Poisson Arrivals See Time Averages: A Generalization

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Abstract

In this note, the result that "Poisson arrivals see time averages" is proved under more general conditions. The limit theorems here require less restrictive assumptions and are shown for a wider class of arrival processes. Applications are presented for a particular cases of discrete-time geometric arrivals and continuous-time Markov-modulated Poisson processes.

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1 Introduction

The purpose of this note is modify Wolff's proof of "Poisson arrivals see time averages", as given in [12], so that it can be applied to a more general class of arrival processes such as the so called 'Markov-modulated Poisson processes'

Let $N(t,\omega)$ denote the cumulative number of arrivals in the time interval $[0,t]$ to some queueing system the state of which is denoted by $Z(t,\omega)$. We shall occasionally suppress the explicit dependence of a stochastic process on ω and write, say, $N(t)$ instead of $N(t,\omega)$. $N(t)$ is a "counting process" and therefore according to the general theory of such processes see, e.g., [1],[6] there exists an increasing process $\Lambda(t,\omega)$,satisfying certain technical conditions, with the property:

$$M(t,\omega) = N(t,\omega) - \Lambda(t,\omega) \text{ is a martingale} \tag{1}$$

Examples:

1. The Poisson Process; here $\Lambda(t) = \lambda t = \int_0^t \lambda \, ds$.
2. The doubly stochastic Poisson process ([1] p.21); here

$$\Lambda(t,\omega) = \int_0^t \lambda(s,\omega)\, ds$$

Remark 1 *$\Lambda(t,\omega) = \langle M \rangle (t,\omega)$ is also called the "compensator" - see [6], Definition 2, p.239, vol.II.*

Here is a simple example of a doubly stochastic Poisson process. Let $Y(t,\omega)$ denote a continuous time Markov chain with two states denoted by $1,2$. The infinitesimal generator matrix Q has the following form [3,8]:

$$Q = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix}$$

Define the function $f(s_i), s_i \in S$ (the state space of the Markov chain) as follows: $f(1) = \lambda_1, f(2) = \lambda_2$ and $\lambda(t,\omega) = f(Y(t,\omega))$. Note that this is an example of a 'Markov-modulated Poisson process' [3,8,9]. More generally, one can consider functions of the form: $\lambda(t,\omega) = f(t, Y(t,\omega))$. Thus, the class of arrival processes to which our methods apply includes the particular cases studied by Wolff - who assumed that $\Lambda(t)$ is a deterministic function.

We want to compare the proportion of time that the process $Z(t) \in B$ with the corresponding proportion of customers who, upon arrival, see $Z(t) \in B$. The key observation is that the difference between these two quantities can be expressed as a stochastic integral with respect to the square integrable martingale $M(t)$ defined in equation (1). More precisely, let $U(t) = I_B(Z(t-))$, where $I_B(x) = 1, x \in B, I_B(x) = 0$, otherwise. Thus,

$$W(t) = \int_0^t U(s)\, d\Lambda(s)$$

is a random weighted average of the amount of time during $[0, t]$ that $Z(t) \in B$. In the special case of the Poisson process $W(t) = \lambda \times$ the amount of time during $[0, t]$ that $Z(t) \in B$. Similarly ,

$$S(t) = \int_0^t U(s)\, dN(s)$$

counts the number of times that an arrival sees $Z(t) \in B$ during the interval of time $[0, t]$. Next observe that $R(t) = S(t) - W(t)$ can be written as a stochastic integral with respect to the square integrable martingale $M(t)$. More precisely, it is easy to verify that:

$$\begin{aligned} R(t) &= \int_0^t U(s)\, dM(s) \qquad (2) \\ &= \int_0^t U(s)\, dN(s) - \int_0^t U(s)\, d\Lambda(s) \\ &= S(t) - W(t) \end{aligned}$$

Notice that the random function $U(t)$ is left continuous and is therefore *predictable* with respect to the σ-field $\mathcal{F}(t)$ where $\mathcal{F}(t) = \sigma\,(Y(s), N(s), Z(s),$ $0 \leq s \leq t)$ - see [1,6] for unexplained terminology.

Lemma 1 *Let $U(t)$ be a predictable process satisfying the condition:*

$$E\left(\int_0^T |U(s,\omega)|\, d|\Lambda(s,\omega)|\right) < \infty, \text{ for every } T > 0.$$

Then the process $\{R(t), \mathcal{F}(t), t \geq 0\}$, defined by the stochastic integral above, is a martingale.

Proof: This is an immediate consequence of the general theory of stochastic integration with respect to:(i) square integrable martingales [6], chap.5, section 5.4, or with respect to (ii) martingales of bounded integrable variation - see [1], Theorem T6, p.10.

Our main result is that Wolff's lemma 2 is still valid for the much larger class of arrival processes considered here. More precisely, we have the following result:

Theorem 1 *Assume $N(t)$ is a doubly stochastic Poisson process with bounded intensity function $\lambda(t,\omega)$. Then*

$$\lim_{t\to\infty}\frac{R(t)}{t}=0,\ \text{with probability one.} \tag{3}$$

2 Proof of Theorem

Theorem 1 is a special case of the following strong law of large numbers for martingales of the form :

$$R(t)=\int_0^t v(s)\,dM(s),$$

where $M(t)$ is a right continuous square integrable martingale whose "compensator", denoted by $<M>(t)$, has the representation

$$<M>(t)=A(t)=\int_0^t a(s)\,ds \text{ with } a(t)\geq 0.$$

In addition we assume that $v(t)$ and $a(t)$ are both bounded i.e., $\|v\| = \sup_{t\geq 0,\omega} \mid v(t,\omega)\mid < \infty$ and $\|a\| = \sup_{t\geq 0,\omega}\mid a(t,\omega)\mid < \infty$.

Theorem 2 *Suppose $M(t)$ is a right continuous square integrable martingale with compensator of the form*

$$<M>(t)=A(t)=\int_0^t a(s)\,ds \textit{ with } a(t)\geq 0, \|a(t)\| \textit{ bounded.}$$

Let $v(s)$ be a bounded predictable process.Then $R(t)$ defined above is a square integrable martingale and

$$\lim_{t\to\infty}\frac{R(t)}{t}=0,\ \textit{with probability one.}$$

Proof: It is well known, see e.g. [6], vol.I, p.175, equation (5.72) and 5.4.6 on p.181, that $R(t)$ is a martingale such that:

$$\begin{aligned} ER(t)^2 &= E\left(\int_0^t v(s)\,dM(s)\right)^2 \\ &= E\int_0^t v(s)^2 a(s)\,ds \leq \|v\|^2 \times \|a\| \times t \end{aligned} \tag{4}$$

This proves that $R(t)$ is a square integrable martingale. Next observe that

$$E(R(t+h) - R(t))^2 = E\int_t^{t+h} v(s)^2 a(s)\,ds \leq \|v\|^2 \times \|a\| \times h \tag{5}$$

Note that the inequalities 4 and 5 sharpen and generalize inequalities (7) and (8) in [12]. From here on the proof proceeds in the same manner as outlined in [12]. Pick $h > 0$ and set $X_k = R(kh) - R((k-1)h), k = 1, 2, \ldots$ and put $R(0) = 0$, thus $R(nh) = \sum_{k=0}^{n} X_k$. In addition inequality 5 implies that $EX_k^2 \leq Ch$,where C is independent of h.Now let $b_k, k = 1, 2, \ldots$denote any sequence of constants satisfying the conditions:

(i) $0 < b_1 < b_2 < \ldots < b_k$,
(ii) $\lim_{k\to\infty} b_k = \infty$ and
(iii) $\sum_{k=1}^{\infty} b_k^{-2} < \infty$.

Then

$$Y_n = \sum_{k=1}^{n} \frac{X_k}{b_k^2}$$

is an L_2 bounded martingale with

$$EY_n^2 = \sum_{k=1}^{n} \frac{EX_k^2}{b_k^2} \leq \sum_{k=1}^{\infty} \frac{Ch}{b_k^2} < \infty$$

Consequently, $\lim_{n\to\infty} Y_n = Y_\infty$ exists and is finite with probability one. This implies,see e.g. Neveu [10], Prop.IV-6-1, p.138, that

$$\lim_{n\to\infty} b_n^{-1} \sum_{k=1}^{n} X_k = \lim_{n\to\infty} b_n^{-1} R(nh) = 0.$$

If we choose $b_n = n$ then we see at once that

$$\lim_{n\to\infty} \frac{R(nh)}{n} = 0 \tag{6}$$

In order to prove that $\lim_{t\to\infty}\frac{R(t)}{t}=0$ it suffices to establish this for the special case $v(t)\geq 0$. If not, one can write $R(t)=R^+(t)-R^-(t)$ where $R^\pm(t)=\int_0^t v^\pm(s)\,dM(s)$ and $v^+(t)=\max(v(t),0), v^-(t)=\max(-v(t),0)$.

Lemma 2 *There exists a constant C, independent of h, such that*

$$R(nh)-Ch\leq R(t)\leq R((n+1)h)+Ch, \text{ for } nh\leq t\leq (n+1)h \tag{7}$$

Observe that

$$\begin{aligned}R(t)-R(nh) &= \int_{nh}^t v(s)\,dN(s)-\int_{nh}^t v(s)a(s)\,ds\\ &\geq -\int_{nh}^t v(s)a(s)\,ds\\ &\geq -\|v\|\|a\|h=Ch\end{aligned} \tag{8}$$

And reasoning in exactly the same way as above one can also show that

$$R(t)-R((n+1)h)\leq Ch$$

This completes the derivation of 7. The condition $nh\leq t\leq (n+1)h$ implies that $\frac{n}{t}\leq h^{-1}$ and $\frac{n+1}{t}$ are both bounded for h held fixed. Thus

$$\lim_{t\to\infty}\frac{R((n+1)h)}{t}=\lim_{n\to\infty}\frac{R((n+1)h)}{n+1}\frac{n+1}{t}=0$$

and similarly one can show that

$$\lim_{t\to\infty}\frac{R(nh)}{t}=0.$$

Using these two results, dividing both sides of 7 by t and letting $t\to\infty$ yields the proof of Theorem 1.

3 Examples

3.1 Geometric Arrivals

We may apply Theorem 2 very easily to show that, in the case of a discrete-time system, geometric arrivals (i.e., the interarrival times are geometric random variables [4]) see time averages. Without loss of generality, let arrivals occur with probability p (the parameter of the geometric distribution) and define the step function $[t]$ as $[t]=n, n\leq t<n+1$. Next, for $t\in R$,

let $B(t)$ denote the number of arrivals in $[0,t]$. Then, it is easily shown that $M(t) = B(t) - p[t]$ is a right-continuous martingale. Now,

$$
\begin{aligned}
E\left[(R(t+h) - R(t))^2\right] &\leq \sup_{s\in[t,t+h]} U^2(s) E\left[\left(\int_t^{t+h} dM(s)\right)^2\right] \\
&\leq p(1-p)\lceil h \rceil
\end{aligned}
$$

where the last inequality arises from the variance of the Binomial distribution. Hence, we have proved inequality (5) for geometric arrivals. The rest of the proof of Theorem 2 is actually simplified in the case of discrete-time, but is applicable in its present form and therefore, we obtain our result that geometric arrivals see time averages.

3.2 Markov-modulated Poisson Processes

As an application of Theorem 1, we compute the probability of a system state as seen by arrivals from a K-state Markov Modulated Poisson Process ($MMPP$) [3,9]. This has important consequences for performance metrics such as blocking probabilities in several queueing applications [3,5,7,8,11].

Let $Y(t), 1 \leq Y(t) \leq K$ denote the state of a K-state $MMPP$ and $Z(t) \in \{0,1,2,...\}$ the system state at time t respectively. We assume that $\{Y(t), Z(t)\}$ is an ergodic Markov process; this is the case, for example, when a stable queue is driven by an $MMPP$ in which the service times are exponentially distributed. The arrival rate when the $MMPP$ is in state $Y(t)$ is, as mentioned earlier, $f(Y(t)) = \lambda_{Y(t)}$. Let $\pi(i,j), 1 \leq i \leq K, j \geq 0$, be the limiting distribution of the Markov process $\{Y(t), Z(t)\}$ and for $1 \leq i \leq K$ define

$$
\begin{aligned}
I_i(t) &= 1, \quad \text{if } Y(t) = i \\
&\ 0, \quad \text{otherwise}
\end{aligned}
$$

Next, define the following indicator functions for the states, $j \geq 0$, of the system

$$
\begin{aligned}
U_j(t) &= 1, \quad \text{if } Z(t) = j \\
&\ 0, \quad \text{otherwise}
\end{aligned}
$$

We may now calculate the long term probability of an arrival seeing the event B as

$$
\begin{aligned}
P\,[\text{arrival sees } B] &= \lim_{t\to\infty} \frac{1}{N(t)} \int_0^t \sum_{j\in B} U_j(s)dN(s) \\
&= \lim_{t\to\infty} \frac{t}{N(t)} \lim_{t\to\infty} \frac{1}{t} \int_0^t \sum_{j\in B} U_j(s)dN(s)
\end{aligned}
$$

Now, from Theorem 1,

$$
\begin{aligned}
\lim_{t\to\infty} \frac{1}{t} \int_0^t \sum_{j\in B} U_j(s)dN(s) &= \lim_{t\to\infty} \frac{1}{t} \int_0^t \sum_{j\in B} U_j(s)d\Lambda(s) \\
&= \lim_{t\to\infty} \frac{1}{t} \int_0^t \sum_{j\in B} U_j(s) \left(\sum_{i=1}^{K} \lambda_i I_i(s) \right) ds \\
&= \sum_{j\in B} \sum_{i=1}^{K} \lim_{t\to\infty} \frac{1}{t} \int_0^t U_j(s) I_i(s) \lambda_i ds \\
&= \sum_{j\in B} \sum_{i=1}^{K} \lambda_i \pi(i,j)
\end{aligned}
$$

The last step follows from ergodic theorems for Markov processes (see Doob [2], Theorem 2.1, page 515). Thus the above limit and $\lim_{t\to\infty} \frac{N(t)}{t}$ may be combined to obtain the result. This is illustrated through an example.

<u>**Example:**</u> Consider a single first-come-first-served queue driven by a 2-state $MMPP$ arrival process in which the service times are exponentially distributed with rate μ. The states of the $MMPP$ are labeled 1 and 2 respectively; the transition rate between state 1 and state 2 is denoted by α whereas the corresponding rate between state 2 state 1 is denoted by β. The arrivals to the queue when the $MMPP$ is in state i, $i = 1, 2$, is Poisson with rate λ_i.

Let us compute the fraction of arrivals that see the system in state j. We get [3]

$$
\lim_{t\to\infty} \frac{N(t)}{t} = \frac{\lambda_1\beta + \lambda_2\alpha}{\alpha+\beta}
$$

Hence,

$$
P\,[\text{Arrival sees state } j] = \frac{(\alpha+\beta)(\lambda_1\pi(1,j) + \lambda_2\pi(2,j))}{\lambda_1\beta + \lambda_2\alpha}
$$

which agrees with the corresponding expression stated in [8].

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