TRACTABLE SUBGROUPS OF THE EUCLIDEAN GROUP

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Abstract

In planning for automatic assembly it is useful to associate with features their symmetry groups which are particular subgroups of the Euclidean group. To the combination of features corresponds the intersection of their associated symmetry groups. In this report we define a family of subgroups of the Euclidean group, called tractable groups, which are of practical importance. The main theorem of this report states that this family is closed under intersection and has as a consequence an efficient implementation for computing this intersection.

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1 Introduction

The main theoretical insight that we wish to communicate in this report is a mathematical treatment of spatial reasoning in terms of group theory. While group theory is a standard tool in physics, it has received relatively little attention in engineering disciplines — see [2] [6]. The essential idea is to make the major descriptor of any feature of a body be the symmetry group of that feature, that is to say the set of rigid transformations of 3-space that map that feature into itself. For example, the set of all the rotations about a fixed axis corresponds to the symmetry group of a cone. On the other hand the symmetry group of a regular pyramid (except for the tetrahedron) is the cyclic group whose order is the number of edges of its base.

The use of group theory for spatial reasoning demands that adequate computational representations of the relevant groups be developed. Such representations must support the important operation of subgroup intersection. This arises from the fact that when we consider combinations of features of a single body, they have a symmetry which is the intersection of their individual symmetry groups — complex features have *less* symmetry than simple ones. In [5] a representation for certain important infinite subgroups of the Euclidean group of all rigid transformations was discussed. In this report we introduce the concept of *tractable groups*, which include all of the groups considered heretofore in [5], and show that the intersection of tractable groups is tractable. More precisely the main theorem gives an explicit formula for the intersection which will be used in an efficient implementation.

2 Preliminaries.

Let $\mathbf{R}^3$ denote the euclidean space of dimension three with the standard scalar product. A bijection $f : \mathbf{R}^3 \rightarrow \mathbf{R}^3$ is an isometry if and only if $\|f(x) - f(y)\| = \|x - y\|$ holds for all $x, y \in \mathbf{R}^3$. The set of all the isometries of $\mathbf{R}^3$ is denoted by $E(3)$ and has a group structure with respect to the composition of mappings.

With any vector $t \in \mathbf{R}^3$ it is possible to associate an isometry $\tau(t)$ defined by $\tau(t)(x) =$

$x + t$ for any $x \in \mathbf{R}^3$. Clearly the mapping $\tau : \mathbf{R}^3 \longrightarrow E(3)$ is a monomorphism. Let us denote the image $\tau(\mathbf{R}^3) \subset E(3)$ by $T(3)$. Henceforth, we will identify $T(3)$ with $\mathbf{R}^3$ via the isomorphism $\tau : \mathbf{R}^3 \longrightarrow T(3)$. Consequently a vector $t \in \mathbf{R}^3$ can be considered as a member of $T(3) \subset E(3)$, analogously any subgroup of $\mathbf{R}^3$, and in particular subvector space, may be viewed as a subgroup of $E(3)$.

Definition : A subgroup of $E(3)$ is a *translational* group if and only if it is a subvector space of $\mathbf{R}^3$.

On the other hand the set of all the linear mappings $r : \mathbf{R}^3 \longrightarrow \mathbf{R}^3$ such that $\|r(x)\| = \|x\|$ for all $x \in \mathbf{R}^3$, denoted by $O(3)$ and called the orthogonal group, is also a subgroup of $E(3)$. By definition the special orthogonal group $SO(3)$ is the subset of $O(3)$ consisting of the linear mappings whose determinant is $+1$. Recall that if $H \subset G$ is a subgroup of the group G, the conjugate of H by $g \in G$, denoted by H^g, is the subgroup gHg^{-1}.

Definition : A subgroup of $E(3)$ is a *rotational* group if and only if it is a conjugate of a subgroup of $SO(3)$.

The two subgroups $T(3)$ and $O(3)$ of $E(3)$ are particularly important in the sense that they can be used to describe $E(3)$. In effect it is easy to show that any isometry f can be written uniquely as the product tr where $t \in T(3)$ and $r \in O(3)$. Therefore the euclidean group $E(3)$ is the product $T(3)O(3)$, where if S, S' are any two subsets of a group G, the product SS' is the set $\{ss' | s \in S, s' \in S'\}$. The end of this section is devoted to introducing some definitions and basic concepts which will be used in the following sections.

A subset $\mathbf{A}$ of $\mathbf{R}^3$ is an affine space if and only if there exists a $x_0 \in \mathbf{R}^3$ and $V \subset \mathbf{R}^3$ a subvector space such that $\mathbf{A} = x_0 + V$. The dimension of $\mathbf{A}$ is defined to be the dimension of V. Affine spaces of dimension 0,1 and 2 are called points, lines and planes respectively. Moreover any subgroup $G \subset E(3)$ acts on $\mathbf{R}^3$ by setting $g.x = g(x)$, $g \in G, x \in \mathbf{R}^3$. Here, the reader who wants to familiarize himself with the actions of groups on sets is refered to [7], [3]. Furthermore, for any subset $X \subset \mathbf{R}^3$, the G-orbit of X is defined as the set $G.X = \{g.x | g \in G, x \in X\}$. For example if T is a translational group and $x_0 \in \mathbf{R}^3$ then $T.\{x_0\} = \{x_0 + t | t \in T\}$ is the affine space $x_0 + T$. Finally with $G \subset E(3)$ we associate the set $\mathbf{A}^G = \{x \in \mathbf{R}^3 | g.x = x, \forall g \in G\}$. We will say that G is a point group or a line group according as $\mathbf{A}^G$ is a singleton or an affine line. If G is a translational group then $\mathbf{A}^G$ is the empty set. In the same manner $\mathbf{A}^f = \{x \in \mathbf{R}^3 | f.x = x\}$ is the set of the fixed points of the isometry f. For instance if f is a non-trivial rotation $\mathbf{A}^f$ is the affine line materializing the axis of rotation.

3 Tractable groups

Definition : A subgroup $G \subset E(3)$ is *tractable* if and only if $G = TR$, where T is a translational group and R is a rotational group.

A simple example of a tractable group is obtained by taking the product of a vectorial plane and the group of all the rotations about a fixed axis whose direction is perpendicular to the plane. Obviously this group is a symmetry group of an affine plane. The reader may convince himself that the group of screw displacements [3] is not tractable.

Lemma 3.1 : Let T and R be translational and rotational groups respectively. Then $G = TR$ is tractable if and only if $T^r = T$ for any $r \in R$.

Proof: The sufficiency, which consists of proving that TR is a group, is straight forward and left to the reader. Conversely any rotation $r \in E(3)$ may be written as the product usu^{-1} where u is a translation and s a linear rotation. Hence, given any $t \in T$ and $r \in R$, the computation $rtr^{-1} = usu^{-1}tus^{-1}u^{-1} = usts^{-1}u^{-1} = us(t)u^{-1} = s(t)$ shows that the isometry $rtr^{-1} \in G$ is the translation $s(t)$. But there exists $k \in T$ and $q \in R$ such that $s(t) = kq$, thus $rtr^{-1} = s(t) = k \in T$.

Corollary 3.2 : If TR is a tractable group then $TR^u = TR$ for any $u \in T$.

Proof: Since TR^u is a tractable group it is sufficient to prove that $TR^u \subset TR$, and hence to prove $R^u \subset TR$. But this inclusion is a consequence of lemma 3.1.

To prove the main proposition of this section we need the following lemma.

Lemma 3.3 : If t is a translation and r is a rotation such that $\langle t\rangle \perp \mathbf{A}^r$ and $r^2 = 1$ then $t^2 r = trt^{-1}$.

Proof: If r is linear then, upto an appropriate change of basis, the proof is straight forward. Otherwise it is possible to choose $u \in \mathbf{R}^3$ such that $r = usu^{-1}$ where s is linear and $\langle u\rangle \perp \mathbf{A}^r$. Hence $t^2 r = t^2 usu^{-1} = t^2u^2 s = (tu)^2 s = tusu^{-1}t^{-1} = trt^{-1}$.

Proposition 3.4 : Let $G = TR$ be tractable and $y \in G$ a rotation. There exists $u \in T$ such that $y \in R^u$.

Proof: We know that the rotation y uniquely determines $t \in T$ and $r \in R$ such that $y = tr$ and we have $\mathbf{A}^y \parallel \mathbf{A}^r$. If dim $T = 3$ there exists $u \in T$ such that $u.\mathbf{A}^r = \mathbf{A}^y$ and hence $y \in R^u$. We now may assume that dim T is 1 or 2 since the case dim $T = 0$ is trivial. By lemma 3.1 we know that $yTy^{-1} = T$ and hence $\mathbf{A}^y \parallel T$ or $\mathbf{A}^y \perp T$.
dim $T = 1$. If $\mathbf{A}^y \parallel T$ then $y = tr$ has a fixed point and so $t = 1$. Otherwise $\mathbf{A}^y \perp T$ implies that $y^2 = r^2 = 1$. Now since T is a vector space we can define $u = t^{1/2}$ and we have $y = uru^{-1}$ by lemma 3.3.
dim $T = 2$. Let $u \in \mathbf{R}^3$ such that $\langle u\rangle \perp \mathbf{A}^y$ and $u.\mathbf{A}^r = \mathbf{A}^y$, where $\langle u\rangle$ denotes the subvector space of $\mathbf{R}^3$ generated by u. If $\mathbf{A}^y \perp T$ then $u \in T$ is the vector we are looking for. If $\mathbf{A}^y \parallel T$ then $y^2 = r^2 = 1$. Defining u as above we have $y = uru^{-1} = u^2 r$ by lemma 3.3 and hence $u^2 = t \in T$. But T is a vector space, hence $u \in T$ is such that $y \in R^u$.

4 Intersection of tractable groups

In this section we show that the intersection of two tractable groups is tractable. We first give the definition of the distance between two tractable groups and then go on to prove a proposition which will be used in the proof of the main theorem.

Definition: The distance $d(G,H)$ between two tractable groups $G = TR$ and $H = KQ$ is the distance between the affine spaces $T.\mathbf{A}^R$ and $K.\mathbf{A}^Q$. A pair $(u,v) \in T \times K$ *realizes* $d(G,H)$ if and only if $d(u.\mathbf{A}^R, v.\mathbf{A}^Q) = d(G,H)$.

It is easy to see that the distance $d(G,H)$ is well defined even if we dont have the uniqueness of representation of tractable groups. Infact the affine space $T.\mathbf{A}^R$ does not depend on the representation of G.

Proposition 4.1 : Let $G = TR$ and $H = KQ$ be tractable groups such that $d(G,H) \neq 0$. If $y \in G \cap H$ is a non-trivial rotation then $\mathbf{A}^y \perp T$ and $\mathbf{A}^y \perp K$. Furthermore R and Q are point groups and $d(\mathbf{A}^y \cap T.\mathbf{A}^R, \mathbf{A}^y \cap K.\mathbf{A}^Q) = d(G,H)$.

Proof: If the vector spaces T and K are both of dimension zero the existence of a non-trivial rotation in $G \cap H$ and the hypothesis $d(G,H) \neq 0$ imply that R and Q are point groups. Besides we clearly have $d(\mathbf{A}^R, \mathbf{A}^Q) = d(G,H)$. From now we may assume that atleast one of the translational groups is of dimension ≥ 1, let us say T. Moreover by proposition 3.4 there exists $(u,v) \in T \times K$ such that $y \in R^u \cap Q^v$. To prove the perpendicularity of the axis $\mathbf{A}^y$ with T and K we proceed by contradiction by supposing that $\mathbf{A}^y \parallel T$. We then have $T.\mathbf{A}^y = T.\mathbf{A}^R$ irrespective of whether R is a point or line group, since if R is line group then $\mathbf{A}^y = \mathbf{A}^R$, else there exists $t \in T$ such that $\mathbf{A}^y = \langle t \rangle.\mathbf{A}^R$. But $v.\mathbf{A}^Q \subset \mathbf{A}^y$, therefore $v.\mathbf{A}^Q \subset T.\mathbf{A}^R \cap K.\mathbf{A}^Q$, contradiction. Finally if R is a line group we then have $\mathbf{A}^y = \mathbf{A}^{R^u}$, implying that $v.\mathbf{A}^Q \subset \mathbf{A}^y \cap K.\mathbf{A}^Q \subset \mathbf{A}^{R^u} \cap K.\mathbf{A}^Q \subset T.\mathbf{A}^R \cap K.\mathbf{A}^Q$, contradiction.

Theorem 4.2 : If $G = TR$ and $H = KQ$ are tractable groups then

$$G \cap H = (T \cap K)(R^u \cap Q^v),$$

where $(u,v) \in T \times K$ realizes $d(G,H)$.

As an immediate consequence of this theorem we see by using the lemma 3.1 that the family of tractable groups is closed under intersection.

Proof: Clearly $(T \cap K)(R^u \cap Q^v) \subset G \cap H$, we prove the other inclusion by considering the cases $d(G,H) = 0$ and $d(G,H) \neq 0$ separately.

$d(G,H) = 0$. Let $y \in G \cap H$ and (u,v) realizes $d(G,H)$. By corollary 3.2 we may write y as the product tr where $t \in T$ and $r \in R^u$. Similarly $y = kq$ where $k \in K$ and $q \in Q^v$. Now $u.\mathbf{A}^R \cap v.\mathbf{A}^Q = \mathbf{A}^{R^u} \cap \mathbf{A}^{Q^v} \neq \phi$, hence the rotations r and q leave a point fixed and so does the translation $t^{-1}k$. This implies that $t = k$, $r = q$ and $y \in (T \cap K)(R^u \cap Q^v)$.

$d(G,H) \neq 0$. Any isometry $y \in G \cap H$ uniquely determines $(t,k) \in T \times K$ and $(r,q) \in R \times Q$ such that $y = tr = kq$. We first remark that $r = 1$ if and only if $q = 1$, and that the case $r = q = 1$ is straight forward. Henceforth the rotations r and q will be assumed non-trivial. We now claim that in the remaining cases y is a rotation and that proposition 4.1 completes the proof. In effect if y is a rotation then by proposition 3.4 we have $y \in R^u \cap Q^v$ where (u,v) realizes $d(G,H)$ by proposition 4.1.
Clearly if $t = 1$ or $k = 1$ then y is a rotation and hence we may assume that $y = tr = kq$ where $t \neq 1 \neq k$ and $r \neq 1 \neq q$. Furthermore if $\langle t \rangle \perp \mathbf{A}^r$ or $\langle k \rangle \perp \mathbf{A}^q$ we know that y is a rotation [3]. Finally the case $\langle t \rangle \parallel \mathbf{A}^r$ and $\langle k \rangle \parallel \mathbf{A}^q$ is absurd since y is then a screw displacement implying $t = k$ and $r = q$. In effect let $a \in \mathbf{A}^r$ and $b \in \mathbf{A}^q$ such that $b - a \perp \mathbf{A}^y$. Since y is an isometry we have $\|b-a\|^2 = \|y(b) - y(a)\|^2 = \|kq(b) - tr(a)\|^2 = \|k(b) - t(a)\|^2 = \|b - a + k - t\|^2 = \|b-a\|^2 + \|k-t\|^2$ and hence $k = t$ and $r = q$. This means that we have found a non trivial rotation in $G \cap H$ whose axis is parallel to T and K, contradicting proposition 4.1.

5 Implementation

The theorem 4.2 gives an explicit expression for the intersection of two tractable groups in terms of the intersection of their translational subgroups and closest conjugates of their rotational subgroups. It is sufficient to implement such intersections, as well as to determine a shortest segment between two tractable groups. In this section we will give a description of an implementation, defining the main data structures and explaining the parameters and results of the major procedures. The code, written in POP11, splits naturally into three files named quaternions, geometry and groups respectively, which are independent but complementary.

5.1 Quaternions

In the quaternions file are defined the basic operations of addition, multiplication and conjugation of quaternions. A quaternion, which is a scalar and a vector, is used to compute effective rotations. It is implemented by the following:

```
recordclass quat
    scal_quat                 ;;; scalar component
    vec3_quat ;               ;;; vectorial component
```

It is well known [1] that any rotation may be written by an appropriate quaternionic conjugation. For example the vector vec3(1,0,0) rotated by an angle $\pi/2$ about the axis vec3(0,0,1) is the vectorial component of conj_quat(V,Q), where

```
quat(0,vec3(1,0,0))->V;
quat(cos(pi/4),vec3(0,0,sin(pi/4)))->Q;

vec3_quat(conj_quat(V,Q)) =>
** <vec3 0.0 1.0 0.0>
```

5.2 Geometry

The geometry file takes care of elementary problems in the space of dimension three. By elementary problems we mean membership of a vector (resp a point) in a vector space (resp an affine space), extraction of a basis from a set of vectors, intersections of vector spaces (resp affine spaces) and location of a shortest segment between two affine spaces. A vector space does not need a recordclass to be defined, it is represented by a list of vectors which may be seen as a set of generators. The following procedures are straightforward and obviously useful.

```
member_vect_space(vec3(5/3,0,-1/7),[% vec3(1,0,0),vec3(0,0,1) %]) =>
** <true>

dim_vect_space([% vec3(1,0,0),vec3(0,0,1),vec3(5/3,0,-1/7) %]) =>
** 2

inter_vect_space([% vec3(1,0,0),vec3(0,1,0) %],
                 [% vec3(1,1,1),vec3(0,0,1) %])
=>
** [<vec3 1 1 0>]

ortho_vect_space([% vec3(1,1,1) %]) =>
** [<vec3 0 1 -1> <vec3 -2 1 1>]
```

Unlike a vector space an affine space requires a recordclass.

```
recordclass aff_space
   vect_space_aff            ;;; underlying vector space
   pt3_aff;                  ;;; distinguished element
```

For example the horizontal affine plane passing through the point (0,0,-1) is defined by:

```
aff_space([% vec3(1,0,0),vec3(0,1,0) %],pt3(0,0,-1)) -> P;

P =>
** <aff_space [<vec3 1 0 0> <vec3 0 1 0>] <pt3 0 0 -1>>
```

The main procedure of this file, which finds a shortest segment between two affine spaces, requires other procedures. Determining the cartesian representation of an affine plane, computing the intersection of three affine spaces in general position, finding the closure or the intersection of two affine spaces are some of them. Let us recall that the cartesian representation of an affine plane is given by a normal vector together with its inner product with any element of the plane. We then have the procedure to get a cartesian representation.

```
conv_aff_cart_plane(P) =>
** <cart_plane <vec3 0 0 -1> 1>
```

From this representation we obtain the cartesian equation of an affine plane and hence we can use the Cramers rule to find the intersection point of three planes in general position. This computation, which seems to be very particular in the sense that it deals only with affine spaces of dimension two, happens to be helpful in most cases of the computation of the intersection of any two affine spaces. Let L be the affine line defined by

```
aff_space([% vec3(1,1,1) %], pt3(0,0,0)) -> L;
```

Clearly the procedure

```
inter_aff_space(P,L) =>
** <aff_space [] <pt3 -1 -1 -1>>
```

which computes the intersection of two affine spaces will be used to find the location of a shortest segment between two affine spaces. Let L and P be the affine spaces defined above and M the following affine line

```
aff_space([% vec3(1,-1,0) %], pt3(0,0,1)) -> M;
```

We have the main procedure

```
short_seg(L,M) =>
** <seg <pt3 1_/3 1_/3 1_/3> <pt3 0 0 1>>

short_seg(L,P) =>
** <seg <pt3 -1 -1 -1> <pt3 -1 -1 -1>>
```

whose output are the end points of a shortest segment between two affine spaces.

5.3 Groups

The groups file deals with tractable groups. Clearly in order to represent a tractable group we use representations of translational and rotational groups. By definition a translational group is a subvector space and its recordclass is

```
recordclass trans_group
   name_trans_group              ;;; name of the group
   gen_trans_group ;             ;;; list of generators
```

where the first component reflects the dimension and the second is a basis Given a list of vectors the following procedure first extracts a basis and then gives the corresponding translational group.

```
ident_trans_group([% vec3(0,-1/2,0),vec3(0,1,0) %]) =>
** <trans_group T1 [<vec3 0 1 0>]>
```

Furthermore the computation of intersection of translational groups is immediately available using the geometry file.

As in the case of a translational group a rotational group is represented by its name and a list of rotations which may be viewed as a set of generators.

```
recordclass rot_group
   name_rot_group          ;;; name of the group
   gen_rot_group ;         ;;; list of generators
```

The list of possible rotational groups is SO3, SO2, O2, Cn, D2n, tetra, octo and ico, the last five being platonic groups. A finite rotation is represented by its axis and its order where the axis is given by a direction d and a fixed point a. For programming convenience a generator of SO2 will be written rot(d, a,0), despite the fact that SO2 is not finitely generated and there is no rotation of order zero. The identity group will be considered as being generated by the empty list. As before we have a procedure which associates with a list of rotations its generated group:

```
ident_rot_group([% rot(vec3(1,0,0),pt3(0,0,0),0) %] ) =>
** <rot_group SO2 [<rot <vec3 1 0 0> <pt3 0 0 0> 0>]>
```

```
ident_rot_group([% rot(vec3(1,-1,-1),pt3(0,0,0),3),
                   rot(vec3(1,0,-1),pt3(0,0,0),2) %]) =>

** <rot_group octo [<rot <vec3 1.0 0.0 0.0> <pt3 0 0 0> 4> <rot <vec3 0.0
     -1.0 -0.0> <pt3 0.0 0.0 0.0> 4> <rot <vec3 0.0 0.0 -1.0> <pt3 0.0 0.0
     0.0> 4> <rot <vec3 1.0 -1.0 -1.0> <pt3 0.0 0.0 0.0> 3> <rot <vec3
     1.0 1.0 -1.0> <pt3 0.0 0.0 0.0> 3> <rot <vec3 -1.0 -1.0 -1.0> <pt3
     0.0 0.0 0.0> 3> <rot <vec3 -1.0 1.0 -1.0> <pt3 0.0 0.0 0.0> 3> <rot
     <vec3 1.0 0.0 -1.0> <pt3 0.0 0.0 0.0> 2> <rot <vec3 0.0 -1.0 -1.0>
     <pt3 0.0 0.0 0.0> 2> <rot <vec3 -1.0 0.0 -1.0> <pt3 0.0 0.0 0.0> 2>
     <rot <vec3 0.0 1.0 -1.0> <pt3 0.0 0.0 0.0> 2> <rot <vec3 1.0 -1.0
     -0.0> <pt3 0.0 0.0 0.0> 2> <rot <vec3 -1.0 -1.0 -0.0> <pt3 0.0 0.0
     0.0> 2>]>
```

Another important procedure that will be used in computing the intersection of rotational groups consists of calculating, given an affine line and a rotational group, the subgroup of all the rotations of the group having the affine line as axis.

```
aff_space([% vec3(1,1,0) %],pt3(1,1,1)) -> l;
```

```
ident_rot_group([% rot(vec3(0,0,1),pt3(1,1,1),8),
                   rot(vec3(1,0,0),pt3(1,1,1),2) %]) -> R;

l =>
** <aff_space [<vec3 1 1 0>] <pt3 1 1 1>>

R =>
** <rot_group D2n [<rot <vec3 0 0 1> <pt3 1 1 1> 8> <rot <vec3 1 0 0> <pt3
    1 1 1> 2>]>

sub_group_given_axis(l,R) =>
** <rot_group Cn [<rot <vec3 1 1 0> <pt3 1 1 1> 2>]>
```

Using the above procedures along with the dichotomy between point and line groups we can compute the intersection of two rotational groups.

```
ident_rot_group([% rot(vec3(0,0,1),pt3(0,0,0),4),
                   rot(vec3(1,0,0),pt3(0,0,0),2) %]) -> R;
ident_rot_group([% rot(vec3(0,0,1),pt3(1,1,0),0),
                   rot(vec3(1,0,0),pt3(1,1,0),0) %]) -> Q;

R =>
** <rot_group D2n [<rot <vec3 0 0 1> <pt3 0 0 0> 4> <rot <vec3 1 0 0> <pt3
    0 0 0> 2>]>

Q =>
** <rot_group SO3 [<rot <vec3 1 0 0> <pt3 1 1 0> 0> <rot <vec3 0 1 0> <pt3
    1 1 0> 0>]>

inter_rot_group(R,Q) =>
** <rot_group Cn [<rot <vec3 1 1 0> <pt3 0 0 0> 2>]>
```

As mentioned before tractable groups are represented by their translational and rotational components.

```
recordclass tract_group
   trans_subgroup         ;;; translational subgroup
   rot_subgroup ;         ;;; rotational subgroup
```

Now, to compute the intersection of tractable groups, we will be done if we have a procedure which associates with a tractable group the translational orbit of the fixed points of its rotational subgroup. More precisely let us define a tractable group G and compute its associated affine space

```
ident_trans_group([% vec3(0,0,1) %] ) -> T;
ident_rot_group([% rot(vec3(0,0,1),pt3(0,0,0),0),
                   rot(vec3(1,0,0),pt3(0,0,0),2) %]) -> R;
ident_tract_group(T,R) -> G;

G=>
** <tract_group <trans_group T1 [<vec3 0 0 1>]> <rot_group O2 [<rot <vec3
     0 0 1> <pt3 0 0 0> 0> <rot <vec3 1 0 0> <pt3 0 0 0> 2>]>>

trace_tract_group(G) =>
** <aff_space [<vec3 0 0 1>] <pt3 0 0 0>>
```

Finally the intersection of tractable groups is computed easily now that we have all the necessary procedures. As an example we introduce another tractable group H and then compute the intersection of G with H.

```
ident_trans_group([% vec3(1,0,0),vec3(0,0,1) %] ) -> K;
ident_rot_group([% rot(vec3(0,1,0),pt3(0,0,0),0) %]) -> Q;
ident_tract_group(K,Q) -> H;

H =>
** <tract_group <trans_group T2 [<vec3 0 0 1> <vec3 1 0 0>]> <rot_group
     SO2 [<rot <vec3 0 1 0> <pt3 0 0 0> 0>]>>
```

```
inter_tract_group(G,H) =>
** <tract_group <trans_group T1 [<vec3 0 0 1>]> <rot_group Cn [<rot <vec3
     0 1 0> <pt3 0 0 0> 2>]>>
```

6 Comparison with previous work

The treatment of the relationships of bodies having symmetric features *via* group theory was pioneered by Hervé [2], although he did not specify a computational approach to realising his theory. The evolution of the current work began with the development of the RAPT language [4], in which spatial relationships between bodies were expressed in terms of an algebra of locations. The theory of RAPT was subsequently recast in group theoretical terms in [6]. A first approach to the implementation of this was the use of *characteristic invariants* [5], which described the computational treatment of a number of important continuous subgroups of $E(3)$. The present work, while it bears some resemblance in the kind of computations required to compute group intersections, requires much less tabulation of cases and at the same time treats many subgroups of $E(3)$ which are of practical import, the major exceptions being screw groups and crystallographic groups.

A treatment which combines the approaches of Hervé [2] and RAPT is to be found in Thomas and Torras[8]. Their work is more concerned with group products than intersections, and does not treat the finite rotational groups encompassed in the present report, and is thus complementary to ours.

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