THE CUSTOMER RESPONSE TIMES IN THE PROCESSOR SHARING QUEUE ARE ASSOCIATED

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Abstract

The customer response times in the egalitarian processor sharing queue are shown to be associated random variables under renewal inputs and general independent service times assumptions.

Key Words: Stochastic ordering, association of random variables, processor sharing queues.

1 Problem Description

Association is a type of positive correlation between random variables that allows one to bound certain of their joint statistics, like the maximum or the minimum, by functions that only require the knowledge of their marginal distributions. This notion has proved quite fruitful for obtaining bounds on various queueing systems (see [BM89] for a survey on the matter).

The processor sharing queue is a basic model for representing multiprogramming in computer systems and has been studied in a number of papers, see [FMI80] and the references contained therein. However, the purpose of these studies was to determine the mean response time of a customer given its service requirements, which are known in the case of Poisson arrivals and general independent service times. Besides these computational results, few structural properties of this queueing system are known.

The main result of the present paper consists in proving that the delays incurred by customers in the egalitarian processor sharing queue are associated random variables provided the service times and the negative interarrival times are associated, so that the property holds in particular in the case of independent and i.i.d. service and interarrival times.

A similar property can be proved for other and more elementary queueing systems, including $G/G/s$ FCFS queues ([Bac87]), under the same assumption on the service and interarrival times. However, the proof for the processor sharing case is far more elaborate and requires the definition of several intermediate processes. Basic definitions on associated random variables are sketched in Section 3. The intermediate processes of interest are introduced in Sections 4 and 5. Besides the proof of the main result,

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which is established in Section 6, we illustrate its practical interest by sketching a simple application to the *resequencing queue* ([BGP84]) in Section 7.

2 Notation and Assumptions

In the sequel, we shall consider bi-infinite vectors in $I\!R^\infty$. We shall denote by e_j, $j = \cdots, -2, -1, 0, 1, 2, \cdots$ the natural basis of $I\!R^\infty$. The projection of a vector $x = (\cdots, x_{-2}, x_{-1}, x_0, x_1, x_2, \cdots)$ in $I\!R^\infty$ on the space endowed by the vectors e_j, $j = J, J+1, J+2, \cdots, K$, $J \leq K \in \mathbb{Z}$ will be denoted by $\Pi^{J,K}(x)$. In particular, we shall denote by x^+ the projection $\Pi^{0,\infty}(x)$ and by x^- the projection $\Pi^{-\infty,0}(x)$. The leftshift operator will be denoted by θ with the meaning that $x \circ \theta = \sum_{i=-\infty}^{\infty} x_{i+1} e_i$ if $x = \sum_{i=-\infty}^{\infty} x_i e_i$. The leftshift operator will also be used for one sided infinite vectors with the meaning $x \circ \theta = \sum_{i=0}^{\infty} x_{i+1} e_i$ if $x = \sum_{i=0}^{\infty} x_i e_i$.

Two vectors x and y of $I\!R^\infty$ will be said to satisfy the inequality $x \leq y$ if for all $i = \cdots, -2, -1, 0, 1, 2, \cdots$, the inequality $x_i \leq y_i$ holds. We will write $x < y$ if the inequality is strict for at least one coordinate. Similarly, a function $f : I\!R^\infty \rightarrow I\!R^\infty$ will be said to be (strictly) increasing if $x \leq y$ ($x < y$) implies $f(x) \leq f(y)$ ($f(x) < f(y)$).

For all x in $I\!R^{+\infty}$, let $N(x) \in I\!N$ and $Min(x) \in I\!R^+$ respectively denote the number of non zero coordinates in y and the minimal non zero coordinate of y, namely

$$N(x) = \sum_{i=-\infty}^{\infty} I[x_i > 0]$$

and

$$Min(y) = \inf_{\{i=\cdots,-2,-1,0,1,2,\cdots | x_i > 0\}} x_i$$

where the minimum over an empty set is zero by convention.

$I\!F$ will denote the subset of $I\!R^{+\infty}$ the vectors of which have finitely many non zero coordinates:

$$I\!F = \{x \in I\!R^{+\infty} | N(x) < \infty\}$$

e will denote the vector with all its components equal to 1 and while for all $x \in I\!R^\infty$, $(x)^+$ will denote the vector of $I\!R^\infty$ defined by $(x)^+ = (\cdots, max(x_{-1}, 0), max(x_0, 0), max(x_1, 0), \cdots)$.

3 Association of random variables

The $I\!R$-valued RV's (Random Variables) $\{x_1, ..., x_K\}$ are *associated* if and only if, with the notation $x = (x_1, ..., x_K)$, the inequality

$$E[f(x)g(x)] \geq E[f(x)]E[g(x)]$$

holds for all pair of *monotone non-decreasing* mappings $f, g : I\!R^K \rightarrow I\!R$ for which the expectations $E[f(x)]$, $E[g(x)]$ and $E[f(x)g(x)]$ exist. A set of random vectors will be said to be associated if the collection of their coordinates is a set of associated RV's.

In order to explain the usefulness of this concept, it will be convenient to say that the $I\!R$-valued RV's $\{\bar{x}_1, ..., \bar{x}_K\}$ form *independent versions* of the RV's $\{x_1, ..., x_K\}$ if

(i) : The RV's $\{\bar{x}_1, ..., \bar{x}_K\}$ are *mutually independent*, and

(ii) : For every $1 \leq k \leq K$, the RV's x_k and $\bar{x}_k$ have the *same* probability distribution.

The following proposition [BP81] is an easy consequence of the above definition.

Theorem 1 *If the RV's $\{x_1, ... x_K\}$ are associated, then the inequality*

$$P[\max_{1 \leq k \leq K} x_k \leq t] \geq P[\max_{1 \leq k \leq K} \bar{x}_k \leq t]$$

holds true for all t in $I\!R$.

This result can be viewed as a statement on the stochastic ordering between the maximum of the RV's $\{x_1, ..., x_K\}$ and the corresponding quantity for the independent version. More precisely, if x and y are two $I\!R$-valued RV's, then the (distribution of the) RV x is said to be *greater* than the (distribution of the) RV y in the *stochastic order* if and only if

$$P[y > t] \leq P[x > t]$$

for all t in $I\!R$; this is denoted in short by $y \leq_{st} x$. With this notion, Theorem 1 can be restated simply as saying that

$$\max_{1 \leq k \leq K} x_k \leq_{st} \max_{1 \leq k \leq K} \bar{x}_k.$$

The elements of a "calculus" for associated RV's are provided in [BP81, pp. 29-31]. Some of these facts, which are often used in the discussion, have been collected in the next theorem for easy reference.

Theorem 2 *Association of r.v.'s exhibits the following properties.*

1. *Independent RV's are associated;*
2. *The union of independent collections of associated RV's forms a set of associated RV's;*
3. *Any subset of a family of associated RV's forms a set of associated RV's, and*
4. *Any monotone non-decreasing function of associated RV's generates a set of associated RV's.*

4 Egalitarian Processor Sharing

Consider an egalitarian processor sharing queue with renewal inputs and general independent service times. At time 0, which is supposed to coincide with the arrival date of customer C_0, there is an initial vector of residual service times $y = (\cdots, y_{-2}, y_{-1}, y_0, 0, 0, \cdots) \in I\!F$, where y_{-i}, $i \geq 0$ is the residual workload of C_{-i}, the i-th customer that arrived in the past. In particular, the residual service time of C_0 at its arrival date is exactly its service time, which will be denoted by σ_0.

New customers $C_1, C_2, \cdots$ arrive at dates $0 < t_1 < t_2 < \cdots$, respectively, which define the inter-arrival sequence $\{\tau_n = t_n - t_{n-1}\}_{n=1}^{\infty}$, where $t_0 = 0$ by definition. Customer C_n $(n \geq 1)$ requires service time $\sigma_n \geq 0$. More generally, in the sequel, σ (resp. τ) will denote the bi-infinite vector with coordinates $(\cdots, \sigma_{-2}, \sigma_{-1}, \sigma_0, \sigma_1, \sigma_2, \cdots)$ (resp. $(\cdots, \tau_{-2}, \tau_{-1}, \tau_0, \tau_1, \tau_2, \cdots)$).

4.1 The process with blocked arrivals

For $y = (\cdots, y_{-2}, y_{-1}, y_0, 0, 0, \cdots) \in I\!F$, let $\mathcal{Y}(y,t) \in I\!F$ denote the value of the residual service times vector at time $t \in I\!R^+$ under the assumption that the arrival process is blocked from time 0. Observe that there are initially $N(y)$ customers in the system and that some of the non-zero coordinates of the $\mathcal{Y}$ vector will eventually vanish as time increases and keep on being equal to zero from that time.

Lemma 1 *The function $\mathcal{Y}(y,t) : IF \times IR^+ \to IF$ is increasing in y and decreasing in t.*

Proof

Consider two initial residual workload vectors $y = (\cdots, y_{-2}, y_{-1}, y_0, 0, 0, \cdots)$ and $y' = (\cdots, y'_{-2}, y'_{-1}, y'_0, 0, 0, \cdots)$ in IF satisfying the inequality $y \leq y'$. Consider two processor sharing queues with blocked arrivals and respective residual service times vector y and y'. We shall refer to the system with initial vector y as to the first system while the second system will be the one with y'. The first departure in both systems takes place at time $\min(N(y)Min(y), N(y')Min(y')$. Let z and z' denote the the residual service time vector at this date, in the first and the second system respectively.

If $N(y)Min(y') \leq N(y')Min(y')$,

$$z = (y - Min(y)e)^+$$

and

$$z' = (y' - \frac{N(y)}{N(y')}Min(y)e)^+$$

Since $y \leq y'$ it follows immediately that $N(y) \leq N(y')$, so that the decrease of residual service time is smaller in the second system than in the first one. We have hence $z \leq z'$.

If $N(y)Min(y') > N(y')Min(y')$,

$$z = (y - \frac{N(y')}{N(y)}Min(y')e)^+$$

and

$$z' = (y' - Min(y')e)^+$$

Since $N(y) \leq N(y')$, it follows that the decrease of residual service time is larger in the first system than in the second one. We have hence again $z \leq z'$.

One immediately gets by induction that the residual residual service times vectors are smaller in the first system than in the second one at all departure epochs. Since the components of the residual service times vector decrease linearly in both systems between departure epochs, it also follows that the ordering extends to continuous time as well.

4.2 The embedded residual service times process

For $n = 1, 2, \cdots$, let $Y^n(y, \sigma^*, \tau^*) \in IF$ denote the vector of the residual service times of customers in the system just after the arrival epoch of the n-th customer, with the convention that coordinate j keeps track of the residual service times of customers C_j for all $j \in \mathbb{Z}$.

Lemma 2 *For all $n = 0, 1, 2, \cdots$,*

$$Y^{n+1}(y, \sigma^*, \tau^*) = \mathcal{Y}(Y^n(y, \sigma^*, \tau^*) \circ \theta^n, \tau_{n+1}) \circ \theta^{-n} + \sigma_{n+1}e_{n+1}$$

with $Y^0(y, \sigma^*, \tau^*) = y$ by convention.

Proof

The proof is by induction on n. If the residual service times vector at the n-th arrival is $Y^n(y, \sigma^*, \tau^*)$, it will be equal to $\mathcal{Y}(Y^n(y, \sigma^*, \tau^*), \tau_{n+1})$ just before the $n+1$-st arrival. Hence the formula of the Lemma.

Lemma 3 *For all $n = 1, 2, \cdots$, the random vector Y^n is an increasing function of the vectors y, σ^*, $-\tau^*$.*

Proof

1)Increasingness in y.
The proof is by induction on n. The property trivially holds for $n = 0$. Assume it holds up to rank n. Choose y and y' in IF with $y' \geq y$. From Lemma 2, we get

$$\begin{aligned} Y^{n+1}(y, \sigma^{\cdot}, \tau^{\cdot}) &= \mathcal{Y}(Y^{n}(y, \sigma^{\cdot}, \tau^{\cdot}) \circ \theta^{n}, \tau_{n+1}) \circ \theta^{-n} + \sigma_{n+1} e_{n+1} \\ &\leq \mathcal{Y}(Y^{n}(y', \sigma^{\cdot}, \tau^{\cdot}) \circ \theta^{n}, \tau_{n+1}) \circ \theta^{-n} + \sigma_{n+1} e_{n+1} \\ &= Y^{n+1}(y', \sigma^{\cdot}, \tau^{\cdot}) \end{aligned}$$

where we used the induction assumption in order to get the inequality.

2)Increasingness in $\sigma^{\cdot}$.
The proof is by induction on n. The property trivially holds for $n = 0$. Assume it holds up to rank n. Choose $\sigma^{\cdot}$ and $\sigma'^{\cdot}$ in IF with $\sigma'^{\cdot} \geq \sigma^{\cdot}$. We get

$$\begin{aligned} Y^{n+1}(y, \sigma^{\cdot}, \tau^{\cdot}) &= \mathcal{Y}(Y^{n}(y, \sigma^{\cdot}, \tau^{\cdot}) \circ \theta^{n}, \tau_{n+1}) \circ \theta^{-n} + \sigma_{n+1} e_{n+1} \\ &\leq \mathcal{Y}(Y^{n}(y, \sigma'^{\cdot}, \tau^{\cdot}) \circ \theta^{n}, \tau_{n+1}) \circ \theta^{-n} + \sigma'_{n+1} e_{n+1} \\ &= Y^{n+1}(y, \sigma'^{\cdot}, \tau^{\cdot}) \end{aligned}$$

where we used Lemma 2 and the induction assumption in order to get the inequality.

3)The proof for the decreasingness in $\tau^{\cdot}$ is similar.

4.3 The residual service times process in continuous time

For $n = 0, 1, 2, \cdots$, and $y = (\cdots, y_{-2}, y_{-1}, y_0, 0, 0, \cdots) \in IF$, let $X(n, y, t, \sigma^{\cdot}, \tau^{\cdot}) \in IF$ denote the residual service times vector at time t if the initial workload was y, when taking into account the customers that were initially present in the system and the first n arrivals, with the usual convention that coordinate j is concerned with the residual service time of customer C_j.

Lemma 4 *The function* $X^{-}(n, y, t, \sigma^{\cdot}, \tau^{\cdot}) : IN \times IF \times IR^{+\infty} \times IR^{+\infty} \to IF$ *is increasing in* n, y *and* $\sigma^{\cdot}$ *and decreasing in* t *and* $\tau^{\cdot}$.

Proof

1) Increasingness in y.
We prove by induction on n that the whole function $X(n, y, t, \sigma^{\cdot}, \tau^{\cdot})$ is increasing in y. For $n = 0$, $X(0, y, t, \sigma^{\cdot}, \tau^{\cdot}) = \mathcal{Y}(y, t)$ and does not depend upon the variables $\sigma^{\cdot}, \tau^{\cdot}$. The property is hence a mere rephrasing of Lemma 3.

Assume now the property holds true for all $0 \leq n < m$. Take initial conditions y and y' in IF with $y' \geq y$. For all $0 \leq t \leq t_1$, we have

$$X(m, y, t, \sigma^{\cdot}, \tau^{\cdot}) = \mathcal{Y}(y, t) \leq \mathcal{Y}(y', t) = X(m, y', t, \sigma^{\cdot}, \tau^{\cdot})$$

where we used Lemma 3 again, so that the increasingness is immediately obtained. For $t \geq t_1$, we get

$$\begin{aligned} X(m, y, t, \sigma^{\cdot}, \tau^{\cdot}) &= X(m-1, (\mathcal{Y}(y, t_1) + \sigma_1 e_1) \circ \theta, t - t_1, \sigma^{\cdot} \circ \theta, \tau^{\cdot} \circ \theta) \circ \theta^{-1} \\ &\leq X(m-1, (\mathcal{Y}(y', t_1) + \sigma_1 e_1) \circ \theta, t - t_1, \sigma^{\cdot} \circ \theta, \tau^{\cdot} \circ \theta) \circ \theta^{-1} \\ &= X(m, y', t, \sigma^{\cdot}, \tau^{\cdot}) \end{aligned}$$

where we used the induction hypothesis and Lemma 3 to get the inequality. It then follows that the increasingness property holds for all $t \in I\!R^+$.

2) Increasingness in n.
We prove that the whole function $X(n, y, t, \sigma^*, \tau^*)$ is increasing in n. For all $0 \leq t \leq t_m$, we have

$$X^-(m, y, t, \sigma^*, \tau^*) = X^-(m-1, y, t, \sigma^*, \tau^*)$$

For all $t \geq t_m$, we have

$$\begin{aligned} X(m, y, t, \sigma^*, \tau^*) &= \mathcal{Y}(X(m-1, y, t_m, \sigma^*, \tau^*) + \sigma_m e_m) \circ \theta^m, t - t_m) \circ \theta^{-m} \\ &\geq \mathcal{Y}(X(m-1, y, t_m, \sigma^*, \tau^*) \circ \theta^m, t - t_m) \circ \theta^{-m} \\ &= X(m-1, y, t, \sigma^*, \tau^*) \end{aligned}$$

where we used Lemma 1 to get the inequality. Hence, the property

$$X(m, y, t, \sigma^*, \tau^*) \geq X(m-1, y, t, \sigma^*, \tau^*)$$

holds for all $t \in I\!R^+$.
Observe that the increasingness is strict for $t > t_n$ whenever there exists an index $i = 0, -1, -2, \cdots$ such that

$$X_i(m, y, t, \sigma^*, \tau^*) > 0$$

3) Decreasingness in t.
This property is obvious from the very definition of the policy: the residual service time of a customer cannot increase with time.

4) Increasingness in σ^*.
The proof is by induction on n. We prove that the whole function $X(n, y, t, \sigma^*, \tau^*)$ is increasing in σ^*. For $n = 0$, the property is trivially true. Assume now the property holds true for all $0 \leq n < m$. Take σ^* and σ'^* with $\sigma^* \leq \sigma'^*$ For all $0 \leq t \leq t_1$, we have

$$X(m, y, t, \sigma^*, \tau^*) = \mathcal{Y}(y, t) = X(m, y, t, \sigma'^*, \tau^*)$$

For all $t \geq t_1$, we get

$$\begin{aligned} X(m, y, t, \sigma^*, \tau^*) &= X(m-1, (\mathcal{Y}(y, t_1) + \sigma_1 e_1) \circ \theta, t - t_1, \sigma^* \circ \theta, \tau^* \circ \theta) \circ \theta^{-1} \\ &\leq X(m-1, (\mathcal{Y}(y, t_1) + \sigma'_1 e_1) \circ \theta, t - t_1, \sigma^* \circ \theta, \tau^* \circ \theta) \circ \theta^{-1} \\ &\leq X(m-1, (\mathcal{Y}(y, t_1) + \sigma'_1 e_1) \circ \theta, t - t_1, \sigma'^* \circ \theta, \tau^* \circ \theta) \circ \theta^{-1} \\ &= X(m, y, t, \sigma'^*, \tau^*) \end{aligned}$$

where we used step 1 to get the first inequality and the induction hypothesis to get the second one. The increasingness property holds then for all $t \in I\!R^+$.

5) Increasingness in τ^*.
The proof is by induction on n. For $n = 0$, the property is trivially true. Assume now the property holds true for all $0 \leq n < m$. Take τ^* and τ'^* in $(I\!R^+)^\infty$ where τ^* and τ'^* have the same components but for the k-th one which is such that $\tau'_k = \tau_k - u$, $0 \leq u \leq \tau_k$. Consider two identical queues that differ only in their interarrival vector τ and τ'. We shall refer to the system with the vector τ as the first system while the second system will be the one with τ'.

Consider first the case $k = 1$.

We want to prove that for τ^* and τ'^* as above, where $k = 1$,

$$X^-(m, y, t, \sigma^*, \tau^*) \leq X^-(m, y, t, \sigma^*, \tau'^*)$$

We will prove this by assuming that it is not true and arriving at a contradiction. This can only occur if there is a time t when $X_i(m, y, t, \sigma^*, \tau^*) = X_i(m, y, t, \sigma^*, \tau'^*) > 0$, and $N(X(m, y, t, \sigma^*, \tau^*)) > N(X(m, y, t, \sigma^*, \tau'^*))$. Let $v > 0$ be the first date at which this occurs. If such a date exists, we show that

$$X(m, y, v, \sigma^*, \tau^*) = X(m, y, v, \sigma^*, \tau'^*)$$

which must necessarily contradict the relation between $N(X(m, y, t, \sigma^*, \tau^*))$ and $N(X(m, y, t, \sigma^*, \tau'^*))$. In order to prove this, observe that all the customers that were initially present in the queue receive the same amount of service by time v, both in the first and the second system. Hence, the existence of an index $i = 0, -1, \cdots$ such that the relation

$$X_i(m, y, v, \sigma^*, \tau^*) = X_i(m, y, v, \sigma^*, \tau'^*) \tag{4.1}$$

is satisfied implies that it holds for all $i = 0, -1, \cdots$.

Secondly, we prove that if N and N' respectively denote the number of new customers arrived in the system by v then

$$N = N' \tag{4.2}$$

The only other possibility is $N < N'$. If this holds, we have then

$$\begin{aligned} X^-(m, y, v, \sigma^*, \tau'^*) &= X^-(N', y, v, \sigma^*, \tau'^*) \\ &> X^-(N, y, v, \sigma^*, \tau'^*) \\ &\geq X^-(N, y, v, \sigma^*, \tau^*) \\ &= X^-(N, y, v, \sigma^*, \tau^*) \end{aligned}$$

where we successively used the strict increasingness property of step 2 together with the fact that $X_i(m, y, t, \sigma^*, \tau^*) > 0$ for some $i = 0, -1, \cdots$ for getting the strict inequality, and the induction hypothesis to prove the second one. This now entails that

$$X_i(m, y, v, \sigma^*, \tau^*) < X_i(m, y, v, \sigma^*, \tau'^*)$$

for some $i = 0, -1, \cdots$, which contradicts the fact that relation (4.1) holds for all $i = 0, -1, \cdots$. The property that $N \geq 1$ is immediate. Hence relation (4.2) is established.

The final step consists in proving that the relation

$$X_i(m, y, v, \sigma^*, \tau^*) = X_i(m, y, v, \sigma^*, \tau'^*) \tag{4.3}$$

is also satisfied for all $i = 1, 2, \cdots$. First observe that because $N = N'$, at time v, $X(m, y, 0, \sigma^*, \tau^*) = X(m, y, 0, \sigma^*, \tau'^*)$, and $d\sum_{i=-\infty}^{\infty} X_i(m, y, t, \sigma^*, \tau^*)/dt = d\sum_{i=infty}^{\infty} X_i(m, y, t, \sigma^*, \tau'^*)/dt = 1$, the relation

$$\sum_{i=-\infty}^{\infty} X_i(m, y, v, \sigma^*, \tau^*) = \sum_{i=-\infty}^{\infty} X_i(m, y, v, \sigma^*, \tau'^*)$$

necessarily holds. Hence, if there exists an index $i = 1, 2, \cdots$ such that

$$X_i(m, y, v, \sigma^*, \tau^*) \neq X_i(m, y, v, \sigma^*, \tau'^*)$$

then, we can choose an index $j = 1, 2, \cdots$ such that

$$X_j(m, y, v, \sigma^*, \tau^*) > X_j(m, y, v, \sigma^*, \tau'^*)$$

There must then be a date w, $\tau_1 < w < v$ such that

$$X_j(m, y, w, \sigma^*, \tau^*) = X_j(m, y, w, \sigma^*, \tau'^*)$$

During the time interval (w, v) customer C_j received the amount of service time

$$S = X_j(m, y, w, \sigma^*, \tau^*) - X_j(m, y, v, \sigma^*, \tau^*)$$

in the first system and

$$S' = X_j(m, y, w, \sigma^*, \tau'^*) - X_j(m, y, v, \sigma^*, \tau'^*)$$

in the second where $S > S'$. Observe that all the customers C_i, $i = 0, -1, \cdots$ that are still present in the queue at time v receive the same amount of service in the same interval, namely S in the first system and S' in the second. Property (1) together with the inequality $S > S'$ then entail

$$X_i(m, y, w, \sigma^*, \tau^*) > X_i(m, y, w, \sigma^*, \tau'^*)$$

for all $i = 0, -1, \cdots$ such that $X_i(m, y, v, \sigma^*, \tau^*) > 0$, which contradicts the definition of v.
Under the assumption that such a date exists, the property

$$X(m, y, v, \sigma^*, \tau^*) = X(m, y, v, \sigma^*, \tau'^*) \tag{4.4}$$

is hence established which implies that $dX_i(m, y, t, \sigma^*, \tau^*)/dt|_{t=v} = dX_i(m, y, t, \sigma^*, \tau'^*)/dt|_{t=v}$ which contradicts the definition of v.

Consider now the case where $k > 1$. Assume that the property holds for $0 \le j \le k$.
For all $0 \le t < t_{k-1}$, we have

$$X(m, y, t, \sigma^*, \tau^*) = X(m, y, t, \sigma^*, \tau'^*) \tag{4.5}$$

For all $t \geq t_{k-1}$, we get first that

$$X(1, y, t_{k-1}, \sigma^*, \tau^*) = X(1, y, t_{k-1}, \sigma^*, \tau'^*)$$

We have then

$$\begin{aligned}
X^-(m, y, t, \sigma^*, \tau^*) &= \Pi^{-\infty,0}(X(m-1, X(1, y, t_1, \sigma^*, \tau^*) \circ \theta, t - t_1, \sigma^* \circ \theta, \tau^* \circ \theta) \circ \theta^{-1}) \\
&= \Pi^{-\infty,0}(X(m-1, X(1, y, t_1, \sigma^*, \tau'^*) \circ \theta, t - t_1, \sigma^* \circ \theta, \tau^* \circ \theta) \circ \theta^{-1}) \\
&\le \Pi^{-\infty,0}(X(m-1, X(1, y, t_1, \sigma^*, \tau'^*) \circ \theta, t - t_1, \sigma^* \circ \theta, \tau'^* \circ \theta) \circ \theta^{-1}) \\
&= X(m, y, t, \sigma^*, \tau'^*)
\end{aligned}$$

where we successively used relation (4.5) and then the induction hypothesis.

The proof of the Lemma is thus completed.

Let $X^-(y, t, \sigma^*, \tau^*))$ denote the limit

$$X^-(y, t, \sigma^*, \tau^*)) = \lim_{n \to \infty} X^-(n, y, t, \sigma^*, \tau^*))$$

This limit exists since this is an increasing function of n.

Corollary 1 *If the function $X^-(y, t, \sigma^*, \tau^*) : IF \times IR \times IR^{+\infty} \times IR^{+\infty} \to IF$ is finite, it is increasing in y and σ^* and decreasing in t and τ^*.*

Proof

$X^-(y, t, \sigma^*, \tau^*))$ is the limit of a sequence of functions that are increasing in y and σ^* and decreasing in t and τ^*.

5 Construction of the limiting residual service times vector

In this section, we show how the stationary residual service times vector, y in the preceding sections, can be built from the sequence of interarrival and service times σ^- and τ^-.
In order to do so, we consider the initial service times vectors that are obtained when taking into account the first n customers that arrived in the past only, namely customers C_0, C_{-1}, $C_{-2}, \cdots, C_{-n}$. Denote this vector as $y^n(\sigma^-, \tau^-)$.

Lemma 5 *For all $n = 1, 2, \cdots$,*

$$y^{n+1}(\sigma^-, \tau^-) = (\mathcal{Y}(y^n(\sigma^- \circ \theta^{-1}, \tau^- \circ \theta^{-1}), \tau_0) + \sigma_0 e_1) \circ \theta$$

where $y^0 = \sigma_0 e_0$. Furthermore, the sequence of vectors $y^n(\sigma^-, \tau^-)$ is increasing in n, σ^- and $-\tau$.

Proof

1)Proof of the relation.

The proof of the relation is by induction on n. Assume it holds for $0 \leq n \leq m$. Then, the residual service time vector seen by customer C_{-1}, when taking into account m customers in its past is

$$y^m(\sigma^- \circ \theta^{-1}, \tau^- \circ \theta^{-1})$$

owing to the induction assumption, so that the residual service time vector at the first arrival epoch that follows this date is $\mathcal{Y}(y^n(\sigma^- \circ \theta^{-1}, \tau^- \circ \theta^{-1}), \tau_0) + \sigma_0 e_1$. Shifting this one unit to the left yields the residual service time vector seen by customer C_0, when taking into account $m+1$ customers in its past.

2)Increasingness in n.

The proof of the relation is by induction on n. The property that $y^1 \geq y^0$ is immediate since $y_0^1 = y_0^0 = \sigma_0$ and $y_{-1}^1 \geq y_{-1}^0 = 0$ with all other coordinates being equal to 0. Assume it holds for $0 \leq n \leq m$. Then,

$$\begin{aligned} y^{n+1}(\sigma^-, \tau^-) &= (\mathcal{Y}(y^n(\sigma^- \circ \theta^{-1}, \tau^- \circ \theta^{-1}), \tau_0) + \sigma_0 e_1) \circ \theta \\ &\geq (\mathcal{Y}(y^{n-1}(\sigma^- \circ \theta^{-1}, \tau^- \circ \theta^{-1}), \tau_0) + \sigma_0 e_1) \circ \theta \\ &= y^n(\sigma^-, \tau^-) \end{aligned}$$

where we made use of the induction assumption together with Lemma 1.

3)Increasingness in σ^-.

The proof of the relation is by induction on n. The property holds for $n = 0$ since $y^0 = \sigma_0 e_0$. Assume it holds for $0 \leq n \leq m$. Then, taking $\sigma'^- \geq \sigma^-$, we get

$$\begin{aligned} y^{n+1}(\sigma^-, \tau^-) &= (\mathcal{Y}(y^n(\sigma^- \circ \theta^{-1}, \tau^- \circ \theta^{-1}), \tau_0) + \sigma_0 e_1) \circ \theta \\ &\leq (\mathcal{Y}((y^{n-1}(\sigma'^- \circ \theta^{-1}, \tau^- \circ \theta^{-1}), \tau_0) + \sigma'_0 e_1) \circ \theta \\ &= y^n(\sigma'^-, \tau^-) \end{aligned}$$

where we made use of the induction assumption and Lemma 1.

4)Decreasingness in τ^-.

The proof of the relation is by induction on n. The property is obvious for $n = 0$. Assume it holds for $0 \leq n \leq m$. Then, taking $\tau'^- \geq \tau^-$, we get

$$\begin{aligned} y^{n+1}(\sigma^-, \tau^-) &= (\mathcal{Y}(y^n(\sigma^- \circ \theta^{-1}, \tau^- \circ \theta^{-1}), \tau_0) + \sigma_0 e_1) \circ \theta \\ &\geq (\mathcal{Y}(y^{n-1}(\sigma^- \circ \theta^{-1}, \tau'^- \circ \theta^{-1}), \tau_0) + \sigma_0 e_1) \circ \theta \\ &\geq (\mathcal{Y}(y^{n-1}(\sigma^- \circ \theta^{-1}, \tau'^- \circ \theta^{-1}), \tau'_0) + \sigma_0 e_1) \circ \theta \\ &= y^n(\sigma^-, \tau'^-) \end{aligned}$$

where we made use of the induction assumption and Lemma 1 in order to get the first inequality and Lemma 1 to get the second one.

This completes the proof of the lemma.

Denote by $y(\sigma^-, \tau^-)$ the limit

$$y(\sigma^-, \tau^-) = \lim_{n \to \infty} y^n(\sigma^-, \tau^-)$$

This limit exists since the function is increasing in n.

Corollary 2 *If it is finite, the function $y(\sigma^-, \tau^-)$ is increasing in σ^- and $-\tau^-$.*

Proof
$y(\sigma^-, \tau^-))$ is the limit of a sequence of functions that are increasing in σ^- and $-\tau^-$.

Define $X^-(t, \sigma, \tau)$ to be the residual service time vector of customers $\cdots, C_{-2}, C_{-1}, C_0$ at time $t \geq 0$ when taking into account all customers that arrive both in the past and the future. It is obvious from the preceding construction that

$$X^-(t, \sigma, \tau) = X^-(y(\sigma^-, \tau^-), t, \sigma^+, \tau^+)$$

so that the following theorem is an immediate consequences of Corollaries 1 and 2

Theorem 3 *If it is finite, the function $X^-(t, \sigma, \tau)$ is increasing in σ and decreasing in t and τ.*

6 The Association Property

We make the following assumptions regarding the interarrival and the service times,

H0: $-\tau \cup \sigma$ is a set of associated random variables.

H1: $-\tau \cup \sigma$ is a stationary and ergodic sequence of integrable random variables satisfying the stability condition $E[\sigma_0] < E[\tau_0]$.

Owing to condition H1, the queue is stable in the sense that all the quantities that were defined in Corollary 2 and Theorem 3 are a.s. finite. Indeed, this discipline is work conserving and has hence the same busy periods as the FCFS single server queue. The classical construction that is commonly used for this last type of queues ([Lo62]) immediately entails such finiteness properties.

We define the sojourn time, T_0, of customer C_0, to be

$$T_0 = \sup\{t \;:\; X_0^-(t, \sigma, \tau) > 0\}$$

and more generally, the sojourn time, T_n, of customer C_n, to be

$$T_n = \sup\{t \ : \ X_0^-(t, \sigma \circ \theta^n, \tau \circ \theta^n) > 0\}$$

Theorem 4 *The sojourn times, $\{T_n\}_{-\infty}^{0}$ together with the service and negative interarrival times $\{\sigma_n\}_{n=0}^{\infty}$ and $\{-\tau_n\}_{n=0}^{\infty}$, form a set of associated r.v.'s.*

Proof

From the above theorem, $X_0^-(t, \sigma, \tau)$ and hence T_0 are non-decreasing functions of the variables σ and $-\tau$. Since $\sigma \circ \theta^n$ and $-\tau \circ \theta^n$ are also are non-decreasing functions of the variables σ and $-\tau$, it follows that more generally, T_n is a non-decreasing function of these variables as well. The proof of the Theorem follows immediately from the properties of associated r.v.'s listed in Theorem 2,

7 Application to resequencing queues

Consider the following problem: a stream of customers C_0, $C_1, \cdots$ arrive to a first queueing system at time $t_0 < t_1, \cdots$ respectively, where each customer experiences some delay. Denote by D_n the delay of C_n. Consider the case where this system is such that the order in which the customers leave the system is not necessarily the same as the order in which they entered it, like for instance in a processor sharing queue. This output process then feeds some end *resequencing queue*, where the customers have to be served in the very order of the initial stream. Such resequencing queues are for instance commonly implemented in packet switching communication networks using Datagram type procedures: Voice packets originating from a source are routed independently through a network with several possible routes between the source and the destination nodes, and may hence arrive at their destination in an order that does not correspond to the emission order anymore. It is then necessary to resequence the packets before processing them in the final output device. It was established in [Bac87] that if the delays $\{D_n\}_{n=0}^{\infty}$, $\{t_{n+1} - t_n\}_{n=0}^{\infty}$ are associated random variables, independent of the service times in the end *resequencing queue*, then the end-to-end delays experienced by customers in the whole system can be bounded above by those in a similar system where the first queueing system is replaced by an infinite server queue with i.i.d. service times. The interest of such a bound comes from the fact that this second system is analytically computable. The interested readers should refer to [BM89] for more practical details on the matter incuding the computation of various moments and asymptotics and to [BMT87] for further applications to other and more elaborate queueing systems with synchronization constraints.

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