} &= \bigvee_{i=1}^n \overleftarrow{x_i}, \text{ by theorem 3.13} \\
&= \overleftarrow{\textstyle\sum_{i=1}^n x_i}, \text{ by observation 3.23(1).}
\end{aligned}$$

3.25 OBSERVATION. Let R be a so-ring, and let x, y be elements of R such that $x \leq y$. Then

(1) if $\overleftarrow{x}$ and $\overleftarrow{y}$ exist, then $\overleftarrow{x} \leq \overleftarrow{y}$;

(2) if $\overrightarrow{x}$ and $\overrightarrow{y}$ exist, then $\overrightarrow{x} \leq \overrightarrow{y}$.

PROOF. (1): Let C be the center of R. Let $X = \{r \in C : r'x = 0\}$, and let $Y = \{r \in C : r'y = 0\}$. Since $x \leq y$, there exists h such that $y = x + h$. Thus, $Y = \{r \in C : r'(x+h) = 0\} = \{r \in C : r'x + r'h = 0\} = \{r \in C : r'x = 0 = r'h\}$. Hence, $Y \subseteq X$, and so $\overleftarrow{y}$ is in X. Thus, since $\overleftarrow{x}$ is the least element of X, $\overleftarrow{x} \leq \overleftarrow{y}$.

3.26 OBSERVATION. Let R be a so-ring, and let x, y be in R. Then

(1) if $\overleftarrow{x}$ and $\overleftarrow{xy}$ exist, then $\overleftarrow{xy} \leq \overleftarrow{x}$;

(2) if $\overrightarrow{y}$ and $\overrightarrow{xy}$ exist, then $\overrightarrow{xy} \leq \overrightarrow{y}$.

PROOF. (1): Let C be the center of R. Let $Z = \{r \in C : rxy = xy\}$. By observation 3.21(1), $\overleftarrow{x}\,xy = xy$, so that $\overleftarrow{x}$ is an element of Z. However, $\overleftarrow{xy}$ is the least element of Z. Hence, $\overleftarrow{xy} \leq \overleftarrow{x}$.

3.27 COROLLARY. Let R be a so-ring with commutative multiplicative operation. Let x, y be in R. Then

(1) if $\overleftarrow{x}$, $\overleftarrow{y}$, and $\overleftarrow{xy}$ exist, then $\overleftarrow{xy} \leq \overleftarrow{x}\,\overleftarrow{y}$;

(2) if $\overrightarrow{x}$, $\overrightarrow{y}$, and $\overrightarrow{xy}$ exist, then $\overrightarrow{xy} \leq \overrightarrow{x}\,\overrightarrow{y}$.

PROOF. (1): We immediately have that

$$\begin{aligned}
\overleftarrow{xy} &= \overleftarrow{xy}\,\overleftarrow{xy}, \text{ by theorem 3.8(3)} \\
&= \overleftarrow{xy}\,\overleftarrow{yx}, \text{ since } \circ \text{ is commutative} \\
&\leq \overleftarrow{x}\,\overleftarrow{y}, \text{ by observation 3.26(1).}
\end{aligned}$$

3.28 OBSERVATION. Let R be an adequate so-ring. Then for x, y in R

(1) $\overleftarrow{xy} = \overleftarrow{x\overleftarrow{y}}$;

(2) $\overrightarrow{xy} = \overrightarrow{\overrightarrow{x}y}$.

PROOF. (1): First, we note that

$$\begin{aligned}
\overleftarrow{xy} &= \overleftarrow{x\overleftarrow{y}}\,y \text{ by observation 3.21(1)} \\
&\leq \overleftarrow{x\overleftarrow{y}} \text{ by observation 3.26(1).}
\end{aligned}$$

Let C be the center of R. Let $Z = \{r \in C : r'x\overleftarrow{y} = 0\}$. Then $\overleftarrow{x\overleftarrow{y}}$ is the least element of Z. We show that $\overleftarrow{xy}$ is in Z. By observation 3.21(1), $\overleftarrow{xy}'xy = 0$. Thus,

$$
\begin{aligned}
0 &= \overrightarrow{\overleftarrow{xy}'}x\overleftarrow{y}, \text{ since } R \text{ is adequate} \\
&= \overleftarrow{xy}'x\overrightarrow{\overleftarrow{xy}'}x\overleftarrow{y} \\
&= \overleftarrow{xy}'x\overleftarrow{y}, \text{ by observation 3.21(2).}
\end{aligned}
$$

Hence, $\overleftarrow{xy}$ is in Z, and so $\overleftarrow{x\overleftarrow{y}} \leq \overleftarrow{xy}$. Therefore, by the antisymmetry of $\leq$, $\overleftarrow{xy} = \overleftarrow{x\overleftarrow{y}}$.

3.29 OBSERVATION. Let R be a so-ring with center C. If x is an element of C, then $\overleftarrow{x} = x = \overrightarrow{x}$.

PROOF. Let $X = \{r \in C : r'x = 0\}$. Let r be any element in X. Then $x = (r + r')x = rx + r'x = rx = r \wedge x$, by theorem 3.11. Thus, $x \leq r$ for all r in X. Note that x is in X, and so is the least element of X. Therefore, $x = \overleftarrow{x}$. Similarly, it can be shown that $x = \overrightarrow{x}$.

3.30 OBSERVATION. Let R be a so-ring, and let x be in $[0,1]$. If $\overleftarrow{x}$ and $\overrightarrow{x}$ exist, then $\overleftarrow{x} = \overrightarrow{x}$.

PROOF. Let C be the center of R. Since x is in $[0,1]$ and $\overleftarrow{x}$ and $\overrightarrow{x}$ are in C, we may apply theorem 3.8(3) and observation 3.21(1) to get $x\overleftarrow{x} = \overleftarrow{x}x = x = x\overrightarrow{x} = \overrightarrow{x}x$. Then, by applying observations 3.26 and 3.29, we have that $\overleftarrow{x} = \overleftarrow{\overrightarrow{x}x} \leq \overleftarrow{\overrightarrow{x}} = \overrightarrow{x}$ and that $\overrightarrow{x} = \overrightarrow{x\overleftarrow{x}} \leq \overrightarrow{\overleftarrow{x}} = \overleftarrow{x}$. Therefore, by the antisymmetry of $\leq$, $\overleftarrow{x} = \overrightarrow{x}$.

3.31 OBSERVATION. Let R be an adequate so-ring with center C. If r is in C, then

(1) $\overleftarrow{rx} = r\overleftarrow{x}$;

(2) $\overrightarrow{xr} = \overrightarrow{x}r$.

PROOF. (1): The conditions on R are the same as those in observation 3.28, and so we may draw the conclusions of observation 3.28(1), namely that $\overleftarrow{rx} = \overleftarrow{r\overleftarrow{x}}$. Furthermore, since r and $\overleftarrow{x}$ are in C, then their product $r\overleftarrow{x}$ is also in C. This implies, by observation 3.29, that $\overleftarrow{r\overleftarrow{x}} = r\overleftarrow{x}$. Hence, $\overleftarrow{rx} = r\overleftarrow{x}$.

Inverses

3.32 DEFINITION. An element x of a monoid M with unit 1 is said to be *invertible* if there exists a (necessarily unique) element x^{-1} in M such that $xx^{-1} = 1 = x^{-1}x$. In this case, x^{-1} is called the *inverse* of x.

3.33 DEFINITIONS. An element x of a semigroup S is called *regular* if there exists y in S such that $xyx = x$. An element x^{-1} in S is an *inverse* of x, if $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. If S has a multiplicative unit 1 and if $xx^{-1} = 1 = x^{-1}x$, then x is said to be *invertible*.

In a semigroup S, if x is invertible, then clearly x^{-1} is the unique inverse of x. Furthermore, any element with an inverse is regular. Conversely, every regular element has an inverse, for if $xyx = x$, then define $x^{-1} = yxy$. However, an inverse of an element in S need not be unique.

3.34 COUNTEREXAMPLE. Consider the semigroup of partial functions from D to D under functional composition. Let d_1, d_2 be elements of D. Define two partial functions x and y such that for d in D

$$dx = \begin{cases} d_1, & \text{if } d = d_1 \text{ or } d = d_2; \\ \text{undefined}, & \text{otherwise;} \end{cases}$$

and

$$dy = \begin{cases} d_1, & \text{if } d = d_1; \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

Observe that $xxx = x$ and so x is an inverse to itself. Also, observe that $xyx = x$ and that $yxy = y$. Hence, y is an inverse of x as well.

We now introduce the concept of multiplicative inverse in a so-ring which (see observation 3.36 below) lies between the concepts of inverse in a monoid and inverse in a semigroup in its generality.

3.35 DEFINITIONS. An *inverse* of an element x of a so-ring R with center C is an element x^{-1} of R for which there exist r, s in C such that the following equations are satisfied:

$$\begin{aligned} xx^{-1} + r = 1 = x^{-1}x + s \\ x^{-1}r = rx = 0 = xs = sx^{-1}. \end{aligned} \tag{3-1}$$

If such x^{-1}, r, and s exist, then x is *inversible*.

The motivation for defining inversibility in this way comes from considering matrix invertibility, as follows. For clarity we consider 2×2 matrices over a so-ring R. Let A, B be two such matrices, where in addition, $AB = I$ (the identity matrix) $=$ BA. These two matrix multiplications result in the following equations (where a_{ij} is the entry in row i and column j of A).

$$\begin{aligned} a_{11}b_{11} + a_{12}b_{21} &= 1 = b_{11}a_{11} + b_{12}a_{21} \\ a_{21}b_{12} + a_{22}b_{22} &= 1 = b_{21}a_{12} + b_{22}a_{22} \\ a_{21}b_{11} + a_{22}b_{21} &= 0 = a_{11}b_{12} + a_{12}b_{22} \\ b_{21}a_{11} + b_{22}a_{21} &= 0 = b_{11}a_{12} + b_{12}a_{22}. \end{aligned}$$

These equations imply that

$$\begin{aligned} a_{11}b_{11} + a_{12}b_{21} &= 1 = b_{11}a_{11} + b_{12}a_{21} \\ b_{11}(a_{12}b_{21}) = (a_{12}b_{21})a_{11} &= 0 = a_{11}(b_{12}a_{21}) = (b_{12}a_{21})b_{11} \end{aligned}$$

so that b_{11} is an inverse of a_{11}. In general, b_{ji} is an inverse of a_{ij} for $n \times n$ square matrices A, B over R where $AB = I = BA$.

3.36 OBSERVATION. Let R be a so-ring, and let x be an element of R. Then the following hold:

(1) If x is invertible, then x is inversible.

(2) If x is inversible, then x is regular.

(3) When they exist, x^{-1}, r, and s are unique in equations (3-1).

(4) If x is inversible, then x has a domain and a range; indeed, $\overleftarrow{x} = xx^{-1}$ and $\overrightarrow{x} = x^{-1}x$.

PROOF. (1): If x is invertible, then $xx^{-1} = 1 = x^{-1}x$. To satisfy equations (3-1), let $r = 0 = s$.

(2): Since x is inversible, $xx^{-1} + r = 1 = x^{-1}x + s$ and $xs = 0 = x^{-1}r$. Thus, $x = x(x^{-1}x + s) = xx^{-1}x + xs = xx^{-1}x$, and $x^{-1} = x^{-1}(xx^{-1} + r) = x^{-1}xx^{-1} + x^{-1}r = x^{-1}xx^{-1}$.

(3): Suppose there exist $\bar{x}$, $\bar{r}$, $\bar{s}$ such that $x\bar{x}+\bar{r} = 1 = \bar{x}x+\bar{s}$ and $\bar{x}\bar{r} = x\bar{s} = 0 = \bar{r}x = \bar{s}\bar{x}$. Then

$$\begin{aligned} \bar{x} &= (x^{-1}x + s)\bar{x} \\ &= x^{-1}x\bar{x} + s\bar{x} \\ &= x^{-1}x\bar{x}(xx^{-1} + r) + s\bar{x} \\ &= x^{-1}(x\bar{x}x)x^{-1} + x^{-1}x\bar{x}r + s\bar{x} \\ &= x^{-1}xx^{-1} + x^{-1}x\bar{x}r + s\bar{x} \\ &= x^{-1} + x^{-1}x\bar{x}r + s\bar{x}. \end{aligned}$$

Hence, $x^{-1} \leq \bar{x}$. By a similar argument, $\bar{x} \leq x^{-1}$. Therefore, $\bar{x} = x^{-1}$. Since r is the complement of xx^{-1} and s is the complement of $x^{-1}x$, both r and s are also unique.

(4): Let C be the center of R. Let $X = \{q \in C : q'x = 0\}$. Since r is the complement of xx^{-1}, $(xx^{-1})'x = rx = 0$, and thus xx^{-1} is in X. Let t be any element of X. Then $t'x = 0$, and so $x = (t + t')x = tx + t'x = tx$. This implies that $t \wedge xx^{-1} = (tx)x^{-1} = xx^{-1}$, which in turn implies that $xx^{-1} \leq t$. Hence, $xx^{-1} = \overleftarrow{x}$. Dually, $x^{-1}x = \overrightarrow{x}$.

3.37 EXAMPLES. In $Pfn(D,D)$, $Mfn(D,D)$, and $Mset(D,D)$, the inversible elements correspond to the injective partial functions. The inverse of such an element in $Pfn(D,D)$ or in $Mfn(D,D)$ is the usual functional inverse. In $Mset(D,D)$, the inversible elements are the 0-1 matrices which correspond to the injective partial functions, namely those 0-1 matrices with at most one 1 per row and per column. Hence, by example 2.46, we have that inversible elements of $Pfn(D,D)$, $Mfn(D,D)$, and $Mset(D,D)$ can be represented by $D\times D$ 0-1 matrices with at most one 1 per row and per column.

3.38 OBSERVATION. If x and y are inversible elements in a so-ring R, then xy is inversible and $(xy)^{-1}=y^{-1}x^{-1}$.

PROOF. Let C be the center of R. Since x and y are inversible, the following equations hold for some r, s, u, v in C:

$$xx^{-1}+r=1=x^{-1}x+s \qquad yy^{-1}+u=1=y^{-1}y+v$$
$$x^{-1}r=xs=0=rx=sx^{-1} \qquad y^{-1}u=yv=0=uy=vy^{-1}.$$

Then

$$1=x(yy^{-1}+u)x^{-1}+r=xyy^{-1}x^{-1}+(xux^{-1}+r)$$

and

$$1=y^{-1}(x^{-1}x+s)y+v=y^{-1}x^{-1}xy+(y^{-1}sy+v).$$

We must prove that

$$(xux^{-1}+r)xy=y^{-1}x^{-1}(xux^{-1}+r)=0=(y^{-1}sy+v)y^{-1}x^{-1}=xy(y^{-1}sy+v).$$

However, we only show that $(xux^{-1}+r)xy=0=xy(y^{-1}sy+v)$, since the other two equations can be gotten by symmetry. Now, $(xux^{-1}+r)xy=xux^{-1}xy+(rx)y=x(ux^{-1}xy)$. Since $0=uy=u(x^{-1}x+s)y=ux^{-1}xy+usy$, $ux^{-1}xy=0$. Hence, $xux^{-1}xy=0$. Also, $xy(y^{-1}sy+v)=xyy^{-1}sy+x(yv)=(xyy^{-1}s)y$. Since $0=xs=x(yy^{-1}+u)s=xyy^{-1}s+xus$, $xyy^{-1}s=0$. Hence, $xyy^{-1}sy=0$. Therefore, the inversibility equations for xy are satisfied, and $(xy)^{-1}=y^{-1}x^{-1}$.

3.39 DEFINITION. An *inverse semiring* is a so-ring in which every element is inversible.

3.40 EXAMPLE. The so-ring of nonnegative real numbers, as defined in example 2.30, is an inverse semiring.

Inverse Semigroups

There exist certain relationships between inverse semigroups and the set of inversible elements of a so-ring. We describe some of these below, beginning with

3.41 DEFINITION. (Howie, 1976, V.1) An *inverse semigroup* S is a semigroup in which for every x in S there exists a unique x^{-1} such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$.

Whereas the set of partial functions from a set to itself is an important example of a so-ring, a motivating example for the study of inverse semigroups has been the set of injective partial functions from a set to itself. In this case, the inverse semigroup arises as the set of inversible elements of the so-ring. More generally, we have

3.42 THEOREM. The set of inversible elements of a so-ring R is an inverse semigroup, and the two notions of inverse coincide.

PROOF. From observation 3.38, we conclude that the set of inversible elements of a so-ring R is a submonoid of the monoid $(R, \circ, 1)$, and so it is a semigroup. Let x, y be two inversible elements of R such that $xyx = x$ and $yxy = y$. We must show that such a y is unique. Let x^{-1}, y^{-1} be the so-ring inverses of x and y. Then

$$\begin{aligned} xy &= xx^{-1}xy, \text{ by observation 3.36(2)} \\ &= x(xyx)^{-1}xy \\ &= xx^{-1}y^{-1}x^{-1}xy, \text{ by observation 3.38} \\ &= (xx^{-1})(xy)^{-1}(xy), \text{ by observation 3.38} \\ &\leq (xy)^{-1}(xy), \text{ since } \leq \text{ is compatible} \\ &\leq 1. \end{aligned}$$

Hence, by observation 3.8(3),

$$xx^{-1} = xyxx^{-1} = xx^{-1}xy = xy.$$

Similarly, $yx = yy^{-1}$. Thus, by the compatibility of $\leq$, we have that

$$y = yxy = yxx^{-1} = yy^{-1}x^{-1} \leq x^{-1},$$

and

$$x^{-1} = x^{-1}xx^{-1} = x^{-1}xy \leq y.$$

Therefore, by the antisymmetry of $\leq$, $y = x^{-1}$ and so is unique.

In the study of inverse semigroups, the structure of the semilattice of idempotents of an inverse semigroup is used in classifying the structure of the inverse semigroup itself. The next theorem shows that the center of a so-ring plays a role analogous to the semilattice of idempotents of an inverse semigroup.

3.43 THEOREM. The center C of a so-ring R is the set of its idempotent inversible elements.

PROOF. Let u be in C. By theorem 3.8(2), u is idempotent. The inversibility equations (3-1) can be satisfied by letting $x = u = x^{-1}$ and $r = u' = s$. Thus, u is inversible and $u = u^{-1}$.

Let $u = u^2$, and let u be inversible. By observation 3.38, $u^{-1}u^{-1} = (u^2)^{-1} = u^{-1}$, and so u^{-1} is idempotent. Hence, $(uu^{-1})(u^{-1}u) = u(u^{-1})^2u = uu^{-1}u = u$. By the compatibility of $\leq$, $(uu^{-1})(u^{-1}u) \leq uu^{-1}$. Thus, $u \leq uu^{-1}$. Again, using compatibility, we get $uu^{-1} = u^2u^{-1} = u(uu^{-1}) \leq u$. Therefore, by the antisymmetry of $\leq$, $u = uu^{-1}$, and is thus in C.

3.44 COROLLARY. The set of inversible elements in the unit interval of a so-ring R is a Clifford semigroup, that is, a semigroup in which all idempotents commute with all elements of the semigroup.

PROOF. Clearly, the unit interval $[0,1]$ of R is a subsemigroup of the semigroup $(R,\circ)$. Likewise, the set of inversible elements in $[0,1]$, call it I, is a subsemigroup of $([0,1],\circ)$. Let e be an idempotent in I. Since e is an inversible idempotent, it is in the center of R, by theorem 3.43. Let x be any element of I. Then by theorem 3.8(3), $ex = xe$. Therefore, I is a Clifford semigroup.

We conclude this section with a pair of results on the conditions for inversibility.

3.45 OBSERVATION. Let R be a so-ring. Then for any x in R, the following two statements are equivalent:

(1) x is inversible and $x = x^{-1}$;

(2) $\overleftarrow{x}$, $\overrightarrow{x}$ exist and $\overleftarrow{x} = x^2 = \overrightarrow{x}$.

PROOF. (1) implies (2): By observation 3.36(4), we have that $\overleftarrow{x} = xx^{-1} = x^2 = x^{-1}x = \overrightarrow{x}$.

(2) implies (1): Let C be the center of R. Since $\overleftarrow{x} = x^2 = \overrightarrow{x}$, x^2 is in C. Hence, $u = (x^2)'$ exists. Thus, $x^2 + u = 1$, and $x^2u = 0 = ux^2$. However, to satisfy the inversibility equations, we need to show that $xu = 0 = ux$. Now, $xu = x(x^2)' = x\overrightarrow{x}' = 0 = \overleftarrow{x}'x = (x^2)'x = ux$. Therefore, x is inversible and $x^{-1} = x$.

3.46 OBSERVATION. Let R be a so-ring. Let x, y be inversible elements of R such that $x = x^{-1}$, $y = y^{-1}$, $\overleftarrow{x}\overleftarrow{y} = 0$, and $x+y$ exists. Then $x+y$ is inversible and $(x+y)^{-1} = x+y$.

PROOF. (Note that $\overleftarrow{x}$ and $\overleftarrow{y}$ must exist by observation 3.36(4).) Let C be the center of R. First, $\overleftarrow{x} = x^2 = \overrightarrow{x}$ and $\overleftarrow{y} = y^2 = \overrightarrow{y}$, by observation 3.45. Then, by observation 3.22, $\overleftarrow{x+y} = \overleftarrow{x} \vee \overleftarrow{y} = \overrightarrow{x} \vee \overrightarrow{y} = \overrightarrow{x+y}$. Moreover, since $0 = \overleftarrow{x}\overleftarrow{y} = \overrightarrow{x}\overrightarrow{y}$, $\overleftarrow{x+y} = \overleftarrow{x} \vee \overleftarrow{y} = \overleftarrow{x} + \overleftarrow{y} = \overrightarrow{x} + \overrightarrow{y} = \overrightarrow{x} \vee \overrightarrow{y} = \overrightarrow{x+y}$, by theorem 3.13. Furthermore, $0 = \overleftarrow{x}\overleftarrow{y} = \overleftarrow{x}\overleftarrow{y}y = \overleftarrow{x}y = \overrightarrow{x}y = x\overrightarrow{x}y = xy$. Similarly, $yx = 0$. Thus,

$$\begin{aligned}(x+y)(x+y) &= x^2 + yx + xy + y^2 \\ &= \overleftarrow{x} + \overleftarrow{y} \\ &= \overleftarrow{x+y}.\end{aligned}$$

Hence, $(x+y)^2$ is in C, with $(x+y)^{2'} = \overleftarrow{x+y}$. Thus, $(x+y)^2 + \overleftarrow{x+y}' = 1$, $\overleftarrow{x+y}'(x+y)^2 = 0$, and $(x+y)^2\overleftarrow{x+y}' = 0$. In order to satisfy the inversibility equations, we must show that $\overleftarrow{x+y}'(x+y) = 0 = (x+y)\overleftarrow{x+y}'$. The first equality is immediate; the second follows from the fact that $\overleftarrow{x+y} = \overrightarrow{x+y}$. Thus, the inversibility equations for $x+y$ have been satisfied, and $(x+y)^{-1}=x+y$.

The Multiplicative Ordering

Up until this point, we have only considered the sum-ordering on the elements of a so-ring. There are, however, other orderings which we may impose on the elements of a so-ring, one of which we describe below. This ordering is a generalisation to so-rings of the natural ordering on the elements of an inverse semigroup [Howie, 1976, V.2].

3.47 DEFINITION. The *multiplicative ordering* on a so-ring R with center C is the binary relation $\sqsubseteq$ such that if x, y are in R, then $x \sqsubseteq y$ if and only if there exists e in C such that $x = ey$.

3.48 OBSERVATION. In a so-ring R, the multiplicative ordering $\sqsubseteq$ is a right compatible partial order.

PROOF. Right compatibility is proved as follows. Let C be the center of R. Let x, y be elements of R such that $x \sqsubseteq y$. Then there exists e in C such that $x = ey$. For any z in R, $xz = eyz$ and so $xz \sqsubseteq yz$. It is also easily shown that the multiplicative ordering is a partial order. Clearly, for any x in R, $x = 1x$ and so $\sqsubseteq$ is reflexive. Antisymmetry and transitivity are a direct consequence of these properties of the natural ordering in an inverse semigroup [Howie, 1976, V.2.1], since neither property depends upon the existence of inverses.

We note that in an inverse semiring the multiplicative ordering is left compatible as well as right compatible. This follows directly from the compatibility of $\sqsubseteq$ in an inverse semigroup [Howie, 1976, V.2.4]. However, in general, the multiplicative ordering need not be left compatible.

3.49 EXAMPLES. In $Mfn(D,D)$ and in $Mset(D,D)$, the multiplicative ordering fails to be left compatible. Let d_1, d_2, and d_3 be three elements of D, and define

x, y in $Mfn(D,D)$ such that

$$dx = \begin{cases} \{d_1\}, & \text{if } d = d_1; \\ \emptyset, & \text{otherwise;} \end{cases}$$

and

$$dy = \begin{cases} \{d_1\}, & \text{if } d = d_1; \\ \{d_2\}, & \text{if } d = d_2; \\ \emptyset, & \text{otherwise.} \end{cases}$$

Recall (example 3.4(2)) that the center of $Mfn(D,D)$ is isomorphic to the set of subsets of D. Hence, both x and y are in the center, and furthermore, $x = xy$ which implies that $x \sqsubseteq y$. Define z in $Mfn(D,D)$ such that

$$dz = \begin{cases} \{d_1, d_2\}, & \text{if } d = d_3; \\ \emptyset, & \text{otherwise.} \end{cases}$$

Thus,

$$dzx = \begin{cases} \{d_1\}, & \text{if } d = d_3; \\ \emptyset, & \text{otherwise;} \end{cases}$$

and

$$dzy = \begin{cases} \{d_1, d_2\}, & \text{if } d = d_3; \\ \emptyset, & \text{otherwise.} \end{cases}$$

However, we see that there is no e in the center such that $zx = ezy$. Therefore, $\sqsubseteq$ is not left compatible in $Mfn(D,D)$. A similar example can be constructed for $Mset(D,D)$.

The next few observations in part answer the question of how the multiplicative ordering is related to the sum-ordering. In $Pfn(D,D)$, $x \leq y$ and $x \sqsubseteq y$ are each equivalent to saying that x is obtained from y by restricting the domain of y to that of x. Hence, the multiplicative ordering and the sum-ordering coincide in $Pfn(D,D)$. In $Mfn(D,D)$ and in $Mset(D,D)$, the multiplicative ordering is not left compatible (example 3.49) whereas the sum-ordering is left compatible (observation 2.37). Thus, the two orderings are not identical in $Mfn(D,D)$ and in $Mset(D,D)$.

3.50 OBSERVATION. For x, y in a so-ring R, $x \sqsubseteq y$ implies that $x \leq y$.

PROOF. Let C be the center of R. Since $x \sqsubseteq y$, there exists e in C such that $x = ey$. Thus, $y = (e + e')y = ey + e'y = x + e'y$, which implies that $x \leq y$.

However, the converse need not hold. Although it does hold in $Pfn(D, D)$, it does not hold in $Mfn(D, D)$ nor in $Mset(D, D)$. Referring again to example 3.49, we note that $zx \leq zy$, but that $zx \not\sqsubseteq zy$. The following result makes precise how the sum-ordering needs to be strengthened so as to imply the multiplicative ordering.

3.51 OBSERVATION. Let R be a so-ring with center C. Let x, y be two elements of R. Then the following two conditions are equivalent:

(1) $x \sqsubseteq y$;

(2) there exist h in R and e in C such that $y = x + h$ and $eh = 0 = e'x$.

PROOF. (1) implies (2): If $x \sqsubseteq y$, then in the proof of observation 3.50, let $h = e'y$.

(2) implies (1): Suppose there exists h in R and e in C with $y = x + h$ and $eh = 0 = e'x$. Then $y = (e+e')y = (e+e')(x+h) = ex + eh + e'x + e'h = ex + e'h$. Now, $ey = e(ex + e'h) = e^2x + ee'h = ex$. But $ex = ex + e'x = (e + e')x = x$. Hence, $x = ey$ and so $x \sqsubseteq y$.

3.52 THEOREM. Let R be an inverse semiring. Then the following three conditions are equivalent:

(1) The multiplicative and sum-orderings coincide.

(2) For all x, y, and h in R with $y = x + h$, $x^{-1} + h^{-1}$ is defined and equals $(x + h)^{-1}$.

(3) For all x, y in R, $x \leq y$ implies $x^{-1} \leq y^{-1}$.

PROOF. (1) implies (2): Suppose $y = x + h$. Then $x \leq y$ and $h \leq y$, and so by assumption $x \sqsubseteq y$ and $h \sqsubseteq y$. Thus, from [Howie, 1976, V.2.2], we have that

$$y^{-1}x = x^{-1}x \qquad y^{-1}h = h^{-1}h$$

$$xy^{-1} = xx^{-1} \qquad hy^{-1} = hh^{-1}.$$

Hence,

$$\begin{aligned}y^{-1} &= y^{-1}yy^{-1}\\ &= y^{-1}(x+h)y^{-1}\\ &= y^{-1}xy^{-1}+y^{-1}hy^{-1}\\ &= x^{-1}xy^{-1}+h^{-1}hy^{-1}\\ &= x^{-1}xx^{-1}+h^{-1}hh^{-1}\\ &= x^{-1}+h^{-1}.\end{aligned}$$

(2) implies (3): This is obvious.

(3) implies (1): Let $y=x+h$. Then by assumption, there exists k in R such that $y^{-1}=x^{-1}+k$. It suffices to show, by [Howie, 1976, V.2.2], that there exists an idempotent e in R such that $x=ye$. By the inversibility of y, there exists z in R such that $yy^{-1}+z=1$ and $zy=0=y^{-1}z$. Then $x=(yy^{-1}+z)x=yy^{-1}x+zx$. But as $zx\le zy=0$, $zx=0$. Thus, $x=ye$ with $e=y^{-1}x$. We must show that e is idempotent, that is, $e^2=e$. Now,

$$\begin{aligned}e &= y^{-1}x\\ &= y^{-1}(yy^{-1}+z)x\\ &= y^{-1}((x+h)y^{-1}+z)x\\ &= y^{-1}xy^{-1}x+(y^{-1}hy^{-1}x+y^{-1}zx)\\ &\ge y^{-1}xy^{-1}x\\ &= e^2.\end{aligned}$$

Conversely,

$$\begin{aligned}e^2 &= y^{-1}xy^{-1}x\\ &= y^{-1}x(x^{-1}+k)x\\ &= y^{-1}xx^{-1}x+y^{-1}xkx\\ &\ge y^{-1}xx^{-1}x\\ &= y^{-1}x\\ &= e.\end{aligned}$$

Therefore, by the antisymmetry of $\le$, $e^2=e$, and so e idempotent.

We conclude this section with conditions for inversibility in a so-ring based upon the given ordering. More specifically,

3.53 OBSERVATION. If x, y are elements of a so-ring R, $x \sqsubseteq y$, and y is inversible, then x is inversible.
PROOF. Let C be the center of R. Since $x \sqsubseteq y$, there exists e in C such that $x = ey$. By theorem 3.43, e is inversible, and by observation 3.38, ey is inversible. Hence, x is inversible.

The identical statement with $\sqsubseteq$ replaced by $\leq$ is not true. For instance,

3.54 COUNTEREXAMPLE. Let R be the so-ring of real numbers from 0 to 1 as defined in counterexample 3.6. Then, although $0.5 \leq 1$ and 1 is inversible, it is clear that 0.5 is not inversible.

However, we do have the following result:

3.55 OBSERVATION. Let R be so-ring. Let x, y, and h be elements of R such that $y = x + h$ is inversible and such that $\overleftarrow{x}$ exists with $\overleftarrow{x}h = 0$. Then x is inversible and $x^{-1} = y^{-1}\overleftarrow{x}$.
PROOF. Let C be the center of R. First, $\overleftarrow{x}y = \overleftarrow{x}(x + h) = \overleftarrow{x}x + \overleftarrow{x}h = x$. Since $\overleftarrow{x}$ is in C, $\overleftarrow{x}^{-1} = \overleftarrow{x}$. Thus, by observation 3.38, $\overleftarrow{x}y$ is inversible, since both $\overleftarrow{x}$ and y are inversible. Therefore, x is inversible, and $x^{-1} = (\overleftarrow{x}y)^{-1} = y^{-1}\overleftarrow{x}^{-1} = y^{-1}\overleftarrow{x}$.

CHAPTER IV

MATRICES OVER SO-RINGS

In this chapter, we investigate the extent to which some of the basic concepts from linear algebra can be generalised or reformulated so as to apply to matrices over partial semirings and, in particular, to matrices over so-rings.

We begin by defining exactly what we mean by a matrix over a partial semiring R.

4.1 DEFINITION. Let R be a partial semiring. An $I \times J$ *matrix* over R is a family $(x_{ij}: i \in I, j \in J)$ in R such that $(x_{ij}: j \in J)$ is supersummable (definition 2.41) for every i in I.[1]

Following the usual conventions, we think of a matrix as a planar array for which I indexes rows and J indexes columns. Supersummability of the rows is added to ensure that the product of two matrices exists and is itself a matrix. Let X be an $I \times J$ matrix over R, and let Y be a $J \times K$ matrix over R. Multiplying any row i of X by any column k of Y gives $\sum_{j \in J} x_{ij} y_{jk} = z_{ik}$, which exists since each row of X is a supersummable family. Thus, the matrix product of X and Y is Z, an $I \times K$ array of elements in R. We now show that Z is a matrix. Let $(t_k: k \in K)$ be a K-indexed family in R. Since each row of Y is supersummable, $\sum_{k \in K} y_{jk} t_k$ exists for each j in J, and thus, since each row of

[1] When I, J are of finite cardinality, we often write $m \times n$ for $I \times J$, where m is the cardinality of I and n is the cardinality of J.

X is supersummable, $\sum_{j\in J} x_{ij}(\sum_{k\in K} y_{jk}t_k)$ exists for each i in I. Hence,

$$\begin{aligned}\sum_{j\in J} x_{ij}(\sum_{k\in K} y_{jk}t_k) &= \sum_{j\in J}(\sum_{k\in K} x_{ij}y_{jk}t_k)\\ &= \sum_{k\in K}(\sum_{j\in J} x_{ij}y_{jk}t_k)\\ &= \sum_{k\in K}(\sum_{j\in J} x_{ij}y_{jk})t_k\\ &= \sum_{k\in K} z_{ik}t_k\end{aligned}$$

for each i in I. Therefore, each row of Z is supersummable, and so Z is a matrix.

Supersummability of the rows of a matrix is not the only way to ensure that the product of two matrices is well-defined. For example, by altering the definition of supersummability so that in a so-ring R, a family $(x_i: i \in I)$ is supersummable if for any family $(y_i: i \in I)$ in R, $\sum_i y_i x_i$ exists, we obtain an alternate definition of matrix. In this case, a matrix over a so-ring R is an $I \times J$ array of elements over R such that the elements in each column of the array form supersummable families.

Although both definitions of matrix guarantee that the product of two matrices is well-defined, we have chosen definition 4.1 for the following reasons. Recall from the introductory section that any iterative flowscheme can be written in terms of a recursive equation of arrays of partial functions, namely, $\bar{x} = A\bar{x} + \bar{b}$. In addition, remember that each row of A contains the coefficients for all loop paths from a given cutpoint in the flowscheme; each coefficient in a row corresponds to the loop path passing through a given cutpoint. These loop paths are disjoint and so as partial functions they are summable, and hence supersummable. Thus, A is an array of partial functions, each of whose rows is supersummable. Because of this important example, we have chosen to make supersummability of rows a property of matrices over so-rings in general. There is a secondary reason for selecting definition 4.1, and that has to do with partially-additive categories, which are an important universe for program semantics [Manes and Arbib, 1985]. Arbib and Manes [1980, 7.8] showed that if R is a partially-additive semiring and if a matrix is a rectangular array with supersummable rows, then the set of matrices over R is a partially-additive category, and hence potentially useful for doing program semantics from a category-theoretic perspective.

Note that the transpose of a matrix over a so-ring is not necessarily a matrix, since the columns of a matrix need not be supersummable, as we see in the following:

4.2 COUNTEREXAMPLE. Let f, g be two non-zero elements of $Pfn(D, D)$ whose domains are disjoint. Then (f, g) is a supersummable family. Thus,

$$\begin{pmatrix} f & g \\ f & g \end{pmatrix}$$

is a matrix over $Pfn(D, D)$, whose transpose

$$\begin{pmatrix} f & f \\ g & g \end{pmatrix}$$

is not a matrix.

Returning to the product of two matrices over a partial semiring R, we observe that if we let $I = J = K$, then we have already shown that the set of square matrices over R is closed under matrix multiplication. By introducing the appropriate additive operation, we can make this set of matrices a partial semiring itself as follows.

4.3 EXAMPLE. The set $Mat_{R,D}$ of $D \times D$ matrices over R is a partial semiring in which the multiplicative operation is the usual matrix multiplication, the multiplicative unit is the identity matrix, and the additive operation is defined as follows. A family $(x^k: k \in K)$ of matrices is *summable* if for all families $(y_j: j \in D)$ in R and for all i in D, the family $({x^k}_{ij}y_j: j \in D, k \in K)$ is summable in R. In this case, we define the matrix $\sum_k x^k$ by $(\sum_k x^k)_{ij} = \sum({x^k}_{ij}: k \in K)$. If, in addition,

R is a so-ring, then $Mat_{R,D}$ must also be a so-ring. To verify that $Mat_{R,D}$ is a partial semiring, and in particular a so-ring, is a straightforward exercise left to the reader.

We now turn our attention to R^n, the set of n-vectors of elements over a partial semiring R, where $0 < n < \infty$. Each vector is denoted by $\bar{x}_j$, where $\bar{x}_j = [x_{1j} \cdots x_{nj}]$. A set of such n-vectors is a kind of module as we show below.

4.4 DEFINITIONS. Let $(R, \sum, \circ, 1)$ be a partial semiring. A *(right) partial semimodule* over R is a partial monoid $(M, \widehat{\sum})$ together with a function

$$M \times R \to M: (x, y) \mapsto x \star y$$

which satisfies the following axioms for x, $(x_i : i \in I)$ in M and y, z, $(y_j : j \in J)$ in R:

(1) if $\widehat{\sum}_i x_i$ exists, $(\widehat{\sum}_i x_i) \star y = \widehat{\sum}_i x_i \star y$;

(2) if $\sum_j y_j$ exists, $x \star (\sum_j y_j) = \widehat{\sum}_j x \star y_j$;

(3) $x \star (y \circ z) = (x \star y) \star z$;

(4) $x \star 0 = 0$;

(5) $x \star 1 = x$.

A map $\phi: M_1 \to M_2$ between two (right) partial semimodules over a so-ring R is called an *R-map* if for any x in M_1 and any y in R, $(x \star y)\phi = x\phi \star y$. A *homomorphism* of two (right) partial semimodules over a so-ring R is an additive map which is also an R-map.

4.5 EXAMPLES.

(1) Let $(R, \sum, \circ, 1)$ be a partial semiring. Then $(R, \sum)$ is a partial semimodule over R, with $\star = \circ$.

(2) Let R be the so-ring $\{0, 1\}$ with trivial addition and trivial multiplication. Any partial monoid $(M, \widehat{\sum})$ is uniquely a partial semimodule over R, with $\star$ defined such that $x \star 1 = x$ and $x \star 0 = 0$.

(3) If $(R, \sum, \circ, 1)$ is a partial semiring, then for $0 < n \leq \infty$, $(R^n, \widehat{\sum})$ is a

partial semimodule over R with $\star$ and $\widehat{\sum}$ defined as follows. Let $(\bar{x}_j: j \in J)$ be a set of n-tuples in R^n. Then $\widehat{\sum}_j \bar{x}_j$ exists and equals $[\sum_j x_{1j} \cdots \sum_j x_{nj}]$ exactly when $\sum_j x_{ij}$ exists for all $1 \leq i \leq n$. Let y be in R. Define $[x_{1j} \cdots x_{nj}] \star y$ to equal $[x_{1j} \circ y \cdots x_{nj} \circ y]$.

For the remainder of the chapter we will be dealing with so-rings, except where noted to the contrary. Having provided the basic definitions, we move on to matrix invertibility.

Invertibility

When is a matrix over a so-ring invertible? Is the inverse of a matrix over a so-ring a matrix itself? How are its entries characterized? Before we turn our attention to matrices, however, we first investigate invertibility for arbitrary square arrays over a so-ring.

4.6 THEOREM. (Manes and Benson, 1985, 6.2) Let R be a so-ring and let A be an $n \times n$ array of elements of R. Then the following two conditions are equivalent.

(1) There exists an $n \times n$ array B of elements of R such that the products AB and BA are defined and $AB = I = BA$, where I is the $n \times n$ identity matrix. Such a B is referred to as the *inverse* of A.

(2) Each a_{ij} in A is inversible, and row domains and column ranges partition – that is, for each i, $(a_{i1}a_{i1}^{-1}, \ldots, a_{in}a_{in}^{-1})$ is an n-partition of R, and for each j, $(a_{1j}^{-1}a_{1j}, \ldots, a_{nj}^{-1}a_{nj})$ is an n-partition of R.

PROOF. First, we observe that (1) is equivalent to the equations

$$\sum_k a_{ik}b_{ki} = 1 = \sum_k b_{ik}a_{ki}$$

for each i and

$$\sum_k a_{ik}b_{kj} = 0 = \sum_k b_{ik}a_{kj} \tag{4-1}$$

for all $i \neq j$, the second of which is equivalent to

$$a_{ik}b_{kj} = 0 = b_{ik}a_{kj} \tag{4-2}$$

for all k and for all $i \neq j$ by positivity.

(1) implies (2): To show that each a_{ij} is inversible, we must show that the equations for inversibility in a so-ring are satisfied. This follows directly from the equivalent characterisation of (1):

$$\begin{aligned}
a_{ij}b_{ji} + x_{ij} = 1, \text{ if } x_{ij} &= \sum_{k \neq j} a_{ik}b_{ki} \\
b_{ji}a_{ij} + y_{ij} = 1, \text{ if } y_{ij} &= \sum_{k \neq i} b_{jk}a_{kj} \\
b_{ji}x_{ij} &= \sum_{k \neq j} b_{ji}a_{ik}b_{ki} = 0 \\
a_{ij}y_{ij} &= \sum_{k \neq i} a_{ij}b_{jk}a_{kj} = 0 \\
x_{ij}a_{ij} &= \sum_{k \neq j} a_{ik}b_{ki}a_{ij} = 0 \\
y_{ij}b_{ji} &= \sum_{k \neq i} b_{jk}a_{kj}b_{ji} = 0.
\end{aligned}$$

Hence, ${a_{ij}}^{-1}$ exists and equals b_{ji}. Thus, $\overleftarrow{a_{ij}}$ and $\overrightarrow{a_{ij}}$ exist for each i and j. For each i, if $j \neq k$

$$\overleftarrow{a_{ij}}\,\overleftarrow{a_{ik}} = a_{ij}a_{ij}^{-1}a_{ik}a_{ik}^{-1} = a_{ij}(b_{ji}a_{ik})b_{ki} = 0,$$

and

$$\sum_j \overleftarrow{a_{ij}} = \sum_j a_{ij}a_{ij}^{-1} = \sum_j a_{ij}b_{ji} = 1.$$

Hence, row domains partition, and similarly we can show that column ranges partition.

(2) implies (1): Since row domains and column ranges form n-partitions,

$$\sum_k a_{ik}{a_{ik}}^{-1} = 1 = \sum_k {a_{kj}}^{-1}a_{kj}$$

for each i, j. If we define $b_{ji} = a_{ij}^{-1}$ and substitute into the above equations, we obtain, for each i, j,

$$\sum_k a_{ik}b_{ki} = 1 = \sum_k b_{jk}a_{kj}.$$

Thus, we have half of the equations (4-1). Using the fact that row domains partition, we have that $a_{ij}a_{ij}^{-1}a_{ik}a_{ik}^{-1} = 0$ for all i and for $k \neq j$. This implies that

$$\begin{aligned} 0 &= a_{ij}a_{ij}^{-1}a_{ik}a_{ik}^{-1} \\ &= a_{ij}^{-1}\left(a_{ij}a_{ij}^{-1}a_{ik}a_{ik}^{-1}\right)a_{ik} \\ &= \left(a_{ij}^{-1}a_{ij}a_{ij}^{-1}\right)\left(a_{ik}a_{ik}^{-1}a_{ik}\right) \\ &= a_{ij}^{-1}a_{ik}. \end{aligned}$$

Substituting b_{ji} for a_{ij}^{-1} yields $b_{ji}a_{ik} = 0$. Similarly, using the fact that column ranges partition, we have that

$$0 = a_{ij}a_{kj}^{-1} = a_{ij}b_{jk}.$$

Thus, we have equations (4-2). Therefore, the equivalent characterization of (1) is satisfied.

Theorem 4.6 characterizes invertibility for $n \times n$ arrays over a so-ring R, but does not conclude whether or not A and B are matrices given that they are invertible arrays. In fact, it is not true in general that invertible arrays are necessarily matrices (refer to counterexample 4.9 below), but we can provide a set of conditions under which this conclusion may be drawn. First, we make the following

4.7 OBSERVATION. Let R be a so-ring, and let $(x_i : i \in I)$ be a family in R. Then

(1) if $\overleftarrow{x_i}$ exists for each i in I and if $(\overleftarrow{x_i} : i \in I)$ is supersummable, then $(x_i : i \in I)$ is also supersummable.

Furthermore, if each x_i is inversible, then

(2) if $(\overrightarrow{x_i} : i \in I)$ is supersummable, then $(x_i^{-1} : i \in I)$ is also supersummable;

(3) if $(x_i : i \in I)$ is supersummable, then $(\overleftarrow{x_i} : i \in I)$ is also supersummable;

(4) if $(x_i^{-1} : i \in I)$ is supersummable, then $(\overrightarrow{x_i} : i \in I)$ is also supersummable.

PROOF. Let $(y_i : i \in I)$ be any family in R.

(1): The supersummability of $(\overleftarrow{x_i} : i \in I)$ implies the existence of

$$\sum_i \overleftarrow{x_i}\,(x_i y_i) = \sum_i \left(\overleftarrow{x_i}\, x_i\right) y_i = \sum_i x_i y_i,$$

and so $(x_i : i \in I)$ is supersummable.

(2): The supersummability of $(\overrightarrow{x_i} : i \in I)$ implies the existence of

$$\sum_i \overrightarrow{x_i}\, x_i^{-1} y_i = \sum_i x_i^{-1} x_i x_i^{-1} y_i = \sum_i x_i^{-1} y_i,$$

and so $(x_i^{-1} : i \in I)$ is supersummable.

(3): The supersummability of $(x_i : i \in I)$ implies the existence of

$$\sum_i x_i \left(x_i^{-1} y_i\right) = \sum_i \left(x_i x_i^{-1}\right) y_i = \sum_i \overleftarrow{x_i}\, y_i,$$

and so $(\overleftarrow{x_i} : i \in I)$ is supersummable.

(4): Note that $\overrightarrow{x_i} = x_i^{-1} x_i = \overleftarrow{x_i^{-1}}$. Hence, the result follows from (3), interchanging x_i and x_i^{-1}.

Observation 4.7 implies that for so-rings in which all n-partitions are supersummable, an invertible $n \times n$ array A and its inverse B are both matrices. To see this, recall from theorem 4.6 that since A is invertible, the domains in each row form an n-partition, which by assumption is supersummable. Then by observation 4.7(1), each row must form a supersummable family. Thus A is a matrix and, symmetrically, B is a matrix. (For the converse, see theorem 4.10 below.)

It is not true, however, that in all so-rings each n-partition is supersummable. To demonstrate this, we first make the following

4.8 OBSERVATION. In the so-ring $Pfn(D, D)^{op}$, the only supersummable families are those with at most one nonzero member.

PROOF. Clearly, all families in $Pfn(D, D)^{op}$ containing at most one nonzero member are supersummable. To prove the converse, let $(x_i : i \in I)$ be a family in

$Pfn(D,D)^{op}$ containing at least two nonzero members x_j and x_k. Thus, there exist d_1 and d_2 in D such that $d_1 \in dom(x_j)$ and $d_2 \in dom(x_k)$. Let $(y_i: i \in I)$ be the family in $Pfn(D,D)^{op}$ such that $y_i = 0$ for i different from j and k, and such that for $d \in D$

$$dy_j = \begin{cases} d_1, & \text{if } d = d_1; \\ \text{undefined}, & \text{otherwise;} \end{cases}$$

and

$$dy_k = \begin{cases} d_2, & \text{if } d = d_1; \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

Then $dom(x_j \bullet y_j) = dom(y_j \circ x_j) = \{d_1\} = dom(y_k \circ x_k) = dom(x_k \bullet y_k)$.[2] Therefore, $(x_i \bullet y_i: i \in I)$ is not a summable family, and so $(x_i: i \in I)$ is not supersummable.

One immediate consequence of this observation is that the only matrices over $Pfn(D,D)^{op}$ are those arrays in which each row contains at most one nonzero member. Another consequence is that not all n-partitions in $Pfn(D,D)^{op}$ are supersummable. In particular, if $(x_i: 1 \le i \le n)$ is an n-partition which contains at least two nonzero members, then $(x_i: 1 \le i \le n)$ is not supersummable.

Using the so-ring $Pfn(D,D)^{op}$, we can construct an invertible array such that neither itself nor its inverse is a matrix.

4.9 COUNTEREXAMPLE. Let D_1, D_2 be a partition of D such that $D_1 \ne \emptyset \ne D_2$. Then (f_1, f_2) where $f_i = \overleftarrow{f_i}$ and $\overleftarrow{f_i} = D_i$ is a 2-partition of $Pfn(D,D)^{op}$ which is not supersummable by observation 4.8, since $f_1 \ne 0 \ne f_2$. Therefore, the array

$$\begin{pmatrix} f_1 & f_2 \\ f_2 & f_1 \end{pmatrix}$$

is not a matrix, but it is invertible and its inverse is itself.

In the next theorem, we show several equivalent conditions on a so-ring under which invertible arrays are necessarily matrices.

[2] As in definition 2.32, we represent the multiplicative operation in R by $\circ$ and the multiplicative operation in R^{op} by $\bullet$.

4.10 THEOREM. For any so-ring R, the following three conditions are equivalent:

(1) If A, B are $n \times n$ arrays of elements over R such that AB and BA exist and $AB = I = BA$, then A and B are matrices.

(2) Every n-partition is supersummable.

(3) If $(r_1, \ldots, r_n)$ is an n-partition and $(y_1, \ldots, y_n)$ is any family in R, then there exists y in R with $r_i y = r_i y_i$ for all i.

Furthermore, if every element in R has a domain, then the above three conditions are equivalent to

(4) Every finite family with pairwise disjoint domains is summable.

PROOF. (1) implies (2): Let $(r_1, \ldots, r_n)$ be an n-partition. If we define

$$A = \begin{pmatrix} r_1 & r_2 & \cdots & r_{n-2} & r_{n-1} & r_n \\ r_2 & r_3 & \cdots & r_{n-1} & r_n & r_1 \\ r_3 & r_4 & \cdots & r_n & r_1 & r_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ r_n & r_1 & \cdots & r_{n-3} & r_{n-2} & r_{n-1} \end{pmatrix} = B,$$

then $AB = I = BA$, and so by assumption, A is a matrix. Thus, each of its rows is supersummable, and in particular, $(r_1, \ldots, r_n)$ is supersummable.

(2) implies (1): We have shown this previously.

(2) implies (3): Let $(r_1, \ldots, r_n)$ be an n-partition, and let $(y_1, \ldots, y_n)$ be a family in R. By assumption, $(r_1, \ldots, r_n)$ is supersummable. Hence, $\sum_j r_j y_j = y$ exists. Since $r_i r_j = 0$ for $j \neq i$, $r_i y = r_i(\sum_j r_j y_j) = \sum_j r_i r_j y_j = r_i^2 y_i = r_i y_i$.

(3) implies (2): Again, let $(r_1, \ldots, r_n)$ be an n-partition, and let $(y_1, \ldots, y_n)$ be a family in R. By assumption, there exists y with $r_i y = r_i y_i$. Since $\sum_j r_j = 1$, $y = (\sum_j r_j) y = \sum_j r_j y = \sum_j r_j y_j$. Hence, $(r_1, \ldots, r_n)$ is supersummable.

Now suppose that $\overleftarrow{x}$ exists for each x in R.

(2) implies (4): Let $(x_1, \ldots, x_n)$ be a family in R satisfying $\overleftarrow{x_i}\,\overleftarrow{x_j} = 0$ for $j \neq i$. By theorem 3.13, $\overleftarrow{x_1} + \cdots + \overleftarrow{x_n}$ exists and equals $\overleftarrow{x_1} \vee \cdots \vee \overleftarrow{x_n} = t$. Hence, $(\overleftarrow{x_1}, \ldots, \overleftarrow{x_n}, t')$ is an $n+1$-partition, which is by assumption supersummable. Therefore, $\overleftarrow{x_1}x_1 + \cdots + \overleftarrow{x_n}x_n + t'1 = x_1 + \cdots + x_n + t'$ exists, and by partition-

associativity the subsum $x_1 + \cdots + x_n$ also exists.

(4) implies (2): Let $(r_1, \ldots, r_n)$ be an n-partition, and let $(y_1, \ldots, y_n)$ be a family in R. Each r_i is in C, implying by observation 3.29 that $\overleftarrow{r_i} = r_i$ for all i. By observation 3.26(1), $\overleftarrow{r_i y_i} \leq \overleftarrow{r_i}$. Hence, $\overleftarrow{r_i y_i}\,\overleftarrow{r_j y_j} \leq \overleftarrow{r_i}\,\overleftarrow{r_j} = r_i r_j = 0$, for $j \neq i$. Therefore, the family $(r_1 y_1, \ldots, r_n y_n)$ has pairwise disjoint domains, and is by assumption summable.

Consider the following modification of condition (1) of theorem 4.10.

(1'): If A, B are $n \times n$ arrays of elements over R such that AB and BA exist and $AB = I = BA$ and if A is a matrix, then B is a matrix.

Thus far, we have not been able to construct any so-ring for which condition (1') fails to be true. Even the so-ring $Pfn(D,D)^{op}$, one of the few which does not satisfy the conditions of theorem 4.10, does satisfy condition (1') as we show below.

4.11 EXAMPLE. We know from counterexample 4.9 that theorem 4.10 does not apply to the so-ring $Pfn(D,D)^{op}$. Recall from observation 4.8 that the only matrices over $Pfn(D,D)^{op}$ are those whose rows each contain at most one nonzero member. Hence, all $n \times n$ invertible matrices over $Pfn(D,D)^{op}$ are composed of rows containing all zeroes save a totally-defined bijection from D to D. Furthermore, since the column ranges of an invertible matrix must form n-partitions, each column must have at least one nonzero member. But since there are n rows and n columns, there must be exactly one nonzero member per column. Is the inverse B of such a matrix A also a matrix? The answer is "yes" because of the fact that $b_{ji} = {a_{ij}}^{-1}$. This implies that each row and each column of B contains exactly one nonzero entry, and that this entry is a totally-defined bijection from D to D.

In the next theorem, we supply another criterion which if satisfied by an invertible matrix over a so-ring implies that the inverse of this matrix is also a matrix. First, however, we need the following definitions and observation.

4.12 DEFINITIONS. We say that a family $(x_i : i \in I)$ in a partial semimodule $(M, \widehat{\sum})$ over a so-ring R *spans* M if for each z in M there exists a family

$(y_i : i \in I)$ in R such that $z = \widehat{\sum}_i x_i y_i$. In this case, $(x_i : i \in I)$ is a *spanning set* for M.

Our decision to adopt the convention of multiplying by coefficients on the right instead of on the left was motivated by the form of the flowscheme matrix equation discussed in the introduction, namely, $\bar{x} = A\bar{x} + \bar{b}$. Here, each column of A contains the coefficients of the loop paths passing through a given cutpoint of the flowscheme. Hence, the recursive portion of a flowscheme is described by a "linear" combination of the columns of A, written $A\bar{x}$. The convention on the order of coefficient multiplication in $Pfn(D,D)^n$ in this example thus determined the convention on the order of coefficient multiplication in partial semimodules in general.

Recall from example 4.5(1) that any so-ring is a partial semimodule over itself. For this reason, we often apply the term "spanning" to so-rings directly.

4.13 OBSERVATION. Let R be a so-ring, and let $(x_i : i \in I)$ be a family in R. Then
(1) if $(x_i : i \in I)$ is an n-partition, it spans R.
Furthermore, if x_i^{-1} exists for each i in I, then
(2) $(x_i : i \in I)$ spans R if and only if $(\overleftarrow{x_i} : i \in I)$ spans R;
(3) $(x_i^{-1} : i \in I)$ spans R if and only if $(\overrightarrow{x_i} : i \in I)$ spans R.

PROOF. (1): Since $(x_i : i \in I)$ is an n-partition, $\sum_i x_i = 1$. Let z be any element of R. Then $z = (\sum_i x_i)z = \sum_i x_i z$. Thus, $(x_i : i \in I)$ spans R.

(2): Suppose $(x_i : i \in I)$ spans R. Then for any z in R there exists $(y_i : i \in I)$ in R such that $z = \sum_i x_i y_i$. Thus, since x_i^{-1} exists for each i, $z = \sum_i x_i y_i = \sum_i (x_i x_i^{-1} x_i) y_i = \sum_i (x_i x_i^{-1}) x_i y_i = \sum_i \overleftarrow{x_i}(x_i y_i)$. Hence, $\overleftarrow{x_i}$ spans R. Now suppose $\overleftarrow{x_i}$ spans R. Then given z in R, there exists $(y_i : i \in I)$ in R such that $z = \sum_i \overleftarrow{x_i} y_i$. Again, using the inversibility of x_i, $z = \sum_i \overleftarrow{x_i} y_i = \sum_i (x_i x_i^{-1}) y_i = \sum_i x_i (x_i^{-1} y_i)$. Hence, $(x_i : i \in I)$ spans R.

(3): This is a consequence of interchanging x_i and x_i^{-1} in (2).

4.14 THEOREM. Let R be a so-ring. If A is an invertible $n \times n$ matrix over R and B is an $n \times n$ array over R such that $AB = I = BA$, then the following two conditions are equivalent:

(1) $a_{ij} = a_{ij}^{-1}$ for each a_{ij} in A;

(2) $B = A^T$ and B is a matrix.

PROOF. Theorem 4.6 shows that since $AB = I = BA$, $b_{ij} = a_{ji}^{-1}$ for each i, j such that $1 \leq i \leq n$ and $1 \leq j \leq n$.

(1) implies (2): First, observe that $B = A^T$, since $b_{ij} = a_{ji}^{-1} = a_{ji} = a^T{}_{ij}$. Hence, by showing that each column of A is supersummable, we can show that B is a matrix. Let $(a_{ij}: 1 \leq i \leq n)$ be column j of A, and let $(y_i: 1 \leq i \leq n)$ be a family in R. First, we show that $a_{uj}y_u + a_{vj}y_v$ exists for $u \neq v$. By observation 3.45, $\overleftarrow{a_{ij}} = \overrightarrow{a_{ij}}$, since $a_{ij} = a_{ij}^{-1}$. Hence, $\overleftarrow{a_{uj}}\,\overleftarrow{a_{vj}} = \overrightarrow{a_{uj}}\,\overrightarrow{a_{vj}} = 0$, since the ranges in the columns of A partition. By observation 4.13(1) and (2), each row of A spans R, since the domains in each row form an n-partition and since a_{ij}^{-1} exists for each i, j. Thus, for row v, there exists a family $(z_k: 1 \leq k \leq n)$ in R such that $a_{uj}y_u = \sum_k a_{vk}z_k = \sum_{k \neq j} a_{vk}z_k$, because $\overleftarrow{a_{uj}}\,\overleftarrow{a_{vj}} = 0$. Each row of the matrix A is supersummable, and so $\sum_{k \neq j} a_{vk}z_k + a_{vj}y_v = a_{uj}y_u + a_{vj}y_v$ exists. Now we must show that the whole of column j is supersummable. Suppose $\sum_{i=1}^{m} a_{ij}y_i$ exists for some $m < n$. Substituting $\sum_{i=1}^{m} a_{ij}y_i$ for $a_{uj}y_u$ and $m+1$ for v, we can show by the method above that $\sum_{i=1}^{m+1} a_{ij}y_i$ exists. First, we find $(w_k: 1 \leq k \leq n)$ in R such that $\sum_{i=1}^{m} a_{ij}y_i = \sum_k a_{m+1k}w_k = \sum_{k \neq j} a_{m+1k}w_k$, since $\overleftarrow{a_{m+1j}}\,\overleftarrow{a_{ij}} = 0$ for $i \neq m+1$. The supersummability of row $m+1$ of A gives the existence of $\sum_{k \neq j} a_{m+1k}w_k + a_{m+1j}y_{m+1} = \sum_{i=1}^{m} a_{ij}y_i + a_{m+1j}y_{m+1} = \sum_{i=1}^{m+1} a_{ij}y_i$ for any $m < n$. Therefore, $\sum_{i=1}^{n} a_{ij}y_i$ exists, and so each column of A is supersummable.

(2) implies (1): Immediately, $a_{ij}^{-1} = b_{ji} = a^T{}_{ji} = a_{ij}$.

Independence

4.15 DEFINITION. A family $(x_i: i \in I)$ in a partial semimodule $(M, \widehat{\sum})$ over a so-ring R is said to be *independent* if for any two families $(b_i: i \in I)$, $(c_i: i \in I)$ in R such that $\widehat{\sum}_i x_i b_i = \widehat{\sum}_i x_i c_i$, $b_i = c_i$ for each i in I.

Our formulation of the definition of independence for partial semimodules not only is reminiscent of the usual definition of linear independence in linear algebra, but also is a special case of the definition of independence for universal algebras [Grätzer, 1968, Theorem 5.31.3].

4.16 DEFINITIONS. A *basis* for a partial semimodule $(M, \widehat{\sum})$ is a family in M which is an independent spanning set. The *standard basis* for the partial semimodule R^n over the so-ring R consists of the n-vectors $(\bar{e}_i : 1 \leq i \leq n)$, where $\bar{e}_i$ is the n-vector whose i^{th} component is 1 and whose other components are zeroes.

We note that to be consistent with the universal-algebraic treatment of bases, it is necessary for coefficients to act on the same side for both spanning sets and independent sets.

The next two theorems give some conditions on independent sets and on spanning sets over the partial semimodule R^n.

4.17 THEOREM. Let R be a so-ring in which $\overrightarrow{x}$ exists for each x in R. If $(\bar{x}_j : j \in J)$ is a family of independent column vectors over R^n, then $\bigvee_{i=1}^{n} \overrightarrow{x_{ij}} = 1$ for each j in J.

PROOF. For each j in J, let $a_j = \bigwedge_{i=1}^{n} \overrightarrow{x_{ij}}'$. For each k, $1 \leq k \leq n$, and for each j, $\bigwedge_{i=1}^{n} \overrightarrow{x_{ij}}' \leq \overrightarrow{x_{kj}}'$. Hence, $x_{kj} a_j = x_{kj} \bigwedge_{i=1}^{n} \overrightarrow{x_{ij}}' \leq x_{kj} \overrightarrow{x_{kj}}' = 0$. This implies that $\sum_j \bar{x}_j a_j = \bar{0}$. If $b_j = 0$ for each j in J, then $\sum_j \bar{x}_j b_j = 0$ as well. Hence, $a_j = b_j$, because the vectors $(\bar{x}_j : j \in J)$ are independent. Therefore, $0 = \bigwedge_{i=1}^{n} \overrightarrow{x_{ij}}' = (\bigvee_{i=1}^{n} \overrightarrow{x_{ij}})'$, which implies that $\bigvee_{i=1}^{n} \overrightarrow{x_{ij}} = 1$.

4.18 THEOREM. Let R be a so-ring in which $\overleftarrow{x}$ exists for each x in R. If $(\bar{x}_j : j \in J)$ is a family of column vectors spanning R^n, then $\bigvee_{j \in J} \overleftarrow{x_{ij}} = 1$ for each i such that $1 \leq i \leq n$.

PROOF. Let $\bar{e}_i$ denote the i^{th} standard basis vector in R^n. Since $(\bar{x}_j : j \in J)$ span R^n, there exists a family $(a_j : j \in J)$ in R such that $\sum_j \bar{x}_j a_j = \bar{e}_i$. This implies that $\sum_j x_{ij} a_j$ exists and equals 1. Thus, $\overleftarrow{\sum_j x_{ij} a_j}$ exists and equals 1. Hence, by observation 3.23(1), $\bigvee_{j\in J}\overleftarrow{x_{ij}a_j}$ exists and equals $\overleftarrow{\sum_j x_{ij}a_j}$. By observation 3.26(1), $\overleftarrow{x_{ij}a_j} \leq \overleftarrow{x_{ij}}$ for each j in J. Therefore, $1 = \bigvee_{j\in J}\overleftarrow{x_{ij}a_i} \leq \bigvee_{j\in J}\overleftarrow{x_{ij}} \leq 1$, so that $\bigvee_{j\in J}\overleftarrow{x_{ij}} = 1$.

Bases and Matrix Invertibility

Recall from linear algebra that if F is a field and A is an $n \times n$ matrix over F, then A is invertible if and only if the columns of A form a basis for F^n. We show that under certain constraints, an analogous result holds for matrices over so-rings. One direction of the proof closely follows the classical proof in linear algebra.

4.19 THEOREM. Let R be a so-ring in which the inverse of any invertible matrix is itself a matrix. If X is an $n \times n$ invertible matrix over R, then the columns of X form a basis for R^n.

PROOF. Let $\bar{b} = [b_1 \cdots b_n]$ be a vector in R^n. To show that the columns of X are a basis for R^n, it suffices to show that there exists a unique vector $\bar{a} = [a_1 \cdots a_n]$ in R^n such that $\sum_j x_{ij}a_j = b_i$ for each i. Writing this as a matrix equation, we obtain $X\bar{a} = \bar{b}$. By assumption, X^{-1} is an $n \times n$ matrix. Hence, $X^{-1}\bar{b}$ exists. Now, $X(X^{-1}\bar{b}) = (XX^{-1})\bar{b} = I\bar{b} = \bar{b}$, and so $X^{-1}\bar{b}$ is a solution to the equation $X\bar{a} = \bar{b}$. Furthermore, if $\bar{c}$ is any solution to the equation $X\bar{a} = \bar{b}$, then $X^{-1}\bar{b} = X^{-1}(X\bar{c}) = (X^{-1}X)\bar{c} = I\bar{c} = \bar{c}$. Therefore, $X^{-1}\bar{b}$ is the unique solution to the equation $X\bar{a} = \bar{b}$.

However, if $(\bar{x}_j : 1 \leq j \leq n)$ is a basis for R^n, it is not necessarily the case that the $\bar{x}_j$s form the columns of an invertible matrix.

4.20 COUNTEREXAMPLE. Let R be the so-ring $Pfn(D, D)^{op}$ where $D = \mathbf{N}$. For d in D, define f_1, f_2 as follows:

$$df_1 = \begin{cases} 0, & \text{if } d = 0; \\ d-1, & \text{otherwise}; \end{cases} \qquad df_2 = \begin{cases} 1, & \text{if } d = 0; \\ \text{undefined}, & \text{otherwise}. \end{cases}$$

Now construct the following vectors in R^2:

$$\bar{x}_1 = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \quad \bar{x}_2 = \begin{pmatrix} f_2 \\ f_1 \end{pmatrix}.$$

It is somewhat tedious but not difficult to show that the vectors $\bar{x}_1$ and $\bar{x}_2$ are an independent spanning set, and hence a basis for R^2. (The details are left to the reader.) Consider the 2×2 array formed by $\bar{x}_1$ and $\bar{x}_2$:

$$X = \begin{pmatrix} f_1 & f_2 \\ f_2 & f_1 \end{pmatrix}.$$

Since f_1 is not injective, it is not an invertible element of $Pfn(D,D)^{op}$. Thus, X is not an invertible array. Since neither f_1 nor f_2 is equal to 0, the family (f_1, f_2) is not supersummable by observation 4.8. Thus, X is not a matrix. Therefore, we have constructed a basis for R^2 which when viewed as a 2×2 array over $Pfn(D,D)^{op}$, is neither invertible nor a matrix.

However, if we impose the constraints that R is adequate and that n-partitions in R are supersummable, then the converse of theorem 4.19 can be proved to be true. Most so-rings satisfy both of these constraints, the major exceptions being certain distributive lattices which fail to be adequate and $Pfn(D,D)^{op}$ which fails to have supersummability of n-partitions for D containing more than one element. Among the so-rings which do satisfy these constraints are $Pfn(D,D)$, $Mfn(D,D)$, and $Mset(D,D)$. (We have demonstrated this fact for $Pfn(D,D)$ in example 3.20 and in observation 2.42; the proofs that $Mfn(D,D)$ and $Mset(D,D)$ also satisfy these constraints are similar and so are not given.)

4.21 THEOREM. Let R be an adequate so-ring such that every n-partition in R is supersummable. If X is an $n \times n$ array over R whose columns form a basis for R^n, then X is an invertible matrix whose inverse A is a matrix as well.

PROOF. Let $\bar{x}_j = [x_{1j} \cdots x_{nj}]$ be the j^{th} column of X. For each standard basis vector $\bar{e}_i$ in R^n, there exist $\bar{a}_i = [a_{j1} \cdots a_{jn}]$ in R such that

$$\sum_j \bar{x}_j a_{ji} = \bar{e}_i \tag{4-3}$$

since the $\bar{x}_j$s span R^n. Define A to be the matrix with row i equal to $\bar{a}_i$. To show that X is invertible, we show that $XA = I = AX$; to show that X and A are matrices, we use the fact that all n-partitions are supersummable.

From equation (4-3), we have that

$$\sum_j x_{ij} a_{ji} = 1 \tag{4-4}$$

for all i and that

$$x_{kj} a_{ji} = 0 \tag{4-5}$$

for all j and for $k \neq i$. Hence, $XA = I$.

As $(\bar{x}_j : 1 \leq j \leq n)$ is an independent set, $\bigvee_i \overrightarrow{x_{ij}} = 1$ for each j, by theorem 4.17. Thus,

$$\begin{aligned} \overleftarrow{a_{ji}} &= \Big(\bigvee_k \overrightarrow{x_{kj}}\Big)\overleftarrow{a_{ji}} \\ &= \Big(\bigvee_k \overrightarrow{x_{kj}}\Big) \wedge \overleftarrow{a_{ji}}, \text{ by theorem 3.11} \\ &= \bigvee_k (\overrightarrow{x_{kj}} \wedge \overleftarrow{a_{ji}}) \\ &= \bigvee_k (\overrightarrow{x_{kj}}\overleftarrow{a_{ji}}), \text{ by theorem 3.11.} \end{aligned} \tag{4-6}$$

For $k \neq i$, $x_{kj} a_{ji} = 0$ by equation (4-5). This implies that $\overrightarrow{x_{kj}}\,\overleftarrow{a_{ji}} = 0$, since R is adequate. Hence,

$$\overleftarrow{a_{ji}} = \overrightarrow{x_{ij}}\,\overleftarrow{a_{ji}}.$$

Thus, for $k \neq i$

$$\begin{aligned} \overleftarrow{a_{ji}}\overleftarrow{a_{jk}} &= \overleftarrow{a_{ji}}(\overrightarrow{x_{kj}}\overleftarrow{a_{jk}}) \\ &= (\overrightarrow{x_{jk}}\overleftarrow{a_{ji}})\overleftarrow{a_{jk}}, \text{ by theorem 3.8(3)} \\ &= 0. \end{aligned}$$

Therefore, $(\overleftarrow{a_{j1}}, \ldots, \overleftarrow{a_{jn}}, (\bigvee_i \overleftarrow{a_{ji}})')$ is an $n+1$-partition in R for each j, and so is supersummable by assumption. Thus, $(\overleftarrow{a_{ji}} : 1 \le i \le n)$ is a supersummable family, and so by observation 4.7(1), $(a_{ji} : 1 \le i \le n)$ is a supersummable family as well. Therefore, A is a matrix, since each of its rows is supersummable.

In particular, this means that $\sum_i a_{ji} x_{ik}$ exists for each j and each k. Thus, for each k,

$$\begin{aligned} \bar{x}_k &= \sum_i \bar{e}_i x_{ik} \\ &= \sum_i \Big(\sum_j \bar{x}_j a_{ji}\Big) x_{ik}, \text{ by equation (4-3)} \\ &= \sum_j \bar{x}_j \Big(\sum_i a_{ji} x_{ik}\Big). \end{aligned}$$

But since the $\bar{x}_j$s are independent, this means that

$$\sum_i a_{ki} x_{ik} = 1$$

for all k and that

$$a_{ji} x_{ik} = 0 \qquad (4-7)$$

for all i and for $j \ne k$. Hence, $AX = I$.

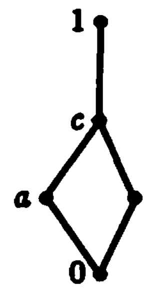
So far, we have shown that $XA = I = AX$, and thus that X and A are invertible $n \times n$ arrays over R. We have also shown that A is a matrix. Since each n-partition is supersummable and since $XA = I = AX$, the fact that X is a matrix is a direct consequence of the equivalence of conditions (1) and (2) of theorem 4.10. Thus, we have demonstrated that for an adequate so-ring R with supersummable n-partitions, any $n \times n$ array X over R whose columns form a basis for R^n is an invertible matrix over R whose inverse is also a matrix.

In particular, we have shown that if R is one of $Pfn(D,D)$, $Mfn(D,D)$, or $Mset(D,D)$, then an $n \times n$ matrix over R is invertible if and only if its columns form a basis for R^n.

In theorem 4.21, we made the assumption that the so-ring R was adequate.

However, this assumption is not necessary in order to show that any basis for R^n must form the columns of an invertible matrix. For instance,

4.22 EXAMPLE. The distributive lattice



is a so-ring R which is not adequate, since $a \wedge b = 0$ but $\overrightarrow{a} = 1 = \overleftarrow{b}$ and so $\overrightarrow{a} \wedge \overleftarrow{b} = 1$. The reader may easily show that the only basis vectors for R^n are the standard basis vectors, which clearly form the columns of an invertible matrix.

Below we will examine so-rings R such that for finite n, R^n has bases of different cardinalities. We will also consider the case in which n is infinite. Note that in theorem 4.21, although we obtained results for a basis of R^n of cardinality n only, we can easily generalise some of the results to bases of arbitrary cardinality. In particular, equations (4-3) through (4-7) in theorem 4.21 do not require X to have only n columns. However, we note that equations (4-6) involve one of the lattice distributive laws, which always holds in the finite case but which may fail to hold in the infinite case. Thus, if we wish to consider bases of R^n of infinite cardinality, we must make sure that R is a so-ring in which its center C satisfies the infinite distributive law in question. The simplest way to ensure that the law holds is to require that C be a complete Boolean lattice.

4.23 COROLLARY. Let R be an adequate so-ring with supersummable n-partitions and with center C. Let $(\bar{x}_j : j \in J)$ be a basis for R^n. If

(1) J is finite, or

(2) J is countably infinite and C is a complete Boolean lattice,

then $(\overleftarrow{x_{ij}} : j \in J)$ partitions R for each i.

PROOF. Since the x_js span R^n, $\bigvee_j \overleftarrow{x_{ij}} = 1$ for each i, as a consequence of theorem 4.18. We must also show that $\overleftarrow{x_{ij}}\,\overleftarrow{x_{ik}} = 0$ for $k \neq j$. First, we have from theorem 4.21 equation (4-7) that $a_{ji}x_{ik} = 0$. (Note that equation (4-7) depends upon equations (4-6). This is why we require C to be a complete Boolean lattice, if we are to consider infinite bases.) Thus, since R is adequate,

$$\overrightarrow{a_{ji}}\,\overleftarrow{x_{ik}} = 0 \tag{4-8}$$

for all i and for $j \neq k$. Then,

$$\begin{aligned}
\overleftarrow{x_{ik}} &= \Big(\sum_j x_{ij}a_{ji}\Big)\overleftarrow{x_{ik}} \text{ by equation (4-4)} \\
&= \sum_j x_{ij}a_{ji}\overleftarrow{x_{ik}} \\
&= \sum_j x_{ij}a_{ji}\overrightarrow{a_{ji}}\,\overleftarrow{x_{ik}} \text{ by observation 3.21(2)} \\
&= x_{ik}a_{ki}\overrightarrow{a_{ki}}\,\overleftarrow{x_{ik}} \text{ by equation (4-8)} \\
&= x_{ik}a_{ki}\overleftarrow{x_{ik}} \text{ by observation 3.21(2)} \\
&= \overleftarrow{x_{ik}}\,x_{ik}a_{ki} \text{ by theorem 3.8(3)} \\
&= x_{ik}a_{ki} \text{ by observation 3.21(1).}
\end{aligned}$$

Hence,

$$\overleftarrow{x_{ij}}\,\overleftarrow{x_{ik}} = x_{ij}a_{ji}x_{ik}a_{ki} = x_{ij}a_{ji}\,\overrightarrow{a_{ji}}\,\overleftarrow{x_{ik}}\,x_{ik}a_{ki} = 0$$

by equation (4-8). Thus, from theorem 3.23(1) we have that $\sum_j \overleftarrow{x_{ij}} = \bigvee_j \overleftarrow{x_{ij}} = 1$. Therefore, $(\overleftarrow{x_{ij}} : j \in J)$ partitions R for each i.

Similarly, we have

4.24 COROLLARY. Let R be a an adequate so-ring with supersummable n-partitions. If $(x_j : j \in J)$ is a basis for R^n, then $(\overrightarrow{x_{ij}} : 1 \leq i \leq n)$ is an n-partition for each j in J.

Cardinality of Bases

In what ways does the structure of the so-ring R affect the cardinality of the bases for R^n, where $0 < n \leq \infty$?

4.25 OBSERVATION. Let R be an adequate so-ring in which all n-partitions are supersummable. In addition, assume that R has countable cardinality ≥ 2, and that its center is a complete Boolean lattice. If the continuum hypothesis holds, then

(1) Every basis of R^∞ has countably infinite cardinality.

(2) If $0 < n < \infty$, then every basis of R^n has finite cardinality.

(3) If $0 < n < \infty$ and R is finite, then every basis of R^n has cardinality n.

PROOF. Let the cardinality of R be denoted by k, and assume that $0 < n \leq \infty$. The cardinality of R^n is thus k^n. Clearly, there exists a basis for R^n of cardinality n, namely the standard basis $(\bar{e}_i: 1 \leq i \leq n)$. Suppose $(\bar{x}_j: j \in J)$ is a basis for R^n of cardinality m. By corollary 4.22, for each i, $(\overline{\overline{x_{ij}}}: j \in J)$ is an m-partition of R, which by assumption is supersummable. This in turn implies, by observation 4.7(1), that for each i, $(x_{ij}: j \in J)$ is supersummable. Thus, for any family $(y_j: j \in J)$ in R, $(\bar{x}_j y_j: j \in J)$ is summable in R^n. The independence of the $\bar{x}_j$s implies that for two different families $(b_j: j \in J)$ and $(c_j: j \in J)$ in R, $\sum_j \bar{x}_j b_j \neq \sum_j \bar{x}_j c_j$. Furthermore, since the $\bar{x}_j$s span R^n, the cardinality of R^n is also equal to k^m.

(1): The cardinality of R^∞ is $k^{\aleph_0}$. As k is countable and ≥ 2 and as the continuum hypothesis is assumed, $k^{\aleph_0} = \aleph_1$. If m is finite, then k^m is countable, since k is countable. If $m > \aleph_0$, then $k^m > \aleph_1$, since $k \geq 2$. Thus, $m = \aleph_0$.

(2): The cardinality of R^n is k^n. As k is countable and n is finite, k^n is countable. If $m \geq \aleph_0$, then $k^m \geq \aleph_1$, since $k \geq 2$. Thus, m is finite.

(3): As k is finite and ≥ 2, $m = n$.

Hence, for finite D and for $0 < n < \infty$, all bases of $Pfn(D,D)^n$, $Mfn(D,D)^n$, and $Mset(D,D)^n$ have cardinality n. However, if D has infinite cardinality, the results are quite different. For example, let $n = 2$, and let $D = \mathbf{N}$. Clearly, $Pfn(D,D)^2$ has a basis of cardinality 2, namely the standard basis. But $Pfn(D,D)^2$ also has a basis of cardinality 1. This basis is the vector with components f_1 and f_2, where for all d in D, $df_1 = 2d$ and $df_2 = 2d+1$. More generally, we have the following:

4.26 THEOREM. Let D be a set of cardinality $\aleph_0$. Let R be any one of $Pfn(D,D)$, $Mfn(D,D)$, $Mset(D,D)$, and let $0 < n < \infty$. Then for each $0 < m < \infty$ there exists a basis for R^n of cardinality m.

PROOF. We begin with the proof for $Pfn(D,D)$. For $m \leq n$, we construct a basis $(\bar{x}_j : 1 \leq j \leq m)$ as follows. For $1 \leq j < m$, let $\bar{x}_j = \bar{e}_j$, the j^{th} standard basis vector for $Pfn(D,D)^n$. To define $\bar{x}_m$ requires more work. For $1 \leq i < m$, let $x_{im} = 0$. Since the $\bar{x}_j$s are to form a basis, they satisfy corollaries 4.22 and 4.23. In particular, this implies for $m \leq i \leq n$, that $\overleftarrow{x_{im}} = 1$, $\bigvee_{i=m}^{n} \overrightarrow{x_{im}} = 1$, and $\overrightarrow{x_{im}}\,\overrightarrow{x_{lm}} = 0$ for $l \neq i$. Hence, we must find a way of defining x_{mm} through x_{nm} such that each is a totally-defined injective partial function, the ranges of any two of these functions are disjoint, and the union of the ranges is D. To make explicit the method of doing this, we use the characterisation of partial functions by $D \times D$ 0-1 matrices. For $m \leq i \leq n$, we define x_{im} such that for $1 \leq k < \infty$, row k of the 0-1 matrix representing x_{im} contains a 1 in column $(i+1-m)+(k-1)\cdot(n+1-m)$.

For $m \geq n$, we construct a basis $(\bar{x}_j : 1 \leq j \leq m)$ as follows. For $1 \leq j < n$, let $\bar{x}_j = \bar{e}_j$. Defining $\bar{x}_j$ for $n \leq j \leq m$ requires more work. For $1 \leq i < n$, let $x_{ij} = 0$. Again, corollaries 4.22 and 4.23 are satisfied by the $\bar{x}_j$s, implying for $n \leq j \leq m$, that $\overrightarrow{x_{nj}} = 1$, $\bigvee_{j=n}^{m} \overleftarrow{x_{nj}} = 1$, and $\overleftarrow{x_{nj}}\,\overleftarrow{x_{nl}} = 0$ for $l \neq j$. Hence, we must define x_{nn} through x_{nm} such that each is a partially-defined bijection from D to D, the domains of any two are disjoint, and the union of the domains is D. Again, we return to the $D \times D$ 0-1 matrix characterisation of partial functions, using an algorithm similar to the one used to define $\bar{x}_m$ for $m \leq n$. For $n \leq j \leq m$, we define x_{nj} such that for $1 \leq k < \infty$, row k of the 0-1 matrix representing x_{nj} is nonzero if $k \equiv (j+1-n) \bmod (m+1-n)$. In this case, row k contains a 1 in column $((k-(j+1-n))/(m+1-n))+1$.

The proofs for $Mfn(D,D)$ and $Mset(D,D)$ are identical to the proof for $Pfn(D,D)$ for the following reason. In the proof for $Pfn(D,D)$, each of the components of the x_js is an injective partial function and therefore inversible. From examples 3.37, we know that the inversible elements for $Pfn(D,D)$, $Mfn(D,D)$, and $Mset(D,D)$ coincide. Hence, we can define the x_js in the same way for both $Mfn(D,D)$ and $Mset(D,D)$ as we did for $Pfn(D,D)$.

Thus we see that the cardinality of D influences the cardinality of the bases of $Pfn(D,D)^n$, $Mfn(D,D)^n$, and $Mset(D,D)^n$.

Eigenvalues and Eigenvectors

In this section, we present a preliminary result on eigenvalues in the context of matrices over so-rings. We hope that a future detailed study of eigenvalues and eigenvectors will provide some insight into transformations and canonical forms of matrices over so-rings.

4.27 DEFINITIONS. Let A be an $n \times n$ matrix over a so-ring R. Then an element λ of R is an *eigenvalue* of A if and only if there exists a vector x in R^n such that:

(1) $Ax = x\lambda$, and

(2) the vector x is a column of some $n \times n$ invertible matrix.

A vector x satisfying both (1) and (2) is called an *eigenvector* for the eigenvalue λ.

In classical linear algebra, condition (2) is replaced by the condition that $x \neq \bar{0}$, which in turn implies that x is a column of some invertible matrix. However, in the context of matrices over so-rings, the fact that a vector is nonzero is not sufficient to guarantee that it is a column of some invertible matrix. Hence, it is necessary to explicitly state condition (2) (or a set of equivalent conditions).

Eigenvalues and eigenvectors as they relate to matrix similarity appear to have applications in algorithm transformation. In particular, considering the matrix equation of an algorithm, $\bar{x} = A\bar{x} + \bar{b}$, it may be possible to perform a similarity transform on the matrix A so that the solution $\sum_{n\geq 0} A^n\bar{b}$ of the equation is put into a form which is easier to compute. Refer to chapter VII and to [Manes, to appear] for details. Knowing something about the nature of the eigenvalues and eigenvectors of the matrix A may give us information about how to construct the similarity transform needed. Below, we show that invertibility of a matrix implies inversibility of its eigenvalues.

4.28 THEOREM. Let R be a so-ring such that $0 \neq 1$ and such that the inverse of any invertible matrix over R is also a matrix over R. If A is an $n \times n$ invertible matrix over R, then any eigenvalue of A is inversible and nonzero.

PROOF. Let λ be an eigenvalue of A, and let $\bar{x} = [x_1 \dots x_n]$ be an eigenvector for λ. Suppose that $\lambda = 0$. Then

$$\begin{aligned}\bar{x} &= I\bar{x}\\ &= (A^{-1}A)\bar{x}\\ &= A^{-1}(A\bar{x})\\ &= A^{-1}(\bar{x}\lambda)\\ &= A^{-1}\bar{0}\\ &= \bar{0},\end{aligned}$$

which implies that $x_i = 0$ for each i. Since $\bar{x}$ is an eigenvector, it is a column of some invertible matrix. Hence, each x_i is inversible, and $(x_1{}^{-1}x_1, \dots, x_n{}^{-1}x_n)$ is an n-partition, which implies that $\sum_i x_i{}^{-1}x_i = 1$. This in turn implies, since $0 \neq 1$ in R, that for some i, $x_i \neq 0$. Hence, we have arrived at a contradiction. Therefore, $\lambda \neq 0$.

The next step is to show that λ is an inversible element of R. Let C be the center of R. We now present a set of observations that we will use in proving the inversibility of λ.

(1) If $[v_1 \dots v_n]$ is a column of an invertible matrix V, then $[v_1^{-1} \dots v_n^{-1}]$ is a row of the matrix V^{-1}. Hence, $(v_1^{-1}, \dots, v_n^{-1})$ is a supersummable family.

(2) If $(v_1 v_1^{-1}, \dots, v_n v_n^{-1})$ is an n-partition, then for $j \neq k$,

$$0 = v_j v_j^{-1} v_k v_k^{-1} = v_j^{-1} v_k.$$

(3) If $(\sum_i v_i)(\sum_i v_i^{-1}) = \sum_i v_i v_i^{-1}$ is in C, then for r in C such that $r(\sum_i v_i v_i^{-1}) = 0 = (\sum_i v_i v_i^{-1})r$, we must have that $r(\sum_i v_i) = 0 = (\sum_i v_i^{-1})r$. This we can prove as follows: $r(\sum_i v_i v_i^{-1}) = 0$ implies that for each i,

$$0 = r v_i v_i^{-1} = r v_i v_i^{-1} v_i = r v_i.$$

Hence, $r(\sum_i v_i) = 0$. Similarly, we can show that $(\sum_i v_i^{-1})r = 0$.

Now, for each i, we have that $x_i \lambda = \sum_j a_{ij} x_j$, since $A\bar{x} = \bar{x}\lambda$. Thus,

$$\begin{aligned}
\lambda &= 1\lambda \\
&= \left(\sum_i x_i^{-1} x_i\right) \lambda \\
&= \sum_i x_i^{-1}(x_i \lambda) \\
&= \sum_i x_i^{-1} \left(\sum_j a_{ij} x_j\right) \\
&= \sum_{i,j} x_i^{-1} a_{ij} x_j.
\end{aligned}$$

To show that λ is inversible, we first show that $\sum_{i,j}(x_i^{-1} a_{ij} x_j)^{-1}$ exists and then that it equals λ^{-1}. Since A is an invertible matrix, a_{ij} is inversible for each i, j. Hence, $x_i^{-1} a_{ij} x_j$ is inversible for each i, j, and

$$(x_i^{-1} a_{ij} x_j)^{-1} = x_j^{-1} a_{ij}^{-1} x_i.$$

By (1), $(a_{1j}^{-1}, \dots, a_{nj}^{-1})$ is a supersummable family for each j. Thus, $\sum_i a_{ij}^{-1} x_i$ exists for each j. Again, using (1), we have that $(x_1^{-1}, \dots, x_n^{-1})$ is a supersummable family. Hence, $\sum_j x_j^{-1} \sum_i a_{ij}^{-1} x_i$ exists, and thus

$$\sum_j x_j^{-1} \sum_i a_{ij}^{-1} x_i = \sum_{i,j} x_j^{-1} a_{ij}^{-1} x_i$$
$$= \sum_{i,j} (x_i^{-1} a_{ij} x_j)^{-1}.$$

We now must show that $\sum_{i,j} (x_i^{-1} a_{ij} x_j)^{-1} = \lambda^{-1}$. We have that

$$\lambda \left(\sum_{k,l} (x_k^{-1} a_{kl} x_l)^{-1} \right) = \left(\sum_{i,j} x_i^{-1} a_{ij} x_j \right) \left(\sum_{k,l} (x_k^{-1} a_{kl} x_l)^{-1} \right)$$
$$= \sum_{i,j,k,l} (x_i^{-1} a_{ij} x_j)(x_k^{-1} a_{kl} x_l)^{-1}$$
$$= \sum_{i,j,k,l} (x_i^{-1} a_{ij} x_j)(x_l^{-1} a_{kl}^{-1} x_k)$$
$$= \sum_{i,j} (x_i^{-1} a_{ij} x_j)(x_j^{-1} a_{ij}^{-1} x_i), \text{ by (2)}$$
$$= \sum_{i,j} (x_i^{-1} a_{ij} x_j)(x_i^{-1} a_{ij} x_j)^{-1}.$$

Clearly, each $(x_i^{-1} a_{ij} x_j)(x_i^{-1} a_{ij} x_j)^{-1}$ is a member of C. For $k \neq i$, we have that $(x_i^{-1} a_{ij} x_j)(x_i^{-1} a_{ij} x_j)^{-1}(x_k^{-1} a_{kl} x_l)(x_k^{-1} a_{kl} x_l)^{-1}$

$$= (x_i^{-1} a_{ij} x_j)(x_j^{-1} a_{ij}^{-1} x_i)(x_k^{-1} a_{kl} x_l)(x_l^{-1} a_{kl}^{-1} x_k)$$
$$= 0, \text{ by (2)}.$$

Thus, by theorem 3.13, $\sum_{i,j} (x_i^{-1} a_{ij} x_j)(x_i^{-1} a_{ij} x_j)^{-1}$ is in C. Hence, it has a complement z in C such that

$$\sum_{i,j} (x_i^{-1} a_{ij} x_j)(x_i^{-1} a_{ij} x_j)^{-1} + z = 1$$

and such that

$$z \left(\sum_{i,j} (x_i^{-1} a_{ij} x_j)(x_i^{-1} a_{ij} x_j)^{-1} \right) = 0 = \left(\sum_{i,j} (x_i^{-1} a_{ij} x_j)(x_i^{-1} a_{ij} x_j)^{-1} \right) z.$$

This implies by (3) that

$$z\left(\sum_{i,j} x_i^{-1}a_{ij}x_j\right) = 0 = \left(\sum_{i,j}(x_i^{-1}a_{ij}x_j)^{-1}\right) z.$$

Hence, we have that

$$\lambda\left(\sum_{i,j}(x_i^{-1}a_{ij}x_j)^{-1}\right) + z = 1$$

and that

$$z\lambda = 0 = \sum_{i,j}\left(x_i^{-1}a_{ij}x_j)^{-1}\right) z.$$

Thus, half of the inversibility equations needed to show that λ is inversible have been satisfied; the other half may be obtained in a similar manner. Therefore, λ is inversible, and

$$\lambda^{-1} = \left(\sum_{i,j} x_i^{-1}a_{ij}x_j\right)^{-1} = \sum_{i,j}(x_i^{-1}a_{ij}x_j)^{-1}.$$

CHAPTER V

REPRESENTATIONS AND COMPATIBLE PARTIAL ORDERS

In this chapter, we provide generalizations of some results in semigroup theory. First, we develop the rudiments of a representation theory for so-rings. Of particular interest is representation by partial functions, since we may relate our results to those that already exist in semigroup theory. Second, we demonstrate how to adapt some of the orderings on semigroups to orderings on so-rings. Recall that in chapter IV, we discussed the multiplicative ordering, which is a generalization of the natural ordering on the elements of an inverse semigroup. Here, we introduce two other orderings from semigroup theory, fundamentally representable orderings and amenable orderings, and we show how these orderings may be applied to so-rings.

Representations of So-rings

5.1 DEFINITION. A *representation* of a so-ring R_1 by a so-ring R_2 is a so-ring homomorphism $\theta: R_1 \to R_2$. Furthermore, if θ is injective, then the representation is called *faithful*.

We say that a representation θ of so-ring R *reflects* a property p of R if $R\theta$ has property p implies that R has property p. We demonstrate some properties of a so-ring R which are preserved by a representation of R and which are reflected if the representation is faithful.

5.2 OBSERVATION. The sum-ordering is reflected by a representation if and only if the representation is faithful.

PROOF. This is a consequence of observations 2.25 and 2.27.

5.3 OBSERVATION. Let R be a so-ring with center C, and let θ be a representation of R. If r is in C, then $r\theta$ is in the center of the so-ring $R\theta$.

PROOF. If r is in C, then there exists r' in C such that $r + r' = 1$ and $rr' = 0 = r'r$. Since θ is additive, $1\theta = (r + r')\theta = r\theta + r'\theta$, and since θ is a monoid homomorphism, $1\theta = 1$. Again, using the fact that θ is a monoid homomorphism, $(r\theta)(r'\theta) = (rr')\theta = 0\theta$, and by observation 2.22, $0\theta = 0$. Similarly, it can be shown that $(r'\theta)(r\theta) = 0$. Thus, $(r\theta)' = r'\theta$, and so $r\theta$ is in the center of $R\theta$.

5.4 OBSERVATION. Let R be a so-ring with domains and ranges and with center C, and let θ be a representation of R.

(1) If $\overleftarrow{x\theta}$ exists, then $\overleftarrow{x\theta} \leq \overleftarrow{x}\,\theta$;

(2) if $\overrightarrow{x\theta}$ exists, then $\overrightarrow{x\theta} \leq \overrightarrow{x}\,\theta$.

Furthermore, if θ is faithful, then

(3) $\overleftarrow{x\theta} = \overleftarrow{x}\,\theta$;

(4) $\overrightarrow{x\theta} = \overrightarrow{x}\,\theta$.

PROOF. (We prove (1) and (3) only; (2) and (4) are dual.) (1): Observation 5.3 implies that $\overleftarrow{x}\,\theta$ is in the center of $R\theta$, since $\overleftarrow{x}$ is in C. As θ is a monoid homomorphism, $(\overleftarrow{x}\,\theta)(x\theta) = (\overleftarrow{x}\,x)\theta = x\theta$. But $\overleftarrow{x\theta}$, if it exists, is the least element r in the center of $R\theta$ such that $r(x\theta) = x\theta$. Therefore, $\overleftarrow{x\theta} \leq \overleftarrow{x}\,\theta$, if $\overleftarrow{x\theta}$ exists.

(3): Assume that θ is a faithful representation. Let $u\theta$ be an element in the center of $R\theta$. Then by observation 5.3, u must be in C. Suppose that u is such that $(u\theta)(x\theta) = x\theta$. Since θ is a monoid homomorphism, $x\theta = (u\theta)(x\theta) = (ux)\theta$, and thus, since θ is injective, $x = ux$. Now, $\overleftarrow{x}$ is the smallest element r in C such that $rx = x$. Hence, $\overleftarrow{x} \leq u$. By (1), $\overleftarrow{x}\,\theta$ is an element r in the center of $R\theta$ such that $r(x\theta) = x\theta$. Furthermore, by observation 2.25, $\overleftarrow{x}\,\theta \leq u\theta$, and so $\overleftarrow{x}\,\theta$ is the least element r in the center of $R\theta$ such that $r(x\theta) = x\theta$. Therefore, $\overleftarrow{x\theta}$ exists and equals $\overleftarrow{x}\,\theta$.

5.5 OBSERVATION. Let R be a so-ring with domains and ranges, and let θ be a representation of R. If sums are disjoint in R, then sums are disjoint in $R\theta$.

PROOF. Let $(x_i\theta : i \in I)$ be a summable family in $R\theta$. By observation 2.24, this implies that $(x_i : i \in I)$ is a summable family in R. Then,

$$\begin{aligned}
\overleftarrow{x_i\theta}\,\overleftarrow{x_j\theta} &\leq (\overleftarrow{x_i}\theta)(\overleftarrow{x_j}\theta), \text{ by observation 5.4(1)} \\
&= (\overleftarrow{x_i}\,\overleftarrow{x_j})\theta, \text{ since } \theta \text{ is a monoid homomorphism} \\
&= 0\theta, \text{ since sums are disjoint in } R \\
&= 0, \text{ by observation 2.22.}
\end{aligned}$$

Therefore, sums are disjoint in $R\theta$.

5.6 OBSERVATION. Let R be a so-ring, and let θ be a representation of R. If x is an inversible element of R, then $x\theta$ is an inversible element of $R\theta$ and $(x\theta)^{-1} = x^{-1}\theta$.

PROOF. Since x is inversible in R, then there exist x^{-1}, y in R such that $xx^{-1} + y = 1$ and $yx = 0 = x^{-1}y$. Thus,

$$\begin{aligned}
1 &= 1\theta, \text{ since } \theta \text{ is a monoid homomorphism} \\
&= (xx^{-1} + y)\theta \\
&= (xx^{-1})\theta + y\theta, \text{ since } \theta \text{ is additive} \\
&= (x\theta)(x^{-1}\theta) + y\theta, \text{ since } \theta \text{ is a monoid homomorphism.}
\end{aligned}$$

Also, using first the fact that θ is a monoid homomorphism and then observation 2.22, we have that $(y\theta)(x\theta) = (yx)\theta = 0\theta = 0$ and $(x^{-1}\theta)(y\theta) = (x^{-1}y)\theta = 0\theta = 0$. This gives us half of the inversibility equations needed to show that $x\theta$ is inversible. The other half of the equations can be obtained similarly. Therefore, $(x\theta)^{-1}$ exist in $R\theta$ and equals $x^{-1}\theta$.

Not all properties of a so-ring R are preserved under an arbitrary representation θ of R. For instance, if R is an adequate so-ring, then $R\theta$ is guaranteed to be adequate only if θ is faithful.

5.7 COUNTEREXAMPLE. Let $R = \{u, v, w, 0, 1\}$ with $\sum$ defined as the trivial addition and with $\circ$ defined as

$$x \circ y = \begin{cases} 0, & \text{if } y = 0 \text{ or } x = 0; \\ x, & \text{if } y = 1 \text{ or } y = x; \\ y, & \text{if } x = 1 \text{ or } x = y; \\ w, & \text{otherwise.} \end{cases}$$

The reader may easily verify that R is a so-ring with domains and ranges. Moreover, since $xy = 0$ only when one of $x = 0$ or $y = 0$, it is clear that R is adequate. Define a map $\theta: R \to R\theta$ such that $w\theta = 0$. Endow $R\theta$ with an additive operation as in observation 2.24, and a multiplicative operation such that $(x\theta)(y\theta) = (xy)\theta$. Then $R\theta$ is a so-ring, and θ is a representation of R. Now, $(u\theta)(v\theta) = (uv)\theta = w\theta = 0$. But $\overrightarrow{u\theta} = 1 = \overleftarrow{v\theta}$, and so $R\theta$ is not adequate.

Later on, we will show some other properties which are not necessarily preserved under so-ring representations.

Representation by Right Translations

We now demonstrate that any so-ring R may be faithfully represented by the set of additive maps of R to itself, denoted by $Add(R)$. In particular, any so-ring R may be faithfully represented by the set $Tr(R)$ of right translations of R. The set $Tr(R)$ consists of mappings $\tau_y: R \to R: x \mapsto xy$ for each y in R. First, however, we show that $Add(R)$ is a so-ring, and that $Tr(R)$ is a sub-so-ring of $Add(R)$.

5.8 OBSERVATION. For any so-ring R the set of additive maps of R to itself, $Add(R)$, is a so-ring.

PROOF. Let $\sum$ denote the additive operation in R. For a family $(f_i: i \in I)$ in $Add(R)$, define the additive operation $\widehat{\sum}$ such that $(f_i: i \in I)$ is summable if for all x in R, $(xf_i: i \in I)$ is summable in R, in which case $x\widehat{\sum}_i f_i = \sum_i xf_i$. Let $(x_j: j \in J)$ be a summable family in R. Let $(f_i: i \in I)$ be a summable family in

$Add(R)$. Then

$$\begin{aligned}(\sum_j x_j)(\widehat{\sum_i} f_i) &= \sum_i(\sum_j x_j)f_i \\ &= \sum_i(\sum_j x_j f_i), \text{ since each } f_i \text{ is additive} \\ &= \sum_j(\sum_i x_j f_i) \\ &= \sum_j x_j(\widehat{\sum_i} f_i).\end{aligned}$$

Hence, $\widehat{\sum}_i f_i$ is also additive, and so $Add(R)$ is closed under the operation $\widehat{\sum}$. The unary sum axiom and the partition-associativity axiom are satisfied by $Add(R)$, since they are satisfied by R. Therefore, $(Add(R), \widehat{\sum})$ is a partial monoid.

Define the multiplicative operation $\hat{\circ}$ as the usual functional composition. Hence, $\hat{\circ}$ is associative. Again, let $(x_j : j \in J)$ be a summable family in R. Let f, g be two functions in $Add(R)$. Then, making use of the additivity of f and of g, we have that $(\sum_j x_j)fg = (\sum_j x_j f)g = \sum_j x_j fg$, and so fg is also additive. Thus, $Add(R)$ is closed under the operation $\hat{\circ}$. Therefore, $(Add(R), \hat{\circ}, 1)$, where 1 is the identity function on R, is a monoid.

We need only show that the distributive laws hold. Let $(f_i : i \in I)$ be a summable family in $Add(R)$, and let g be any function in $Add(R)$. Let x be any element of R. Then $xg(\widehat{\sum}_i f_i) = \sum_i xgf_i = x\widehat{\sum}_i gf_i$, and so the left distributive law holds. Now, using the additivity of g, $x(\widehat{\sum}_i f_i)g = (\sum_i xf_i)g = \sum_i xf_i g = x\widehat{\sum}_i f_i g$, and so the right distributive law holds. Since we have distributivity on both sides, $Add(R)$ is a partial semiring.

The sum-ordering $\hat{\leq}$ on $Add(R)$ is derived from the sum-ordering $\leq$ on R, in that for f, g in $Add(R)$, $f \hat{\leq} g$ if there exists h in $Add(R)$ such that $xf + xh = xg$ for all x in R. Thus, $Add(R)$ is a so-ring.

To show that $Tr(R)$ is a sub-so-ring of $Add(R)$ we need the following lemmas.

5.9 LEMMA. For any so-ring R, $Tr(R)$ is a subset of $Add(R)$.

PROOF. Let τ_y be an element of $Tr(R)$, and let $(x_i : i \in I)$ be a summable family in R. Then $(\sum_i x_i)\tau_y = (\sum_i x_i)y = \sum_i x_i y = \sum_i x_i \tau_y$. Thus, τ_y is an additive map, and so $T(R)$ is contained in $Add(R)$.

5.10 LEMMA. The family $(x_i : i \in I)$ in a so-ring R is summable if and only if the family $(\tau_{x_i} : i \in I)$ in $Tr(R)$ is summable in $Add(R)$. Furthermore, $Tr(R)$ is closed under the additive operation of $Add(R)$.

PROOF. Let $(x_i : i \in I)$ be a summable in R. Then for any y in R, $y\sum_i x_i = \sum_i yx_i = \sum_i y\tau_{x_i}$, which implies that $(\tau_{x_i} : i \in I)$ is summable in $Add(R)$.

Now, let $(\tau_{x_i} : i \in I)$ be summable in $Add(R)$. Then for any y in R, $y\widehat{\sum_i}\tau_{x_i} = \sum_i y\tau_{x_i} = \sum_i yx_i$, and in particular, for $y = 1$. Hence, $(x_i : i \in I)$ is summable in R.

Thus, we have that $\sum_i x_i$ exists in R if and only if $\widehat{\sum_i}\tau_{x_i}$ exists in $Add(R)$. Hence, for any y in R, $y\widehat{\sum_i}\tau_{x_i} = \sum_i y\tau_{x_i} = \sum_i yx_i = y\sum_i x_i = y\tau_{\Sigma x_i}$. Therefore, $\widehat{\sum_i}\tau_{x_i} = \tau_{\Sigma x_i}$, and so $\widehat{\sum_i}\tau_{x_i}$ is an element of $Tr(R)$.

5.11 OBSERVATION. For any so-ring R, $Tr(R)$ is a sub-so-ring of $Add(R)$.

PROOF. In lemma 5.9, we showed that $Tr(R)$ is a subset of $Add(R)$, and in lemma 5.10 we showed that $Tr(R)$ is closed under the additive operation of $Add(R)$. Hence, $Tr(R)$ is a sub-so-monoid of $Add(R)$.

Now let τ_y, τ_z be elements of $Tr(R)$. For any x in R, $x\tau_y\tau_z = xy\tau_z = xyz = x\tau_{yz}$. Thus, $\tau_y\tau_z = \tau_{yz}$, and so $\tau_y\tau_z$ is an element of $Tr(R)$. Furthermore, for any x in R, $x\tau_1 = x1 = x$, and so the identity function is a member of $Tr(R)$. Hence, $Tr(R)$ is a sub-monoid of $Add(R)$, and therefore, $Tr(R)$ is a sub-so-ring of $Add(R)$.

5.12 THEOREM. Any so-ring R can be faithfully represented by $Tr(R)$.

PROOF. Define a mapping $\theta : R \to Tr(R) : x \mapsto \tau_x$. Let $(x_i : i \in I)$ be a summable family in R. Then

$$(\sum_i x_i)\theta = \tau_{\Sigma_i x_i}$$

$$= \widehat{\sum}_i \tau_{x_i}, \text{ by lemma 5.10}$$

$$= \widehat{\sum}_i x_i\theta.$$

Hence, θ is an additive map.

Let y, z be elements of R. Then

$$(y\theta)(z\theta) = \tau_y \tau_z$$

$$= \tau_{yz}, \text{ by observation 5.11}$$

$$= (yz)\theta.$$

Also, $1\theta = \tau_1$. Hence, θ is a monoid homomorphism. Therefore, θ is a so-ring homomorphism, and thus a so-ring representation.

Suppose that y, z are elements of R such that $y\theta = z\theta$. This implies that $\tau_y = \tau_z$, which in turn implies that $xy = xz$ for all x in R. In particular, for $x = 1$, $y = 1y = 1z = z$. Therefore, θ is injective and thus a faithful representation of R.

Representation by Partial Functions

Although every so-ring may be faithfully represented by the set of right translations of the elements of that so-ring, not all so-rings may be faithfully represented by the set of partial functions from some set D to itself.

5.13 COUNTEREXAMPLES. Any so-ring R which contains a nonzero element x such that $x + x$ is defined cannot be faithfully represented by $Pfn(D, D)$ for any set D. Suppose that $\theta: R \to Pfn(D, D)$ were a faithful representation. Let $y = x + x$. Since θ is additive, $y\theta = (x + x)\theta = x\theta + x\theta$. Now, the only element of $Pfn(D, D)$ which may be added to itself is 0. Hence, $x\theta = 0$. By observation 2.22, $0\theta = 0$. But since θ is injective, $x = 0$, which is a contradiction. Therefore, a faithful representation of R by $Pfn(D, D)$ cannot exist.

Even if we extend summability in $Pfn(D, D)$ to families whose members agree on domain overlaps (as defined in example 2.7), we find that not all so-rings may be faithfully represented by this so-ring of partial functions. Adding $x \neq y$ to the above conditions on R, we make the argument that $y\theta = (x+x)\theta = x\theta + x\theta = x\theta$. But since θ is injective, $x = y$ which is a contradiction. Therefore, such an R may not be faithfully represented by $Pfn(D, D)$ with or without overlap summability.

There are, however, many so-rings which do admit a faithful representation by partial functions on some set D. We give such representations a special name.

5.14 DEFINITION. Let R be a so-ring, and let $\theta: R \to Pfn(D, D)$ be a representation of R by the so-ring of partial functions from a set D to itself. Then θ is called a *D-representation* of R.

Special Properties of $Pfn(D, D)$. In attempting to determine which so-rings admit a D-representation, we looked for characteristics of $Pfn(D, D)$ which would help to distinguish it from other so-rings. Four of these characteristics are described below.

Property 1: Domains and ranges exist.

Property 2: Sums are disjoint.

Property 3: Adequacy.

Property 4: Atomicity (defined below).

These properties fail to characterize $Pfn(D, D)$; refer to counterexample 5.18 below. However, in the following section, we show that a so-ring possessing these four properties can be faithfully represented by $Pfn(D, D)$ for some set D. In fact, we will not even require adequacy for a D-representation to be constructed.

5.15 DEFINITION. A nonzero element x of a so-ring R is called an *atom* if there does not exist any element y in R such that $0 < y < x$. (The set of atoms in R is denoted by $A(R)$.) A so-ring R is *atomic* if each nonzero x in R is the sum of all elements in $\{y \in A(R): y \leq x\}$.

The properties of being an atom of a so-ring and of being an atomic so-ring are not necessarily preserved under a so-ring representation. If x is an atom in a so-ring R and θ is a representation of R, then $x\theta$ is an atom in $R\theta$ only if θ is faithful.

5.16 COUNTEREXAMPLE. Let $R = \{u, v, w, z, 0, 1\}$ with $\sum$ defined as

$$\sum(x_i : i \in I) = \begin{cases} x_j, & \text{if } x_i = 0 \text{ for } i \neq j; \\ w, & \text{if } x_j = u,\ x_k = v, \text{ and } x_i = 0 \text{ for } i \neq j, k; \\ \text{undefined}, & \text{otherwise;} \end{cases}$$

and with $\circ$ defined as the trivial multiplication. The reader may easily verify that R is a so-ring. Define a map $\theta: R \to R\theta$ such that for all nonzero x in R, $x\theta \neq 0$, and such that $z\theta = w\theta$. Endow $R\theta$ with an additive operation as in observation 2.24, and a multiplicative operation such that $(x\theta)(y\theta) = (xy)\theta$. Then $R\theta$ is a so-ring, and θ is a representation of R. Now, z is clearly an atom of R. But $z\theta = w\theta = (u+v)\theta = u\theta + v\theta$, and so $z\theta$ is not an atom of $R\theta$.

Moreover, if R is atomic, then $R\theta$ is atomic only if θ is faithful.

5.17 COUNTEREXAMPLE. Let $R = \{a, b, c, e, f, g, 0, 1\}$ with $\sum$ defined as

$$\sum(x_i : i \in I) = \begin{cases} x_j, & \text{if } x_i = 0 \text{ for } i \neq j; \\ c, & \text{if } x_j = a,\ x_k = b, \text{ and } x_i = 0 \text{ for } i \neq j, k; \\ g, & \text{if } x_j = e,\ x_k = f, \text{ and } x_i = 0 \text{ for } i \neq j, k; \\ \text{undefined}, & \text{otherwise;} \end{cases}$$

and with $\circ$ defined as the trivial multiplication. The reader may easily verify that R is an atomic so-ring. Define a map $\theta: R \to R\theta$ such that for all nonzero x in R, $x\theta \neq 0$, and such that $c\theta = g\theta$. Endow $R\theta$ with an additive operation as in observation 2.24, and with a multiplicative operation such that $(x\theta)(y\theta) = (xy)\theta$. Then $R\theta$ is a so-ring, and θ is a representation of R. Now, $a\theta$, $b\theta$, $e\theta$, and $f\theta$ are all atoms of $R\theta$. Furthermore, $a\theta + b\theta = (a+b)\theta = c\theta = g\theta = (e+f)\theta = e\theta + f\theta$, and so $a\theta$, $b\theta$, $e\theta$, and $f\theta$ are all less than or equal to $c\theta$. But $a\theta + b\theta + e\theta + f\theta$ is not defined and hence does not equal $c\theta$. Therefore, $R\theta$ is not atomic.

It is easy to see that $Pfn(D,D)$ is atomic, since the atoms are all partial functions with domains of cardinality 1. However, properties 1 through 4 are not sufficient in order to characterize $Pfn(D,D)$.

5.18 COUNTEREXAMPLE. Let $R = \{u, v, 0, 1\}$ with $\sum$ defined as

$$\sum(x_i : i \in I) = \begin{cases} 1, & \text{if } x_j = u,\ x_k = v, \text{ and } x_i = 0 \text{ for } i \neq j, k; \\ x_j, & \text{if } x_i = 0 \text{ for } i \neq j; \\ \text{undefined}, & \text{otherwise;} \end{cases}$$

and with $\circ$ defined as

$$x \circ y = \begin{cases} x, & \text{if } y = 1 \text{ or } y = x; \\ y, & \text{if } x = 1 \text{ or } x = y; \\ 0, & \text{otherwise.} \end{cases}$$

The reader may easily verify that R is a so-ring. Clearly, for all x in R, $\overleftarrow{x} = x = \overrightarrow{x}$. Hence, R has domains and ranges. Thus, from the definitions of $\sum$ and $\circ$ we immediately have that sums are disjoint in R and that R is adequate. Furthermore, u and v are atoms R and $1 = u + v$. Hence, R is atomic. But there does not exist any set D such that $R = Pfn(D,D)$. (However, R is a sub-so-ring of $Pfn(D,D)$ for many sets D.)

Furthermore, properties 1 through 4 need not all hold in an arbitrary sub-so-ring of $Pfn(D,D)$.

5.19 COUNTEREXAMPLE. Let $D = \{0,1,2\}$ and consider the subset $R = \{x, y, u, v, z, 1, 0\}$ of elements of $Pfn(D,D)$ defined below:

$$dx = \begin{cases} 0, & \text{if } d = 0 \text{ or } d = 1; \\ \text{undefined}, & \text{otherwise;} \end{cases} \qquad dy = \begin{cases} 0, & \text{if } d = 2; \\ \text{undefined}, & \text{otherwise;} \end{cases}$$

$$du = \begin{cases} 0, & \text{if } d = 0; \\ \text{undefined}, & \text{otherwise;} \end{cases} \qquad dv = \begin{cases} 0, & \text{if } d = 1 \text{ or } d = 2; \\ \text{undefined}, & \text{otherwise;} \end{cases}$$

$$dz = 0, \text{ for all } d \in D;$$

together with the identity function on D and the nowhere-defined function. It is easy to show that these seven partial functions form a monoid under the multiplicative operation, since each element of R is idempotent and since the product of any two elements f and g of R must be one of f, g, or 0. Hence, R is a sub-monoid of $Pfn(D,D)$. If we define the additive operation such that $x+y=u+v=z$ and $0+f=f+0=f$ for any f in R, then it is easy to show that R is a sub-so-monoid of $Pfn(D,D)$, since all of these sums are also defined in $Pfn(D,D)$. Hence, R is a sub-so-ring of $Pfn(D,D)$.

However, S does not possess properties 2, 3, and 4. For all $f \neq 0$ in S, $\overleftarrow{f} = 1 = \overrightarrow{f}$, and for $f = 0$, $\overleftarrow{f} = 0 = \overrightarrow{f}$. Thus, S has domains and ranges. But sums are not disjoint in S, since $x+y$ exists and $\overleftarrow{x} = 1 = \overleftarrow{y}$. Neither is S adequate, since $xy = 0$ and $\overrightarrow{x} = 1 = \overleftarrow{y}$. Last of all S is not atomic, since x, y, u, and v are all atoms which are less than or equal to z but $z \neq x+y+u+v$. Thus, even though S is a sub-so-ring of $Pfn(D,D)$, it does not satisfy properties 2, 3, and 4 which are satisfied by $Pfn(D,D)$ itself.

This counterexample shows that a so-ring possessing a D-representation need not satisfy properties 2, 3, and 4. In fact, at this writing, we know of only one necessary constraint on a so-ring R admitting a D-representation: there can be no nonzero x in R such that $x+x$ is defined (see counterexample 5.13). We do not know whether or not this constraint is sufficient. However, as we show below, properties 1, 2, and 4 are sufficient conditions on a so-ring to allow us to construct a D-representation.

Constructing a D-Representation. For a so-ring R satisfying properties 1 through 4 (in fact, property 3 is not necessary), we show how to choose a set D and how to construct partial functions from D to D, such that each partial function corresponds to an element of R and such that the mapping from R to $Pfn(D,D)$ is a faithful representation. In order to prove this, however, we first need the following two results.

5.20 OBSERVATION. Let R be a so-ring in which sums are disjoint. Then x is an atom of R if and only if $\overleftarrow{x}$ is an atom of R. Dually, x is an atom of R if and only if $\overrightarrow{x}$ is an atom of R.

PROOF. We prove the first assertion only. Let C be the center of R. Let x be in $A(R)$. Let y be any element of R such that $0 \leq y \leq \overleftarrow{x}$. By observation 3.25(1), $0 \leq \overleftarrow{y} \leq \overleftarrow{\overleftarrow{x}} = \overleftarrow{x}$. This in turn implies that $0 \leq \overleftarrow{y}x \leq \overleftarrow{x}x = x$, since $\leq$ is compatible. But since x is an atom, either $\overleftarrow{y}x = 0$ or $\overleftarrow{y}x = x$. If $\overleftarrow{y}x = 0$, then $\overleftarrow{y} = 0$ since $\overleftarrow{y} \leq \overleftarrow{x}$. Thus, $y = 0$ and we are done. If $\overleftarrow{y} \neq 0$, then $\overleftarrow{y}x = x$. Since $\overleftarrow{x}$ is the smallest element r of C such that $rx = x$, we must have that $\overleftarrow{y} = \overleftarrow{x}$. Now, $y \leq \overleftarrow{x}$ implies that there exists h in R such that $y + h = \overleftarrow{x}$. Since sums are disjoint in R, $\overleftarrow{x} = \overleftarrow{\overleftarrow{x}} = \overleftarrow{y+h} = \overleftarrow{y} + \overleftarrow{h}$ as a result of observation 3.24. Thus,

$$
\begin{aligned}
0 &= \overleftarrow{y}\,\overleftarrow{h}, \text{ since sums are disjoint in } R \\
&= \overleftarrow{x}\,\overleftarrow{h} \\
&= (\overleftarrow{y} + \overleftarrow{h})\overleftarrow{h} \\
&= \overleftarrow{y}\,\overleftarrow{h} + \overleftarrow{h}\,\overleftarrow{h} \\
&= 0 + \overleftarrow{h} \\
&= \overleftarrow{h}.
\end{aligned}
$$

Hence, $h = 0$ and so $y = \overleftarrow{x}$. Therefore, $\overleftarrow{x}$ is in $A(R)$.

Now, let $\overleftarrow{x}$ be in $A(R)$. Let y be any element of R such that $0 \leq y \leq x$. By observation 3.25(1), $0 \leq \overleftarrow{y} \leq \overleftarrow{x}$. But since $\overleftarrow{x}$ is in $A(R)$, then $\overleftarrow{y} = 0$ or $\overleftarrow{y} = \overleftarrow{x}$. If $\overleftarrow{y} = 0$, then $y = 0$ and we are done. If $\overleftarrow{y} = \overleftarrow{x}$, then we have more work to do. Now, $y \leq x$ implies that there exists h in R such that $y + h = x$. Since sums are disjoint in R, $\overleftarrow{x} = \overleftarrow{y+h} = \overleftarrow{y} + \overleftarrow{h}$, as a result of observation 3.24. But $\overleftarrow{y} = \overleftarrow{x}$, and so $\overleftarrow{h} = 0$, which implies that $h = 0$ and thus that $y = x$. Therefore, x is in $A(R)$.

Notice that in the above observation, the condition that sums be disjoint in R is necessary. Refering back to counterexample 5.19, we observe that sums are not disjoint. Here, z is not an atom, yet $\overleftarrow{z} = 1$ which is an atom.

5.21 OBSERVATION. Let R be a so-ring in which sums are disjoint. If x and z are atoms of R such that $xz \neq 0$, then xz is also an atom of R.

PROOF. Let x, z be elements of $A(R)$ such that $xz \neq 0$. Let y be an element of R such that $0 \leq y \leq xz$. Then,

$$0 \leq \overleftarrow{y} \leq \overleftarrow{xz}, \text{ by observation 3.25(1)}$$
$$\leq \overleftarrow{x}, \text{ by observation 3.26(1).}$$

But by observation 5.20, $\overleftarrow{x}$ is in $A(R)$, and so $\overleftarrow{xz} = \overleftarrow{x}$. Thus, $\overleftarrow{y} = 0$ or $\overleftarrow{y} = \overleftarrow{x}$. If $\overleftarrow{y} = 0$, then $y = 0$ and we are done. If $\overleftarrow{y} = \overleftarrow{x}$, then we have more work to do. Now, $y \leq xz$ implies that there exists h in R such that $y + h = xz$. Since sums are disjoint in R, $\overleftarrow{x} = \overleftarrow{xz} = \overleftarrow{y+h} = \overleftarrow{y} + \overleftarrow{h}$, as a result of observation 3.24. But $\overleftarrow{y} = \overleftarrow{x}$, and so $\overleftarrow{h} = 0$, which implies that $h = 0$ and thus that $y = xz$. Therefore, xz is in $A(R)$.

Having established the preliminary results, we are now ready to construct a D-representation of a so-ring which satisfies properties 1 through 4.

To construct the D-representation, we require that the center C of R satisfy the following infinite distributive laws:

$$r \wedge \left(\bigvee_i x_i\right) = \bigvee_i r \wedge x_i;$$

$$\left(\bigvee_i x_i\right) \wedge r = \bigvee_i x_i \wedge r.$$

These infinite distributive laws hold in any complete Boolean lattice [Birkhoff, 1967, V.5.16], and so we will make the assumption that C is a complete Boolean lattice. Hence, R automatically has domains and ranges.

5.22 THEOREM. Let R be an atomic so-ring in which sums are disjoint and whose center C is a complete Boolean lattice. Then R has a faithful D-representation.

PROOF. Let $D = \{d \in A(R): d \neq 1\}$, and let $D' = D \cup \{0,1\}$. For x in D', define $\tau_x: D \to D: d \mapsto dx$. If $dx = 0$, we say that τ_x is undefined at d. Hence, for $x = 0$, τ_x is the totally undefined function. For $x = 1$, τ_x is the identity function, since 1 is the identity element of R. For x in D, τ_x is well-defined as a result of observation 5.21. For now, define $\theta: D' \to Pfn(D,D): x \mapsto \tau_x$; we will extend θ to include all of R later on. First, we prove the following lemmas.

5.23 LEMMA. The mapping θ is a monoid homomorphism from D' to $Pfn(D,D)$.

PROOF. For x, y in D', xy is also in D', since

$$xy = \begin{cases} x, & \text{if } y = 1; \\ y, & \text{if } x = 1; \\ \text{an atom}, & \text{if } x, y \in A(R) \text{ and } xy \neq 0, \text{ by observation 5.21;} \\ 0, & \text{otherwise.} \end{cases}$$

Hence, $(xy)\theta$ is well-defined. We now show that $(x\theta)(y\theta) = (xy)\theta$ by the following set of equivalences:

$$e \in \{d \in D: d(x\theta)(y\theta) \text{ is defined}\}$$

if and only if

$$e \in \{d \in D: d(x\theta) \text{ is defined}\} \text{ and } ex \in \{d \in D: d(y\theta) \text{ is defined}\}$$

if and only if

$$e \in \{d \in D: d((xy)\theta) \text{ is defined.}\}$$

Also, θ maps 1 to the identity function from D to D. Therefore, θ is a monoid homomorphism from D' to $Pfn(D,D)$.

5.24 LEMMA. If x, y are elements of D' such that $\overleftarrow{x}\,\overleftarrow{y} = 0$, then $\{d \in D: dx \text{ is defined}\} \cap \{d \in D: dy \text{ is defined}\} = \emptyset$.

PROOF. Suppose $e \in \{d \in D: dx \text{ is defined}\} \cap \{d \in D: dy \text{ is defined}\}$. Then $ex \neq 0$ and $ey \neq 0$. This implies that $\overrightarrow{e}\,\overleftarrow{x} \neq 0$ and $\overrightarrow{e}\,\overleftarrow{y} \neq 0$. Since e, x, and y are atoms, $\overrightarrow{e}$, $\overleftarrow{x}$, and $\overleftarrow{y}$ are also atoms, by observation 5.20. Then, by observation

5.21, $\overrightarrow{e}\overleftarrow{x}$ is an atom. Thus, since $\overrightarrow{e}\overleftarrow{x} \leq \overrightarrow{e}$, $\overrightarrow{e}\overleftarrow{x} \leq \overleftarrow{x}$, and $\overrightarrow{e}\overleftarrow{x} \neq 0$, we must have that $\overrightarrow{e} = \overleftarrow{x}$. Similarly, $\overrightarrow{e} = \overleftarrow{y}$. But this is a contradiction, since $\overleftarrow{x}\overleftarrow{y} = 0$. Therefore, $\{d \in D: dx \text{ is defined}\} \cap \{d \in D: dy \text{ is defined}\} = \emptyset$.

Now, for any x in R, let $S_x = \{0\} \cup \{y \in A(R): y \leq x\}$. Since R is atomic, any x in R may be written as $x = \sum(y: y \in S_x)$. Since sums are disjoint in R, this implies that for any v, w in S_x such that $v \neq w$, $\overleftarrow{v}\overleftarrow{w} = 0$. This permits us to extend θ to all of R as follows. For any x in R, define $x\theta = \sum(y: y \in S_x)\theta = \sum(y\theta: y \in S_x)$. This sum is well-defined in $Pfn(D, D)$ as a result of lemma 5.24.

We now show that $\theta: R \to Pfn(D, D)$ is a monoid homomorphism. Let x, z be elements of R. Since R is atomic, $x = \sum(y: y \in S_x)$, $z = \sum(w: w \in S_z)$, and $xz = \sum(v: v \in S_{xz})$. Also, $xz = \sum(y: y \in S_x)\sum(w: w \in S_z) = \sum(yw: y \in S_x, w \in S_z)$. Hence, $yw \leq xz$ for any y in S_x and any w in S_z. Furthermore, since each of y and w is either 0 or an atom of R, $yw = 0$ or yw is an atom of R, by observation 5.21. Thus, $(yw: y \in S_x, w \in S_z)$ is contained in S_{xz}. If v is in S_{xz}, then by observation 3.26(1) $\overleftarrow{v} \leq \overleftarrow{xz}$, which implies that $\overleftarrow{v}\,\overleftarrow{xz} = \overleftarrow{v}$. Suppose v is not in $(yw: y \in S_x, w \in S_z)$. Then $v \neq 0$. Thus,

$$
\begin{aligned}
\overleftarrow{v} &= \overleftarrow{v}\,\overleftarrow{xz} \\
&= \overleftarrow{v}\,\overleftarrow{\textstyle\sum(yw: y \in S_x, w \in S_z)} \\
&= \overleftarrow{v}\left(\bigvee(\overleftarrow{yw}: y \in S_x, w \in S_z)\right), \text{ by observation 3.23(1)} \\
&= \bigvee(\overleftarrow{v}\,\overleftarrow{yw}: y \in S_x, w \in S_z), \text{ since } C \text{ is complete} \\
&= 0, \text{ since sums are disjoint in } R.
\end{aligned}
$$

But this is a contradiction, since $v \neq 0$. Hence, $S_{xz} = (yw: y \in S_x, w \in S_z)$.

Therefore,

$$\begin{aligned}(x\theta)(z\theta) &= \left(\sum(y: y \in S_x)\theta\right)\left(\sum(w: w \in S_z)\theta\right) \\ &= \sum(y\theta: y \in S_x)\sum(w\theta: w \in S_z) \\ &= \sum((y\theta)(w\theta): y \in S_x, w \in S_z) \\ &= \sum((yw)\theta: y \in S_x, w \in S_z), \text{ by lemma 5.23} \\ &= \sum(yw: y \in S_x, w \in S_z)\theta \\ &= (xz)\theta.\end{aligned}$$

We have already shown, in lemma 5.23, that $1\theta = 1$. Therefore, $\theta: R \to Pfn(D, D)$ is a monoid homomorphism.

Next, we must show that θ is additive. Let $(x_i: i \in I)$ be a summable family in R. Then $\sum_i x_i = \sum(y: y \in S_{\Sigma_i x_i})$ and $\sum_i x_i = \sum_i(\sum(y: y \in S_{x_i}))$ as well. For each i in I and for each y in S_{x_i}, y is either 0 or an atom of R, $y \leq x_i$, and thus $y \leq \sum_i x_i$. Hence, $\bigcup_i(y: y \in S_{x_i})$ is contained in $S_{\Sigma_i x_i}$. If v is in $S_{\Sigma_i x_i}$, then by observation 3.26(1), $\overleftarrow{v} \leq \overleftarrow{\sum_i x_i}$ and so $\overleftarrow{v} = \overleftarrow{v}\,\overleftarrow{\sum_i x_i}$. Suppose v is not in $\bigcup_i(y: y \in S_{x_i})$. Then $v \neq 0$. Thus,

$$\begin{aligned}\overleftarrow{v} &= \overleftarrow{v}\,\overleftarrow{\textstyle\sum_i x_i} \\ &= \overleftarrow{v}\,\overleftarrow{\textstyle\sum_i(\sum(y: y \in S_{x_i}))} \\ &= \overleftarrow{v}\left(\bigvee_i(\overleftarrow{\textstyle\sum(y: y \in S_{x_i})})\right), \text{ by observation 3.23(1)} \\ &= \overleftarrow{v}\left(\bigvee_i\left(\bigvee(\overleftarrow{y}: y \in S_{x_i})\right)\right), \text{ by observation 3.23(1)} \\ &= \bigvee_i \overleftarrow{v}\left(\bigvee(\overleftarrow{y}: y \in S_{x_i})\right), \text{ since } C \text{ is complete} \\ &= \bigvee_i\left(\bigvee(\overleftarrow{v}\,\overleftarrow{y}: y \in S_{x_i})\right), \text{ since } C \text{ is complete} \\ &= 0, \text{ since sums are disjoint in } R.\end{aligned}$$

But this is a contradiction, since $v \neq 0$. Hence, $S_{\Sigma_i x_i} = \bigcup_i(y: y \in S_{x_i}) = (y: y \in$

$\bigcup_i S_{x_i})$. Thus,

$$\begin{aligned}\left(\sum_i x_i\right)\theta &= \left(\sum(y: y \in \bigcup_i S_{x_i})\right)\theta \\ &= \sum(y\theta: y \in \bigcup_i S_{x_i}) \\ &= \sum_i \left(\sum(y\theta: y \in S_{x_i})\right) \\ &= \sum_i \left(\sum(y: y \in S_{x_i})\theta\right) \\ &= \sum_i x_i\theta,\end{aligned}$$

and so $\theta: R \to Pfn(D, D)$ is an additive map.

Therefore, θ is a D-representation of R. We now show that it is faithful. Let x, z be two elements of R such that $x\theta = z\theta$. Then $\tau_x = \tau_z$. Let $S_1 = \{0\} \cup \{y \in A(R): y \leq 1\}$. Then,

$$\begin{aligned}x &= 1x \\ &= \left(\sum(y: y \in S_1)\right)x, \\ &= \sum(yx: y \in S_1) \\ &= \sum(yz: y \in S_1), \text{ since } \tau_x = \tau_z \\ &= \left(\sum(y: y \in S_1)\right)z \\ &= 1z \\ &= z.\end{aligned}$$

Therefore, θ is injective, and so θ is a faithful D-representation of R.

As a corollary, we exhibit a wider class of so-rings such that each one possesses a full sub-so-ring for which a D-representation may be constructed as in theorem 5.22. Each so-ring R in this class satisfies the following two constraints.

The first constraint, as in theorem 5.22, is that the center C of R must satisfy the following infinite distributive laws:

$$r \wedge \left(\bigvee_i x_i \right) = \bigvee_i r \wedge x_i;$$

$$\left(\bigvee_i x_i \right) \wedge r = \bigvee_i x_i \wedge r.$$

Hence, we assume that C is a complete Boolean lattice. Thus, R automatically has domains and ranges.

The second constraint which we impose on R is that if r, s are in C and if $rs = 0$, then for any x in R, $\overleftarrow{xr}\,\overleftarrow{xs} = 0$. This constraint is satisfied, for example, by all so-rings in which the multiplicative operation is commutative, and by all so-rings in which sums are disjoint. We prove the latter claim. Assume R is a so-ring in which sums are disjoint, and let r, s be elements of C such that $rs = 0$. Then, $s \leq r'$, which implies that $r + s$ exists, since $r + r' = 1$ exists. Hence, for any x in R, $x(r+s) = xr + xs$, and since sums are disjoint in R, $\overleftarrow{xr}\,\overleftarrow{xs} = 0$.

Thus, we see that both of these constraints are met by so-rings which satisfy the hypotheses of theorem 5.22.

Before we can show that each so-ring satisfying the above two constraints possesses a full sub-so-ring for which a D-representation may be constructed, we need to show that for each such so-ring R, restricting the additive operation so that sums are disjoint yields a full sub-so-ring of R.

5.25 OBSERVATION. Let R be a so-ring such that its center C is a complete Boolean lattice, and such that if r, s are in C and $rs = 0$, then for any x in R, $\overleftarrow{xr}\,\overleftarrow{xs} = 0$. Let $\sum$ be the additive operation on R, and restrict $\sum$ to disjoint families, that is, define

$$\widehat{\sum}(x_i : i \in I) = \begin{cases} \sum(x_i : i \in I), & \text{if } (x_i : i \in I) \text{ is a summable family in } R \\ & \text{and if } \overleftarrow{x_i}\,\overleftarrow{x_j} = 0 \text{ for } j \neq i; \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

Then R with $\widehat{\sum}$ as the additive operation is a so-ring.

PROOF. First, we must show that $(R, \widehat{\sum})$ is a partial monoid. For any x in R, $\widehat{\sum}x = \sum x = x$, and so the unary sum axiom is satisfied.

To show that the partition-associativity axiom holds requires more work. Suppose that $(x_i : i \in I)$ is a family in R such that $\widehat{\sum}_i (x_i : i \in I)$ exists. This implies that $\sum (x_i : i \in I)$ exists and that $\overleftarrow{x_i}\,\overleftarrow{x_j} = 0$ for $j \neq i$. Let $(I_j : j \in J)$ be a partition of I. Since partition-associativity holds in $(R, \sum)$, $\sum (x_i : i \in I_j)$ exists for each j in J, $\sum (\sum (x_i : i \in I_j) : j \in J)$ exists, and $\sum (x_i : i \in I) = \sum (\sum (x_i : i \in I_j) : j \in J)$. Now, for each j in J, $(x_i : i \in I_j)$ is contained in $(x_i : i \in I)$. Thus, for x_k, x_l in $(x_i : i \in I_j)$, we have that $\overleftarrow{x_k}\,\overleftarrow{x_l} = 0$ for $l \neq k$. Hence, $\widehat{\sum} (x_i : i \in I_j)$ exists for each j in J. Thus, for $k \neq j$, $\overleftarrow{\widehat{\sum} (x_i : i \in I_j)}\;\overleftarrow{\widehat{\sum} (x_l : l \in I_k)}$

$$= \bigvee \left(\overleftarrow{x}_i : i \in I_j\right) \wedge \bigvee \left(\overleftarrow{x}_l : l \in I_k\right), \text{ by observation 3.23(1)}$$
$$= \bigvee \left(\bigvee \left(\overleftarrow{x}_i : i \in I_j\right) \wedge \overleftarrow{x_l} : l \in I_k\right), \text{ since } C \text{ is complete}$$
$$= \bigvee \left(\bigvee \left(\overleftarrow{x_i} \wedge \overleftarrow{x_l} : i \in I_j\right) : l \in I_k\right), \text{ since } C \text{ is complete}$$
$$= \bigvee \left(\bigvee \left(\overleftarrow{x_i}\,\overleftarrow{x_l} : i \in I_j\right) : l \in I_k\right), \text{ by theorem 3.11}$$
$$= 0, \text{ since } \overleftarrow{x_i}\,\overleftarrow{x_l} = 0 \text{ for } l \neq i.$$

Therefore, $\widehat{\sum}\left(\widehat{\sum} (x_i : i \in I_j) : j \in J\right)$ exists and equals $\widehat{\sum} (x_i : i \in I)$.

Now, suppose that for each j in J, $\widehat{\sum} (x_i : i \in I_j)$ exists and that $\widehat{\sum}\left(\widehat{\sum} (x_i : i \in I_j) : j \in J\right)$ exists. This implies that for each j in J, $\sum (x_i : i \in I_j)$ exists and that $\sum (\sum (x_i : i \in I_j) : j \in J)$ exists, which in turn imply that $\sum (x_i : i \in I)$ exists and equals $\sum (\sum (x_i : i \in I_j) : j \in J)$, since partition-associativity holds in $(R, \sum)$. Thus,

$$
\begin{aligned}
0 &= \overleftarrow{\widehat{\sum}(x_i\colon i \in I_j)}\,\overleftarrow{\widehat{\sum}(x_l\colon l \in I_k)} \\
&= \bigvee\left(\overleftarrow{x}_i\colon i \in I_j\right) \wedge \bigvee\left(\overleftarrow{x}_l\colon l \in I_k\right), \text{ by observation 3.23(1)} \\
&= \bigvee\left(\bigvee\left(\overleftarrow{x}_i\colon i \in I_j\right) \wedge \overleftarrow{x_l} : l \in I_k\right), \text{ since } C \text{ is complete} \\
&= \bigvee\left(\bigvee\left(\overleftarrow{x_i} \wedge \overleftarrow{x_l} : i \in I_j\right) : l \in I_k\right), \text{ since } C \text{ is complete} \\
&= \bigvee\left(\bigvee\left(\overleftarrow{x_i}\,\overleftarrow{x_l} : i \in I_j\right) : l \in I_k\right), \text{ by theorem 3.11.}
\end{aligned}
$$

Thus, for $l \neq i$, $\overleftarrow{x_i}\,\overleftarrow{x_l} = 0$, which in conjunction with the existence of $\sum(x_i\colon i \in I)$, implies that $\widehat{\sum}(x_i\colon i \in I)$ exists and equals $\widehat{\sum}\left(\widehat{\sum}(x_i\colon i \in I_j) : j \in J\right)$. Therefore, the partition-associativity axiom is satisfied by $(R, \widehat{\sum})$, and so $(R, \widehat{\sum})$ is a partial monoid.

Clearly, $(R, \widehat{\sum})$ satisfies conditions (1), (2), and (3) of definition 2.18. Hence, $(R, \widehat{\sum})$ is a partial sub-monoid of $(R, \sum)$, and by observation 2.20, $(R, \widehat{\sum})$ is a sub-so-monoid of $(R, \sum)$.

Now, we must demonstrate that the distributive laws hold in order to show that $(R, \widehat{\sum}, \circ, 1)$ is a so-ring. Suppose that $\widehat{\sum}_i x_i$ exists. Then $\sum_i x_i$ exists and $\overleftarrow{x_i}\,\overleftarrow{x_j} = 0$ for $j \neq i$. Since the distributive laws hold in $(R, \sum, \circ, 1)$, $(\sum_i x_i)y = \sum_i x_i y$ for any y in R. Furthermore, $(\widehat{\sum}_i x_i)y$ exists for any y in R. By observation 3.26(1), $\overleftarrow{x_i y} \leq \overleftarrow{x_i}$. Thus, $\overleftarrow{x_i y}\,\overleftarrow{x_j y} \leq \overleftarrow{x_i}\,\overleftarrow{x_j} = 0$ for $j \neq i$. Therefore, $\widehat{\sum}_i x_i y$ exists and equals $(\widehat{\sum}_i x_i)y$.

The existence of $\widehat{\sum}_i x_i$ also implies that $y(\widehat{\sum}_i x_i)$ exists for any y in R, while the existence of $\sum_i x_i$ implies that $y(\sum_i x_i) = \sum_i y x_i$ for any y in R, since the distributive laws hold in $(R, \sum, \circ, 1)$. Now,

$$
\begin{aligned}
\overleftarrow{y x_i} &= \overleftarrow{y\,\overleftarrow{x}_i x_i}, \text{ by observation 3.21(1)} \\
&\leq \overleftarrow{y\,\overleftarrow{x_i}}, \text{ by observation 3.26(1).}
\end{aligned}
$$

Hence, for $j \neq i$, $\overleftarrow{y x_i}\,\overleftarrow{y x_j} \leq \overleftarrow{y\,\overleftarrow{x_i}}\,\overleftarrow{y\,\overleftarrow{x_j}} = 0$, since $\overleftarrow{x_i}$, $\overleftarrow{x_j}$ are in C and $\overleftarrow{x_i}\,\overleftarrow{x_j} = 0$. Therefore, $\widehat{\sum}_i y x_i$ exists and equals $y(\widehat{\sum}_i x_i)$. We have shown that the right and left distributive laws hold in $(R, \widehat{\sum}, \circ, 1)$, and therefore that $(R, \widehat{\sum}, \circ, 1)$ is a sub-so-ring of $(R, \sum, \circ, 1)$.

5.26 COROLLARY. Let R be so-ring such that its center C is a complete Boolean lattice, and such that if r, s are in C and $rs = 0$, then for any x in R, $\overleftarrow{xr}\,\overleftarrow{xs} = 0$. If the full sub-so-ring $(R, \widehat{\sum}, \circ, 1)$ is atomic, then it has a faithful D-representation.

Fundamentally Representable Orderings

In the study of semigroups, orderings and representations are closely connected. In this section, we present a simple characterisation (due to Schein) of any quasi-order on a semigroup S, for which there exists a D-representation of S preserving and reflecting the quasi-order. (By a *D-representation* of a semigroup S, we mean a semigroup homomorphism $\phi: S \to Pfn(D, D)$ for some set D. Note that this definition is analogous to definition 5.14 for so-rings.) We then show that the same characterisation applied to a quasi-order on a so-ring R is necessary but not sufficient to show that there exists a D-representation of R which preserves and reflects the quasi-order. To describe the characterisation for semigroups, we need the following definitions. [1] The terminology in the first two definitions follows that of Vagner [1956].

5.27 DEFINITION. If S is a semigroup with a D-representation ϕ, then the *fundamental quasi-order relation* for ϕ is the binary relation

$$\preceq_\phi = \{(x, y): x, y \in S \text{ and } x\phi \leq y\phi\}.$$

[1] These definitions are given in the context of semigroups but may easily be extended to the context of so-rings simply by replacing each occurrence of "semigroup" with "so-ring" and each occurrence of "ϕ" with "θ" with the understanding that ϕ is a semigroup homomorphism and θ is a so-ring homomorphism.

5.28 DEFINITION. Let S be a semigroup, and let $\preceq$ be a quasi-order defined on S. Then $\preceq$ is said to be *fundamentally representable* if there exists a D-representation ϕ of S such that $x \preceq y$ if and only if $x\phi \leq y\phi$. Such a semigroup is said to be *fundamentally ordered* by $\preceq$.

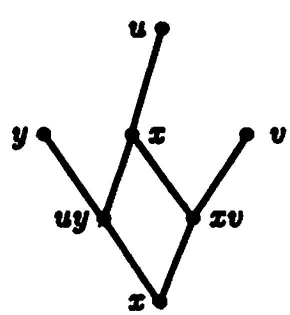
Hence, as a consequence of observation 5.2, any so-ring which possesses a faithful D-representation is fundamentally ordered by its sum-ordering.

5.29 DEFINITION. A quasi-order $\preceq$ on a semigroup S is said to be *weakly steady* if $z \preceq xv$, $z \preceq uy$, and $x \preceq u$ imply that $z \preceq xy$.

At first glance, this definition appears inscrutable. However, as we see in the examples below, weak steadiness is a property of the sum-ordering on any lattice and of the sum-ordering on partial functions. It was the sum-ordering on partial functions which provided the motivation for the construction of this definition in semigroup theory.

5.30 EXAMPLES.

(1) The sum-ordering on any lattice L is weakly steady. Let x, u, v, y, and z be elements of L such that $z \leq xv$, $z \leq uy$, and $x \leq u$.



Then $zxv = z$ and $xu = x$, since L is a lattice. The sum-ordering is of course compatible by observation 2.37, and since $z \leq uy$, we have that $xz \leq xuy = xy$. Using the facts that the multiplicative (meet) operation in a lattice is both commutative and idempotent, we find that since $zxv = z$, $xz = xzxv = zxxv = zxv = z$. Thus, $z = xz \leq xy$, and so $\leq$ is weakly steady in L.

(2) The sum-ordering on $Pfn(D,D)$ is weakly steady. To see this, let x, u, v, y, z, k, m, and h be elements of $Pfn(D,D)$ such that $z+k=xv$, $z+m=uy$, and $x+h=u$. Then $z \le xv$, $z \le uy$, and $x \le u$. Thus,

$$\begin{aligned} \overleftarrow{z} &\le \overleftarrow{xv}, \text{ by observation 3.25(1)} \\ &\le \overleftarrow{x}, \text{ by observation 3.26(1).} \end{aligned}$$

Also, by observation 3.26(1), $\overleftarrow{hy} \le \overleftarrow{h}$. Since sums are disjoint in $Pfn(D,D)$ and $x+h$ exists, $\overleftarrow{x}\,\overleftarrow{h} = 0$. Hence, $\overleftarrow{z}\,\overleftarrow{hy} \le \overleftarrow{x}\,\overleftarrow{hy} \le \overleftarrow{x}\,\overleftarrow{h} = 0$. Since $z+m$ exists, $\overleftarrow{z}\,\overleftarrow{m} = 0$. Thus, $z = \overleftarrow{z}\,z + \overleftarrow{z}\,\overleftarrow{m}\,m = \overleftarrow{z}\,(z+m) = \overleftarrow{z}\,uy = \overleftarrow{z}\,(x+h)y = \overleftarrow{z}\,xy + \overleftarrow{z}\,\overleftarrow{hy}\,hy = \overleftarrow{z}\,xy \le xy$. Therefore, $\le$ is a weakly steady partial order on $Pfn(D,D)$.

The sum-ordering on an arbitrary semigroup or so-ring need not be weakly steady, however.

5.31 COUNTEREXAMPLES.

(1) In $Mfn(D,D)$, the sum-ordering is not weakly steady. Let a, b, c, d, and e be elements of D, and define x, u, v, y, and z in $Mfn(D,D)$ as follows.

$$dx = \begin{cases} \{b\}, & \text{if } d=a; \\ \emptyset, & \text{otherwise;} \end{cases} \qquad du = \begin{cases} \{b,c\}, & \text{if } d=a; \\ \emptyset, & \text{otherwise;} \end{cases}$$

$$dv = \begin{cases} \{e\}, & \text{if } d=b; \\ \emptyset, & \text{otherwise;} \end{cases} \qquad dy = \begin{cases} \{e\}, & \text{if } d=c; \\ \emptyset, & \text{otherwise;} \end{cases} \qquad dz = \begin{cases} \{e\}, & \text{if } d=a; \\ \emptyset, & \text{otherwise.} \end{cases}$$

Then $x \le u$, $z \le xv$, and $z \le uy$. But $axy = by = \emptyset$, whereas $az = \{e\}$. Thus, $z \not\le xy$, and so $\le$ is not weakly steady on $Mfn(D,D)$.

(2) In $Mset(D,D)$, the sum-ordering is not weakly steady. Define x, u, v, y, and z in $Mset(D,D)$ as follows.

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad u = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$$

$$v = \begin{pmatrix} 0 & 0 \\ 2 & 2 \end{pmatrix} \qquad y = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \qquad z = \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}.$$

Then $x \le u$, $z \le xv$, and $z \le uy$. But

$$xy = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

Thus, $z \not\le xy$, and so $\le$ is not weakly steady on $Mset(D,D)$.

(3) In the so-ring **N**, with the usual finite addition and multiplication as the two so-ring operations, the sum-ordering is not weakly steady. Let $x = 1$, $u = 4$, $v = 6$, $y = 3$, and $z = 5$. Then $x \le u$, $z \le xv$, and $z \le uy$. But $z \not\le xy$, and so $\le$ is not weakly steady on **N**.

Having provided some motivation, we now state but do not prove Schein's result. (A complete version of the proof (in English) may be found in [Schein, 1979, pp. 136-145].)

5.32 THEOREM. (Schein, 1964, and 1979, Section 5) A quasi-order $\preceq$ on a semigroup S is fundamentally representable if and only if it is compatible and weakly steady.

An immediate consequence of this theorem is that for any so-ring R and any compatible, weakly steady, quasi-order $\preceq$ defined on R, the semigroup $(R, \circ)$ composed of the elements of R under the multiplicative operation $\circ$ on R is fundamentally ordered by $\preceq$. However, Schein's theorem does not guarantee that the so-ring $(R, \sum, \circ, 1)$ is fundamentally ordered by $\preceq$. For instance, consider the following:

5.33 COUNTEREXAMPLE. Let R be the so-ring $\{0,1\}$ with the following additive operation

$$\sum(x_i : i \in I) = \begin{cases} 0, & \text{if } x_i = 0 \text{ for all } i \in I; \\ 1, & \text{otherwise;} \end{cases}$$

and with the trivial multiplicative operation. The reader may easily verify that R has center $C = R$ and domains and ranges, and that the sum-ordering $\le$ is compatible and weakly steady. If R were to have a D-representation θ such that $x \le y$ if and only if $x\theta \le y\theta$, then θ would have to be faithful by observation 5.2.

But by counterexample 5.13, R cannot have a faithful D-representation, since 1 is a non-zero additive idempotent in R. Therefore, R is not fundamentally ordered by $\leq$.

However, we can show that compatibility and weak steadiness are necessary attributes of any fundamentally representable quasi-order on a so-ring R.

5.34 OBSERVATION. Let R be a so-ring which is fundamentally ordered by $\preceq$. Then $\preceq$ is a compatible and weakly steady quasi-order.

PROOF. Since $\preceq$ is fundamentally representable, there exists a D-representation θ of R such that $x \preceq y$ if and only if $x\theta \leq y\theta$. Let x, y be elements of R such that $x \preceq y$. Then $x\theta \leq y\theta$. Let z be any element of R. Then $z\theta$ is an element of $R\theta$. Since $\leq$ is left compatible, $(z\theta)(x\theta) \leq (z\theta)(y\theta)$. But this implies that $(zx)\theta \leq (zy)\theta$, as θ is a monoid homomorphism. Since $\preceq$ is fundamentally representable, $(zx)\theta \leq (zy)\theta$ implies that $zx \preceq zy$. Hence, $\preceq$ is left compatible. Similarly, it can be shown that the right compatibility of $\leq$ implies the right compatibility of $\preceq$, and so $\preceq$ is a compatible quasi-order.

Let x, u, v, y, and z be elements of R such that $z \preceq xv$, $z \preceq uy$, and $x \preceq u$. Using the facts both that $\preceq$ is fundamentally representable and that θ is a monoid homomorphism, we obtain that $z\theta \leq (xv)\theta = (x\theta)(v\theta)$, $z\theta \leq (uy)\theta = (u\theta)(y\theta)$, and $x\theta \leq u\theta$. Since $\leq$ is a weakly steady ordering on $Pfn(D, D)$ as we showed in example 5.30(2), $\leq$ is weakly steady on the sub-so-ring $R\theta$. Thus, $z\theta \leq (x\theta)(y\theta) = (xy)\theta$. This in turn implies that $z \preceq xy$, since $\preceq$ is fundamentally representable. Therefore, $\preceq$ is a weakly steady quasi-order.

Although the properties of compatibility and weak steadiness are not sufficient to prove that a quasi-order on a so-ring is fundamentally representable, they are, nevertheless, necessary as we demonstrated above. It is for this reason that we now examine some of the quasi-orders on a so-ring R which are compatible and weakly steady.

5.35 OBSERVATION. The smallest weakly steady, compatible, quasi-ordering on a so-ring is the identity ordering Δ.

5.36 OBSERVATION. For any so-ring R, the multiplicative ordering $\sqsubseteq$ is weakly steady.

PROOF. Let C be the center of R. Suppose that x, u, v, y, and z are elements of R such that $z \sqsubseteq xv$, $z \sqsubseteq uy$, and $x \sqsubseteq u$. Now, $z \sqsubseteq xv$ implies that there exists f in C such that $z = fxv$, $z \sqsubseteq uy$ implies that there exists g in C such that $z = guy$, and $x \sqsubseteq u$ implies that there exists e in C such that $x = eu$. Putting this all together, we have that $z = fxv = feuv = fe^2uv = efeuv = efxv = ez = eguy = geuy = gxy$. Hence, $z \sqsubseteq xy$, and so $\sqsubseteq$ is weakly steady.

5.37 OBSERVATION. If R is a so-ring in which sums are disjoint, then the sum-ordering $\leq$ on R is weakly steady.

PROOF. Suppose that z, x, v, u, and y are elements of R such that $z \leq xv$, $z \leq uy$, and $x \leq u$. Then there exist k, m, and h in R such that $z + k = xv$, $\overleftarrow{z}\,\overleftarrow{k} = 0$, $z + m = uy$, $\overleftarrow{z}\,\overleftarrow{m} = 0$, $x + h = u$, and $\overleftarrow{x}\,\overleftarrow{h} = 0$. Thus, $z + m = uy = (x + h)y = xy + hy$. Now, $z \leq xv$ implies that

$$\begin{aligned} \overleftarrow{z} &\leq \overleftarrow{xv}, \text{ by observation 3.25(1)} \\ &\leq \overleftarrow{x}, \text{ by observation 3.26(1).} \end{aligned}$$

Also, by observation 3.26(1), $\overleftarrow{hy} \leq \overleftarrow{h}$. Thus, $\overleftarrow{z}\cdot\overleftarrow{hy} \leq \overleftarrow{z}\,\overleftarrow{h} \leq \overleftarrow{x}\,\overleftarrow{h} = 0$. Therefore, $z = \overleftarrow{z}z + \overleftarrow{z}m = \overleftarrow{z}(z+m) = \overleftarrow{z}(xy+hy) = \overleftarrow{z}xy + \overleftarrow{z}hy = \overleftarrow{z}xy \leq xy$, and so $\leq$ is weakly steady on R.

Although for an arbitrary so-ring, there may not be a greatest weakly steady, compatible, partial order, we can show that for any inverse semiring, there does exist a largest such ordering. This fact is a consequence of the following theorem from semigroup theory.

5.38 THEOREM. (Goberstein, 1980, Theorem 4.1) The largest compatible, weakly steady, partial order on an inverse semigroup S is the multiplicative ordering $\sqsubseteq$.

We can show that for $Pfn(D, D)$, the sum-ordering is a maximal weakly steady, compatible, partial order containing the multiplicative ordering $\sqsubseteq$. However, we postpone the proof of this until the next section (observation 5.54), since we need results from that section in order to prove this result.

Amenable Orderings

5.39 DEFINITION. (McAlister, 1980, Definition 1.1) Let S be an inverse semigroup, and let $\preceq$ be a compatible partial order on S. Then S is said to be *left (right) amenably ordered* if for x, y in S, the relation $x \preceq y$ implies that $x^{-1}x \preceq y^{-1}y$ ($xx^{-1} \preceq yy^{-1}$). In such a case, the ordering $\preceq$ is said to be *left (right) amenable.* If $\preceq$ is both left and right amenable, then it is said to be *amenable* and S is said to be *amenably ordered.*

This definition may be stated identically for inverse semirings. However, we desire a notion of amenable which is applicable to a wider class of so-rings. Recall from observation 3.36(4) that if x is an inversible element of a so-ring R, then $x^{-1}x = \overrightarrow{x}$ and $xx^{-1} = \overleftarrow{x}$. Hence, rewriting the above definition to include so-rings with domains and ranges, we obtain:

5.40 DEFINITION. A compatible partial order $\preceq$ on a so-ring R with domains and ranges is *left (right) amenable* if for x, y in R, the relation $x \preceq y$ implies that $\overrightarrow{x} \preceq \overrightarrow{y}$ ($\overleftarrow{x} \preceq \overleftarrow{y}$). An ordering which is both left and right amenable is said to be *amenable.*

We now give several examples of amenable orderings on a so-ring R with domains and ranges.

5.41 OBSERVATION. In any so-ring R with domains and ranges, the sum-ordering $\leq$ is an amenable ordering.

PROOF. By definition, the sum-ordering on a so-ring is a partial order, and by observation 2.37, it is also compatible. Suppose x, y are elements of R such that $x \leq y$. The existence of both $\overrightarrow{x}$ and $\overrightarrow{y}$ implies that $\overrightarrow{x} \leq \overrightarrow{y}$, by observation 3.25(2). Hence, $\leq$ is left amenable. Similarly, the existence of $\overleftarrow{x}$ and $\overleftarrow{y}$ implies that $\overleftarrow{x} \leq \overleftarrow{y}$ by observation 3.25(1), and so $\leq$ is right amenable as well. Therefore, $\leq$ is an amenable ordering.

By observation 3.48, we know that the multiplicative ordering $\sqsubseteq$ on a so-ring R is a right compatible partial order. Furthermore, if R is an inverse so-ring, then we have that $\sqsubseteq$ is left compatible as well [Howie, 1976, V.2.4]. However, as we observed in example 3.49, in an arbitrary so-ring $\sqsubseteq$ need not be left compatible, and thus, $\sqsubseteq$ need not be amenable. But, in any case, we do have the following:

5.42 OBSERVATION. Let R be a so-ring with domains and ranges. If x, y are elements of R such that $x \sqsubseteq y$, then $\overrightarrow{x} \sqsubseteq \overrightarrow{y}$.
PROOF. Let C be the center of R. Let x, y be elements of R such that $x \sqsubseteq y$. Then there exists e in C such that $x = ey$. By observation 3.26(2), the existence of both $\overrightarrow{y}$ and $\overrightarrow{x} = \overrightarrow{ey}$ implies that $\overrightarrow{x} = \overrightarrow{ey} \leq \overrightarrow{y}$. This in turn implies that $\overrightarrow{x} = \overrightarrow{x}\,\overrightarrow{y}$, since $\overrightarrow{x}$ and $\overrightarrow{y}$ are in C. Therefore, $\overrightarrow{x} \sqsubseteq \overrightarrow{y}$.

Furthermore, if R is a so-ring in which $\sqsubseteq$ is left compatible, then R is amenably ordered, as we show below.

5.43 OBSERVATION. Let R be a so-ring with domains and ranges. If the multiplicative ordering $\sqsubseteq$ is a left compatible ordering on R, then $\sqsubseteq$ is an amenable ordering on R.
PROOF. In observation 3.48, it was established that $\sqsubseteq$ is a right compatible partial order on any so-ring. But we are assuming that $\sqsubseteq$ is also left compatible on R. Hence, $\sqsubseteq$ is a compatible partial order on R. It immediately follows, from observation 5.42, that $\sqsubseteq$ is left amenable. To show right amenability, we proceed as follows. Let C be the center of R. Let x, y be elements of R such that $x \sqsubseteq y$.

Then there exists e in C such that $x = ey$. Noting that

$$\begin{aligned} x &= ey \\ &= e\overleftarrow{y}y, \text{ by observation 3.21(1)} \\ &= \overleftarrow{y}ey, \text{ by observation 3.8(3)} \end{aligned}$$

and that both $\overleftarrow{y}$ and $\overleftarrow{x} = \overleftarrow{\overleftarrow{y}ey}$ exist, we may apply observation 3.26(1), yielding $\overleftarrow{x} = \overleftarrow{\overleftarrow{y}ey} \leq \overleftarrow{y}$. Hence, $\overleftarrow{x} = \overleftarrow{x}\overleftarrow{y}$, since $\overleftarrow{x}$ and $\overleftarrow{y}$ are in C. This implies that $\overleftarrow{x} \sqsubseteq \overleftarrow{y}$, and thus, $\sqsubseteq$ is right amenable as well. Therefore, $\sqsubseteq$ is an amenable ordering on R.

5.44 OBSERVATION. For a so-ring R with domains and ranges, the smallest amenable ordering is the identity ordering Δ.

5.45 OBSERVATION. Let R be a so-ring with domains and ranges and with center $C = \{0, 1\}$. Then the only amenable orderings between Δ and $\sqsubseteq$ are of the form:[2]

$$\preceq_A = \Delta \cup \{(0,1)\} \cup \{(0,x) : x \in A \subseteq R\}.$$

PROOF. Since $C = \{0,1\}$, $\sqsubseteq = \Delta \cup \{(0,x) : x \in R\}$. Thus, for any subset A of R, $\preceq_A = \Delta \cup \{(0,1)\} \cup \{(0,x) : x \in A\}$ is contained in $\sqsubseteq$. One can easily verify that $\preceq_A$ is a compatible partial order. We know by observation 5.44 that Δ is amenable, and we must show that $\{(0,1)\} \cup \{(0,x) : x \in A\}$ is also amenable. Since $C = \{0,1\}$, $\overleftarrow{x} = 1 = \overrightarrow{x}$ for all nonzero x in R. As $(0,1)$ is in $\preceq_A$, $0 \preceq_A x$ implies both that $0 \preceq_A \overrightarrow{x}$ and that $0 \preceq_A \overleftarrow{x}$. Hence, $\{(0,1)\} \cup \{(0,x) : x \in A\}$ is amenable. Therefore, $\preceq_A$ is amenable, and $\Delta \subseteq \preceq_A \subseteq \sqsubseteq$.

For an inverse semigroup S, it is easily shown that the multiplicative ordering $\sqsubseteq$ is an amenable ordering. Although this fact may be deduced from observation 5.43, it may also be proved directly in the following way. Suppose x, y are elements

[2] Here and in subsequent places we use the ordering symbol $\preceq$ not only to indicate the relation between two elements of R as in "$x \preceq y$" but also to represent the set of ordered pairs defining the ordering as in "$\preceq = \{(x,y) : x, y \in R \text{ and } x \preceq y\}$".

of S such that $x \sqsubseteq y$. Then by [Howie, 1976, V.2.4], $x^{-1} \sqsubseteq y^{-1}$. Furthermore, by [Howie, 1976, V.2.4], $\sqsubseteq$ is compatible and so both $x^{-1}x \sqsubseteq y^{-1}y$ and $xx^{-1} \sqsubseteq yy^{-1}$, which imply that $\sqsubseteq$ is an amenable ordering on S. It is possible to characterise all amenable orderings on S which contain the multiplicative ordering.

5.46 THEOREM. (McAlister, 1980, Theorem 4.1) Let S be an inverse semigroup with semilattice of idempotents E, and let $Czr(E) = \{x \in S: ex = xe$ for all $e \in E\}$ be the centralizer of E. Suppose that Q is contained in $Czr(E)$ and that Q is a subsemigroup of S such that
(1) $Q \cap Q^{-1} = E$ and
(2) $xQx^{-1} \subseteq Q$ for all x in S.
If $\preceq$ is a relation on S defined by $x \preceq y$ if and only if $x^{-1}x \sqsubseteq y^{-1}y$ and yx^{-1} is in Q, then $\preceq$ is an amenable ordering on S which extends $\sqsubseteq$. Conversely, every amenable ordering on S which extends $\sqsubseteq$ has this form for a unique subsemigroup Q of $Czr(E)$ which satisfies (1) and (2).

5.47 COROLLARY. (McAlister, 1980, Corollary 4.2) If S is a fundamental inverse semigroup, that is , $x^{-1}ex = y^{-1}ey$ for all e in E implies that $x = y$, then the multiplicative ordering $\sqsubseteq$ is a maximal amenable ordering on S.

We note that both theorem 5.46 and corollary 5.47 apply equally well to inverse semirings. Generalizing to so-rings with domains and ranges, we obtain the following results.

5.48 OBSERVATION. Let R be a so-ring with domains and ranges, and let $\preceq$ be a compatible partial order on R, which contains $\sqsubseteq$.
(1) If $\preceq$ is left amenable, then $x \preceq y$ implies that $\overrightarrow{x} \sqsubseteq \overrightarrow{y}$.
(2) If $\preceq$ is right amenable, then $x \preceq y$ implies that $\overleftarrow{x} \sqsubseteq \overleftarrow{y}$.
PROOF. (We prove (1) only; (2) is dual.) (1): The left amenability of $\preceq$ implies that if $x \preceq y$, then $\overrightarrow{x} \preceq \overrightarrow{y}$, which in turn implies that $\overrightarrow{x} = \overrightarrow{x}\overrightarrow{x} \preceq \overrightarrow{x}\overrightarrow{y}$, since $\preceq$ is left compatible. Now, $\overrightarrow{x}\overrightarrow{y} = \overrightarrow{y}\overrightarrow{x}$, since $\overrightarrow{x}$, $\overrightarrow{y}$ are in C. Thus, $\overrightarrow{x}\overrightarrow{y} \sqsubseteq \overrightarrow{x}$, and so $\overrightarrow{x}\overrightarrow{y} \preceq \overrightarrow{x}$, since $\sqsubseteq$ is contained in $\preceq$. By the antisymmetry

of $\preceq$, $\overrightarrow{x} = \overrightarrow{x}\overrightarrow{y}$, and therefore, $\overrightarrow{x} \sqsubseteq \overrightarrow{y}$.

5.49 OBSERVATION. Let R be an adequate so-ring. If $\preceq$ is a relation on R such that $x \preceq y$ if and only if $\overrightarrow{x} \sqsubseteq \overrightarrow{y}$, then $\preceq$ is a right compatible quasi-order which contains $\sqsubseteq$, such that $x \preceq y$ implies $\overrightarrow{x} \preceq \overrightarrow{y}$.

PROOF. It is easily seen that $\preceq$ is a quasi-order, since reflexivity and transitivity follow directly from the fact that $\sqsubseteq$ is a partial order. If $x \sqsubseteq y$, then by observation 5.42, $\overrightarrow{x} \sqsubseteq \overrightarrow{y}$, which in turn implies that $x \preceq y$. Hence, $\sqsubseteq$ is contained in $\preceq$. If $x \preceq y$, then $\overrightarrow{x} \sqsubseteq \overrightarrow{y}$, which implies that $\overrightarrow{x}c \sqsubseteq \overrightarrow{y}c$ for any c in R, since by observation 3.48, $\sqsubseteq$ is right compatible. But this in turn implies, by observation 5.42, that $\overrightarrow{\overrightarrow{x}c} \sqsubseteq \overrightarrow{\overrightarrow{y}c}$. Furthermore, by observation 3.28(2), $\overrightarrow{xc} = \overrightarrow{\overrightarrow{x}c} \sqsubseteq \overrightarrow{\overrightarrow{y}c} = \overrightarrow{yc}$. Hence, $xc \preceq yc$, and so $\preceq$ is right compatible. Assume that $x \preceq y$. Then $\overrightarrow{x} \sqsubseteq \overrightarrow{y}$. But by observation 3.29, $\overrightarrow{x} = \overrightarrow{\overrightarrow{x}}$ and $\overrightarrow{y} = \overrightarrow{\overrightarrow{y}}$. Hence, $\overrightarrow{\overrightarrow{x}} \sqsubseteq \overrightarrow{\overrightarrow{y}}$, which implies that $\overrightarrow{x} \preceq \overrightarrow{y}$.

We note that $\preceq$ as defined in observation 5.49 is not necessarily a left amenable ordering, since it need not be antisymmetric. Furthermore, $\preceq$ need not be left compatible. We illustrate both of these facts with examples from $Pfn(D,D)$. Let d_1 and d_2 be two elements of D and define two partial functions x and y as follows:

$$dx = \begin{cases} d_1, & \text{if } d = d_1; \\ \text{undefined}, & \text{otherwise;} \end{cases} \qquad dy = \begin{cases} d_1, & \text{if } d = d_2; \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

Now, $\overrightarrow{x} = \overrightarrow{y}$, and so $x \preceq y$ and $y \preceq x$. But clearly $x \neq y$, and so $\preceq$ is not antisymmetric. If $z = x$, then $zx = x$ and $zy = 0$. Hence, $\overrightarrow{zx} = \overrightarrow{x} \neq 0$ and $\overrightarrow{zy} = 0$, and so $\overrightarrow{zx} \not\sqsubseteq \overrightarrow{zy}$. This means that $zx \not\preceq zy$, and therefore that $\preceq$ is not left compatible.

Although we do not have a complete characterisation of all amenable orders which contain $\sqsubseteq$ on an arbitrary so-ring with domains and ranges, we do have a complete characterisation of such orders on particular so-rings.

5.50 OBSERVATION. For $Pfn(D,D)$, $Mfn(D,D)$, and $Mset(D,D)$, any compatible partial order containing the multiplicative order $\sqsubseteq$ is an amenable ordering.

PROOF. (We give the proof for $Pfn(D,D)$ only. The proofs for $Mfn(D,D)$ and $Mset(D,D)$ are similar.) Let $\preceq$ be a compatible partial order containing $\sqsubseteq$, and let x, y be elements of $Pfn(D,D)$ such that $x \preceq y$.

Case (1): $\overrightarrow{x} \not\sqsubseteq \overrightarrow{y}$: This implies that there exists d' in D such that $d' \in \overrightarrow{x}$ but $d' \notin \overrightarrow{y}$. Define z in $Pfn(D,D)$ as follows:

$$dz = \begin{cases} d', & \text{if } d = d'; \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

Thus, $yz = 0$ but $xz \neq 0$. Since $\preceq$ is right compatible, $x \preceq y$ implies that $xz \preceq yz = 0$. Now $xz \neq 0$, and so $0 \sqsubseteq xz$ which implies that $0 \preceq xz$, since $\preceq$ contains $\sqsubseteq$. By the antisymmetry of $\preceq$, we have that $0 = xz$, which is clearly a contradiction. Thus, $\overrightarrow{x} \sqsubseteq \overrightarrow{y}$ which implies that $\overrightarrow{x} \preceq \overrightarrow{y}$.

Case (2): $\overleftarrow{x} \not\sqsubseteq \overleftarrow{y}$: This implies that there exists d' in D such that $d' \in \overleftarrow{x}$ but $d' \notin \overleftarrow{y}$. Define z as in case (1). Then $zy = 0$ but $zx \neq 0$. Since $\preceq$ is left compatible, $x \preceq y$ implies that $zx \preceq zy = 0$. Now $zx \neq 0$, and so $0 \sqsubseteq zx$ which implies that $0 \preceq zx$, since $\preceq$ contains $\sqsubseteq$. By the antisymmetry of $\preceq$, we have that $0 = zx$, which is clearly a contradiction. Thus, $\overleftarrow{x} \sqsubseteq \overleftarrow{y}$ which implies that $\overleftarrow{x} \preceq \overleftarrow{y}$.

Therefore, $\preceq$ is an amenable ordering on $Pfn(D,D)$.

Recall from corollary 5.47 that the multiplicative ordering is a maximal amenable ordering on a fundamental inverse semigroup and hence on a fundamental inverse semiring. Generalising to so-rings with domains and ranges, we see that maximal amenable orderings may be much larger than the multiplicative ordering.

5.51 OBSERVATION. Let R be a so-ring with domains and ranges. Then the sum-ordering $\leq$ on R is a maximal amenable ordering containing the multiplicative ordering $\sqsubseteq$, if $\leq$ is a total ordering.

PROOF. By observation 3.50, $\leq$ contains $\sqsubseteq$, and by observation 5.41, $\leq$ is an amenable ordering on R. Furthermore, any total order is a maximal partial order on R. Thus, $\leq$ is a maximal amenable ordering on R, which contains $\sqsubseteq$.

5.52 OBSERVATION. Let R be a so-ring which is a distributive lattice. Then the sum-ordering $\leq$ (equal to the lattice ordering) is a maximal amenable ordering containing the multiplicative ordering $\sqsubseteq$.

PROOF. Let $\preceq$ be an amenable ordering on R containing $\leq$, and let x, y be two elements of R such that $x \preceq y$. Define $z = x \wedge y$, the lattice meet of x and y. Hence, $z \leq x$ which implies that $z \preceq x$. Since $\preceq$ is left compatible, $x = x \wedge x \preceq x \wedge y = z$, and so by the antisymmetry of $\preceq$, $x = z$. But this implies that $x \wedge y = x$, which in turn implies that $x \leq y$. Therefore, $x \preceq y$ implies that $x \leq y$ and thus that $\preceq = \leq$. Hence, $\leq$ is a maximal amenable ordering containing $\sqsubseteq$.

5.53 OBSERVATION. For $Pfn(D,D)$, $Mfn(D,D)$, and $Mset(D,D)$ the sum-ordering $\leq$ is a maximal amenable ordering containing the multiplicative ordering $\sqsubseteq$.

PROOF. (We prove the result only for $Pfn(D,D)$; the proofs for $Mfn(D,D)$ and $Mset(D,D)$ are similar.) By observation 5.41, $\leq$ is an amenable ordering, and by observation 3.50, $\leq$ contains $\sqsubseteq$. Now suppose that $\preceq$ is an amenable ordering which contains $\leq$, and let x, y be elements of $Pfn(D,D)$ such that $x \preceq y$.

Case (1): $\overrightarrow{x} \not\subseteq \overrightarrow{y}$. Then there exists $d' \in \overrightarrow{x}$ such that $d' \notin \overrightarrow{y}$. Define a partial function z such that

$$dz = \begin{cases} d', & \text{if } d = d'; \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

Then $yz = 0$, but $xz \neq 0$. Since $\preceq$ is right compatible, $x \preceq y$ implies that $xz \preceq yz = 0$. Now, since $xz \neq 0$, $0 \leq xz$, and since $\preceq$ extends $\leq$, $0 \preceq xz$. But by the antisymmetry of $\preceq$, $xz = 0$, which is a contradiction. Thus, $\overrightarrow{x} \leq \overrightarrow{y}$.

Case (2): $\overleftarrow{x} \not\subseteq \overleftarrow{y}$. Then there exists $d' \in \overleftarrow{x}$ such that $d' \notin \overleftarrow{y}$. Define a partial function z as in case (1). Then $zy = 0$, but $zx \neq 0$. Since $\preceq$ is left compatible, $x \preceq y$ implies that $zx \preceq zy = 0$. Now, since $zx \neq 0$, $0 \leq zx$, and since $\preceq$ extends $\leq$, $0 \preceq zx$. But by the antisymmetry of $\preceq$, $zx = 0$, which is a contradiction. Thus, $\overleftarrow{x} \leq \overleftarrow{y}$.

Now, suppose that $x \preceq y$ but that $x \not\leq y$. In cases (1) and (2), we have shown that $x \preceq y$ implies both that $\overrightarrow{x} \leq \overrightarrow{y}$ and that $\overleftarrow{x} \leq \overleftarrow{y}$. Thus, since $x \not\leq y$, there exists $d' \in \overleftarrow{x}$ and $d' \in \overleftarrow{y}$ such that $d'x \neq d'y$. Define z in $Pfn(D,D)$ as in case (1). Since $\preceq$ is left compatible, $x \preceq y$ implies that $zx \preceq zy$. This in turn implies that $\overrightarrow{zx} \preceq \overrightarrow{zy}$, since $\preceq$ is left amenable. But $\overrightarrow{zx} = d'x \neq d'y = \overrightarrow{zy}$, and so $\overrightarrow{zx} \not\leq \overrightarrow{zy}$. This, of course, a contradiction, by case (1). Thus, $x \leq y$.

Therefore, $x \preceq y$ implies that $x \leq y$, and so $\leq$ is a maximal amenable ordering containing $\sqsubseteq$.

5.54 OBSERVATION. For $Pfn(D,D)$, the sum-ordering $\leq$ is a maximal weakly steady, compatible, partial order containing $\sqsubseteq$.

PROOF. We know, by observation 5.50, that any compatible partial order on $Pfn(D,D)$ containing $\sqsubseteq$ is amenable. Hence, any weakly steady, compatible, partial order on $Pfn(D,D)$ containing $\sqsubseteq$ is amenable. Furthermore, by observation 5.53, $\leq$ is a maximal amenable ordering containing $\sqsubseteq$ for $Pfn(D,D)$. By example 5.30(2), $\leq$ is weakly steady on $Pfn(D,D)$. Therefore, for $Pfn(D,D)$, $\leq$ is a maximal weakly steady, compatible, partial order which contains $\sqsubseteq$.

Note that the above result does not hold for $Mfn(D,D)$ or $Mset(D,D)$, since by counterexamples 5.31, the sum-ordering is not weakly steady on either so-ring.

Although the sum-ordering is always an amenable ordering on a so-ring with domains and ranges, it need not be in general a maximal amenable ordering, as we demonstrate below.

5.55 COUNTEREXAMPLE. Let R be a so-ring with trivial addition and trivial multiplication, and with domains and ranges. In such a so-ring, $C = \{0,1\}$ and also $\sqsubseteq \; = \; \leq \; = \; \preceq_R \; = \Delta \cup \{(0,x): x \in R\}$. Furthermore, suppose that R contains at least three elements. Define $\preceq \; = \; \preceq_R \cup \{(y,z)\}$, where y, z are two elements of R

such that $y \neq 0 \neq z$ and $y \neq z$. It is easily shown that $\preceq$ is a compatible partial order. Furthermore, $\overrightarrow{y} = \overleftarrow{y} = 1 = \overrightarrow{z} = \overleftarrow{z}$, and so $\overrightarrow{y} \preceq \overrightarrow{z}$ and $\overleftarrow{y} \preceq \overleftarrow{z}$. Thus, $\preceq$ is an amenable ordering which properly contains $\sqsubseteq$ and $\leq$. Therefore, for this so-ring, $\leq$ is not a maximal amenable ordering.

In those cases in which the sum-ordering is a maximal amenable ordering, it need not be the greatest such ordering containing the multiplicative ordering. There may be several maximal amenable orderings which contain $\sqsubseteq$, and thus no greatest such ordering, as we now show.

5.56 COUNTEREXAMPLE. Let R be a so-ring with domains and ranges and with center $C = \{0, 1\}$. Then $\sqsubseteq = \preceq_R = \Delta \cup \{(0, x) : x \in R\}$. Furthermore, suppose that R contains at least three elements and that $\leq$ is a total order on R. (The real numbers $\mathbf{R}$ with the usual finite addition and multiplication is such a so-ring.) By observation 5.51, $\leq$ is a maximal amenable ordering containing $\sqsubseteq$. By the Duality Principle for partial orders, $\geq$, the converse of $\leq$ is also a maximal amenable ordering on R. But $\geq$ does not contain $\sqsubseteq$. We define a new relation on R as follows. For any x, y in R,

$$x \geq' y \text{ if } x = 0 \text{ or if } x \geq y \text{ and } y \neq 0.$$

It is easily demonstrated that $\geq'$ is a total order which contains $\sqsubseteq$, and that $\geq'$ is amenable. Therefore, by observation 5.51, $\geq'$ is a maximal amenable order on R containing $\sqsubseteq$, but $\geq'$ clearly does not contain $\leq$. Hence, there is no greatest amenable ordering on R which contains $\sqsubseteq$.

CHAPTER VI

OTHER INFINITE PARTIAL SUMS

In this chapter, we describe three different algebraic structures which possess infinite partial sums. These structures – generalised cardinal algebras, $\sum$-structures, and infinite sums in topological groups – are interesting as points of comparison with partial monoids, but are not cental to our study of so-rings.

Generalised Cardinal Algebras

Tarski's motivation for developing cardinal algebras stems from work on the arithmetic of cardinal numbers. He states, in the preface to his book *Cardinal Algebras* [1949], that results in this area appear to be of two types. The first are the very general theorems which have been established by invoking the axiom of choice. The second are results which have been arrived at by construction; these tend to be of a more restrictive nature, but are no less interesting. Using the constructive method, it is possible to derive results about cardinal addition using an arithmetic approach based on certain basic theorems. These derivations do not tend to be any more involved than derivations by other methods. Tarski notes that the basic theorems are rather like formal laws, which can be applied to other mathematical systems, in addition to cardinal arithmetic. Thus, he suggests developing a set of algebras for which these basic theorems are actually the defining postulates - the cardinal algebras.

D. S. Scott suggested that parallels might be drawn between cardinal algebras and partially-additive monoids,[1] although the motivation for developing each of these algebraic structures was indeed different. Following his suggestion, we investigated some relationships between the two structures.

6.1 DEFINITION. A *cardinal algebra* (or *CA*) $(A, +, \sum)$ is a set A together with a binary operation $+$ and an infinitary operation $\sum$, subject to the following postulates:[2]

(1) *(Finite Closure Postulate.)* If $a, b \in A$, then $a + b \in A$.

(2) *(Infinite Closure Postulate.)* If $a_0, a_1, \ldots, a_i, \ldots \in A$, then $\sum_{i<\infty} a_i \in A$.

(3) *(Associative Postulate.)* If $a_0, a_1, \ldots, a_i, \ldots \in A$, then $\sum_{i<\infty} a_i = a_0 + \sum_{i<\infty} a_{i+1}$.

(4) *(Commutative-Associative Postulate.)* If $a_0, a_1, \ldots, a_i, \ldots, b_0, b_1, \ldots, b_i, \ldots \in A$, then $\sum_{i<\infty}(a_i + b_i) = \sum_{i<\infty} a_i + \sum_{i<\infty} b_i$.

(5) *(Postulate of the Zero Element.)* There is an element $z \in A$ such that $a + z = z + a = a$ for every $a \in A$.

(6) *(Refinement Postulate.)* If $a, b, c_0, c_1, \ldots, c_i, \ldots \in A$ and $a + b = \sum_{i<\infty} c_i$, then there are elements $a_0, a_1, \ldots, a_i, \ldots, b_0, b_1, \ldots, b_i, \ldots \in A$ such that $a = \sum_{i<\infty} a_i$, $b = \sum_{i<\infty} b_i$, and $c_n = a_n + b_n$ for $n = 0, 1, 2, \ldots$.

(7) *(Remainder, or Infinite Chain, Postulate.)* If $a_0, a_1, \ldots, a_i, \ldots, b_0, b_1, \ldots, b_i, \ldots \in A$ and if $a_n = b_n + a_{n+1}$ for $n = 0, 1, 2, \ldots$, then there is an element $c \in A$ such that $a_n = c + \sum_{i<\infty} b_{n+i}$ for $n = 0, 1, 2, \ldots$.

6.2 EXAMPLES. Although the motivating example for cardinal algebras is indeed the cardinal numbers under cardinal addition, there are many other examples, among them the extended set of natural numbers as defined in example 2.12, the set of nonnegative real-valued functions over an arbitrary domain, and any countably complete Boolean algebra.

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[1] Personal communication, 1982.

[2] We use $\sum_{i<\infty}$ to denote the sum over a family indexed by N, and $\sum_{i<n}$ to denote the sum over a family indexed by the set $\{0, \ldots, n-1\}$. In addition, we use "$\in A$", in reference to $+$ and $\sum$, as an abbreviation for "defined and in A".

Note that in a CA all sums are defined. However, we are interested in a wider class of algebras – those for which $+$ and $\sum$ need not be totally defined. These algebras are referred to as *generalized cardinal algebras*, and they essentially obey the postulates for a CA with added premises and conclusions about existence of sums. More formally,

6.3 DEFINITION. A *generalised cardinal algebra* (or GCA) $(A, +, \sum)$ is a set A together with a binary operation $+$ and an infinitary operation $\sum$, both of which may be only partially defined, subject to the following postulates:[3]

(1) *(Associative Postulate.)* If $a_i \in A$ for every $i < \infty$ and $\sum_{i<\infty} a_i \in A$, then $\sum_{i<\infty} a_{i+1} \in A$ and $\sum_{i<\infty} a_i = a_0 + \sum_{i<\infty} a_{i+1}$.

(2) *(Commutative-Associative Postulate.)* If a_i, b_i, $a_i + b_i \in A$ for every $i < \infty$ and $\sum_{i<\infty}(a_i + b_i) \in A$, then $\sum_{i<\infty} a_i$, $\sum_{i<\infty} b_i \in A$ and $\sum_{i<\infty}(a_i + b_i) = \sum_{i<\infty} a_i + \sum_{i<\infty} b_i$.

(3) *(Postulate of the Zero Element.)* The statement remains unchanged for GCAs.

(4) *(Refinement Postulate.)* The statement is identical to that for CAs, with the additional hypothesis that $a + b \in A$.

(5) *(Remainder Postulate.)* The statement is identical to that for CAs, with the additional conclusion that $\sum_{i<\infty} b_{n+i} \in A$ for all $n < \infty$.

6.4 THEOREM. (Tarski, 1949, 5.24) An algebra $U = (A, +, \sum)$ is a CA if and only if it is a GCA which satisfies the infinite closure postulate.

There is a natural relation on the elements of a GCA that is analogous to the sum-ordering on the elements of a partial monoid.

6.5 DEFINITION. We say that $a \leq b$ if $a, b \in A$ and if there is a $c \in A$ such that $a + c = b$. The relation $\leq$ is always a partial order.

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[3] For the sake of continuity, we have retained the names of the CA postulates for the GCA postulates, even though the statements differ slightly between algebras. In the future, when we refer to a postulate by name, the context of the algebra under discussion will make clear the version of the postulate to be applied.

Since in a GCA all summable families have countable support and the relation $\leq$ is always a partial order, we can try to compare GCAs with ω-so-monoids. Note, however, that $\sum$ acts only on countable families in a GCA but acts on arbitrary families in an ω-so-monoid. Hence we cannot directly compare GCAs with ω-so-monoids. Our solution to this problem is to extend the definition of $\sum$ in a GCA to $\widehat{\sum}$ as follows. Let $(a_i : i \in I)$ be any family in a GCA and let $(a_i : i \in J)$ be its support. Define

$$\widehat{\sum}(a_i : i \in I) = \begin{cases} \sum(a_i : i \in J), & \text{if } J \text{ is countable;} \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

In a sense, all we have added to the definition of a GCA is the axiom that the sum of an arbitrary number of zeroes is still zero. Henceforth, we assume that all GCAs satisfy this property. We may now compare such GCAs with ω-so-monoids.

First, we ask whether all ω-so-monoids are also GCAs. The partition-associativity axiom together with the unary sum axiom immediately imply both the associative postulate and the commutative-associative postulate. The existence of the empty sum in an ω-so-monoid serves to satisfy the postulate of the zero element. However, we observe that neither the refinement postulate nor the remainder postulate need be obeyed in an ω-so-monoid. For instance,

6.6 COUNTEREXAMPLE. Let X be any set, and let A contain all subsets of X of cardinality different from 1. Define $\sum$ over A so that a family of such subsets is summable only if it contains countably many non-empty pairwise disjoint members, in which case $\sum$ is set union. Then $(A, \sum)$ is an ω-so-monoid. In particular, let $X = \{0, \ldots, 5\}$. Substituting for the variables in the statement of the refinement postulate, let $a = \{0, 1, 2, 3\}$, $b = \{4, 5\}$, $c_0 = \{0, 1, 2\}$, $c_1 = \{3, 4, 5\}$, and $c_n = \emptyset$ for $n > 1$. Then $a + b = \sum_{i<\infty} c_i$, and so the premises of the postulate are satisfied. Suppose that the conclusion of the postulate is also satisfied. That is, there exist $a_0, a_1, \ldots, a_i, \ldots, b_0, b_1, \ldots, b_i, \ldots \in A$ such that $a = \sum_{i<\infty} a_i$, $b = \sum_{i<\infty} b_i$, and $c_n = a_n + b_n$ for all $n < \infty$. Then it must be true that for $n > 1$, $a_n = \emptyset = b_n$.

Hence, $a = a_0 + a_1$, $b = b_0 + b_1$, $c_0 = a_0 + b_0$, and $c_1 = a_1 + b_1$.

Case (i): $a_0 = \emptyset$. This implies that $a_1 = a$, which in turn implies that $c_1 = a + b_1$. This is impossible since $a \not\subseteq c_1$.

Case (ii): a_0 contains at least two elements. Since c_0 contains three elements, and b_0 cannot have cardinality 1, then it must be true that $b_0 = \emptyset$. This in turn implies that $a_0 = c_0$. But since the cardinality of a is 4 and the cardinality of a_0 is 3, then a_1 must contain only a single element. This is by assumption impossible. Therefore, the ω-so-monoid $(A, \sum)$ does not obey the refinement postulate.

In attempting to satisfy the refinement postulate, one would like to construct the set of a_n s and the set of b_n s by decomposing each c_n into its contribution from a and its contribution from b. As we showed in the example above, it is not always possible to do this in an ω-so-monoid. However, it appears that a wide variety of ω-so-rings do obey the refinement postulate. In an ω-so-ring, one might hope to determine the contributions from a and from b in each c_n by finding elements r_n, s_n, t_n, and u_n, where $0 \le r_n, t_n \le 1$ and $0 \le s_n, u_n \le 1$, such that:

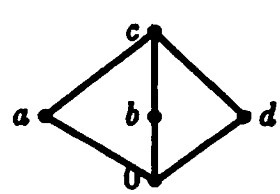
(1) $r_n c = a_n = t_n a$ and $s_n c = b_n = u_n b$.

(2) $r_n + s_n = 1$.

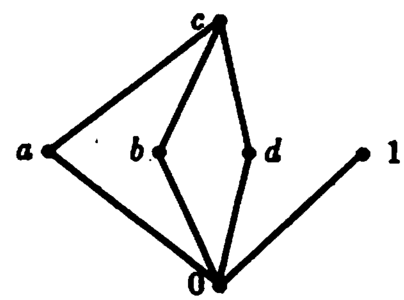
(3) $\sum_{i<\infty} t_i = 1 = \sum_{i<\infty} u_i$.

It is not sufficient, however, simply to be an ω-so-ring in order to satisfy the refinement postulate. It appears that it is also important to have a nontrivial multiplication as well as a rich structure on the set of x for which $0 \le x \le 1$. (Note, however, that both of these attributes need not be present simultaneously. For instance, refer back to example 2.34.)

6.7 COUNTEREXAMPLE. Consider the following poset $(P, \le)$ consisting of five distinct elements – 0, a, b, c, and d.



Note that this poset is a finite upper semilattice under the operation of supremum and as such is an ω-so-monoid. Using observation 2.49, we can extend this ω-so-monoid to an ω-so-ring



such that the multiplication is trivial.

Note first that $c = a \vee b$. Let $c_0 = b, c_1 = d$, and $c_n = 0$ for $n > 1$. Then, $c = c_0 \vee c_1$, and so the premises of the refinement postulate are satisfied. Let us suppose that the conclusion is true. Then $a = a_0 \vee a_1$, $b = b_0 \vee b_1$, $c_0 = a_0 \vee b_0$, $c_1 = a_1 \vee b_1$, and for $n > 1$, $a_n = 0 = b_n$. Now, $a_1 \vee b_1 = c_1 = d$ implies that $a_1 = 0$ or $a_1 = d$, since the only elements of P that are $\leq d$ are 0 and d itself. Since $a_0 \vee a_1 = a$ and $d \not\leq a$, $a_1 \neq d$. Thus, $a_1 = 0$ which in turn implies that $b_1 = d$. However, $b_0 \vee b_1 = b$, and since $d \not\leq b$, $b_1 \neq d$. Hence, we have a contradiction, and so such an ω-so-ring does not satisfy the refinement postulate.

It is a bit tricky to construct an ω-so-monoid which does not obey the remainder postulate. We first show that any ω-so-monoid which obeys the countable limit axiom "almost" satisfies the remainder postulate. Then, we give an example of an ω-so-monoid which fails to satisfy the remainder postulate.

6.8 THEOREM. Let $(A, \sum)$ be an ω-so-monoid which satisfies the countable limit axiom. If $a_0, a_1, \ldots, a_i, \ldots,\ b_0, b_1, \ldots, b_i, \ldots \in A$ and if $a_n = b_n + a_{n+1}$ for $n = 0, 1, 2, \ldots$, then $\sum_{i<\infty} b_{n+i}$ is defined and in A for all $n < \infty$, and there are elements c_n in A such that $a_n = c_n + \sum_{i<\infty} b_{n+i}$ for $n = 0, 1, 2, \ldots$.

PROOF. First, we show by induction that $\sum_{i<k} b_{n+i}$ is defined for all $n, k < \infty$. For $k = 1$, $\sum_{i<k} b_{n+i} = b_n$, by the unary sum axiom. Assume that for $k < m$,

$\sum_{i<k} b_{n+i}$ exists. Then, for $k < m+1$,

$$\begin{aligned} a_n &= b_n + a_{n+1} \\ &= b_n + b_{n+1} + a_{n+2} \\ &= \ldots \\ &= b_n + b_{n+1} + \ldots + b_{n+m-2} + a_{n+m-1} \\ &= b_n + b_{n+1} + \ldots + b_{n+m-2} + b_{n+m-1} + a_{n+m} \\ &= \left(\sum_{i<m} b_{n+i}\right) + a_{n+m} \\ &= \left(\sum_{i<m} b_{n+i}\right) + b_{n+m} + a_{n+m+1} \\ &= \left(\sum_{i<m+1} b_{n+i}\right) + a_{n+m+1}. \end{aligned}$$

We have thus shown that for all $n < \infty$, given that $\sum_{i<k} b_{n+i}$ is defined for all $k < m$, then it is defined for all $k < m+1$. Hence, it is defined for all $k < \infty$.

The next step is to show that $\sum_{i<\infty} b_{n+i}$ is defined. Since all subfamilies of a summable family are summable, and since $(b_{n+i} : i < k)$ is summable for all $n, k < \infty$, all finite subfamilies of $(b_{n+i} : i < \infty)$ are summable. Therefore, by the countable limit axiom, $(b_{n+i} : i < \infty)$ is summable.

Lastly, we must show that for each $n < \infty$, there is a c_n in A such that $a_n = c_n + \sum_{i<\infty} b_{n+i}$. Since $a_n = \sum_{i<k} b_{n+i} + a_{n+k}$ for all $n, k < \infty$, $\sum_{i<k} b_{n+i} \leq a_n$ for all $n, k < \infty$. Hence, for any $i < \infty$, b_{n+i} is a summand of a_n. Therefore, $\sum_{i<\infty} b_{n+i} \leq a_n$, which implies that for each $n < \infty$, there exists c_n in A such that $a_n = c_n + \sum_{i<\infty} b_{n+i}$.

However, there may fail to exist a single c in A such that $a_n = c + \sum_{i<\infty} b_{n+i}$ for all $n < \infty$, as is shown by the following:

6.9 COUNTEREXAMPLE. Let X be a set, and let A be the collection of all subsets of X with cardinality different from 1. If we define $\sum$ to be set union over families of countable support, then $(A, \sum)$ is an ω-so-monoid. In particular, let

$X = \mathbf{N}$. For each $n < \infty$, let $a_n = \{1\} \cup \{m \in \mathbf{N}: m$ is even and $m \geq 2n\}$, and let $b_n = \{2n, 2(n+1)\}$. Suppose there exists c in A such that $a_n = c + \sum_{i<\infty} b_{n+i}$ for each $n < \infty$. Then 1 is in c, since for all $n < \infty$, 1 is in a_n but is not in b_{n+i} for any $i < \infty$. Since c is a member of A, it must contain more than one element, and so, in addition to the number 1, it must contain some even number m, because for each $n < \infty$, the elements of a_n are all the even numbers $\geq 2n$ together with the number 1. Hence, $m = 2j$ for some $j < \infty$. But $2j \notin a_k$ for any $k > j$. Hence, m cannot be in c, and so $c = \{1\}$, which implies that c is not in A, a contradiction. Therefore, $(A, \sum)$ does not obey the remainder postulate.

We have thus shown that an ω-so-monoid need not satisfy all the postulates for a GCA. However, as we demonstrate below, all GCAs are ω-so-monoids.

6.10 OBSERVATION. The unary sum axiom is satisfied by every GCA.
PROOF. Suppose that $b_i = a$ for $i = 0$. Then $1 \cdot a = \sum_{i<1} a = \sum_{i<1} b_i = b_0 = a$.[4]

Before we can show that the partition-associativity axiom is also satisfied by any GCA, it is necessary to look more closely at generalized cardinal algebras.

Many theorems which apply to CAs can also be made to apply to GCAs, often with additional hypotheses and conclusions about the existence of sums. Instead of re-proving each of these theorems for GCAs, we can automatically extend the results for CAs to GCAs. We now show the manner in which this is done.

6.11 DEFINITION. An algebra $\overline{U} = (\overline{A}, \overline{+}, \overline{\sum})$ is a *closure* of an algebra $U = (A, +, \sum)$ if the following conditions hold:

(1) U is a GCA, $\overline{U}$ is a CA, and A is a subset of $\overline{A}$;

(2) for any elements $a, a_0, a_1, \ldots, a_i, \ldots \in A$, the formulas $a = \sum_{i<\infty} a_i$ and $a = \overline{\sum}_{i<\infty} a_i$ are equivalent;

(3) for every element $a \in \overline{A}$ there are elements $a_0, a_1, \ldots, a_i, \ldots \in A$ such that $a = \overline{\sum}_{i<\infty} a_i$.

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[4] In a GCA A, the notation $n \cdot a$ is equivalent to $\sum_{i<n} a$ for each $a \in A$.

6.12 THEOREM. (Tarski, 1949, 7.3,7.4) If $\overline{U} = (\overline{A}, \overline{+}, \overline{\sum})$ is a closure of $U = (A, +, \sum)$, then for every $n \leq \infty$ and for any elements $a, b, c, a_0, a_1, \ldots, a_i, \ldots \in A$

(1) $a = \sum_{i<n} a_i$ and $a = \overline{\sum}_{i<n} a_i$ are equivalent;

(2) $a = b + c$ and $a = b \overline{+} c$ are equivalent;

(3) $a = n \cdot b$ and $a = n \bar{\cdot} b$ are equivalent;

(4) $d \in A$ and $d \leq a$ is equivalent to $d \in \overline{A}$ and $d \overline{\leq} a$.

6.13 THEOREM. (Tarski, 1949, 7.8) *(Imbedding Theorem)* For every GCA $U = (A, +, \sum)$, there exists a CA $\overline{U}$ which is a closure of U.

If we have a particular arithmetic result for CAs, we can immediately extend the result to GCAs. First, we imbed a GCA U into its closure $\overline{U}$ which is a CA. Since the given result holds in any CA, it holds in $\overline{U}$. This in turn implies, by the above equivalences, that the result also holds in U.

We apply the imbedding theorem to the following theorem for CAs, in order to obtain a version for GCAs.

6.14 THEOREM. (Tarski, 1949, 1.40) Let $n, p \leq \infty$, and let $k_0, k_1, \ldots, k_i, \ldots$ with $i < n+p$, $l_0, l_1, \ldots, l_i, \ldots$ with $i < n$, and $m_0, m_1, \ldots, m_i, \ldots$ with $i < p$ be three sequences of finite nonnegative integers, each without repeating terms, such that every term of the first sequence occurs in one and only one of the remaining two sequences, and conversely. Then $\sum_{i<n+p} a_{k_i} = \sum_{i<n} a_{l_i} + \sum_{i<p} a_{m_i}$.

Note that the above theorem guarantees a restricted form of partition-associativity for CAs.

6.15 THEOREM. The restatement of the above theorem as it applies to a GCA $U = (A, +, \sum)$ contains the same set of hypotheses, with the following conclusions: $\sum_{i<n+p} a_{k_i} \in A$ if and only if $\sum_{i<n} a_{l_i}$, $\sum_{i<p} a_{m_i}$, and $\sum_{i<n} a_{l_i} + \sum_{i<p} a_{m_i} \in A$, in which case, $\sum_{i<n+p} a_{k_i} = \sum_{i<n} a_{l_i} + \sum_{i<p} a_{m_i}$.

PROOF. First, we imbed U in its closure $\overline{U} = (\overline{A}, \overline{+}, \overline{\sum})$. Hence, by the imbedding theorem, theorem 6.14 becomes true in $\overline{U}$. We prove the "only if" portion, leaving the "if" portion to the reader. Suppose that $\sum_{i<n+p} a_{k_i} \in A$. By applying theorem 6.11(2) if $n+p=\infty$ (theorem 6.12(1) if $n+p<\infty$) and then theorem 6.14, we obtain

$$\sum_{i<n+p} a_{k_i} = \overline{\overline{\sum_{i<n+p}}} a_{k_i} = \overline{\overline{\sum_{i<n}}} a_{l_i} \mp \overline{\overline{\sum_{i<n}}} a_{m_i}.$$

This implies that

$$\overline{\overline{\sum_{i<n}}} a_{l_i} \lesseqgtr \sum_{i<n+p} a_{k_i} \text{ and } \overline{\overline{\sum_{i<p}}} a_{m_i} \lesseqgtr \sum_{i<n+p} a_{k_i},$$

which by theorem 6.12(4), in turn implies that

$$\overline{\overline{\sum_{i<n}}} a_{l_i} \in A \text{ and } \overline{\overline{\sum_{i<p}}} a_{m_i} \in A.$$

Hence,

$$\begin{aligned}
\sum_{i<n+p} a_{k_i} &= \overline{\overline{\sum_{i<n}}} a_{l_i} \mp \overline{\overline{\sum_{i<p}}} a_{m_i} \\
&= \sum_{i<n} a_{l_i} \mp \sum_{i<p} a_{m_i} \text{ by theorem 6.11(2) and/or 6.12(1)} \\
&= \sum_{i<n} a_{l_i} + \sum_{i<p} a_{m_i} \text{ by theorem 6.12(2)}
\end{aligned}$$

in the *GCA* U.

In addition to the above theorem, we need the following theorem in order to show that any *GCA* satisfies the partition-associativity axiom.

6.16 THEOREM. (Tarski, 1949, 2.21, 7.10) *(Fundamental Law of Infinite Addition)* If $\sum_{i<n} a_i \in A$ and $\sum_{i<n} a_i \leq b$ for every $n<\infty$, then $\sum_{i<\infty} a_i \in A$ and $\sum_{i<\infty} a_i \leq b$.

Having prepared the necessary background, we are ready to show that

6.17 THEOREM. Any *GCA* $U=(A,+,\sum)$ satisfies the partition-associativity axiom.

The theorem is proved as a set of lemmas.

6.18 LEMMA. In any $GCA\ \ U = (A, +, \sum)$, the partition-associativity axiom holds for all finite partitions.

PROOF. We prove the "only if" part of partition-associativity, leaving the "if" part to the reader. Theorem 6.15 demonstrates that partition-associativity holds for all partitions of cardinality ≤ 2. Assume that partition-associativity holds for all partitions of finite cardinality $\leq m$. Let $(a_i : i \in I)$ be a summable family in A, and let $(I_j : j < m)$ be a partition of I of cardinality m. Then, by the inductive assumption, $\sum(a_i : i \in I_j) \in A$ for each $j < m$, $\sum_{j<m}(\sum(a_i : i \in I_j)) \in A$, and

$$\sum(a_i : i \in I) = \sum_{j<m}\Big(\sum(a_i : i \in I_j)\Big).$$

Choose $k < m$, and let $\{I_{k_0}, I_{k_1}\}$ be a partition of I_k. By theorem 6.15, we know that the existence of $\sum(a_i : i \in I_k)$ implies the existence of $\sum(a_i : i \in I_{k_q})$ for $q = 0, 1$ and the existence of $\sum_{q<2}(\sum(a_i : i \in I_{k_q}))$. Furthermore,

$$\sum(a_i : i \in I_k) = \sum_{q<2}\Big(\sum(a_i : i \in I_{k_q})\Big).$$

Hence,

$$\begin{aligned}\sum(a_i : i \in I) &= \sum_{j<m}\Big(\sum(a_i : i \in I_j)\Big)\\ &= \sum_{\substack{j<m\\ j\neq k}}\Big(\sum(a_i : i \in I_j)\Big) + \sum(a_i : i \in I_k)\\ &= \sum_{\substack{j<m\\ j\neq k}}\Big(\sum(a_i : i \in I_j)\Big) + \sum_{q<2}\Big(\sum(a_i : i \in I_{k_q})\Big).\end{aligned}$$

Now, $\{I_0, \ldots, I_{k-1}, I_{k_0}, I_{k_1}, I_{k+1}, \ldots, I_{m-1}\}$ is a partition of I of cardinality $m+1$. Since we arbitrarily selected the original partition, the k^{th} member of it, and the partition of the k^{th} member, we have shown that partition-associativity holds for any partition of cardinality $m+1$, and thus for any finite partition.

We must now show that partition-associativity holds for all countable partitions, and to do this we use the fundamental law of infinite addition. Let $(a_i : i \in I)$ be a summable family in A of infinite cardinality. Without loss of generality, we assume $I = \mathbf{N}$. Again, we prove the "only if" portion of partition-associativity, leaving the "if" portion to the reader.

6.19 LEMMA. If $\sum_{i<\infty} a_i \in A$ and $(I_j : j < \infty)$ is a partition of $\mathbf{N}$, then for any $n < \infty$, $\sum_{j<n}(\sum(a_i : i \in I_j)) \in A$ and $\sum_{j<n}(\sum(a_i : i \in I_j)) \leq \sum_{i<\infty} a_i$.
PROOF. For any $n < \infty$, $\{\bigcup_{j<n} I_j, \bigcup_{j\geq n} I_j\}$ is a partition of $\mathbf{N}$ of cardinality 2. Applying theorem 6.15 with this partition yields $\sum\left(a_i : i \in \bigcup_{j<n} I_j\right) \in A$ and

$$\sum_{i<\infty} a_i = \sum\left(a_i : i \in \bigcup_{j<n} I_j\right) + \sum\left(a_i : i \in \bigcup_{j\geq n} I_j\right).$$

Note that $(I_j : j < n)$ is a finite partition of $\bigcup_{j<n} I_j$. By applying lemma 6.18 to this partition, we obtain

$$\sum\left(a_i : i \in \bigcup_{j<n} I_j\right) = \sum_{j<n}\left(\sum(a_i : i \in I_j)\right).$$

Hence,

$$\begin{aligned}\sum_{i<\infty} a_i &= \sum\left(a_i : i \in \bigcup_{j<n} I_j\right) + \sum\left(a_i : i \in \bigcup_{j\geq n} I_j\right)\\ &= \sum_{j<n}\left(\sum(a_i : i \in I_j)\right) + \sum\left(a_i : i \in \bigcup_{j\geq n} I_j\right).\end{aligned}$$

Therefore, $\sum_{j<n}(\sum(a_i : i \in I_j)) \in A$ and $\sum_{j<n}(\sum(a_i : i \in I_j)) \leq \sum_{i<\infty} a_i$.

We observe that lemma 6.19 establishes the premises of the fundamental law of infinite addition (substituting $\sum(a_i : i \in I_j)$ for a_i and $\sum_{i<\infty} a_i$ for b). Thus, we may conclude that $\sum_{j<\infty}(\sum(a_i : i \in I_j)) \in A$ and that $\sum_{j<\infty}(\sum(a_i : i \in I_j)) \leq \sum_{i<\infty} a_i$.

We still need to show that $\sum_{i<\infty} a_i \le \sum_{j<\infty}(\sum(a_i: i \in I_j))$ in order to complete the proof that $\sum_{i<\infty} a_i = \sum_{j<\infty}(\sum(a_i: i \in I_j))$. Let $K_j = \{j\}$ for each $j < \infty$. Then $(K_j: j < \infty)$ is an infinite partition of $\mathbf{N}$. Applying lemma 6.19 to this partition yields

$$\sum_{j<n}\left(\sum(a_i: i \in K_j)\right) = \sum_{j<n} a_j$$

for any $n < \infty$. Hence, $\sum_{j<n} a_j \in A$.

Now, for each $n < \infty$, there exists $j < \infty$ such that $a_n \in (a_i: i \in I_j)$, since $(I_j: j < \infty)$ is a partition of $\mathbf{N}$. Let k be the maximum element of $\{j: \exists m \le n \text{ such that } a_m \in (a_i: i \in I_j)\}$. Substituting $\sum(a_i: i \in I_j)$ for a_i and K_j for I_j in lemma 6.19, we obtain $\sum_{j<k+1}((\sum(a_i: i \in I_j))_q: q \in K_j) = \sum_{j<k+1}(\sum(a_i: i \in I_j)) \in A$. From the proof of lemma for 6.19 we also find that

$$\sum_{j<k+1}\left(\sum(a_i: i \in I_j)\right) = \sum\left(a_i: i \in \bigcup_{j<k+1} I_j\right).$$

Now $(a_i: i < n)$ is a subfamily of $\left(a_i: i \in \bigcup_{j<k+1} I_j\right)$. Let $K_0 = \{0, 1, \ldots, n-1\}$, and let $K_1 = (\bigcup_{j<k+1} I_j) - K_0$. Then $\{K_0, K_1\}$ is a partition of $\bigcup_{j<k+1} I_j$ of cardinality 2. Thus, we can apply theorem 6.15 to this partition to obtain

$$\sum\left(a_i: i \in \bigcup_{j<k+1} I_j\right) = \sum(a_i: i \in K_0) + \sum(a_i: i \in K_1).$$

Hence,

$$\begin{aligned}
\sum_{i<n} a_i &= \sum(a_i: i \in K_0) \\
&\le \sum\left(a_i: i \in \bigcup_{j<k+1} I_j\right) \\
&= \sum_{j<k+1}\left(\sum(a_i: i \in I_j)\right) \\
&\le \sum_{j<\infty}\left(\sum(a_i: i \in I_j)\right).
\end{aligned}$$

We have thus again established the premises of the fundamental law of infinite addition (this time substituting $\sum_{j<\infty}(\sum(a_i : i \in I_j))$ for b). Therefore, we conclude that $\sum_{i<\infty} a_i \leq \sum_{j<\infty}(\sum(a_i : i \in I_j))$, and by the antisymmetry of the relation $\leq$, that $\sum_{i<\infty} a_i = \sum_{j<\infty}(\sum(a_i : i \in I_j))$. This shows that the partition-associativity axiom holds for all infinite partitions. Thus, we have proved that the partition-associativity axiom is satisfied by any GCA U.

We have shown that any GCA is an ω-so-monoid, but that an ω-so-monoid needn't be a GCA. We conclude by showing that the countable limit axiom is not necessarily satisfied by a GCA.

6.20 COUNTEREXAMPLE. The set **N** with $+$ defined as the usual addition and with $\sum$ defined only for families of finite support, in which case $\sum$ is also the usual addition, is a GCA. However, **N** does not satisfy the countable limit axiom since a family of countably infinite support is not summable even though every one of its finite subfamilies is summable.

$\sum$-Structures

Higgs [1980] has developed the notion of $\sum$-structure in order to motivate integration theory from an algebraic standpoint. He considers an integral as a linear function which preserves the algebraic structure of the underlying vector space, in this case, a $\sum$-*vector space* – a vector space together with a $\sum$-*structure* which provides infinite sums subject to a set of axioms. Higgs's axioms for a $\sum$-structure are similar to the axioms on infinite sums in topological groups [Bourbaki, 1966] (also discussed below). They are related to but not equivalent to the axioms for a partial monoid.

6.21 DEFINITION. A $\sum$-*monoid* is a non-empty set A together with a partially defined operation $\sum$ such that for $(a_i : i \in I)$ a family in A, the following axioms hold:

(1) *Unary Sum Axiom.* If $(a_i : i \in I)$ is a one-element family in A and $I = \{j\}$, then $\sum(a_i : i \in I)$ is defined and equals a_j.

(2) If $\sum(a_i : i \in I)$ exists and $f : I \to J$ is any function, then $\sum(a_i : f(i) = j)$ exists for each $j \in J$, and $\sum(\sum(a_i : f(i) = j) : j \in J)$ exists and equals $\sum(a_i : i \in I)$.

(3) If $f : I \to J$ is any function with J finite, and $\sum(a_i : f(i) = j)$ exists for each $j \in J$, then $\sum(a_i : i \in I)$ exists and $\sum(\sum(a_i : f(i) = j) : j \in J)$ exists and equals $\sum(a_i : i \in I)$.

6.22 EXAMPLES. The motivating examples are the basic $\sum$-monoids of analysis – the real numbers $\mathbf{R}$ and the complex numbers $\mathbf{C}$ – where in each case $\sum$ is defined so that $(a_i : i \in I)$ is summable if and only if $\sum(a_i : i \in I)$ is absolutely convergent.

In the above examples, each element of each $\sum$-monoid possesses an additive inverse. By observation 2.4, this implies that a $\sum$-monoid need not be a partial monoid. Must a partial monoid necessarily be a $\sum$- monoid? Note that the unary sum axiom is axiom (1) and that the "only if" direction of the partition-associativity axiom gives axiom (2). However,

6.23 OBSERVATION. In a $\sum$-monoid A, all finite families are summable.

PROOF. Let $(a_i : i \in I)$ be a finite family in A. Let $J = I$ and define $f : I \to J$ to be the identity function on I. For each j in J, $\sum(a_i : f(i) = j) = a_j$. Hence, the premises of axiom (3) are satisfied, and so $\sum(a_i : i \in I)$ exists.

Observation 6.23 need not hold in a partial monoid. For example, we know that it fails in $Pfn(D, D)$. Thus, we observe that a partial monoid need not be a $\sum$-monoid.

We now examine other algebraic objects which are constructed from $\sum$-monoids – $\sum$-groups and $\sum$-semirings.

6.24 DEFINITION. A $\sum$-*group* is a $\sum$-monoid which is a (necessarily abelian) group under the operation $a_1 + a_2 = \sum(a_i : i = 1, 2)$, and in which, if $\sum(a_i : i \in I)$ exists, then so does $\sum(-a_i : i \in I)$ with $\sum(-a_i : i \in I) = -\sum(a_i : i \in I)$.

6.25 DEFINITION. A $\sum$-*bimorphism* of $\sum$-monoids A, B, and C is a function $f: A \times B \to C$ such that if $\sum(a_i: i \in I)$ and $\sum(b_j: j \in J)$ exist, then $\sum(f(a_i, b_j): i \in I, j \in J)$ exists and equals $f(\sum(a_i: i \in I), \sum(b_j: j \in J))$.

6.26 OBSERVATION. A function $\circ: A \times A \to A$ is a $\sum$-bimorphism of the $\sum$-monoid A if and only if $\sum$ and $\circ$ obey the distributive laws for a partial semiring.

PROOF. If we assume that $\circ: A \times A \to A$ is a $\sum$-bimorphism, then it follows immediately that $\sum$ and $\circ$ obey the distributive laws.

Conversely, let $(x_i: i \in I)$ and $(y_j: j \in J)$ be two summable families in A. Suppose that $\sum$ and $\circ: A \times A \to A$ obey the distributive laws. Then

$$\begin{aligned}\sum(x_i: i \in I) \circ \sum(y_j: j \in J) &= \sum\left(\left(\sum(x_i: i \in I)\right) \circ y_j: j \in J\right)\\ &= \sum\left(\sum(x_i \circ y_j: i \in I): j \in J\right)\\ &= \sum(x_i \circ y_j: i \in I, j \in J).\end{aligned}$$

Thus, $\circ$ is a $\sum$-bimorphism.

6.27 DEFINITION. A $\sum$-*semiring* is a $\sum$-monoid which is also a monoid $(A, \circ)$ with $A \times A \xrightarrow{\circ} A$ a $\sum$-bimorphism.

If the $\sum$-monoid of a $\sum$-semiring is a partial monoid, then the $\sum$-semiring is a partial semiring. Alternatively, if the partial monoid of a partial semiring is a $\sum$-monoid, then the partial semiring is a $\sum$-semiring. Both of these statements are a consequence of definition 6.27 and observation 6.26.

This concludes our discussion of $\sum$-structures. We now turn our attention to the last infinitary partial addition.

Infinite Sums in Commutative Groups

The discussion of infinite sums in Bourbaki [1966] is confined to Hausdorff commutative topological groups in which the group operation is written additively. The results obtained for infinite sums are used in determining series convergence. Since every group element has an additive inverse, a nontrivial group is not a partial monoid. However, these groups do satisfy the axioms for a $\sum$-monoid, as we shall see presently.

6.28 DEFINITION. Let $(x_i : i \in I)$ be a family in a Hausdorff commutative group G. For each finite subset J of I let $s_J = \sum(x_i : i \in J)$. The family $(x_i : i \in I)$ is *summable* and its sum is s if, for each neighborhood V of the origin in G, there is a finite subset J_0 of I such that for each finite subset $J \supset J_0$ of I we have $s_J \in s + V$.

Hence, since all finite families are summable, the unary sum axiom is satisfied. But because such a group is not a partial monoid, this implies that the partition-associativity axiom is not satisfied.

If we add the constraint that G is also a complete group (in the sense that G is a group which is also a uniform topological space endowed with a left and a right uniformity, each of which is a structure of a complete topological space), then we have the following characterization of summability.

6.29 THEOREM. (Bourbaki, 1966, III.5.2, Theorem 1) *(Cauchy's Criterion)* A family $(x_i : i \in I)$ is summable if and only if for each neighborhood V of the origin in G, there is a finite subset J_0 of I such that $\sum(x_i : i \in K) \in V$ for all finite subsets K of I which do not meet J_0.

In the remainder of this discussion, we make the assumption that the group G is complete.

6.30 PROPOSITION. (Bourbaki, 1966, III.5.3, Proposition 2) Every subfamily of a summable family is summable.

6.31 THEOREM. (Bourbaki, 1966, III.5.3, Theorem 2) *(Associativity of the Sum)* Let $(x_i : i \in I)$ be a summable family in G, and let $(I_\lambda : \lambda \in L)$ be any partition of

I. If s_λ denotes $\sum(x_i : i \in I_\lambda)$, then the family $(s_\lambda : \lambda \in L)$ is summable and has the same sum as the family $(x_i : i \in I)$.

6.32 PROPOSITION. (Bourbaki, 1966, III.5.3, Proposition 3) Let $(x_i : i \in I)$ be a family in G, and let $(I_\lambda : \lambda \in L)$ be a finite partition of I. If each of the subfamilies $(x_i : i \in I_\lambda)$ is summable, then the family $(x_i : i \in I)$ is summable and

$$\sum\left(\sum(x_i : i \in I_\lambda) : \lambda \in L\right) = \sum(x_i : i \in I).$$

6.33 PROPOSITION. (Bourbaki, 1966, III.5.5, Proposition 5) Let f be a continuous homomorphism of a commutative group G into a commutative group G'. If $(x_i : i \in I)$ is a summable family in G, then $(f(x_i) : i \in I)$ is a summable family in G', and we have $\sum(f(x_i) : i \in I) = f(\sum(x_i : i \in I))$.

6.34 PROPOSITION. (Bourbaki, 1966, III.5.5, Proposition 6) If $(x_i : i \in I)$ and $(y_i : i \in I)$ are two summable families in G, then
(1) $(-x_i : i \in I)$,
(2) $(nx_i : i \in I)$ for $n \in \mathbf{Z}$, and
(3) $(x_i + y_i : i \in I)$
are summable families in G. Furthermore,
(1') $\sum(-x_i : i \in I) = -\sum(x_i : i \in I)$,
(2') $\sum(nx_i : i \in I) = n\sum(x_i : i \in I)$, and
(3') $\sum(x_i + y_i : i \in I) = \sum(x_i : i \in I) + \sum(y_i : i \in I)$.

Definition 6.28, theorem 6.31, and proposition 6.32 imply axioms (1), (2), and (3) for a $\sum$-monoid, and proposition 6.34 shows that the additional $\sum$-group axiom is satisfied. Hence, we see that a complete Hausdorff commutative topological group is a $\sum$-group.

CHAPTER VII

CONCLUSION

This chapter is divided into two parts. The first part contains a summary of the principal results in the thesis together with open questions and suggestions for future work. The second part is a discussion of the relevance of these results to theoretical computer science.

Summary

In chapter II, we presented the definition of a so-ring and provided several examples of so-rings, the principal ones being the natural numbers, the distributive lattices, and the so-rings important in program semantics – $Pfn(D, D)$, $Mfn(D, D)$, and $Mset(D, D)$. We also defined the concept of so-ring homomorphism and sub-so-ring, and we provided general constructions of so-rings, namely products, quotients, and free so-rings. Unknown at this writing is the general construction of the co-product of so-rings.

In Chapter III, we presented basic properties and substructures of so-rings: the center, domains and ranges, adequacy, and inversibility, and we described their interrelationships. Begun in this chapter and continued throughout the remainder of the thesis, we noted the similarities between semigroups and so-rings, and we generalised results in semigroup theory to so-rings. Current research in semigroup theory continues to demonstrate that that the structure of the set of idempotents of a semigroup greatly influences the semigroup itself. The center of a so-ring, being the set of its inversible idempotents, appears to be a natural analogue of the set of idempotents of a semigroup, and as such perhaps plays a similar role for so-rings. Two related open questions are the following:

(1) To what extent does the structure of the center of a so-ring influence the structure of the so-ring?

(2) Is there a close relationship between so-rings and semigroups whose idempotents form a Boolean algebra?

Matrices over so-rings were introduced in chapter IV. A matrix over a so-ring was defined as an array each of whose rows is supersummable. We discussed invertibility as it applies to arrays over so-rings and gave equivalent conditions for array invertibility (due to Manes and Benson [1985]). At this writing, the following question remains unanswered: does the inverse of a matrix over a so-ring ever fail to be a matrix? Following the section on invertibility, we discussed the concepts of independence and basis. We demonstrated that for a wide class of so-rings, including $Pfn(D,D)$, $Mfn(D,D)$, and $Mset(D,D)$, a matrix over a so-ring R is invertible if and only if its columns form a basis for R^n, but the proof was not reminiscent of the same theorem in classical linear algebra. For R equal to any one of $Pfn(D,D)$, $Mfn(D,D)$, or $Mset(D,D)$, and for n finite and nonzero, we showed that if D has countably infinite cardinality, then R^n has a basis of cardinality m for each finite, nonzero m. If D is finite, then R^n has a basis of cardinality n only. Many areas pertaining to matrices over so-rings have not yet been explored. These include eigenvalues and eigenvectors (some introductory remarks appeared at the end of chapter IV), and similarity, equivalence, and canonical forms of matrices.

In the first section of chapter V, which was concerned with representations of so-rings, we showed that any so-ring R may be embedded in the so-ring of additive maps from R to itself. We also gave a set of conditions on a so-ring which guarantee that it can be embedded in a so-ring of partial functions. Although these conditions are sufficient, they are not all necessary. A complete characterization of all so-rings which may be embedded in the partial functions has yet to be found. In the second part of chapter V, we generalised several partial orders studied in semigroup theory to so-rings, namely, the multiplicative order (introduced at the end of chapter III), amenable orders, and fundamentally representable orders, and we related each to the sum-ordering. There are no doubt other interesting partial orders in the semigroup literature whose so-ring analogues have yet to be explored.

In chapter VI, we described the relationships between four algebraic structures with infinite partial additive operations. Three of these – generalised cardinal algebras, $\sum$-structures, and infinite sums in topological groups – were used as points of comparison with partial monoids (and partial semirings).

Implications for Computer Science

Although a primary motivation for performing a detailed investigation of the structure of so-rings was to provide a framework and a set of mathematical tools with which to build a matrix theory of algorithm transformation, we did not attempt to develop such a theory. We chose instead to concentrate on the mathematics, thinking that providing a more thorough treatment at this stage would make the development of a matrix theory of algorithm transformation possible, it being clear, in any case, that imitation of proofs from classical linear algebra was not fruitful. In addition, the structure of so-rings is interesting in its own right and hence deserves a careful treatment. We do, however, present a set of questions for further study pertaining to a matrix theory of algorithm transformation; these are outlined below.

Recall from the introductory chapter of this thesis, that any iterative algorithm can be written as a matrix equation $\bar{x} = A\bar{x} + \bar{b}$ over the partial functions from some set D to itself, whose least solution is $\bar{x} = \sum_{n\geq 0} A^n\bar{b}$, the first component of which is the partial function denoting the original algorithm.

The first question we ask is the following: is there any way to simplify the computation of the solution of $\bar{x} = A\bar{x} + \bar{b}$? If D is a finite set (possibly approximating an infinite one), then we can pursue an answer along the following lines. Recall from chapter II, that any partial function can be embedded in the ring of complex matrices, where each partial function is represented as a matrix of 0s and 1s. Under this embedding, A and $\bar{b}$ become 0-1 matrices. Since we are now in a ring, the matrix equation to solve can be expressed as $(I - A)\bar{x} = \bar{b}$. It is not difficult to show that the least solution can be determined simply by row reduction and substituting 0s instead of 1s, when there is a choice of either; this solution is identical to $\sum_{n\geq 0} A^n\bar{b}$. However, what we have gained by being able to use row reduction, we have lost by having to translate each partial function into its 0-1 matrix equivalent. Thus, at this point, it is not clear whether there are any practical advantages in this approach, but even so, this device has some potential for theoretical investigation.

Another direction to pursue is to determine if it is possible to perform a similarity transform on the matrix A which simplifies the computation of the solution of $\bar{x} = A\bar{x} + \bar{b}$. In other words, for some invertible matrix P, it may be easier to compute $(P^{-1}AP)^n$ than A^n. Hence, the solution would take the form $P\sum_{n\geq 0}(P^{-1}AP)^n P^{-1}\bar{b}$. Some results have already been obtained by Manes [to appear].

This leads us to the study of eigenvalues and eigenvectors of A. In classical linear algebra, the eigenvalues of a matrix are the roots of the characteristic polynomial, which has no analogue here. However, if we embed partial functions (as 0-1 matrices) in the ring of complex matrices, then we can compute the eigenvalues of A in the classical manner. In any case, it is possible that the form of the eigenvalues of A can tell us something qualitative about the nature of the algorithm in question. If this is so, then we may be able to use this information to determine the form of the similarity tranformation of A and to determine other canonical forms of A which might simplify the computation of $\sum_{n\geq 0} A^n\bar{b}$. In addition, the nature of the eigenvalues may give us clues as to how to construct equivalent forms of the original algorithm.

We hope that our study of so-rings, and in particular matrices over so-rings, will form the basis for the development of a matrix theory of algorithm transformation which will provide answers to the above questions.

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