HAMILTONIAN CIRCUITS IN CAYLEY DIGRAPHS*

Fred Annexstein
Marc Baumslag
Computer and Information Science Department
University of Massachusetts

COINS Technical Report 88-40

*This research was supported in part by NSF Grant no. DCI-87-96236

HAMILTONIAN CIRCUITS IN CAYLEY DIGRAPHS *

Fred Annexstein and *Marc Baumslag*
Department of Computer and Information Science
University of Massachusetts
Amherst, MA 01003

April 20, 1988

ABSTRACT. If the Cayley digraphs of two groups G and H contain Hamiltonian circuits, then so also does the Cayley digraph of the wreath product $G \otimes_{wr} H$, for a particular generating set. We characterize some conditions under which the chosen generating set is minimal.

1. INTRODUCTION

Given a group G and a generating set S for G, the associated Cayley digraph $Cay(G : S)$ is defined as follows: The vertex set is the group G. There is a directed arc from g_1 to g_2 whenever $g_2 = g_1 h$ where $h \in S$. A digraph $\Gamma = (V, A)$ is Hamiltonian if there is a listing $v_1, v_2, \ldots, v_n$ of the vertex set so that $(v_i, v_{i+1}) \in A$ for $i = 1, \ldots, n-1$ and $(v_n, v_1) \in A$. A digraph is ***strongly Hamiltonian*** if each arc of the digraph lies on some Hamiltonian circuit. We say that a finite group G possesses a ***Hamiltonian generating set*** if there exists a minimal generating set S for G such that the Cayley digraph $Cay(G : S)$ is Hamiltonian. Klerlein [Kl] has shown that all finite abelian groups have a Hamiltonian generating set and conjectured that all finite groups do. In this paper, we show that the members of a class of non-abelian groups, the wreath product of cyclic groups, possess Hamiltonian generating sets. Furthermore, we exhibit sufficient conditions for the wreath product of arbitrary finite groups, $G \otimes_{wr} H$, to have this property. Finally, we show that the Cayley digraph of a group with respect to a Hamiltonian generating set is strongly Hamiltonian.

We will consider the Cayley digraph of the wreath product of two groups with respect to a specific non-standard generating set. We prove that a directed Hamiltonian circuit exists in the Cayley digraph of the standard wreath product $G \otimes_{wr} H$, provided the Cayley digraphs of two

*This research was supported in part by NSF Grant no. DCI-87-96236.

groups G and H contain Hamiltonian circuits. We give sufficient conditions for when the chosen generating set is minimal.

This first section will introduce definitions and related work. Section 2 contains the proof of our main result. Section 3 characterizes conditions for the minimality of the generating set. We show also that a Cayley digraph that is Hamiltonian with respect to a minimal generating set is strongly Hamiltonian. Finally, Section 4 provides some further observations, conclusions, and some open questions.

1.1. Basic Definitions

A *group* is given by a set G, together with an associative binary *multiplication* on G (denoted by a "$\cdot$", omitted when it is clear from context) that has

1. an *identity* $e \in G$ for which $ge = eg = g$, for all $g \in G$.
2. *inverses* – for each $s \in G$, an element $s^{-1} \in G$ for which $ss^{-1} = s^{-1}s = e$.

A subset $S \subseteq G$ is a *generating set* for G if each element of G can be written as a product of elements of S. A generating set is *minimal* if no proper subset of S generates the group.

Let S be a generating set for a group G. The *Cayley digraph* of G with respect to S is the directed graph whose vertex set is the group G; there is a directed arc from g_1 to g_2 whenever $g_1^{-1}g_2 \in S$. We denote by $Cay(G : S)$, or simply $Cay(S)$, the Cayley digraph induced by the set S of generators.

In this paper, the term "wreath product" refers to the *standard wreath product* (see [Su]) of a pair of groups G and H. Let $|H| = n$ and let Υ be the cartesian product of n copies of G. Choose an ordering of the elements of H, say $h_1 < h_2 < \cdots < h_n$. Use this order as an index into the elements of Υ.

The set of elements of the wreath product of G by H, written $G \otimes_{wr} H$, is the cartesian product $H \times \Upsilon$. Thus a given element $\pi \in G \otimes_{wr} H$ can be written as an ordered pair

$$\pi = \langle \alpha; \beta_{h_n}, \ldots, \beta_{h_1} \rangle$$

where $\alpha \in H$ and each $\beta_{h_i} \in G$. Multiplication within the wreath product is defined as follows:

$$\langle \alpha; \beta_{h_n}, \ldots, \beta_{h_1} \rangle \langle \gamma; \beta'_{h_n}, \ldots, \beta'_{h_1} \rangle = \langle \alpha\gamma; (\beta_{h_n}\beta'_{(h_n\alpha)}), \ldots, (\beta_{h_1}\beta'_{(h_1\alpha)}) \rangle$$

Note how the first component of the left multiplicand yields an action on the indices in the resultant product.

We use e_G and e_H to denote the identity elements of G and H, respectively; we omit the subscripts when the group in question is clear.

Suppose that S_G and S_H are generating sets for the groups G and H, respectively. The generating set Π we use for the wreath product $G \otimes_{wr} H$ is the following, which we shall refer to as the *non-standard* generating set:

$$\{\langle h; e, e, \ldots, e\rangle \mid \text{ for each } h \in S_H\} \bigcup \{\langle h; e, e, \ldots, e, g\rangle \mid \text{ for each } h \in S_H, g \in S_G\}$$

Below we define the *standard* generating set for comparison. In order to visualize $Cay(\Pi)$, consider the case where H is the cyclic group Z_n. Let A represent an n-tuple with components in G, and let $u_i(g)$ be the n-tuple consisting of e_G in all places except for the i-th place, in which there is a $g \in S_G$. Then in $Cay(\Pi)$, the vertex (i, A) is adjacent to both $(i+1, A)$ and $(i+1, A \cdot u_i(g))$ for each g in S_G and $A \cdot u_i(g)$ is component-wise vector multiplication. If G is a cyclic group of order two, the Cayley digraph resulting from this generating set is the Butterfly digraph (see Fig.1); if G is an arbitrary cyclic group, the Cayley digraph is a generalization of the Butterfly digraph, and is closely related to the generalized deBruijn digraph [ABR].

A *path* in a directed graph is a sequence of distinct vertices, $v_1, v_2, \cdots, v_\ell$, where each pair (v_i, v_{i+1}) is an arc. A *circuit* is a path such that (v_ℓ, v_1) is an arc. Our focus in this paper is on Hamiltonian circuits – that is, circuits which include every vertex in the digraph.

We exhibit paths in Cayley digraphs by specifying an initial vertex and a sequence of elements of the generating set. We use superscripts to denote concatenation, as in $a^1 = a, a^2 = aa, a^3 = aaa$ and so on.

In Section 2 we show that with the previously defined non-standard generating set we can cycle through all the elements of the wreath product via a "staggered counting" technique, giving us a Hamiltonian circuit for the Cayley digraph.

1.2. Related Work

Finding Hamiltonian circuits in Cayley digraphs has been an area of active research in recent years; for a survey of the results in this field see [WG]. The most famous outstanding conjecture in this area is that all undirected Cayley graphs[1] contain (undirected) Hamiltonian cycles. In the case of Cayley digraphs it is known that the analogue of this conjecture is false. For example, the Cayley digraph of $Z_2 \otimes_{wr} Z_3$ with respect to the standard set of generators (see definition below) does not contain even a Hamiltonian path [WG].

For the direct product of cyclic groups $Z_m \times Z_n$, we call the pair of elements $\{(1,0),(0,1)\}$ the *standard set* of generators. By a result of Trotter and Erdös [TE], there is a Hamiltonian circuit in the Cayley digraph of $Z_m \times Z_n$ with the standard set of generators iff the $gcd(m,n) = d \geq 2$ and there exists positive integers d_1, d_2 so that $d_1 + d_2 = d$ and $gcd(m, d_1) = gcd(n, d_2) = 1$. In such a case, we say that the pair (m, n) satisfies the *Trotter-Erdös* conditions.

[1] A Cayley graph is regarded as undirected when the generating set is closed under inverses.

The *standard set* of generators for the wreath product $G \otimes_{wr} H$, is

$$\{\langle h; e, e, \ldots, e\rangle \mid \text{ for each } h \in S_H\} \bigcup \{\langle e_H; e, \ldots, e, g\rangle \mid \text{ for each } g \in S_G\}$$

Only partial results have been obtained towards characterizing the Hamiltonianicity of the resulting Cayley graphs (See [St]) and these results all pertain to undirected circuits. It remains an open question when the Cayley digraph of the wreath product using the standard set of generators is Hamiltonian.

2. THE MAIN RESULT

In order to prove our main result we first prove a slightly weaker version as a lemma, to simplify exposition. For the lemma, we consider the case where H is the cyclic group Z_n.

Fix a group G and a set S of generators for G such that the Cayley digraph of G has a Hamiltonian circuit. Let $s_1, s_2, \ldots, s_m$ (each $s_i \in S$, $m = |G|$) be such a circuit, starting from the identity element of G. Note that $s_1 s_2 \cdots s_m = e_G$.

We now introduce a shorthand to enhance the readability of our results. Let $Cay(\Pi)$ denote the Cayley digraph of $G \otimes_{wr} Z_n$ with the non-standard generating set Π. Within the set Π, let π_0 denote the element

$$\langle 1; e, e, \ldots, e\rangle$$

and let π_j denote the element

$$\langle 1; e, e, \ldots, e, s_j\rangle$$

where $s_j \in S$.

We first exhibit a path in $Cay(\Pi)$ and then verify its correctness as a Hamiltonian circuit. The path is constructed (starting at the identity) as the concatenation of n distinct abutting paths $P_0, P_1, \ldots, P_{n-1}$ where each path is defined recursively as follows :

$$P_i \equiv \begin{cases} [\pi_j\ \pi_0^{n-1}]_{j=1}^{m-1}\ \pi_m & \text{if } i = 0 \\ [\pi_j\ \pi_0^{n-i-1}, P_0, P_1, \ldots, P_{i-1}]_{j=1}^{m-1}\ \pi_m & \text{if } i = 1, 2, \ldots, n-1 \end{cases}$$

for example :

$$P_0 \equiv \pi_1\ \pi_0^{n-1}\ \pi_2\ \pi_0^{n-1}\ \ldots\ \pi_{m-1}\ \pi_0^{n-1}\ \pi_m$$

Lemma 1 *The Cayley digraph $Cay(\Pi)$ contains a Hamiltonian circuit. This circuit is the concatenation of n distinct abutting paths $P_0, P_1, \ldots, P_{n-1}$ as defined above.*

Proof. The proof follows directly from an inductive application of the following claim.

Claim : Given any integer i such that $0 \leq i \leq n-1$, choose $n-1-i$ elements of G, say

$$\beta_{n-1}, \beta_{n-2}, \ldots, \beta_{i-1}$$

The path in $Cay(\Pi)$ starting at

$$\alpha = \langle i; \beta_{n-1}, \beta_{n-2}, \ldots, \beta_{i+1}, e, e, \ldots, e \rangle$$

and traversing the arc sequence defined by P_i passes uniquely through all vertices labelled by

$$\langle *; \beta_{n-1}, \beta_{n-2}, \ldots, \beta_{i+1}, +, *, \ldots, * \rangle$$

where $*$ represents any element of G, and $+$ represents any element of $G \setminus \{e\}$; the path terminates at the vertex

$$\omega = \langle i+1; \beta_{n-1}, \beta_{n-2}, \ldots, \beta_{i+1}, e, e, \ldots, e \rangle$$

Proof of claim. The proof is by induction. For the basis of the induction consider the path P_0.

Case i=0:

$$\left.\begin{array}{rcl} & & \langle 0; \beta_{n-1}, \beta_{n-2}, \ldots, \beta_1, e \rangle \\ \pi_1 & \rightarrow & \langle 1; \beta_{n-1}, \beta_{n-2}, \ldots, \beta_1, s_1 \rangle \\ \pi_0^{n-1} & \rightarrow & \langle 0; \beta_{n-1}, \beta_{n-2}, \ldots, \beta_1, s_1 \rangle \\ \pi_2 & \rightarrow & \langle 1; \beta_{n-1}, \beta_{n-2}, \ldots, \beta_1, (s_1 \cdot s_2) \rangle \\ \pi_0^{n-1} & \rightarrow & \langle 0; \beta_{n-1}, \beta_{n-2}, \ldots, \beta_1, (s_1 \cdot s_2) \rangle \\ & \cdot & \\ & \cdot & \\ & \cdot & \\ \pi_{m-1} & \rightarrow & \langle 1; \beta_{n-1}, \beta_{n-2}, \ldots, \beta_1, (s_1 \cdot s_2 \cdot \ldots \cdot s_{m-1}) \rangle \\ \pi_0^{n-1} & \rightarrow & \langle 0; \beta_{n-1}, \beta_{n-2}, \ldots, \beta_1, (s_1 \cdot s_2 \cdot \ldots \cdot s_{m-1}) \rangle \\ \pi_m & \rightarrow & \langle 1; \beta_{n-1}, \beta_{n-2}, \ldots, \beta_1, (s_1 \cdot s_2 \cdot \ldots \cdot s_m) \rangle \\ & = & \langle 1; \beta_{n-1}, \beta_{n-2}, \ldots, \beta_1, e \rangle \end{array}\right\} P_0$$

Since the sequence $\{s_i\}_{i=1}^{n}$ defines a Hamiltonian circuit of the digraph $Cay(S)$ by assumption, we see that

i) the path P_0 traverses all vertices of the form

$$\langle *; \beta_{n-1}, \beta_{n-2}, \ldots, \beta_1, + \rangle$$

ii) each vertex on this path is unique.

Thus the basis of the induction holds.

Inductive hypothesis:

Assume the claim holds for all $j < i$.

Inductive step:

Let $\alpha = \langle i; \beta_{n-1}, \beta_{n-2}, \ldots, \beta_{i+1}, e, e, \ldots, e\rangle$. Following the path P_i, we see that

$$\alpha \cdot \pi_1 = \langle i+1; \beta_{n-1}, \beta_{n-2}, \ldots, \beta_{i+1}, s_1, e, \ldots, e\rangle$$

Now applying π_0^{n-i-1}, we reach

$$\delta = \langle 0; \beta_{n-1}, \beta_{n-2}, \ldots, \beta_{i+1}, s_1, e, \ldots, e\rangle$$

having passed through

$$\langle k; \beta_{n-1}, \beta_{n-2}, \ldots, \beta_{i+1}, s_1, e, \ldots, e\rangle$$

for each $i < k < n$, uniquely.

We now repeatedly invoke the inductive hypothesis, starting at δ, achieving a path terminating at

$$\langle i; \beta_{n-1}, \beta_{n-2}, \ldots, \beta_{i+1}, s_1, e, e, \ldots, e\rangle$$

Hence we have a path beginning at

$$\alpha \cdot \pi_1 = \langle i+1; \beta_{n-1}, \beta_{n-2}, \ldots, \beta_{i+1}, s_1, e, \ldots, e\rangle$$

passing uniquely through all vertices labelled by

$$\langle *; \beta_{n-1}, \beta_{n-2}, \ldots, \beta_{i+1}, s_1, +, *, \ldots, *\rangle$$

and terminating at the vertex

$$\langle i; \beta_{n-1}, \beta_{n-2}, \ldots, \beta_{i+1}, s_1, e, e, \ldots, e\rangle$$

We then repeat this process for each of

$$\pi_2, \pi_3, \ldots, \pi_{m-1}$$

ending at

$$\langle i; \beta_{n-1}, \beta_{n-2}, \ldots, \beta_{i+1}, (s_1 \cdots s_{m-1}), e, e, \ldots, e\rangle$$

Finally applying π_m brings us to ω. The proof of the claim then follows.

The lemma now follows by applying the claim for each $i \in \{0, \ldots, n-1\}$, starting at the identity element. ∎

We now graphically demonstrate the Hamiltonian circuit as the repeated application of the claim. Notice how the circuit defines a type of staggered counting through n-tuples of elements of G. The reader is also referred to Figure 2 for a concrete example.

We begin at the identity element of $G \otimes_{wr} Z_n$:

$$
\left.\begin{array}{rr}
 & \langle 0; e, e, \ldots, e\rangle \\
\rightarrow & \langle *; e, e, \ldots, e, +\rangle \\
\rightarrow & \langle 1; e, e, \ldots, e\rangle
\end{array}\right\} P_0
$$

$$
\left.\begin{array}{rr}
 & \langle 1; e, e, \ldots, e\rangle \\
\rightarrow & \langle *; e, e, \ldots, e, +, *\rangle \\
\rightarrow & \langle 2; e, e, \ldots, e\rangle
\end{array}\right\} P_1
$$

$$
\left.\begin{array}{rr}
 & \langle 2; e, e, \ldots, e\rangle \\
\rightarrow & \langle *; e, e, \ldots, e, +, *, *\rangle \\
\rightarrow & \langle 3; e, e, \ldots, e\rangle
\end{array}\right\} P_2
$$

$$\vdots$$

$$
\left.\begin{array}{rr}
 & \langle n-1; e, e, \ldots, e\rangle \\
\rightarrow & \langle *; +, *, *, \ldots, *\rangle \\
\rightarrow & \langle 0; e, e, \ldots, e\rangle
\end{array}\right\} P_{n-1}
$$

Note that each element of $G \otimes_{wr} Z_n$ is traversed once and only once in this list.

Lemma 1 can be extended in a natural way to $G \otimes_{wr} H$ for arbitrary H, as follows.

Theorem 1 *Suppose a pair of groups G and H come equipped with generating sets for which their resulting Cayley digraphs have Hamiltonian circuits. Then the Cayley digraph of the wreath product $G \otimes_{wr} H$ with respect to the non-standard set of generators has a Hamiltonian circuit.*

Proof. This theorem follows directly from Lemma 1. This can be seen if we order the elements of $H = \{h_0, \ldots, h_{n-1}\}$ corresponding to their order in the Hamiltonian circuit in the Cayley digraph of H, starting at the identity. With such an ordering, the element $h_i \in H$ simply "plays the role" of $i \in Z_n$ in the proof of the Lemma. The sequence of vertices traversed is then analogous to that in the proof of Lemma 1. ∎

3. MINIMALITY OF GENERATING SETS

We now give sufficient conditions for when the non-standard generating set for the wreath product $G \otimes_{wr} Z_n$ is minimal. Consider first the wreath product of cyclic groups. The non-standard generating set Π for $Z_m \otimes_{wr} Z_n$ contains two elements. Since the group $Z_m \otimes_{wr} Z_n$ is not cyclic, Π is minimal. By Lemma 1, the Cayley digraph of $Z_m \otimes_{wr} Z_n$ with respect to Π is Hamiltonian. Thus it follows

Corollary 1 *The wreath product of cyclic groups has a Hamiltonian generating set.*

Theorem 2 *Let $S = \{s_1, s_2, \cdots, s_n\}$ be a minimal set of generators for G, and let $\Pi = \{\pi_0, \pi_1, \cdots, \pi_n\}$ be the corresponding set of non-standard generators for $G \otimes_{wr} Z_n$. Π is a minimal set of generators for $G \otimes_{wr} Z_n$ iff for no integer m does $Cay(S)$ have both a circuit of length m and a circuit of length $m+1$.*

Proof. Consider an element $\langle \alpha; \beta_n, \beta_{n-1}, \ldots, \beta_1 \rangle$ of $G \otimes_{wr} Z_n$. For convenience we refer to α as the first component and $(\beta_n, \beta_{n-1}, \ldots, \beta_1)$ as the second component. Recall that the second component is an element of the cartesian product of n copies of G.

Suppose Π is not a minimal set of generators for $G \otimes_{wr} Z_n$. Since S is minimal for G and π_0 does not affect the values in the second component it is not possible to have a product of elements in $\Pi \setminus \{\pi_k\}$ equal to π_k for $k \neq 0$. Thus it follows that there exists a sequence $\{\pi_{k_1}, \pi_{k_2}, \ldots, \pi_{k_\ell}\}$ of elements in $\Pi \setminus \{\pi_0\}$ for which

$$\pi_{k_1} \cdot \pi_{k_2} \cdots \pi_{k_\ell} = \pi_0$$

If we now dissect this product we observe the following.

- Consider the first component of the wreath product. In order to obtain the generator 1 of Z_n as the first component in the result of the above product, it must be the case that $k_\ell = mn+1$ for some natural number m.
- Now consider the second component of the wreath product $G \otimes_{wr} Z_n$ (i.e., an element of the cartesian product of n copies of G). Each consecutive block of n generators contributes exactly one element to the product being accrued in each component of the cartesian product.

Thus after m blocks (each of length n) have been multiplied, each product that has accrued will be of length m. The final multiplication by π_{k_ℓ} will make the length of the product in the first position (of the cartesian product) equal to $m+1$. The forward implication follows from this observation.

Conversely, suppose that there is a product of m elements of S which multiplies out to the identity: say $s_{k_1} s_{k_2} \cdots s_{k_m} = e_G$, and similarly, there is a product of length $m+1$ say $s_{l_1} s_{l_2} \cdots s_{l_m} s_{l_{m+1}} = e_G$. Then we can obtain π_0 in terms of the other generators, as is borne out by the following product.

$$\pi_{l_1} \cdot \pi_{k_1}^{n-1} \cdot \pi_{l_2} \cdot \pi_{k_2}^{n-1} \cdots \pi_{l_m} \cdot \pi_{k_m}^{n-1} \cdot \pi_{l_{m+1}} = \pi_0$$

This establishes the non-minimality of Π. ∎

Corollary 2 *The iterated wreath product $Z_{i_0} \otimes_{wr} Z_{i_1} \otimes_{wr} \cdots \otimes_{wr} Z_{i_n}$ has a Hamiltonian generating set.*

Proof. Since the wreath product is an associative operator on groups, we can iteratively apply Theorem 1 to obtain a Hamiltonian circuit in the Cayley digraph of this group using the non-standard generating set. We only need to show the minimality of this generating set. Consider $G \otimes_{wr} Z_n$ (where G is arbitrary) with the non-standard generating set Π. Any product of elements of Π equal to the identity must have cardinality equal to a multiple of n; otherwise the first component of the product will not be the identity element of Z_n. The result follows by applying Theorem 2 inductively. ∎

As an example of Theorem 2, we note that the non-standard set of generators for $(Z_2 \times Z_3) \otimes_{wr} Z_2$ is not minimal (when taking the set of standard generators for $Z_2 \times Z_3$). This is true since

$$(1,0) \cdot 2 + (0,1) \cdot 0 = (1,0) \cdot 0 + (0,1) \cdot 3 = (0,0)$$

This generalizes to the following result:

Corollary 3 *Suppose we choose the standard set of generators for $Z_p \times Z_q$, and let Π be the non-standard set of generators for $(Z_p \times Z_q) \otimes_{wr} Z_n$. Then Π is minimal iff $gcd(p,q) \geq 2$.*

Proof. The proof proceeds by contradiction. By Theorem 2, we need only find an integer m for which there exist products of m and $m+1$ standard generators for $Z_p \times Z_q$ that are both equal to the identity. In this case, there exist integers x_1, y_1, x_2, y_2 such that

$$(1,0) \cdot x_1 + (0,1) \cdot y_1 = (1,0) \cdot x_2 + (0,1) \cdot y_2 = (0,0)$$

where $x_1 + y_1 = m$ and $x_2 + y_2 = m + 1$

$\Leftrightarrow$ $\exists$ integers n_1, n_1', n_2, n_2' such that $x_1 + y_1 = n_1 p + n_1' q = m$ and $x_2 + y_2 = n_2 p + n_2' q = m + 1$

$\Leftrightarrow$ $(n_2 - n_1)p + (n_2' - n_1')q = 1$

$\Leftrightarrow$ $gcd(p,q) = 1$. ∎

Corollary 4 *If a pair of positive integers (p,q) satisfies the Trotter-Erdős conditions then the group $(Z_p \times Z_q) \otimes Z_n$ has a Hamiltonian generating set.*

Proof. The Cayley digraph of $Z_p \times Z_q$ with the standard generating set has a Hamiltonian circuit if (p, q) satisfies the Trotter-Erdös conditions. By Corollary 3 the non-standard generating set for $(Z_p \times Z_q) \otimes_{wr} Z_n$ is minimal, and by Theorem 1 the respective Cayley digraph contains a Hamiltonian circuit. ∎

We show that a Cayley digraph with respect to a minimal generating set is Hamiltonian iff it is strongly Hamiltonian.

Proposition 1 *If a group possesses a Hamiltonian set of generators then the Cayley digraph with respect to that set is strongly Hamiltonian.*

Proof. Suppose the set of generators is Π and there is a Hamiltonian circuit $\pi_1, \pi_2, \cdots, \pi_n$ originating at the identity element. Since Π is minimal every generator must appear at least once in the circuit. Choose an arbitrary arc, say $(g, g\pi_i)$. If we wish to obtain a circuit through the arc $(g, g\pi_i)$ then we simply start the given circuit from the element

$$g \cdot \pi_{i-1}^{-1} \cdot \pi_{i-2}^{-1} \cdots \pi_1^{-1}$$

and apply the sequence of arcs defined by $\pi_1, \pi_2, \cdots, \pi_n$. We have simply translated the circuit to a new starting vertex. Observe that a translation of a circuit in a Cayley digraph is again a circuit. ∎

4. CONCLUSION

We have exhibited classes of Cayley digraphs which possess Hamiltonian circuits. Negative results in this area appear harder to come by. Towards this direction, we have verified by computer search that if we remove the redundant generator (of the non-standard set) for $(Z_2 \times Z_3) \otimes_{wr} Z_2$, then the Cayley digraph *does not* have a Hamiltonian circuit. This is interesting in light of the fact that $Z_2 \times Z_3$ does not have a Hamiltonian circuit; note that $(2, 3)$ does not satisfy the Trotter-Erdös conditions. This fact also answers a question posed in [CW] as to whether there exists a non-abelian group with a minimal set of generators whose Cayley digraph does not have a Hamiltonian circuit.

The obvious generalization of this situation leads to the following question:

Open Question: Suppose the pair of integers (p, q) does not satisfy the Trotter-Erdös conditions. When, if ever, does the Cayley digraph of $(Z_p \times Z_q) \otimes_{wr} Z_n$, with respect to the non-standard set of generators, have a Hamiltonian circuit?

In general, it would be of interest to know whether or not it is necessary for G to have a circuit in order for $G \otimes_{wr} Z_n$ to have a circuit.

Acknowledgements: We would like to express our thanks to Arnold Rosenberg and Dave Barrington for their support and useful advice during this research project.

5. References

[**ABR**] F. Annexstein, M. Baumslag, A.L. Rosenberg (1987): Group-action graphs and parallel architectures. *UMASS-COINS Tech. Rpt. 87-133*

[**CW**] S.J. Curran and D. Witte (1984): Hamiltonian paths in cartesian products of directed cycles. Ann. Discrete Math. 27, 35-74.

[**Kl**] J.B. Klerlein (1978): Hamiltonian cycles in Cayley color graphs, *J. Graph Theory 2*, 65-68.

[**St**] R. Stong (1987) : On Hamiltonian cycles in Cayley graphs of wreath products. *Discrete Math. 65*, 75-80.

[**Su**] M. Suzuki (1982): *Group Theory I.* Springer-Verlag, New York.

[**TE**] W.T. Trotter, Jr., and P. Erdös (1978): When the cartesian product of directed cycles is hamiltonian. *J. Graph Theory 2*, 137-142.

[**WG**] D. Witte and D. Gallian (1984): A survey: Hamiltonian cycles in Cayley graphs. *Discrete Math. 51*, 293-304.

[**Wi**] D. Witte (1982): On Hamiltonian circuits in Cayley diagrams. *Discrete Math. 38*, 99-108.

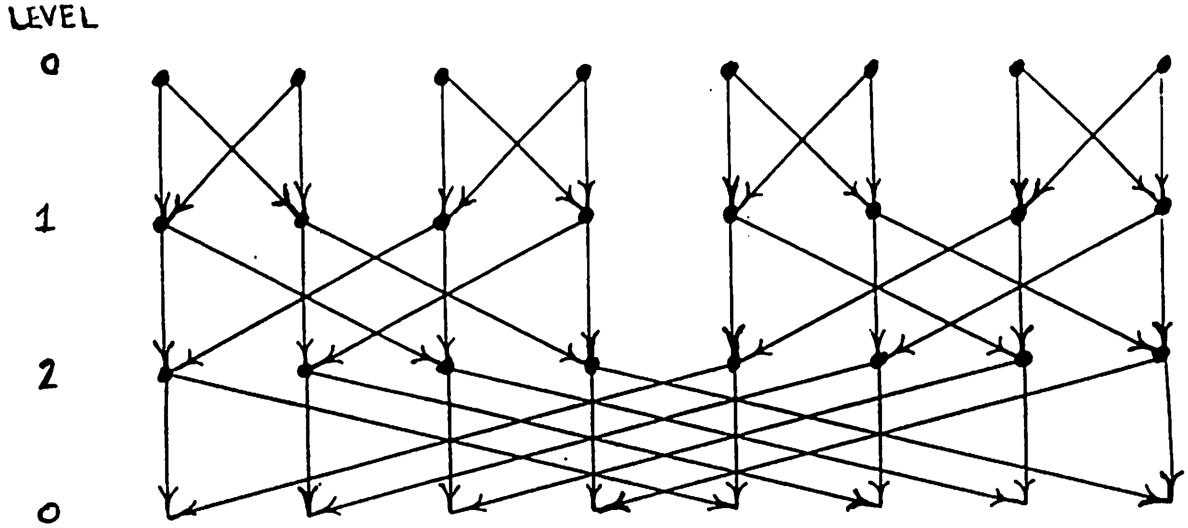

Figure 1. The Butterfly graph, $(Z_2 \otimes_{wr} Z_3, \Pi)$.

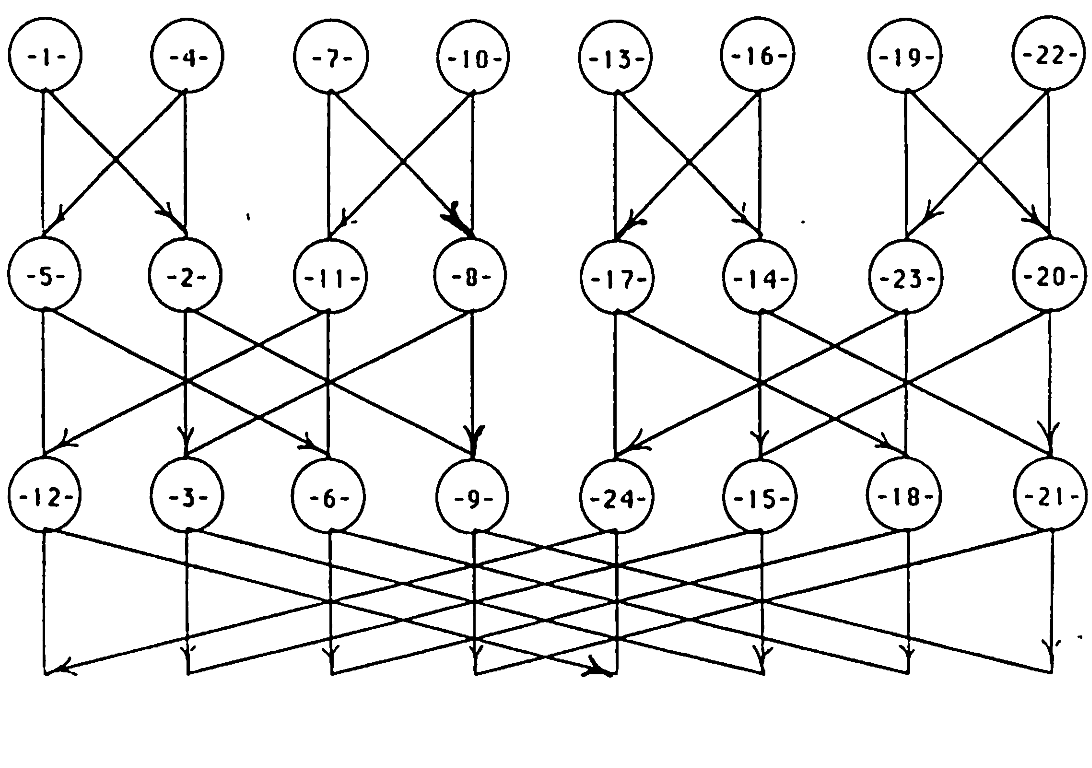

Figure 2. The circuit in $(Z_2 \otimes_{wr} Z_3, \Pi)$.