RECONSTRUCTION OF SURFACES FROM PROFILES

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ABSTRACT

This paper presents an algorithm for computing a depth map for a smooth surface from a sequence of profile curves. The algorithm requires that the viewing directions be coplanar. In addition formulae are derived for computing directly the Gauss and mean curvatures without first computing a depth map. We have used the algorithm to reconstruct curves from their profiles with a high degree of accuracy from synthetic, noise-free data.

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1. INTRODUCTION

We can tell a lot about the shape of an object from a single profile, and with multiple views we can often determine the shape uniquely. In this paper we analyze this reconstruction process mathematically and derive an algorithm to produce a depth map. For some applications it may not necessary to produce a depth map at all (for example in recognition problems). The Gauss and mean curvatures may be sufficient by themselves to solve this problem, and in any case it will be useful to decompose surfaces into patches according to whether they are convex, concave, hyperbolic, parabolic, or planar (Besl and Jain [1], Brady et al. [2], Ferrie and Levine [4]). This gives a method for describing surfaces of objects and a basis for matching with standard surfaces such as spheres, planes, and cylinders. Koenderink and van Doorn [7] have derived a formula which could be used to compute the Gauss curvature from a sequence of intensity images without depth. We have extended these results to produce formulae for the mean curvature, principal curvatures, and principal directions.

The use of profiles for the recognition and description of surfaces has been explored by many people (Koenderink and van Doorn [7], Callahan and Weiss [3], Hoffman and Richards[5]), but most investigations have not combined information from multiple views. In general, there is no way to identify a point on one profile with a corresponding point on a profile from a different view since for smooth surfaces they will not have any points in common. In fact, most stereo algorithms which are based on correspondence find the most similar point and assume it is the same. However, if the camera motion is known, then there is a method to identify points on two different profiles. In our work, we have restricted the camera to planar motion, so that planes parallel to the plane of motion induce a correspondence between the profiles. However, it is possible for the profile to change qualitatively between views, and in order to understand this, we have analyzed the analogous problem for a curve in the plane. These transitions create ambiguities in the reconstruction process. The criterion used to resolve this ambiguity is that the most likely solution is the one which minimizes the change in depth between adjacent views.

A mathematical approach to the reconstruction o surfaces is based on the fact that a smooth surface without inflection points is the envelope of all of its tangent planes. However, there are two problems with this; how to compute the envelope of a family of planes and how to handle inflection points. With the assumption of planar camera motion, we have been able to reduce the envelope of planes problem to that of computing the envelope of a family of lines in a plane, which we were able to solve. The problem of inflection points requires interpolation of the surface. We have a simple approach to this which would work in most cases, and we plan to extend this to polyhedral surfaces where every points is either an inflection point or a crease point (i.e. non-smooth).

In Section 2, we give the mathematical framework for discussing profiles. Then for simplicity in Section 3, we take the case of reconstructing a plane curve. In Section 4, this is generalized to surfaces. In Section 5, we present the experimental results using simulated data.

2. THE PROFILES OF A SURFACE

Let u be a unit vector in 3-space $\mathbf{R}^3$ and let M be a smooth surface in $\mathbf{R}^3$. We regard u as defining a viewing direction. On the surface M there is a locus of points p for which the tangent plane contains the viewing direction. This locus of points is called the *critical set* corresponding to u. If this curve is projected parallel to u onto the viewing plane which is perpendicular to u and passes through the origin, the image is called the *profile* (Figure 1). For future reference we note the following standard facts about critical sets and profiles:

- The critical set is a smooth curve at p unless p is a parabolic point and u is the asymptotic direction.
- The profile near q in the viewing plane arising from p on the critical set is a smooth curve unless the viewing direction is an asymptotic direction.
- When the critical set is smooth at p, its tangent direction is parallel to the viewing plane if and only if the viewing direction is a principal direction at p.
- When the profile is smooth at q, its tangent at q is always in the tangent plane to the surface at p.

In addiiton, the Gauss curvature can be computed from the profiles. Consider a point p of the critical set. The viewing direction at p together with the normal to the surface there determine a plane, whose intersection with the surface is a smooth curve. The curvature of this curve at p on the critical set is called the radial curvature. The projection of p onto q lies on the profile, and the curvature of the profile at q is called the transverse curvature. Koenderink and van Doorn [7] showed that the Gauss curvature is the product of the transverse curvature and the radial curvature. Brady et al. [2] has also given an independent proof of this fact. The curvature of the profile can be computed directly from an image, and the radial curvature could be computed from a sequence of images.

3. RECONSTRUCTING PLANE CURVES

Consider a smooth plane curve C, which we usually take to be closed, so that it is given by a parametrization $\gamma(t) = (X(t), Y(t))$ such that the derivative is never zero, i.e. $X'(t)$ and $Y'(t)$ are never zero for the same t. As shown in Figure 2, u is the *viewing direction.* The *viewing line* is the line perpendicular to u through the origin. The profile in this context is the set of points on the viewing line for which the tangent line is parallel to u. The position of a profile point can be expressed as $w \cdot (\sin\theta, -cos\theta)$, where w is the signed distance from the origin, O, to the profile point. For example, in Figure 2, there are three points for which $w > 0$ and one for which $w < 0$. Note that if θ produces a value w, then $\theta + \pi$ produces $-w$. Thus we restrict to $0 \leq \theta < \pi$. The collection of all profile points forms a curve in the plane classically called the *pedal curve* of C with respect to O. If θ changes in such a way that a pair of values of w is annihilated and disappear, or a pair is created, then the pedal curve has a singular point (usually a cusp). This happens when u passes through the direction of an inflexional tangent to C, as shown in Figure 3. In a neighborhood of that point, w cannot be a smooth function of θ. In general, there will be parts of C for which w is a function of θ for some range of values of θ. Suppose C_1 is such a subset of C, then the following proposition states that C_1 can be reconstructed from the function $w = w(\theta)$. Any part of the curve which has no inflexions satisfies this property. Another way to say this is that the Gauss map of C_1 has an inverse, i.e. for each direction on the unit circle there is at most one point of C_1 whose normal points in that direction. Such curves possibly with singularities have been studied by Langevin, Levitt, and Rosenberg [9] and are called *herissons* (hedgehogs)

Proposition 1 *1. The curve C_1 consists of points*

$$x = w\sin\theta + w'\cos\theta$$

$$y = -w\cos\theta + w'\sin\theta$$

and $w' = dw/d\theta$ is the distance from the profile point to the corresponding point of C_1

2. The radius of curvature of C_1 at the point corresponding to θ is $w + w''$. (Here C_1 is oriented by increasing θ.)

proof: The line through the profile point $w \cdot (\sin\theta, \cos\theta)$ parallel to the viewing direction $u = (\cos\theta, \sin\theta)$ has the equation

$$((x, y) - (w\sin\theta, -w\cos\theta)) \cdot (\sin\theta, -\cos\theta) = 0$$

i.e.

$$x\sin\theta - y\cos\theta = w \tag{1}$$

The curve C_1 is the *envelope* of the lines obtained by eliminating θ between equation (1) and the following equation:

$$x\cos\theta + y\sin\theta = w' \tag{2}$$

This gives the unique solution (x, y) in part 1 of the Proposition. Rewriting this as

$$(x, y) = w(\sin\theta, -\cos\theta) + w'(\cos\theta, \sin\theta)$$

proves that w' is the distance. The formula for the radius of curvature for a curve is given by:

$$\rho = (x'^2 + y'^2)^{3/2}/(x'y'' - x''y')$$

Differentiating the equations in part 1 and substituting into the formula for ρ gives part 2 of the Proposition.

It is not possible for $w + w''$ to be zero while C_1 is smooth: zero radius of curvature is only possible at a cusp of a curve. On the other hand $w + w''$ can tend to infinity, making the curvature tend to zero (which is an inflexion). But at an inflexion w is no longer a function of θ so it is not in C_1.

Examples of these two cases are $w = \theta^3$ and $w^2 = \theta^3$ respecively (see Figure 4). For the former, $x = 3\theta^2+$higher order terms, and $y = 2\theta^3+$ higher order terms. Thus C_1 has a cusp and the pedal curve has an inflexion at $\theta = 0$. For the latter, $x = \pm\frac{3}{2}\theta^{1/2}+$ higher order terms, and $y = \pm\frac{1}{2}\theta^{3/2}+$ higher order terms. Thus, C_1 has an inflexion and the pedal curve has a cusp at $\theta = 0$.

In practice if we start with a curve C, and measure its profile data, i.e. for each viewing direction $u = (\cos\theta, \sin\theta)$ for some range of values for θ we obtain the various values of w for the profile points. Some values of θ will give more values of w than others, unless C has no inflexions, in which case each value of θ will have two values of w. Starting at some value $\theta = \theta_0$, we choose one of the corresponding values to be w_0 and increase θ following w as a function of θ until an inflexion is encountered. This is detected by a change in the number of w-values. In fact θ_0 can be chosen so that it immediately follows after an inflexion. It turns out that one only needs to consider the case when the number of w-values decreases by two. The parts of the curve C lying between inflexions can be reconstructed using part 1 of the Proposition. The computational problem to be solved is this: having chosen w_0 for $\theta = \theta_0$, what value corresponds to $\theta_0 + \delta\theta$? It is not sufficient to simply choose the closest value of w. Since it is possible for the pedal curve to have crossings (See Figure 5), there is the danger of starting on one branch and switching to the other. Note that we are approximating the function $w = f(\theta)$ at discrete points, and we have assumed that this function is continuous since C is smooth. In addition, since w' is the distance from the viewing line to the point of tangency on the curve, we also want w' to be continuous. This can be viewed as a constraint on choosing successive values of w such that w'' is minimized.

4. PERSPECTIVE PROJECTION OF CURVES

There is no essential difference in the mathematics of reconstructing curves from profiles obtained from perspective projection. We derive the formulas here for completeness. Assume that the curve C is contained in a circle of radius r. From each point A on the circle we define the viewing line to be the line parallel to the tangent to the circle at A and a distance d away from it. Each tangent line to C through A will intersect the viewing line at a profile point. (See Figure 6) The profile points are identified by their distance w in a counterclockwise direction from the origin of the viewing line, which is the closest point to A. The values of w for θ and $\theta + \pi$ are no longer directly related.

Proposition 2 *For regions of the curve in which w is a function of θ,*

$$x = \frac{rwd\sin\theta + rw^2\cos\theta + rw'd\cos\theta}{d^2 + w^2 + dw'}$$

$$y = \frac{rwd\cos\theta + rw^2\sin\theta + rw'd\sin\theta}{d^2 + w^2 + dw'}$$

Thus in this case also the parts of C between inflexions can be recovered from a knowledge of the function w. However, in this case the formula for the curvature is rather more complicated.

5. SURFACES

We would like to reconstruct a surface from its profiles in a way analogous to the method used for curves in the previous section. The situation for surfaces differs in two respects: each profile is a curve and there is a two-parameter family of viewing directions which as unit vectors are points on the sphere S^2. We seek to find all the tangent planes to M, and for this purpose, it is sufficient (and necessary) to know the profiles for a "great circle" of viewing directions, i.e. all directions which lie in a plane. Consider the tangent plane to M at p. The unit tangent vectors in this plane form a great circle on S^2, and this circle intersects the great circle of viewing directions. Hence some tangent line to M at p is in the viewing direction u, so that p contributes a point q to the profile for the viewing direction u. Of course, for opaque surfaces the situation is complicated by occlusion, and it is possible that for some points, none of the singular sets would be visible.

How do we find the tangent plane to M at p? One line in that plane is, of course, the line through q parallel to u. Another line is the *tangent line to the profile* at q (see Figure 7). If the profile is singular at q then the tangent line is interpreted as a limit of tangent lines at nearby smooth points of the profile. If the profile has a crossing at q, then both branches will contribute a tangent plane to M but at different points p of M ([7], [6], [3]).

We shall now reconstruct M from the profiles corresponding to a circle of viewing directions (except where the tangent plane is parallel to the plane of the circle of viewing directions). The calculation which follows is local in that it assumes a parametrization of profiles near each point.

Consider a family of viewing directions, $u = (0, \cos\theta, \sin\theta)$ and corresponding viewing planes $y\cos\theta + z\sin\theta = 0$. Each viewing plane will contain a profile of M; we use orthonormal coordinates (x, w) in the viewing plane, where the w-axis is in the direction $(0, \sin\theta, \cos\theta)$ and the x-axis is the same for all of the planes. A point (x, w) in a viewing plane then is the point

$$x(1,0,0) + w(0, \sin\theta, -\cos\theta) = (x, w\sin\theta, -w\cos\theta) \tag{3}$$

in (x,y,z) space. In practice it is the numbers (x, w) which will be measured from an image.

In general, the profile will be be a curve with isolated cusps, and we present the theory for reconstructing the surface from smooth points of profiles. The case of those cusp points is still under investigation. In practice, this should not be a problem since one can compute the surface at points in a neighborhood of a cusp where the profile is smooth. Assume that the profiles have the form $w = w(x, \theta)$ over a range of values of θ and x, where $w(x, \theta)$ is a smooth function. Thus we assume that over some range of values of x and θ the profiles are smooth and not tangent to the w-axis. So starting with a point on a given profile of M, we choose an axis of rotation which is not parallel to the normal to the profile at that point. It is intuitively reasonable that if the viewing direction remains in the tangent plane, no additional information will be obtained.

Now consider a fixed x and θ, i.e. a fixed point on a particular profile curve. The tangent plane to M determined by this x and θ passes through $(x, \sin\theta, -w\cos\theta)$ and contains the directions:

$(0, \cos\theta, \sin\theta)$ (the viewing direction)

$(1, w_x \sin\theta, -w_x \cos\theta)$ (tangent to the profile)

where $w_x = \partial w/\partial x$. The equation of the plane is therefore

$$w_x X - \sin\theta Y + \cos\theta Z = x w_x - w \tag{4}$$

where we temporarily use (X,Y,Z) as current coordinates in $\mathbb{R}^3$ to avoid the double use of x.

Remark: If the profiles are parametrized as $x = x(t, \theta)$, $w = w(t, \theta)$ for some parameter t, then the tangent plane is

$$w_t X - x_t \sin\theta Y + x_t \cos\theta Z = x w_t - x_t w \tag{5}$$

The surface M is the *envelope* of all the tangent planes described by (4), that is we obtain the point (X, Y, Z) of M corresponding to (x, θ) by eliminating x and θ between (4) and its derivatives with respect to x and θ, viz.

$$\partial/\partial x: \quad w_{xx} X = x w_{xx} \tag{6}$$

$$\partial/\partial\theta: \quad w_{x\theta} X - \cos\theta Y - \sin\theta Z = x w_{x\theta} - w_\theta \tag{7}$$

This amounts to finding the intersection of three tangent planes given by (x, θ), $(x + \delta x, \theta)$, and $(x, \theta + \delta\theta)$. The intersection of these three planes will approach the point on M in the limit as δx and $\delta\theta$ go to zero. This runs into problems when one or the other direction produces a stationary tangent plane. According to equation (6), $X = x$ (the line through a profile point in a viewing direction is always in a plane X = a constant), but (6) also indicates that if the profile has an inflexion ($w_{xx} = 0$), then there will be problems distinguishing one tangent plane to M from the "next". This is similar to the case for curves in which the envelope of tangent lines to a plane curve with an inflexion contains the whole inflexional tangent line. In fact (4), (6), and (7) determine a unique point (X,Y,Z) if and only if $w_{xx} \neq 0$, namely the point

$$f(x, \theta) = (x, w\sin\theta + w_\theta \cos\theta, -w\cos\theta + w_\theta \sin\theta) \in M \tag{8}$$

Note that w_θ is the distance from a profile point to the corresponding point on M.

Remark In the general case where w is not always a function of x, we can parametrize the family of profile curves as follows:

$$x = x(t, \theta)$$
$$w = w(t, \theta)$$

In this case, except where x_t is zero, equation (8) becomes

$$f(t,\theta) = (x, w\sin\theta, -w\cos\theta) + \left(\frac{-x_\theta w_t + x_t w_\theta}{x_t}\right)(0,\cos\theta,\sin\theta) \tag{9}$$

and $(-x_\theta w_t + x_t w_\theta)/x_t$ is the distance from a profile point to a point on M.

The formula (8) tells us how to reconstruct M from its profile data, as long as w can be expressed as a function of x and θ. The condition for f to give a smooth piece of surface (i.e. the condition that the differential of f has rank 2) is $w + w_{\theta\theta} \neq 0$; note that this also arose in the case of curves above.

On M at any point $f(x,\theta)$, we have coordinate directions corresponding to:

$$\partial f/\partial x = f_x = (1, w_x \sin\theta + w_{x\theta}\cos\theta, -w_x\cos\theta + w_{x\theta}\sin\theta)$$

where f_x is in the direction of the *critical set* and f_θ is in the *viewing direction* so they are not in general orthogonal. Nevertheless, they are *conjugate* with respect to the second fundamental form (see Figure 8). Note that f_x and f_θ will not coincide at a smooth point of the profile. Geometrically, this means that with respect to the ellipse determined by this quadratic form, each direction is tangent to the ellipse at the point of intersection by the axis determined by the other. Algebraically, the matrix associated with the second fundamental form is diagonal with respect to the basis of these two directions and (using $n = (-w_x, \sin\theta, -\cos\theta)/(1 + w_x^2)^{1/2}$ as the unit normal to M) can be written as:

$$\begin{pmatrix} \dfrac{w_{xx}}{(1+w_x^2)^{1/2}} & 0 \\ 0 & -\dfrac{w + w_{\theta\theta}}{(1+w_x^2)^{1/2}} \end{pmatrix}$$

Thus, it is possible to derive simple formulae the Gaussian and mean curvatures of M in terms of the profile data w. As noted above, the consequent formula for the Gaussian curvature as the product of the radial curvature, κ_c and the transverse curvature, κ_θ, was known,but the mean curvature cannot be expressed in terms of only these two.

6. RELATIONSHIP BETWEEN SECTIONAL AND SURFACE CURVATURES

There are three curvatures which enter into the equations for the surface curvatures. κ_c is the curvature of the profile or the radial curvature. Its formula is

$$\kappa_c = w_{xx}/(1 + w_x^2)^{3/2}$$

κ_x is the sectional curvature of M in the f_x direction (the direction of the tangent to the critical set). We find (see below) that

$$\kappa_x = w_{xx}/[(1+w_x^2)^{1/2}(1+w_x^2+w_{x\theta}^2)]$$

κ_θ is the sectional curvature of M in the f_θ direction, which is the viewing direction u when $w + w_{\theta\theta} > 0$ and $-u$ when $w + w_{\theta\theta} < 0$. This is also known as the transverse curvature. We find (see below) that

$$\kappa_\theta = -1/[(1+w_x^2)^{1/2}(w+w_{\theta\theta})]$$

Proposition 3 *1. The Gauss curvature K and the mean curvature H of M at $f(x,\theta)$ are given by*

$$K = -w_{xx}/[(1+w_x^2)^2(w+w_{\theta\theta})] = \kappa_c\kappa_\theta$$

$$H = \frac{w_{xx}(w+w_{\theta\theta}) - 1 - w_{x\theta}^2}{2(w+w_{\theta\theta})} = \frac{1}{2}\kappa_c\left(1+\frac{\kappa_\theta}{\kappa_x}\right)$$

2. $w_{x\theta} = 0$ if and only if the viewing direction is a principal direction on M at the corresponding point. In this case f_x and f_θ are the principal directions, $\kappa_c = \kappa_x$, and $H = \frac{1}{2}(\kappa_c + \kappa_\theta)$.

proof: The computation of the Gauss and mean curvatures from a local parametrization f of M can be found in O'Neill [8]. We can obtain f_x and f_θ from equation (8) and compute the first fundamental form as

$$\begin{pmatrix} 1+w_x^2+w_{x\theta}^2 & w_{x\theta}(w+w_{\theta\theta}) \\ w_{x\theta}(w+w_{\theta\theta}) & (w+w_{\theta\theta}) \end{pmatrix}$$

This together with the second fundamental form given above allows us to compute the surface curvatures. Thus K and H can be expressed in terms of the curvatures $\kappa_c, \kappa_x, \kappa_\theta$:

The shape operator, which is the derivative of the Gauss mapping, referred to the basis f_x, f_θ is:

$$\frac{1}{(1+w_x^2)(w+w_{\theta\theta})}\begin{pmatrix} w_{xx}(w+w_{\theta\theta}) & -w_{xx}w_{x\theta} \\ w_{x\theta}(w+w_{\theta\theta}) & -(1+w_x^2+w_{x\theta}^2) \end{pmatrix}$$

The principal directions are the eigenvectors of this matrix. One can see that, given the assumption that $w + w_{\theta\theta}$ is nonzero, the matrix is diagonal if and only if $w_{x\theta} = 0$. The principal curvatures are the eigenvalues, and the Gauss curvature is the product of the eigenvalues and the mean curvature is their sum. Thus, when the matrix is diagonal, f_x and f_θ are the principal directions and they are orthogonal.

When the profile has a cusp, the curvatures K and H will be the limits of the values of the formulae at corresponding smooth points near it. However, K for example cannot be expressed as $\kappa_c\kappa_\theta$, since κ_c is infinite and κ_θ is zero. Finding a formula to replace this is a goal of current investigation.

7. EXPERIMENTAL RESULTS

In order to determine the potential accuracy of the algorithm for reconstructing curves, several experiments were performed. Synthetic data was used to generate profile points for the various values of θ. Figure 8. shows a picture of a curve with two inflexions and the pieces of the curve which are reconstructed from the data. The reconstruction process stops at the inflexion points producing breaks in the curve. In addition, there is a break in the curve where the reconstruction process started.

In principle, one could start with the profile data and trace the curve from start to finish, reversing direction at inflexion points, always choosing the value of w which minimizes w''. However, there is no sure test on the values for w which will guarantee that a point is an inflexion point. It is possible to find all of the tangent directions for the inflexion points, which can be called *flex directions*. These are the directions at which the number of tangent lines changes. We use these flex directions to break up the profile data into *heurisson* segments for which w is a function of the viewing direction. The current algorithm constructs arcs corresponding to the *heurisson* segments between inflexion points.

As a practical point, one can choose the place to start drawing the curve to be any w value after a flex direction. Once the pieces of the curve between the inflexions are drawn, they can be linked together with straight lines based on proximity of endpoints (assuming one has started with a closed curve. In theory one would like to minimize the integral of the absolute value of w''. In the current algorithm we only apply this criterion locally to choose the values of w sequentially. It may be necessary to apply standard relaxation techniques when dealing with real data.

The algorithm can be extended easily to surfaces, and we intend to apply this to the problem of model acquisition from physical prototypes.

8. CONCLUSION

We have given a procedure for reconstructing surfaces from a sequence of profiles. In addition we have given a formula for mean curvature in terms of the profile data which extends the similar result for the Gauss curvature. This makes it possible to compute both of these curvatures without first computing a dense depth map. We have used this algorithm on synthetic, noise-free data to reconstruct curves with a high degree of accuracy, and we plan to test it on real data.

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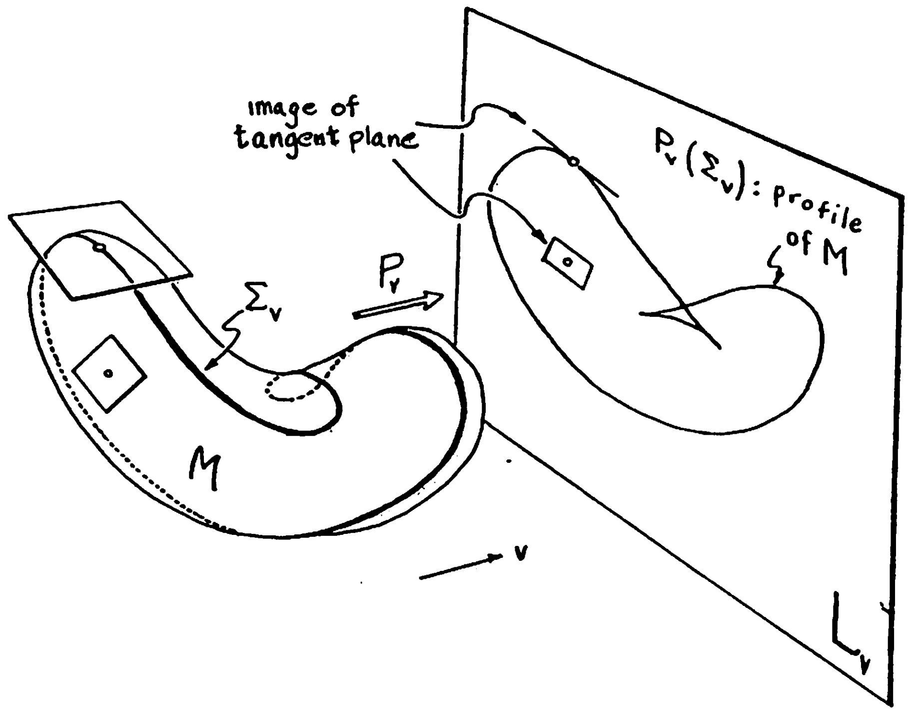

Figure 1: The critical set Σ_M and profile of the projection of M

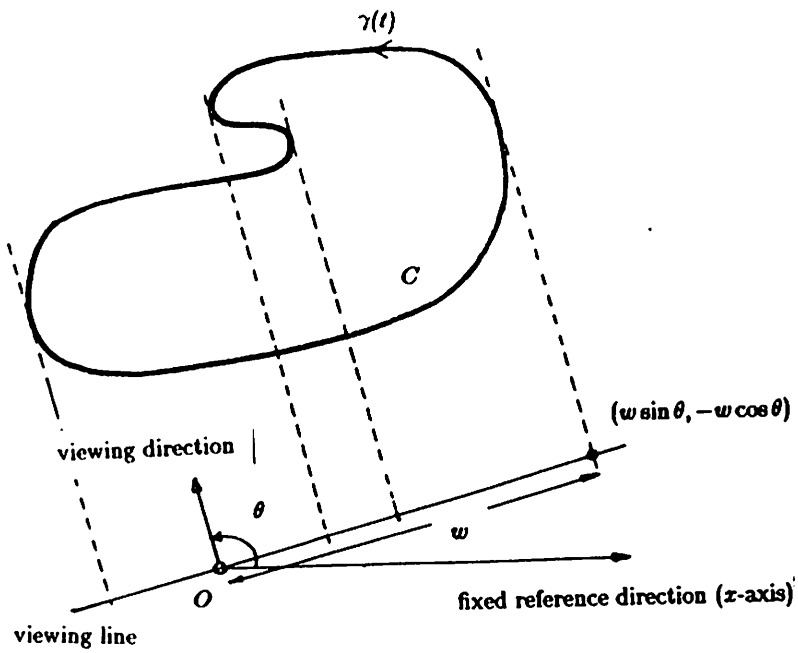

Figure 2: The profile of a plane curve is a set of points on the viewing line

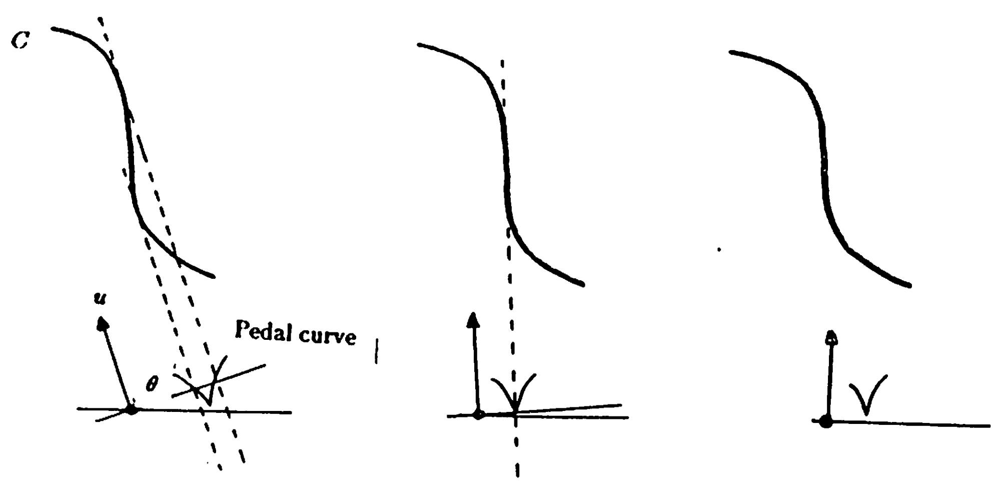

Figure 3: An inflexion in C produces a cusp in the pedal curve

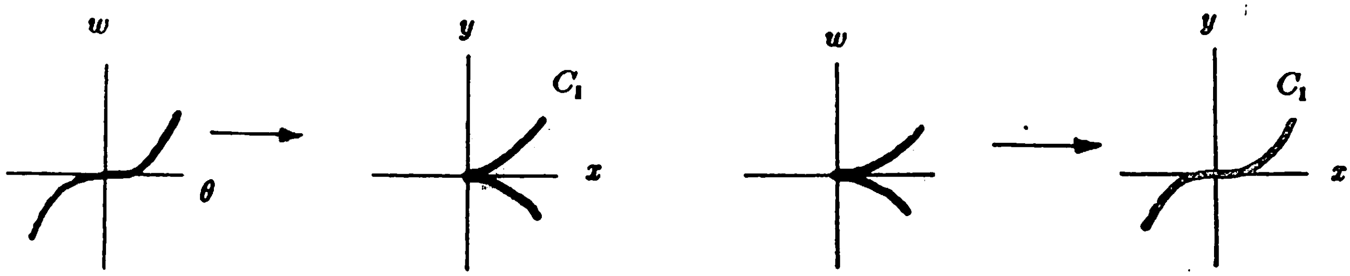

Figure 4: Examples of pedal curves for a cusp and an inflexion

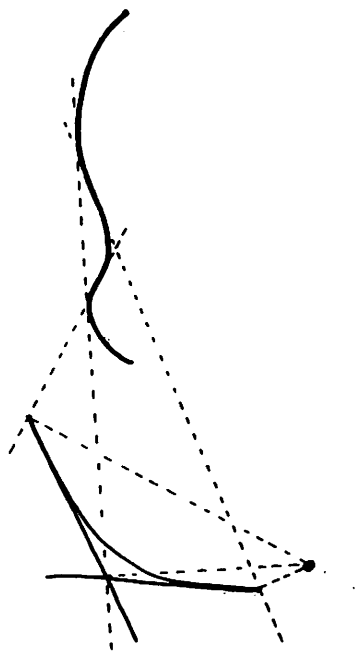

Figure 5: A point where the pedal curve crosses itself

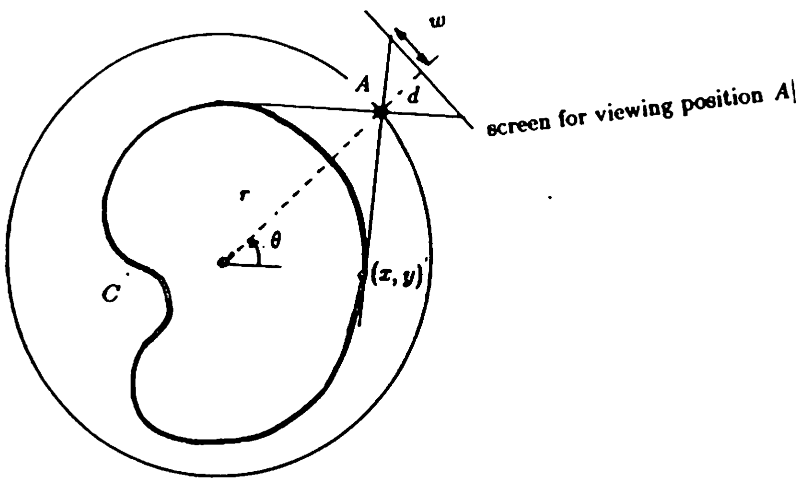

Figure 6: The viewing line for perspective projection

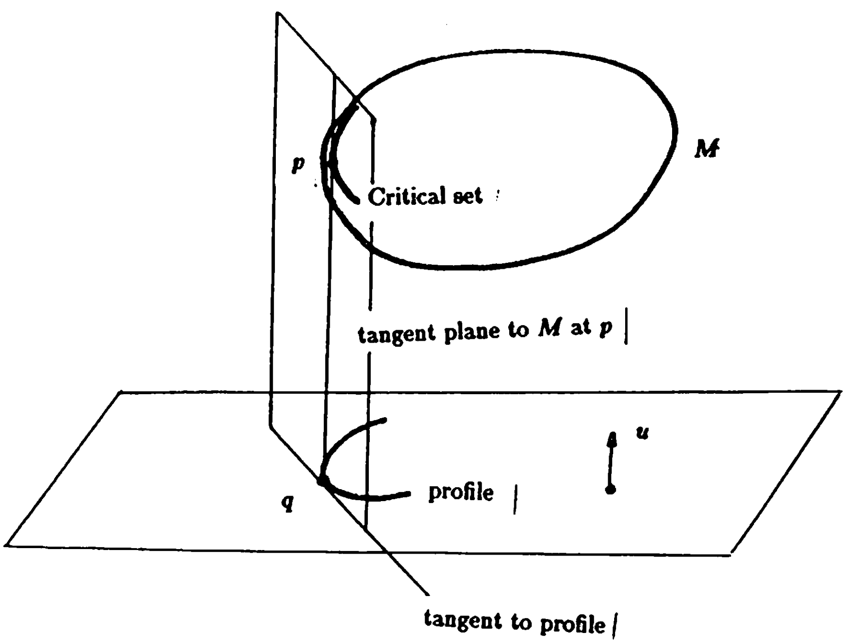

Figure 7: The tangent plane at p contains the viewing direction and a vector parallel to the tangent to the profile curve at q

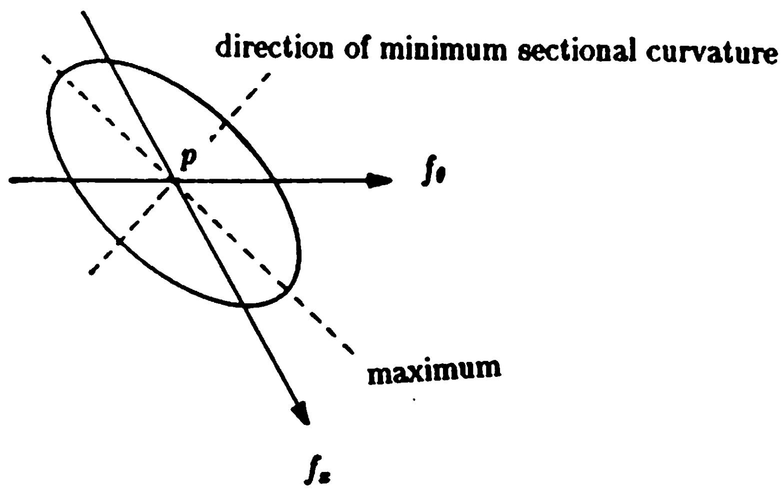

Figure 8: The viewing direction and the tangent to the critical set are conjugate

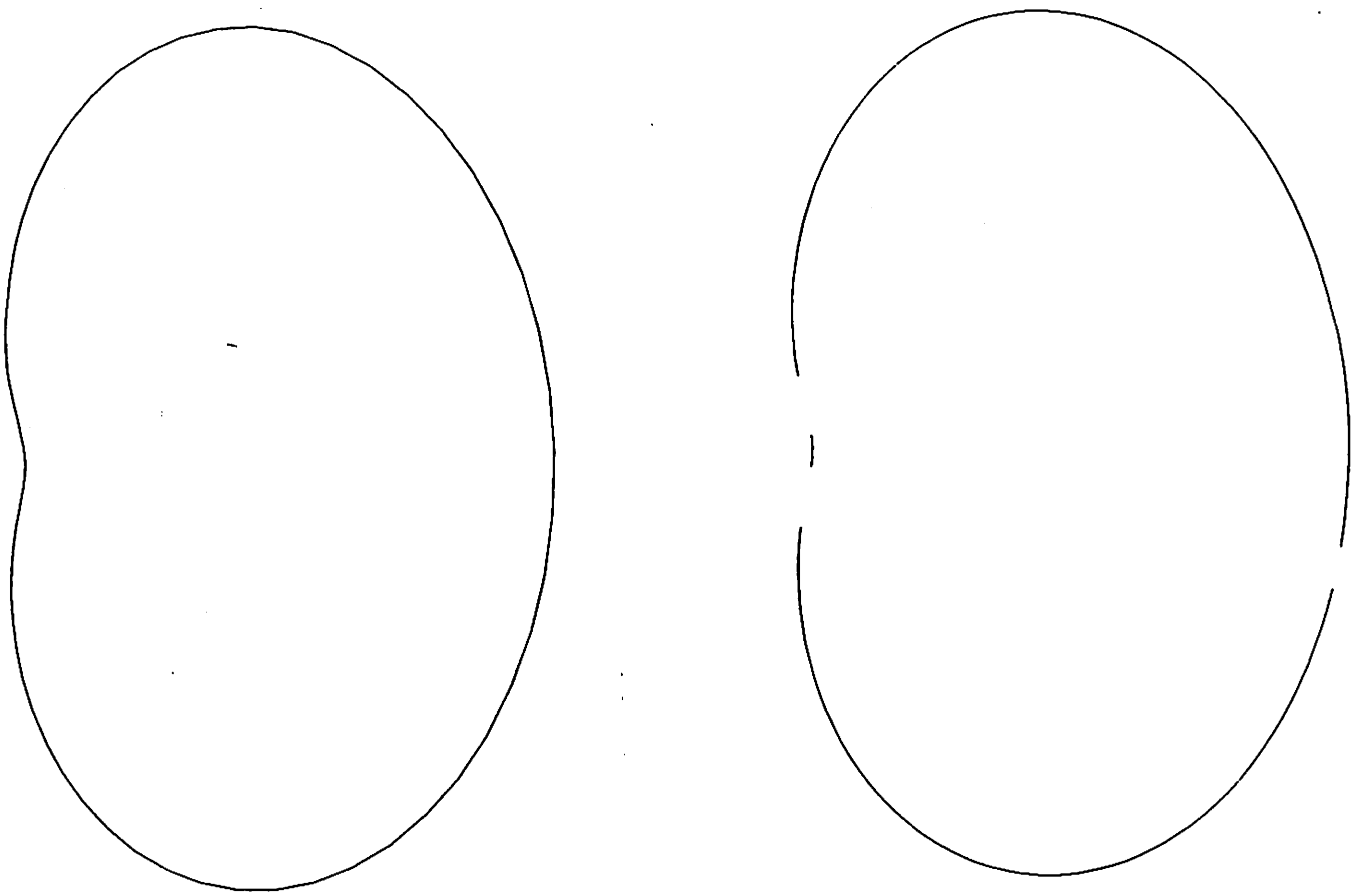

Figure 9: a. A drawing of the limacon from initial data. b. The reconstruction of the limacon.