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Abstract

We consider the calculation of the motion parameters for finite rigid motion from the "essential" matrix $\mathbf{E}$ introduced by Tsai and Huang [Tsai84] and by Longuet–Higgins [Long81]. We give a simple method for calculating the rotational parameters which involves only the multiplication of simple known matrices; it does not use a singular value decomposition of the $\mathbf{E}$ matrix, and hence requires no iterative procedures. We show that, contrary to the assertions of [Tsai84] and [Long81], the direction of translation and the rotation matrix are not uniquely determined from $\mathbf{E}$. We show that there are precisely two rigid motions which give rise to the same essential matrix, and that these motions are related by duality. It is shown that the ambiguity is not at all rare, but vanishes for the case when the translation is in the image plane, which includes stereo as a special case. We use the theory of the rotation group SO(3) and its Lie algebra *so(3)* to elucidate certain of the manipulations of Tsai and Huang [Tsai84], and point out errors in their work and that of Longuet–Higgins[Long81].

1 Introduction

The calculation of the parameters which describe the motion of rigidly moving objects in the environment through which a sensor is moving is a major research area in computational approaches to visual motion. Over the years, many techniques have been devised to find these motion parameters, which describe the rotational and translational motion of objects in the environment. One such approach is to use the 2–D motion of distinctive image structures (tokens) between frames of a sequence of camera images to compute the 3–D motion parameters of the corresponding environmental structures. Such techniques are called correspondence–based techniques, since the fiducial information required for their application is the correspondence between tokens in successive frames of the image sequence.

In this paper we consider one aspect of a particular technique for computing the rotational and translational motion parameters for finite rigid motion using image correspondences, the "eight–point" algorithm discovered independently by Longuet–Higgins [Long81] and Tsai and Huang [Tsai84]. This technique uses the correspondence of eight (or more, for robustness) image points in two image frames to find a 3×3 matrix $\mathbf{E}$, called by Tsai and Huang the "essential" matrix, which encodes the translational and rotational motion which gave rise to these image correspondences. This matrix can be calculated from the eight point correspondences in a linear fashion, so that the questions of existence and uniqueness can be addressed in a straightforward way. The problem we address here is how the rotation and translation can be calculated from the essential matrix, and whether such motion is uniquely determined from $\mathbf{E}$.

In the work of Tsai and Huang [Tsai84], the essential matrix is used to find the axis of the translational vector $\hat{\mathbf{t}}$. Since such an axis determines two directions in space, there will be a two-fold ambiguity in the determination of $\hat{\mathbf{t}}$. Tsai and Huang then find the rotation matrix by making a singular value decomposition of the essential matrix $\mathbf{E}$. They find two possibilities for the rotation matrix, and attempt to prove that only one of the possibilities will lead to positive depth values for both the initial and the final (i.e., after the motion has taken place) set of 3D points. We will call the requirement that the rigid motion be associated with initial and final 3D points which lie in the forward hemisphere the *positivity condition*. Longuet-Higgins [Long81] introduces a matrix which is essentially the transpose of the essential matrix defined by Tsai and Huang. His rather spare technique is quite different from that of Tsai and Huang. He also finds a two-fold ambiguity in the expression for the rotation matrix (given the essential matrix), then states (without proof) that one of these expressions will satisfy the positivity condition, and one will not. We show in Appendix B of the present work that both of these papers are mistaken: both ambiguous solutions satisfy the positivity condition for initial points in a "large" region of the forward hemisphere.

We are aware that the use of linear algorithms such as the eight-point algorithm is currently out of fashion, and that methods which use least-mean-square techniques (such as the algorithm of Faugeras, Lustman, and Toscani [Faug87]) generally give superior results. We note, however, that the essential matrix is of considerable interest in theoretical discussions of motion ambiguity, as is seen from its use in, for example, the work of Faugeras and Maybank [Faug89]. Consequently, we feel that any information about the essential matrix should be of interest to the vision community.

The structure of this paper is as follows. In Section 2, we introduce the essential matrix and discuss its properties. We discuss the meaning of a rigid motion in the context of the rotation group SO(3), and exhibit the two ambiguous rigid motions which give rise to the same essential matrix. In Section 3, we show how the direction of translation and the rotation matrix corresponding to the rigid motion can be calculated in a direct way, using only explicitly known matrices. This section makes extensive use of the group theory of SO(3) and its associated Lie algegra *so(3)*. We show that there can be only three sources of ambiguity in the calculation of the rigid motion from the essential matrix. These are called the *Coset*, the *Parity*, and the *Duality* ambiguities. In Section 4, we discuss these ambiguities, and show that only the duality ambiguity is a real ambiguity. We also discuss the relation between our work and the work of Tsai and Huang, and of Longuet–Higgins. There are three extensive appendices to the paper. In Appendix A, we give a short tutorial on the theory of Lie groups, with particular attention to the rotation group SO(3). The reader unfamiliar with this topic should refer to this appendix to understand the main text. In Appendix B, we show that both ambiguous solutions satisfy the positivity condition for initial points in "large" regions of the forward hemisphere, so that it is generally impossible to decide which is the "correct" rigid motion on this basis alone. We also consider the issue of when the indicated ambiguity *can* be resolved by recourse to the positivity condition, and show that it can be so resolved when the translation is in the image plane, corresponding to the case of stereo. In Appendix C, we present a more general proof of a property of the essential matrix first discovered by Faugeras and Maybank [Faug89].

2 The Essential Matrix and a Remark on Uniqueness

A rigid motion is a linear map from $\mathcal{R}^3$ into itself that preserves both the distance between points, and the orientation of the coordinate axes. It is straightforward to show that such a map can be parametrized by a rigid *motion pair* $(\mathbf{R}, \mathbf{t})$, which takes $\mathbf{r} \in \mathcal{R}^3$ into $\mathbf{r}' \in \mathcal{R}^3$, where

$$(\mathbf{R}, \mathbf{t}) \ : \ \ \mathbf{r} \longrightarrow \mathbf{r}' \equiv \mathbf{R}\mathbf{r} + \mathbf{t}. \tag{2.1}$$

Here $\mathbf{R}$ is an element of SO(3), the Lie group of 3×3 real matrices with determinant $+1$, and $\mathbf{t}$ is the translation vector. This map therefore defines the action of the Euclidean group ISO(3) on $\mathcal{R}^3$. The initial point $\mathbf{r}$ and the motion–transformed final point $\mathbf{r}'$ are called *corresponding* points.

We use bold face symbols to denote both vectors and tensors, and the corresponding 3×1 and 3×3 matrices, respectively. Context will distinguish which is the appropriate viewpoint. We can, if $\mathbf{t} \neq 0$, choose to measure distances in units of $|\mathbf{t}|$, so that $\mathbf{t}$ will be taken to be a unit vector $\hat{\mathbf{t}}$, with $\hat{\mathbf{t}} \cdot \hat{\mathbf{t}} = 1$.

There are many ways of introducing the "essential" matrix [Long81,Tsai84,Faug89]. We note that If $\mathbf{t} \neq 0$, then from equation (2.1)

$$\hat{\mathbf{t}} \times \mathbf{r}' = \hat{\mathbf{t}} \times (\mathbf{R}\mathbf{r}),$$

whence

$$0 \equiv \mathbf{r}' \cdot \hat{\mathbf{t}} \times \mathbf{r}' = \mathbf{r}' \cdot \hat{\mathbf{t}} \times (\mathbf{R}\mathbf{r}). \tag{2.2}$$

We can rewrite (2.2) by introducing the antisymmetric matrix $\mathbf{G}$, given in terms of the infinitesimal generators $\{\mathbf{J}_i \ ; i = 1, 2, 3\}$ of SO(3) by (see Appendix A for a summary of the group theory of SO(3) and its associated Lie algebra *so(3)*):

$$\mathbf{G} \equiv \hat{\mathbf{t}} \cdot \mathbf{J} = \begin{pmatrix} 0 & -t_3 & t_2 \\ t_3 & 0 & -t_1 \\ -t_2 & t_1 & 0 \end{pmatrix}, \tag{2.3}$$

where $\hat{\mathbf{t}} = (t_1, t_2, t_3)^{\mathrm{T}}$. This matrix is an element of the Lie algebra *so(3)*. It has the interesting property that if $\mathbf{A}$ is a vector, then (see (A.44) of Appendix A)

$$\mathbf{GA} = \hat{\mathbf{t}} \times \mathbf{A}. \tag{2.4}$$

Therefore, we can write (2.2) as

$$(\mathbf{r}')^{\mathrm{T}}\mathbf{GRr} = 0.$$

The "essential" matrix $\mathbf{E}$ introduced independently by Tsai and Huang [Tsai84] and by Longuet–Higgins[Long81], is then defined to be

$$\mathbf{E} = \mathbf{GR}, \tag{2.5}$$

so that it satisfies

$$(\mathbf{r}')^{\mathrm{T}}\mathbf{Er} = 0. \tag{2.6}$$

Although (2.6) can be taken as the basis for an algorithm [Tsai84,Long81] for computing $\mathbf{E}$, we do not address that issue here; we concern ourselves only with the question of how the motion parameters $\mathbf{R}$ and $\hat{\mathbf{t}}$ can be computed from $\mathbf{E}$. That is, *given* a matrix $\mathbf{E}$ known to be of the form (2.5), how can we recover its constituent parts $\mathbf{R}$ and $\hat{\mathbf{t}}$? We assume in the remainder of this work that $\mathbf{E}$ is a fixed matrix of the form (2.5). We do not address the ambiguity of $\mathbf{E}$ on the level of this algorithm, i.e., that if $\mathbf{E}$ satisfies (2.6), so does $\lambda\mathbf{E}$, where $\lambda \in \mathcal{R}$.

We first observe that if one can find a motion pair $(\mathbf{R}, \hat{\mathbf{t}})$ that satisfies (2.5), then there are *three* (at least) other pairs that give the same $\mathbf{E}$. Clearly, one such pair is $(-\mathbf{R}, -\hat{\mathbf{t}})$, since

$$\left(\left(-\hat{\mathbf{t}}\right) \cdot \mathbf{J}\right)(-\mathbf{R}) = \left(-\hat{\mathbf{t}} \cdot \mathbf{J}\right)(-\mathbf{R}) = (\hat{\mathbf{t}} \cdot \mathbf{J})\mathbf{R}.$$

However, if $\mathbf{R} \in \mathrm{SO}(3)$ (i.e., $\det \mathbf{R} = +1$), then $-\mathbf{R}$ is *not* in SO(3), since $\det(-\mathbf{R}) = (-1)^3 \det \mathbf{R} = -\det \mathbf{R} = -1$. That is, $-\mathbf{R} \in \mathrm{O}^-(3)$ (see App. A). Therefore, the transformation $\mathbf{r} \longrightarrow \mathbf{r}'$ given by $(-\mathbf{R}, -\hat{\mathbf{t}})$:

$$(-\mathbf{R}, -\hat{\mathbf{t}}) \ : \ \mathbf{r} \longrightarrow \mathbf{r}' = -\mathbf{R}\mathbf{r} - \hat{\mathbf{t}}$$

is *not a rigid motion*, contrary to the assertions on p. 18 of [Tsai84].

To be sure, if $(\mathbf{R}, \hat{\mathbf{t}})$ gives rise to a positive depth for the corresponding point $\mathbf{r}'$ of the initial point $\mathbf{r}$, then $(-\mathbf{R}, -\hat{\mathbf{t}})$ will give rise to a negative depth for $\mathbf{r}'$ (i.e., $\mathbf{r}'$ would be behind the camera plane, and hence not imaged by the camera), but that is quite irrelevant. The real reason for eschewing $(-\mathbf{R}, -\hat{\mathbf{t}})$ in favor of $(\mathbf{R}, \hat{\mathbf{t}})$ is that if the latter is a rigid motion, the former is not (if it were, a (3D) left hand could be rigidly transformed into a right hand, and that is not possible in $\mathcal{R}_3$ (though it is possible in other spaces, e.g., the Möbius strip or the Klein bottle)).

We promised that there were at least four rigid motions that give rise to the same $\mathbf{E}$, but have so far presented only two. The two others can be defined as follows. Let $\mathbf{R}_0 = \exp(\pi\hat{\mathbf{t}} \cdot \mathbf{J}) = \exp(\pi\mathbf{G}) \in \mathrm{SO}(3)$, i.e., $\mathbf{R}_0$ is a rotation by 180° around the direction of translation. Then consider the pair $(\mathbf{R}_0\mathbf{R}, -\hat{\mathbf{t}})$ which, as we will show, is *dual* to the pair $(\mathbf{R}, \hat{\mathbf{t}})$. We claim that this rigid motion gives the same $\mathbf{E}$ as the pair $(\mathbf{R}, \hat{\mathbf{t}})$. We first give a geometrical proof that this is the case, and then a more formal proof.

Consider the two rigid motions in question:

$$\mathbf{r}' = \mathbf{R}\mathbf{r} + \hat{\mathbf{t}}$$
$$\mathbf{r}'' = \mathbf{R}_0\mathbf{R}\mathbf{r} - \hat{\mathbf{t}}.$$

Let the essential matrix defined from the first be $\mathbf{E}$, and that from the second be $\mathbf{E}'$. That is,

$$\mathbf{E}\mathbf{r} = \hat{\mathbf{t}} \times (\mathbf{R}\mathbf{r})$$
$$\mathbf{E}'\mathbf{r} = -\hat{\mathbf{t}} \times (\mathbf{R}_0\mathbf{R}\mathbf{r}).$$

Define, now,

$$\mathbf{R}\mathbf{r} \equiv \mathbf{V} \equiv \mathbf{V}_{\parallel} + \mathbf{V}_{\perp},$$

where $\mathbf{V}_{\parallel}$ and $\mathbf{V}_{\perp}$ are parallel and perpendicular, respectively, to $\hat{\mathbf{t}}$. It then follows that

$$\mathbf{E}\mathbf{r} = \hat{\mathbf{t}} \times (\mathbf{R}\mathbf{r}) = \hat{\mathbf{t}} \times (\mathbf{V}_{\parallel} + \mathbf{V}_{\perp})$$
$$= \hat{\mathbf{t}} \times \mathbf{V}_{\perp},$$

and that

$$\mathbf{E}'\mathbf{r} = \hat{\mathbf{t}} \times (\mathbf{R}_0\mathbf{R}\mathbf{r}) = -\hat{\mathbf{t}} \times (\mathbf{R}_0(\mathbf{R}\mathbf{r})) = -\hat{\mathbf{t}} \times \underbrace{\left(\mathbf{R}_0(\mathbf{V}_{\parallel} + \mathbf{V}_{\perp})\right)}_{=\mathbf{V}_{\parallel} - \mathbf{V}_{\perp}}$$
$$= +\hat{\mathbf{t}} \times \mathbf{V}_{\perp}$$
$$= \mathbf{E}\mathbf{r}.$$

But if $\mathbf{E}'$ and $\mathbf{E}$ have the same effect on an arbitrary vector, that means that $\mathbf{E}$ and $\mathbf{E}'$ are identical, Q.E.D.

For the interested reader, we now give a more formal proof of this result. To whit, let $\mathbf{E}'$ be the essential matrix defined by $(\mathbf{R}_0\mathbf{R}, -\hat{\mathbf{t}})$:

$$\mathbf{E}' = \left(\left(-\hat{\mathbf{t}}\right) \cdot \mathbf{J}\right)(\mathbf{R}_0\mathbf{R}).$$

Then

$$\mathbf{E}' = -\mathbf{G}\mathbf{R}_0\mathbf{R},$$

with $\mathbf{G}$ given by (2.3). But (see equation (A.43) of Appendix A)

$$\mathbf{R}_0 = \exp(\pi\mathbf{G}) = \mathbf{1}_3 + 2\mathbf{G}^2.$$

Hence, using the property $\mathbf{G}^3 = -\mathbf{G}$ (see equation (A.42) of Appendix A), we find that

$$-\mathbf{G}\mathbf{R}_0 = -\mathbf{G}(\mathbf{1}_3 + 2\mathbf{G}^2) = -\mathbf{G} - 2\mathbf{G}^3 = -\mathbf{G} + 2\mathbf{G} = +\mathbf{G}.$$

Therefore,

$$\mathbf{E}' = (+\mathbf{G})\mathbf{R} = \mathbf{G}\mathbf{R} = \mathbf{E}.$$

Therefore both $(\mathbf{R}_0\mathbf{R}, -\hat{\mathbf{t}})$ and $(\mathbf{R}, \hat{\mathbf{t}})$ give rise to the same essential matrix $\mathbf{E}$. The fourth possibility is then just the negative of the former: $(-\mathbf{R}_0\mathbf{R}, +\hat{\mathbf{t}})$. Of course, if $\mathbf{R}$ is a proper rotation, so is $\mathbf{R}_0\mathbf{R}$, but $-\mathbf{R}_0\mathbf{R}$ is *not*. Therefore of these four motions that give rise to $\mathbf{E}$, only two of them, namely

$$(\mathbf{R}, \hat{\mathbf{t}}) \quad \text{and} \quad (\mathbf{R}_0\mathbf{R}, -\hat{\mathbf{t}}),$$

are in fact rigid motions. The analysis of Sections 3 and 4 show that these are the *only* rigid motions that give the same $\mathbf{E}$. If we can then show that for both of these rigid motions there exist situations for which both the initial point $\mathbf{r}$ and the corresponding points $\mathbf{r}'$ for both motions satisfy the positivity condition, we will have shown the claims of [Tsai84,Long81], namely, that the direction of $\hat{t}$ can be determined from the positivity condition, and hence that $\mathbf{E}$ gives unique values for the rotational and translational motion parameters, are false. This is done in Appendix B. We hasten to add, however, that these two motions will in general give different point correspondences. Consequently, the ambiguity can be resolved at the level of correspondences, but not at the level of the essential matrix. It is, in this sense, not an "inherent" ambiguity. Once one has found these two sets of motion parameters, however, one must check which gives the correct point correspondences. It is only when noise, spatial discretization, etc. make it impossible to determine which is the "correct" choice of parameters that the discovered ambiguity is "inherent."

In the next section, we show how to calculate the constituent $(\mathbf{R}, \hat{t})$ pair, given the essential matrix $\mathbf{E}$, in a very direct way which avoids the need for a singular value decomposition of $\mathbf{E}$. We show that there is in principle a one–parameter infinite family of pairs $(\mathbf{R}, \hat{t})$ that give the same $\mathbf{E}$. In the succeeding section, we show that only 2 of these give distinguishable rigid motions. In Appendix B, we show that these 2 motion pairs can in fact be realized with initial and transformed points in the forward hemisphere for both motion pairs, and find when this ambiguity vanishes.

3 The Determination of $\hat{t}$ and R from E

3.1 The Determination of $\hat{t}$

We note that the essential matrix satisfies the relation

$$\mathbf{E}\mathbf{E}^{\mathrm{T}} = \mathbf{G}\mathbf{R}(-\mathbf{R}^{\mathrm{T}}\mathbf{G}) = -\mathbf{G}^2 = \mathbf{1}_3 - \hat{\mathbf{t}}\hat{\mathbf{t}}^{\mathrm{T}}.$$

where we have used the relation (A.46) in Appendix A. This means that $\mathbf{E}\mathbf{E}^{\mathrm{T}}$ is a *projection operator*: $\mathbf{E}\mathbf{E}^{\mathrm{T}} = \mathbf{P}$, where $\mathbf{P}^2 = \mathbf{P}$:

$$\begin{aligned} \mathbf{P}^2 &= (\mathbf{1}_3 - \hat{\mathbf{t}}\hat{\mathbf{t}}^{\mathrm{T}})(\mathbf{1}_3 - \hat{\mathbf{t}}\hat{\mathbf{t}}^{\mathrm{T}}) \\ &= \mathbf{1}_3 - 2\hat{\mathbf{t}}\hat{\mathbf{t}}^{\mathrm{T}} + \hat{\mathbf{t}}(\hat{\mathbf{t}}^{\mathrm{T}}\hat{\mathbf{t}})\hat{\mathbf{t}}^{\mathrm{T}} \\ &= \mathbf{1}_3 - \hat{\mathbf{t}}\hat{\mathbf{t}}^{\mathrm{T}} \equiv \mathbf{P}. \quad \textbf{Q.E.D.} \end{aligned}$$

Physically, $\mathbf{E}\mathbf{E}^{\mathrm{T}}$ projects a vector $\mathbf{A}$ onto its component $\hat{\mathbf{t}} \times \left(\mathbf{A} \times \hat{\mathbf{t}}\right) = -\mathbf{G}^2\mathbf{A}$ in the plane perpendicular to $\hat{\mathbf{t}}$. $\mathbf{E}\mathbf{E}^{\mathrm{T}}$ must therefore have one zero eigenvalue (corresponding to the eigenvector $\hat{\mathbf{t}}$), and two equal (to one) eigenvalues (corresponding to any two vectors which span the plane perpendicular to $\hat{\mathbf{t}}$). Therefore, $\mathbf{E}$ is a singular matrix, with two equal singular values and one zero singular value, as was shown in a different way by Faugeras and Maybank[Faug89]. The argument can be reversed, either by explicit construction [Faug89], or by using the singular value decomposition of $\mathbf{E}$ (see Appendix C).

The translation vector can thus be determined as a solution to the eigenvalue equation $\mathbf{E}\mathbf{E}^{\mathrm{T}}\hat{\mathbf{t}} = 0$, or by the explicit calculation presented by Tsai and Huang[Tsai84]. In either case, however, only the line parallel to $\hat{\mathbf{t}}$ can be determined, so that there is a two–fold ambiguity, in general, corresponding to the two different directions determined by the line in question. If we (arbitrarily) pick $\hat{\mathbf{t}}_0$ to be one of these directions, then the translation vector can be either $\hat{\mathbf{t}} = +\hat{\mathbf{t}}_0$ or $\hat{\mathbf{t}} = -\hat{\mathbf{t}}_0$. We will refer to this as the "duality" ambiguity in the determination of $\hat{\mathbf{t}}$ and $\mathbf{R}$ from $\mathbf{E}$. It is discussed further in Section 4.3. Both [Tsai84] and [Long81] state that the correct choice of $\hat{\mathbf{t}}$ can be determined from the condition that the initial and transformed points satisfy the positivity condition. We show in Appendix B that this is not true.

3.2 The Determination of R from E

We assume that one of the values $\hat{\mathbf{t}} = \pm\hat{\mathbf{t}}_0$ has been chosen. We define the matrix $\mathbf{Q}$ to be an element of SO(3) which rotates the chosen vector $\hat{\mathbf{t}}$ onto the positive Z-axis. As is shown later in Sec. 4.4, the matrix $\mathbf{Q}$ is just the transpose of the matrix of the same name introduced by Tsai and Huang [Tsai84]. In their work, however, $\mathbf{Q}$ was an unknown matrix; here we see exactly what $\mathbf{Q}$ is. Denoting by $\widehat{\zeta}$ the unit vector $(0,0,1)^{\mathrm{T}}$ along the $+Z$-axis, then

$$\mathbf{Q}\hat{\mathbf{t}} \equiv (0,0,1)^{\mathrm{T}} = \widehat{\zeta} \implies \hat{\mathbf{t}} = \mathbf{Q}^{\mathrm{T}}\widehat{\zeta}. \tag{3.7}$$

For instance, if $\hat{\mathbf{t}}$ is parametrized by the polar angles (θ, ϕ) (see Figure),

$$\hat{\mathbf{t}}_0 = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta),$$

where $0 \le \theta < \pi,\ 0 \le \phi < 2\pi$, then the following choice for $\mathbf{Q}$ satisfies (3.7):

$$\mathbf{Q} \equiv \mathbf{Q}_0 = \exp\left(\theta\widehat{\mathbf{N}} \cdot \mathbf{J}\right), \tag{3.8}$$

where (see the Figure) $\widehat{\mathbf{N}}$ is the unit vector

$$\widehat{\mathbf{N}} = (\tilde{\eta}^{\mathrm{T}}, 0)^{\mathrm{T}}; \quad \tilde{\eta} = (\sin\phi, -\cos\phi)^{\mathrm{T}}. \tag{3.9}$$

However, there are an infinity of other elements of SO(3) which satisfy (3.7). To see this, suppose that $\mathbf{Q}$ is an element of SO(3) which satisfies (3.7), and that $\mathbf{Q}_0$ is defined by (3.8). Hence,

$$\mathbf{Q}\hat{\mathbf{t}} \equiv \widehat{\zeta} = \mathbf{Q}_0\hat{\mathbf{t}},$$

and so

$$\mathbf{Q}\mathbf{Q}_0^{\mathrm{T}}\widehat{\zeta} = \mathbf{Q}\hat{\mathbf{t}} = \widehat{\zeta}. \tag{3.10}$$

But any rotation $\mathbf{Q}\mathbf{Q}_0^{\mathrm{T}}$ which leaves $\hat{\zeta}$ invariant must be a rotation $\mathbf{H}$ by some angle β around the Z–axis:

$$\mathbf{H} = \exp(\beta\hat{\zeta}\cdot\mathbf{J}) = \exp(\beta\mathbf{J}_3),$$

where $\beta \in [0, 2\pi)$. The set of all such $\mathbf{H}$ forms an SO(2) subgroup of SO(3) called the *little group*[1] of $\hat{\zeta}$. Equation (3.10) therefore says that $\mathbf{Q}\mathbf{Q}_0^{\mathrm{T}}$ must be in the little group of $\hat{\zeta}$, so that for some $\mathbf{H} \in \mathrm{SO}(2)$

$$\mathbf{Q}\mathbf{Q}_0^{\mathrm{T}} = \mathbf{H} \implies \mathbf{Q} \equiv \mathbf{Q}\mathbf{Q}_0^{\mathrm{T}}\mathbf{Q}_0 \equiv \mathbf{H}\mathbf{Q}_0. \tag{3.11}$$

That is, a $\mathbf{Q}$ which rotates $\hat{\mathbf{t}}$ into $\hat{\zeta}$ must be a product of an element $\mathbf{H}$ of the little group and the matrix $\mathbf{Q}_0$ defined in (3.8). This ambiguity in the determination of $\mathbf{Q}$ will be called the "coset" ambiguity, for reasons that will become clear shortly. We show in Section 4.1 that this ambiguity in the definition of the matrix $\mathbf{Q}$ has no effect on our analysis.

We now find an explicit expression for $\mathbf{R}$ in terms of $\mathbf{Q}$ and $\mathbf{E}$. If we assume that $\mathbf{Q}$ satisfies (3.7), then (using equation (A.48) of Appendix A),

$$\begin{aligned}\mathbf{Q}\mathbf{E}\mathbf{Q}^{\mathrm{T}} &= \mathbf{Q}\left(\hat{\mathbf{t}}\cdot\mathbf{J}\right)\mathbf{Q}^{\mathrm{T}}\mathbf{Q}\mathbf{R}\mathbf{Q}^{\mathrm{T}} = \left(\left(\mathbf{Q}\hat{\mathbf{t}}\right)\cdot\mathbf{J}\right)\left(\mathbf{Q}\mathbf{R}\mathbf{Q}^{\mathrm{T}}\right)\\ &\equiv \left(\hat{\zeta}\cdot\mathbf{J}\right)\left(\mathbf{Q}\mathbf{R}\mathbf{Q}^{\mathrm{T}}\right) = \mathbf{J}_3\left(\mathbf{Q}\mathbf{R}\mathbf{Q}^{\mathrm{T}}\right),\end{aligned} \tag{3.12}$$

where we have used equation (A.43) of Appendix A, and equation (3.7). But

$$\mathbf{J}_3 = \begin{pmatrix} -\mathbf{J} & 0 \\ 0 & 0 \end{pmatrix},$$

where $\mathbf{J}$ is the 2×2 antisymmetric matrix

$$\mathbf{J} = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}. \tag{3.13}$$

[1]Other names for this group are the *stabilizer* or *isotropy group*.

(It will be clear from context whether $\mathbf{J}$ denotes the (2×2) matrix defined in (3.13) or the triplet of (3×3) matrices $\{\mathbf{J}_1, \mathbf{J}_2, \mathbf{J}_3\}$.) Hence, it is easy to see from (3.12) that $\mathbf{QEQ}^{\mathrm{T}}$ is of the form

$$\mathbf{QEQ}^{\mathrm{T}} \equiv \begin{pmatrix} \mathbf{E}_0 & \mathbf{e}_1 \\ 0 & 0 \end{pmatrix}, \tag{3.14}$$

where $\mathbf{E}_0$ is a 2×2 matrix and $\mathbf{e}_1$ is a 2×1 matrix (i.e., a column vector). We now define

$$\mathbf{QRQ}^{\mathrm{T}} \equiv \begin{pmatrix} \mathbf{F} & \mathbf{f}_1 \\ \mathbf{f}_2^{\mathrm{T}} & \varphi \end{pmatrix}. \tag{3.15}$$

By substituting this expression into (3.12) and comparing with (3.14), we find that

$$\mathbf{F} = \mathbf{J}\mathbf{E}_0, \tag{3.16}$$

$$\mathbf{f}_1 = \mathbf{J}\mathbf{e}_1, \tag{3.17}$$

and $\mathbf{f}_2$ and φ are undetermined. However, since $\mathbf{Q}$ and $\mathbf{R}$ are in SO(3), so is $\mathbf{QRQ}^{\mathrm{T}}$, so we must have that

$$\begin{aligned} \mathbf{F}^{\mathrm{T}}\mathbf{f}_1 + \varphi\mathbf{f}_2 &= 0, \\ \mathbf{f}_1^{\mathrm{T}}\mathbf{f}_1 + \varphi^2 &= 1. \end{aligned}$$

Hence,

$$\begin{aligned} \varphi &= \pm\sqrt{1 - \mathbf{f}_1^{\mathrm{T}}\mathbf{f}_1}, \\ \mathbf{f}_2 &= \frac{-1}{\varphi}\mathbf{F}^{\mathrm{T}}\mathbf{f}_1. \end{aligned}$$

Consequently, using (3.16,3.17), and the fact that $\mathbf{J}^2 = -\mathbf{1}_2$,

$$\varphi = \pm\mu \quad ; \quad \mu \equiv \sqrt{1 - \mathbf{e}_1^{\mathrm{T}}\mathbf{e}_1} \tag{3.18}$$

$$\mathbf{f}_2 = \mp\frac{1}{\mu}\mathbf{E}_0^{\mathrm{T}}\mathbf{e}_1.$$

As a consequence, we have *two* implicit expressions $\mathbf{R}_\pm$ for the rotation matrix $\mathbf{R}$:

$$\mathbf{Q}\mathbf{R}_\pm\mathbf{Q}^{\mathrm{T}} = \begin{pmatrix} \mathbf{J}\mathbf{E}_0 & \mathbf{J}\mathbf{e}_1 \\ \mp\frac{\mathbf{e}_1^{\mathrm{T}}\mathbf{E}_0}{\mu} & \pm\mu \end{pmatrix} \equiv \mathbf{R}_\pm^*, \tag{3.19}$$

where μ is defined in (3.18). Note that the " $*$ " superscript does not mean "complex conjugate": everything in our analysis is in the real number field.

We can therefore solve (3.19) immediately for the rotation $\mathbf{R}_\pm$:

$$\mathbf{R}_\pm = \mathbf{Q}^{\mathrm{T}}\mathbf{R}_\pm^*\mathbf{Q}. \tag{3.20}$$

We note the ambiguity in the choice of sign. We call this the "parity" ambiguity. It is discussed further in Section 4.2, where we show that only one choice of sign gives an element of SO(3); the other is not a proper rotation (it is an element of $\mathrm{O}^-(3)$).

In summary, the matrix $\mathbf{E}\mathbf{E}^{\mathrm{T}}$ determines (up to sign) the translation vector $\hat{\mathbf{t}}$. The matrix $\mathbf{Q}$ is determined (up to an SO(2) factor) by $\hat{\mathbf{t}}$. The matrices $\mathbf{E}_0$ and $\mathbf{e}_1$ are determined from $\mathbf{E}$ and $\mathbf{Q}$ via (3.14), which then determines the matrix $\mathbf{R}_\pm^*$, and hence the rotation matrix $\mathbf{R}_\pm$ via (3.20).

We therefore have found expressions for $\mathbf{R}$ and $\hat{\mathbf{t}}$, given only the matrix $\mathbf{E}$. The expression (3.20) for $\mathbf{R}$, in particular, is explicit and simple; no singular value decomposition is necessary. However, there are several ambiguities in the calculation of $\mathbf{R}$ and $\hat{\mathbf{t}}$ from $\mathbf{E}$. We address these ambiguities in the next section, and show that the only real ambiguity is in the choice of sign in the expression $\hat{\mathbf{t}} = \pm\hat{\mathbf{t}}_0$.

4 Resolution of the Ambiguities

There are three sources of ambiguity in our analysis. The first arises from the fact that the rotation matrix $\mathbf{Q}$ which rotates $\hat{\mathbf{t}}$ onto the $+Z$–axis is not uniquely determined, the second from the choice of sign in the quantity $\mathbf{R}^{*}_{\pm}$, and the third from the choice of sign for the translation vector $\hat{\mathbf{t}} = \pm\hat{\mathbf{t}}_0$. We will call the first kind of ambiguity the "coset" ambiguity, the second the "parity" ambiguity, and the third the "duality" ambiguity. We will show that only the last is an actual ambiguity.

4.1 Resolution of the Coset Ambiguity

We refer to the non–uniqueness of $\mathbf{Q}$ (given $\hat{\mathbf{t}}$) as the "coset" ambiguity for the following reason. The set of rotations $\{\mathbf{H} = \exp(\beta \mathbf{J}_3)\ ;\ \beta \in [0, 2\pi)\}$ forms an SO(2) subgroup of the full rotation group SO(3). The relation (3.11) can be expressed as saying that two rotation matrices $\mathbf{Q}$ and $\mathbf{Q}'$ in SO(3) are "equivalent" if they differ by an SO(2) rotation $\mathbf{H}$:

$$\mathbf{Q} \sim \mathbf{Q}' \quad \text{iff} \quad \mathbf{Q}' = \mathbf{H}\mathbf{Q}.$$

It is easily checked that this is indeed an equivalence relation, and hence that the set of equivalence classes (the left cosets of SO(3) mod SO(2)) is just the coset space

$$\frac{\mathrm{SO}(3)}{\mathrm{SO}(2)} \approx \mathrm{S}^2,$$

where S^2 is the unit 2–sphere. The question whether the non–uniqueness of $\mathbf{Q}$ gives rise to an ambiguity in the motion pair $\left(\mathbf{R}, \hat{\mathbf{t}}\right)$ associated with $\mathbf{E}$ is therefore equivalent to whether different elements of the same coset give rise to different motion pairs $\left(\mathbf{R}, \hat{\mathbf{t}}\right)$, whence the name "coset" ambiguity. We now prove that the expression for the rotation matrix $\mathbf{R}$ depends only on the coset to which $\mathbf{Q}$ belongs, i.e., there is no coset ambiguity.

Let $\mathbf{Q}$ and $\overline{\mathbf{Q}}$ be two elements of SO(3) which rotate the translation vector $\hat{\mathbf{t}}$ onto the $+Z$-axis:

$$\mathbf{Q}\hat{\mathbf{t}} = \hat{\zeta} = \overline{\mathbf{Q}}\hat{\mathbf{t}}$$

We showed in Section 3.2 that this implies that

$$\overline{\mathbf{Q}} = \mathbf{H}\mathbf{Q} \tag{4.21}$$

for some element $\mathbf{H}$ of the little group SO(2) of $\hat{\zeta}$. Now any such element of SO(2) can be expressed as

$$\mathbf{H} = \exp(\beta \mathbf{J}_3) = \begin{pmatrix} \mathbf{1}_2 \cos\beta - \mathbf{J}\sin\beta & 0 \\ 0 & 1 \end{pmatrix} \equiv \begin{pmatrix} \mathbf{A} & 0 \\ 0 & 1 \end{pmatrix}, \tag{4.22}$$

where the matrix $\mathbf{A}$ is an element of SO(2):

$$\mathbf{A} = \mathbf{1}_2 \cos\beta - \mathbf{J}\sin\beta \ \ ; \mathbf{A}\mathbf{A}^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}}\mathbf{A} = \mathbf{1}_2 \ ; \ \det \mathbf{A} = +1. \tag{4.23}$$

We note that $[\mathbf{J}, \mathbf{A}] \equiv \mathbf{J}\mathbf{A} - \mathbf{A}\mathbf{J} = 0$, i.e., $\mathbf{J}$ and $\mathbf{A}$ commute.

We define $\mathbf{R}$ as the rotation matrix which arises from the choice $\mathbf{Q}$, and $\overline{\mathbf{R}}$ as that which arises from the choice $\overline{\mathbf{Q}}$ (we will suppress the $\pm$ subscripts in the symbols for the rotation):

$$\mathbf{Q}\mathbf{E}\mathbf{Q}^{\mathrm{T}} = \begin{pmatrix} \mathbf{E}_0 & \mathbf{e}_1 \\ 0 & 0 \end{pmatrix} \quad ; \quad \mathbf{R} = \mathbf{Q}^{\mathrm{T}} \begin{pmatrix} \mathbf{J}\mathbf{E}_0 & \mathbf{J}\mathbf{e}_1 \\ \mp\frac{1}{\mu}\mathbf{e}_1^{\mathrm{T}}\mathbf{E}_0 & \pm\mu \end{pmatrix} \mathbf{Q},$$

$$\overline{\mathbf{Q}}\mathbf{E}\overline{\mathbf{Q}}^{\mathrm{T}} = \begin{pmatrix} \overline{\mathbf{E}}_0 & \overline{\mathbf{e}}_1 \\ 0 & 0 \end{pmatrix} \quad ; \quad \overline{\mathbf{R}} = \overline{\mathbf{Q}}^{\mathrm{T}} \begin{pmatrix} \mathbf{J}\overline{\mathbf{E}}_0 & \mathbf{J}\overline{\mathbf{e}}_1 \\ \mp\frac{1}{\overline{\mu}}\overline{\mathbf{e}}_1^{\mathrm{T}}\overline{\mathbf{E}}_0 & \pm\overline{\mu} \end{pmatrix} \overline{\mathbf{Q}},$$

$$\mu = \sqrt{1 - \mathbf{e}_1^{\mathrm{T}}\mathbf{e}_1} \quad ; \quad \overline{\mu} = \sqrt{1 - \overline{\mathbf{e}}_1^{\mathrm{T}}\overline{\mathbf{e}}_1}.$$

Hence, using (4.21),

$$\begin{aligned}\overline{\mathbf{Q}}\mathbf{E}\overline{\mathbf{Q}}^{\mathrm{T}} &= (\mathbf{HQ})\,\mathbf{E}\,(\mathbf{HQ})^{\mathrm{T}} = \mathbf{H}\left(\mathbf{QEQ}^{\mathrm{T}}\right)\mathbf{H}^{\mathrm{T}} \\ &= \begin{pmatrix}\mathbf{A} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\mathbf{E}_0 & \mathbf{e}_1\\ 0 & 0\end{pmatrix}\begin{pmatrix}\mathbf{A}^{\mathrm{T}} & 0\\ 0 & 1\end{pmatrix} \\ &= \begin{pmatrix}\mathbf{AE}_0\mathbf{A}^{\mathrm{T}} & \mathbf{Ae}_1\\ 0 & 0\end{pmatrix}.\end{aligned}$$

We therefore have the following relations:

$$\begin{aligned}\overline{\mathbf{E}}_0 &= \mathbf{AE}_0\mathbf{A}^{\mathrm{T}}, \\ \overline{\mathbf{e}}_1 &= \mathbf{Ae}_1.\end{aligned}$$

This then gives:

$$\begin{aligned}\overline{\mathbf{e}}_1^{\mathrm{T}}\overline{\mathbf{E}}_0 &= \mathbf{e}_1^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{AE}_0\mathbf{A}^{\mathrm{T}} = \mathbf{e}_1^{\mathrm{T}}\mathbf{E}_0\mathbf{A}^{\mathrm{T}}, \\ \overline{\mathbf{e}}_1^{\mathrm{T}}\overline{\mathbf{e}}_1 &= \mathbf{e}_1^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{Ae}_1 = \mathbf{e}_1^{\mathrm{T}}\mathbf{e}_1,\end{aligned}$$

so that $\overline{\mu} = \mu$. Consequently,

$$\begin{aligned}\overline{\mathbf{R}} &= (\mathbf{HQ})^{\mathrm{T}}\begin{pmatrix}\mathbf{JAE}_0\mathbf{A}^{\mathrm{T}} & \mathbf{JAe}_1\\ \mp\frac{1}{\mu}\mathbf{e}_1^{\mathrm{T}}\mathbf{E}_0\mathbf{A}^{\mathrm{T}} & \pm\mu\end{pmatrix}(\mathbf{HQ}) \\ &\equiv \mathbf{Q}^{\mathrm{T}}\mathbf{FQ},\end{aligned}$$

where, using (4.22),

$$\begin{aligned}\mathbf{F} &\equiv \begin{pmatrix}\mathbf{A}^{\mathrm{T}} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\mathbf{JAE}_0\mathbf{A}^{\mathrm{T}} & \mathbf{JAe}_1\\ \mp\frac{1}{\mu}\mathbf{e}_1^{\mathrm{T}}\mathbf{E}_0\mathbf{A}^{\mathrm{T}} & \pm\mu\end{pmatrix}\begin{pmatrix}\mathbf{A} & 0\\ 0 & 1\end{pmatrix} \\ &= \begin{pmatrix}\mathbf{A}^{\mathrm{T}}\mathbf{JAE}_0\mathbf{A}^{\mathrm{T}}\mathbf{A} & \mathbf{A}^{\mathrm{T}}\mathbf{JAe}_1\\ \mp\frac{1}{\mu}\mathbf{e}_1^{\mathrm{T}}\mathbf{E}_0\mathbf{A}^{\mathrm{T}}\mathbf{A} & \pm\mu\end{pmatrix}\end{aligned}$$

$$= \begin{pmatrix} \mathbf{A}^{\mathrm{T}}\mathbf{J}\mathbf{A}\mathbf{E}_0 & \mathbf{A}^{\mathrm{T}}\mathbf{J}\mathbf{A}\mathbf{e}_1 \\ \mp\frac{1}{\mu}\mathbf{e}_1^{\mathrm{T}}\mathbf{E}_0 & \pm\mu \end{pmatrix}.$$

But $\mathbf{J}$ and $\mathbf{A}$ commute, so that

$$\mathbf{A}^{\mathrm{T}}\mathbf{J}\mathbf{A} = \mathbf{A}^{\mathrm{T}}\mathbf{A}\mathbf{J} \equiv \mathbf{J}.$$

Consequently,

$$\mathbf{F} = \begin{pmatrix} \mathbf{J}\mathbf{E}_0 & \mathbf{J}\mathbf{e}_1 \\ \mp\frac{1}{\mu}\mathbf{e}_1^{\mathrm{T}}\mathbf{E}_0 & \pm\mu \end{pmatrix},$$

and so

$$\overline{\mathbf{R}} = \mathbf{Q}^{\mathrm{T}} \begin{pmatrix} \mathbf{J}\mathbf{E}_0 & \mathbf{J}\mathbf{e}_1 \\ \mp\frac{1}{\mu}\mathbf{e}_1^{\mathrm{T}}\mathbf{E}_0 & \pm\mu \end{pmatrix} \mathbf{Q} \equiv \mathbf{R} \quad \text{Q.E.D.}$$

Therefore there is no coset ambiguity: the expression for $\mathbf{R}$ is independent of the particular $\mathbf{Q}$ which rotates $\hat{\mathfrak{t}}$ onto $\widehat{\zeta}$.

4.2 The Parity Ambiguity

The ambiguity which we call the parity ambiguity arises from the two possibilities for the rotation matrix $\mathbf{R}$: $\mathbf{R}_+$ and $\mathbf{R}_-$, where

$$\mathbf{R}_\pm = \mathbf{Q}^{\mathrm{T}}\mathbf{R}_\pm^\star\mathbf{Q} = \mathbf{Q}^{\mathrm{T}} \begin{pmatrix} \mathbf{J}\mathbf{E}_0 & \mathbf{J}\mathbf{e}_1 \\ \mp\frac{1}{\mu}\mathbf{e}_1^{\mathrm{T}}\mathbf{E}_0 & \pm\mu \end{pmatrix} \mathbf{Q}.$$

We note, however, that

$$\begin{aligned} \det \mathbf{R}_\pm &= \det \mathbf{Q} \ \det \mathbf{Q}^{\mathrm{T}} \ \det \mathbf{R}_\pm^\star \\ &= \det \mathbf{R}_\pm^\star = \det \begin{pmatrix} \mathbf{J}\mathbf{E}_0 & \mathbf{J}\mathbf{e}_1 \\ \mp\frac{1}{\mu}\mathbf{e}_1^{\mathrm{T}}\mathbf{E}_0 & \pm\mu \end{pmatrix} \end{aligned}$$

But clearly,

$$\det \begin{pmatrix} \mathbf{J}\mathbf{E}_0 & \mathbf{J}\mathbf{e}_1 \\ -\frac{1}{\mu}\mathbf{e}_1^T\mathbf{E}_0 & +\mu \end{pmatrix} = (-)\det \begin{pmatrix} \mathbf{J}\mathbf{E}_0 & \mathbf{J}\mathbf{e}_1 \\ +\frac{1}{\mu}\mathbf{e}_1^T\mathbf{E}_0 & -\mu \end{pmatrix},$$

since the two matrices in question differ only by the sign of the third row. Hence,

$$\det\ \mathbf{R}_- = -\det\ \mathbf{R}_+. \tag{4.24}$$

This means that of the two possible choices $\{\mathbf{R}_+, \mathbf{R}_-\}$ for $\mathbf{R}$, one of the matrices has determinant $+1$, while the other has determinant -1. That is, one is a proper rotation (an element of $\mathrm{O}^+(3) = \mathrm{SO}(3)$), while the other is an improper rotation (an element of $\mathrm{O}^-(3)$). Since only proper rotations are associated with a rigid motion, there is in fact no ambiguity in the choice of sign in $\mathbf{R}_\pm$: one *must* choose whatever sign makes the determinant positive—the other choice is not a rigid motion and can be neglected. In the work of Tsai and Huang [Tsai84], this point is confused with the less intrinsic requirement that only points in the forward hemisphere can be viewed.

4.2.1 The Relationship between $\mathbf{R}_+$ and $\mathbf{R}_-$

It is of interest to know the precise nature of the relationship between $\mathbf{R}_+$ and $\mathbf{R}_-$, both for completeness and for the discussion of the "duality" ambiguity addressed in Section 4.3. In the present section we elucidate this relationship.

We note that

$$\mathbf{R}_+ = \mathbf{Q}^T \begin{pmatrix} \mathbf{J}\mathbf{E}_0 & \mathbf{J}\mathbf{e}_1 \\ -\frac{1}{\mu}\mathbf{e}_1^T\mathbf{E}_0 & +\mu \end{pmatrix} \mathbf{Q}.$$

Hence,

$$
\begin{aligned}
-\mathbf{Q}\mathbf{R}_+\mathbf{Q}^{\mathrm{T}} &= \begin{pmatrix} -\mathbf{J}\mathbf{E}_0 & -\mathbf{J}\mathbf{e}_1 \\ +\frac{1}{\mu}\mathbf{e}_1^{\mathrm{T}}\mathbf{E}_0 & -\mu \end{pmatrix} = \begin{pmatrix} -\mathbf{1}_2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{J}\mathbf{E}_0 & \mathbf{J}\mathbf{e}_1 \\ +\frac{1}{\mu}\mathbf{e}_1^{\mathrm{T}}\mathbf{E}_0 & -\mu \end{pmatrix} \\
&= \begin{pmatrix} -\mathbf{1}_2 & 0 \\ 0 & 1 \end{pmatrix} \left(\mathbf{Q}\mathbf{R}_-\mathbf{Q}^{\mathrm{T}}\right) \\
&= \exp\left(\pi \mathbf{J}_3\right) \left(\mathbf{Q}\mathbf{R}_-\mathbf{Q}^{\mathrm{T}}\right),
\end{aligned}
$$

where we have used expression (A.43) of Appendix A. Consequently,

$$
\mathbf{R}_+ = -\left\{\mathbf{Q}^{\mathrm{T}} \exp\left(\pi \mathbf{J}_3\right) \mathbf{Q}\right\} \mathbf{R}_- .
$$

But (equation (A.48) of Appendix A)

$$
\begin{aligned}
\mathbf{Q}^{\mathrm{T}} \exp\left(\pi \mathbf{J}_3\right) \mathbf{Q} &= \mathbf{Q}^{\mathrm{T}} \exp\left(\pi \hat{\zeta} \cdot \mathbf{J}\right) \mathbf{Q} \\
&= \exp\left(\pi \left(\mathbf{Q}^{\mathrm{T}} \hat{\zeta}\right) \cdot \mathbf{J}\right).
\end{aligned}
$$

However, from (3.7), $\mathbf{Q}^{\mathrm{T}}\hat{\zeta} = \hat{\mathbf{t}}$. Therefore,

$$
\begin{aligned}
\mathbf{R}_+ &= -\exp\left(\pi \hat{\mathbf{t}} \cdot \mathbf{J}\right) \mathbf{R}_- \\
&\equiv -\mathbf{R}_0 \mathbf{R}_-, \qquad (4.25)
\end{aligned}
$$

where

$$
\mathbf{R}_0 \equiv \exp\left(\pi \mathbf{G}\right) = \exp\left(\pi \hat{\mathbf{t}} \cdot \mathbf{J}\right) = \mathbf{R}_0^{\mathrm{T}} \in \mathrm{SO}(3), \qquad (4.26)
$$

i.e., $\mathbf{R}_0$ is a rotation by 180° about the direction of translation. This expression for $\mathbf{R}_+$ in terms of $\mathbf{R}_-$ makes clear the earlier remark that these two quantities have opposite determinants: $\mathbf{R}_\pm$ is given by $\mathbf{R}_\mp$, followed by a rotation of 180° around the direction of $\hat{\mathbf{t}}$, followed by an inversion. The inversion operation $\mathbf{I}$ takes a vector $\mathbf{r} \in \mathcal{R}^3$ into its negative $-\mathbf{r}$. It is therefore represented by the matrix $\mathbf{I} \equiv -\mathbf{1}_3$. Hence, $\det \mathbf{I} = -1$. It is the presence of the inversion that accounts for the fact that $\mathbf{R}_-$ and $\mathbf{R}_+$ have determinants of opposite sign, and hence that only one can in fact represent a rigid motion.

4.3 The Duality Ambiguity

As we discussed earlier, only the line along which the translation lies can be found from $\mathbf{E}$. That is, each $\mathbf{E}$ corresponds in principle to two possible translations $\hat{\mathbf{t}} = \pm\hat{\mathbf{t}}_0$. This leads to two definitions of the matrix $\mathbf{Q}$, and hence to two expressions for the motion pair $(\mathbf{R}, \hat{\mathbf{t}})$. We call this the "duality" ambiguity for reasons which will be clear shortly. Suppose that $\hat{\mathbf{t}}_0$ is fixed, and that I have chosen $\hat{\mathbf{t}} = +\hat{\mathbf{t}}_0$, while you, the reader, made (unbeknownst to me) the opposite choice $\hat{\mathbf{t}} = -\hat{\mathbf{t}}_0$. I would therefore have found polar angles θ and ϕ for $\hat{\mathbf{t}}_0$, and defined from that the matrix $\mathbf{Q}$, found the matrices $\mathbf{E}_0$ and $\mathbf{e}_1$ using $\mathbf{Q}$ and $\mathbf{E}$, and calculated the rotation matrix $\mathbf{R}$ (choosing, of course, the appropriate sign so as to make $\det \mathbf{R} = +1$). You, on the other hand, would have proceeded in the same way, but using quantities which (in principle) might differ from mine: I will denote your quantities by a "prime" ($'$). We then have:

$$\textbf{ME} \quad ; \quad \textbf{YOU}$$

$$\hat{\mathbf{t}} = (\theta, \phi) \quad ; \quad \hat{\mathbf{t}}' = (\theta', \phi')$$

$$\mathbf{Q} = \exp\left(\theta\widehat{\mathbf{N}} \cdot \mathbf{J}\right) \quad ; \quad \mathbf{Q}' = \exp\left(\theta'\widehat{\mathbf{N}}' \cdot \mathbf{J}\right)$$

$$\widehat{\mathbf{N}} = \left(\tilde{\eta}^{\mathrm{T}}, 0\right)^{\mathrm{T}} \quad ; \quad \widehat{\mathbf{N}}' = \left(\tilde{\eta}'^{\mathrm{T}}, 0\right)^{\mathrm{T}}$$

$$\tilde{\eta} = (\sin\phi, -\cos\phi)^{\mathrm{T}} \quad ; \quad \tilde{\eta}' = (\sin\phi', -\cos\phi')^{\mathrm{T}}$$

$$\mathbf{QEQ}^{\mathrm{T}} = \begin{pmatrix} \mathbf{E}_0 & \mathbf{e}_1 \\ 0 & 0 \end{pmatrix} \quad ; \quad \mathbf{Q}'\mathbf{EQ}'^{\mathrm{T}} = \begin{pmatrix} \mathbf{E}'_0 & \mathbf{e}'_1 \\ 0 & 0 \end{pmatrix}$$

$$\mathbf{R}_{\pm} = \mathbf{Q}^{\mathrm{T}} \begin{pmatrix} \mathbf{JE}_0 & \mathbf{Je}_1 \\ \mp\frac{1}{\mu}\mathbf{e}_1^{\mathrm{T}}\mathbf{E}_0 & \pm\mu \end{pmatrix} \mathbf{Q} \quad ; \quad \mathbf{R}'_{\pm} = \mathbf{Q}'^{\mathrm{T}} \begin{pmatrix} \mathbf{JE}'_0 & \mathbf{Je}'_1 \\ \mp\frac{1}{\mu'}\mathbf{e}_1'^{\mathrm{T}}\mathbf{E}'_0 & \pm\mu' \end{pmatrix} \mathbf{Q}'$$

$$\mu = \sqrt{1 - \mathbf{e}_1^{\mathrm{T}}\mathbf{e}_1} \quad ; \quad \mu' = \sqrt{1 - \mathbf{e}_1'^{\mathrm{T}}\mathbf{e}'_1}$$

Now the relation between the unprimed (my) quantities and the primed (your) quantities is that *your* translational vector $\hat{\mathbf{t}}'$ is just the negative of my translational vector $\hat{\mathbf{t}}$. A little thought shows that this means that

$$\begin{aligned} \theta' &= \pi - \theta \\ \phi' &= \phi + \pi \end{aligned}$$

so that

$$\widehat{\mathbf{N}}' = -\widehat{\mathbf{N}}.$$

This then gives the relation between $\mathbf{Q}'$ and $\mathbf{Q}$:

$$\begin{aligned} \mathbf{Q}' &\equiv \exp\left(\theta'\widehat{\mathbf{N}}' \cdot \mathbf{J}\right) \\ &= \exp\left((\pi - \theta)\left(-\widehat{\mathbf{N}}\right) \cdot \mathbf{J}\right) \\ &= \exp\left(-\pi\widehat{\mathbf{N}} \cdot \mathbf{J}\right) \exp\left(\theta\widehat{\mathbf{N}} \cdot \mathbf{J}\right) \\ &\equiv \mathbf{Q}_{\mathrm{n}}\mathbf{Q}, \end{aligned} \tag{4.27}$$

where

$$\mathbf{Q}_{\mathrm{n}} \equiv \exp\left(-\pi\widehat{\mathbf{N}} \cdot \mathbf{J}\right) \in \mathrm{SO}(3).$$

The question, then, is if $\mathbf{Q}'$ and $\mathbf{Q}$ are related by (4.27), what is the relation between $\mathbf{R}_\pm$ and $\mathbf{R}'_\pm$? This is easily found.

We first note that (see Appendix A)

$$\mathbf{Q_n} = \exp\left(-\pi\widehat{\mathbf{N}}\cdot\mathbf{J}\right) = \mathbf{1}_3 + 2\left(\widehat{\mathbf{N}}\cdot\mathbf{J}\right)^2 = \mathbf{Q_n^T},$$

so that $\mathbf{Q_n^2} = \mathbf{1}_3$. Explicitly, in terms of the (2×1) matrix $\eta \equiv (\cos\varphi, \sin\varphi)^{\mathrm{T}} = -\mathbf{J}\tilde{\eta}$,

$$\mathbf{Q_n} = \begin{pmatrix} \mathbf{M} & 0 \\ 0 & -1 \end{pmatrix} \quad ; \quad \mathbf{M} = \mathbf{1}_2 - 2\eta\eta^{\mathrm{T}} = \mathbf{M}^{\mathrm{T}}, \tag{4.28}$$

i.e., $\mathbf{M}$ is a Householder matrix. We note that $\mathbf{M}^2 = \mathbf{1}_2$. Now then:

$$\begin{aligned}\begin{pmatrix} \mathbf{E}'_0 & \mathbf{e}'_1 \\ 0 & 0 \end{pmatrix} \equiv \mathbf{Q}'\mathbf{E}\mathbf{Q}'^{\mathrm{T}} &= (\mathbf{Q_nQ})\,\mathbf{E}\,(\mathbf{Q_nQ})^{\mathrm{T}} = \mathbf{Q_n}\left(\mathbf{QEQ^T}\right)\mathbf{Q_n^T} \\ &= \mathbf{Q_n}\begin{pmatrix} \mathbf{E}_0 & \mathbf{e}_1 \\ 0 & 0 \end{pmatrix}\mathbf{Q_n} \\ &= \begin{pmatrix} \mathbf{ME}_0\mathbf{M} & -\mathbf{Me}_1 \\ 0 & 0 \end{pmatrix}.\end{aligned}$$

Hence, we have that

$$\begin{aligned}\mathbf{E}'_0 &= \mathbf{ME}_0\mathbf{M} \\ \mathbf{e}'_1 &= -\mathbf{Me}_1.\end{aligned}$$

and so

$$\begin{aligned}\mu' &= \sqrt{1-\mathbf{e}_1'^{\mathrm{T}}\mathbf{e}'_1} = \sqrt{1-\mathbf{e}_1^{\mathrm{T}}\mathbf{M}^2\mathbf{e}_1} = \sqrt{1-\mathbf{e}_1^{\mathrm{T}}\mathbf{e}_1} = \mu \\ \mathbf{e}_1'^{\mathrm{T}}\mathbf{E}'_0 &= -\mathbf{e}_1^{\mathrm{T}}\mathbf{M}^2\mathbf{E}_0\mathbf{M} = -\mathbf{e}_1^{\mathrm{T}}\mathbf{E}_0\mathbf{M}.\end{aligned}$$

Therefore

$$\begin{aligned} \mathbf{R}'_{\pm} &= \mathbf{Q}'^{\mathrm{T}} \begin{pmatrix} \mathbf{JME_0M} & -\mathbf{JMe_1} \\ \pm\frac{1}{\mu}\mathbf{e}_1^{\mathrm{T}}\mathbf{E_0M} & \pm\mu \end{pmatrix} \mathbf{Q}' \\ &= \mathbf{Q}^{\mathrm{T}}\mathbf{Q_n} \begin{pmatrix} \mathbf{JME_0M} & -\mathbf{JMe_1} \\ \pm\frac{1}{\mu}\mathbf{e}_1^{\mathrm{T}}\mathbf{E_0M} & \pm\mu \end{pmatrix} \mathbf{Q_nQ} \\ &= \mathbf{Q}^{\mathrm{T}} \begin{pmatrix} \mathbf{MJME_0} & +\mathbf{MJMe_1} \\ \mp\frac{1}{\mu}\mathbf{e}_1^{\mathrm{T}}\mathbf{E_0M} & \pm\mu \end{pmatrix} \mathbf{Q} \end{aligned}$$

However, it is easy to check that $\mathbf{M}$ and $\mathbf{J}$ anticommute: $\{\mathbf{M},\mathbf{J}\} \equiv \mathbf{MJ} + \mathbf{JM} = 0$, i.e.,

$$\mathbf{MJM} = -\mathbf{J}.$$

Consequently

$$\mathbf{R}'_{\pm} = -\mathbf{Q}^{\mathrm{T}} \begin{pmatrix} \mathbf{JE_0} & \mathbf{Je_1} \\ \pm\frac{1}{\mu}\mathbf{e}_1^{\mathrm{T}}\mathbf{E_0} & \mp\mu \end{pmatrix} \mathbf{Q}.$$

Therefore,

$$\mathbf{R}'_{\pm} = -\mathbf{R}_{\mp} \tag{4.29}$$

That is, if you choose $\hat{\mathbf{t}} = -\hat{\mathbf{t}}_0$, and I choose $\hat{\mathbf{t}} = +\hat{\mathbf{t}}_0$, then the $\mathbf{R}_{\pm}$ you calculate will be the negative of the $\mathbf{R}_{\mp}$ that I calculate. In each case, you must choose the one with positive determinant, and so must I. Therefore, if $\det \mathbf{R}_+$ is $+1$ (and hence $\det \mathbf{R}_-$ is -1), then I must choose $\mathbf{R} = \mathbf{R}_+$, and you (according to (4.29)) must choose $\mathbf{R}' = -\mathbf{R}_-$. On the other hand if $\det \mathbf{R}_+ = -1$, then I would have to choose $\mathbf{R} = \mathbf{R}_-$, and you would have to choose $\mathbf{R}' = -\mathbf{R}_+$. The possible rigid motion pairs are then:

$$\left(\mathbf{R}_+, +\hat{\mathbf{t}}_0\right) \text{ and } \left(-\mathbf{R}_-, -\hat{\mathbf{t}}_0\right) \quad \textit{if} \quad \det \mathbf{R}_+ = +1$$

$$\left(\mathbf{R}_-, +\hat{\mathbf{t}}_0\right) \text{ and } \left(-\mathbf{R}_+, -\hat{\mathbf{t}}_0\right) \quad \textit{if} \quad \det \mathbf{R}_- = +1$$

But we recall from (4.25) that $\mathbf{R}_+ = -\mathbf{R}_0\mathbf{R}_-$, and so $\mathbf{R}_- = -\mathbf{R}_0\mathbf{R}_+$, where $\mathbf{R}_0 = \exp\left(\pi\hat{\mathbf{t}}\cdot\mathbf{J}\right)$. That is,

$$\mathbf{R}_\pm = -\mathbf{R}_0\mathbf{R}_\mp.$$

Therefore, we conclude that the rigid motions that can be derived from the essential matrix $\mathbf{E}$ are just

$$\left(\mathbf{R},\hat{\mathbf{t}}_0\right) \quad \text{and} \quad \left(\mathbf{R}_0\mathbf{R},-\hat{\mathbf{t}}_0\right), \tag{4.30}$$

where $\mathbf{R}$ is the one matrix from the set $\{\mathbf{R}_+,\mathbf{R}_-\}$ having positive determinant. In Section 2, we checked that each of these rigid motions does indeed give the same essential matrix $\mathbf{E}$. We now note the *duality* of these solutions. If we let

$$\begin{aligned}
\mathbf{R}' &= \mathbf{R}_0\mathbf{R}, \\
\hat{\mathbf{t}}_0' &= -\hat{\mathbf{t}}_0, \\
\Longrightarrow \mathbf{R}_0' &= \exp\{\pi\hat{\mathbf{t}}_0'\cdot\mathbf{J}\} = \exp\{-\pi\hat{\mathbf{t}}_0\cdot\mathbf{J}\} \\
&= \exp\{+\pi\hat{\mathbf{t}}_0\cdot\mathbf{J}\} \\
&\equiv \mathbf{R}_0,
\end{aligned}$$

then it follows that

$$\begin{aligned}
\mathbf{R}_0'\mathbf{R}' &= \mathbf{R}_0\left(\mathbf{R}_0\mathbf{R}\right) = \mathbf{R}_0^2\mathbf{R} = \mathbf{1}_3\mathbf{R} = \mathbf{R} \\
-\hat{\mathbf{t}}_0' &= -(-\hat{\mathbf{t}}_0) = \hat{\mathbf{t}}_0.
\end{aligned} \tag{4.31}$$

Therefore, if let Π be the operation that converts $\left(\mathbf{R},\hat{\mathbf{t}}_0\right)$ into $\left(\mathbf{R}_0\mathbf{R},-\hat{\mathbf{t}}_0\right)$:

$$\Pi\left\{\left(\mathbf{R},\hat{\mathbf{t}}_0\right)\right\} = \left(\mathbf{R}_0\mathbf{R},-\hat{\mathbf{t}}_0\right),$$

then

$$\Pi\left\{\left(\mathbf{R}_0\mathbf{R}, -\hat{\mathbf{t}}_0\right)\right\} = \left(\mathbf{R}, \hat{\mathbf{t}}_0\right).$$

Therefore, Π^2 is just the identity operation. We are therefore justified in referring to the two solutions as *duals* of each other.

We have thus shown that for each $\mathbf{E}$, there are exactly two different rigid motions that give $\mathbf{E}$, and that these two motions are related by duality.

To be complete, there is one more thing that has to be proved, namely, that for the two rigid motions, there exists at least one situation in which the ambiguity can be realized, i.e., that there exist initial and final 3D points associated with each of the rigid motions, all of which lie in the forward hemisphere. This is a necessary condition for the discovered ambiguity to be a real ambiguity, in a practical sense. We give the requisite example, and find the conditions under which the ambiguity vanishes, in Appendix B. We thus conclude that, contrary to the assertions of [Tsai84,Long81], the essential matrix is *not* associated with a unique rigid motion.

4.4 Relation to the Work of Tsai and Huang, and of Longuet–Higgins

In the paper of Tsai and Huang [Tsai84], the central result of their calculation of the rotation matrix from the essential matrix is contained in their Theorem I, which states that if the singular value decomposition of the matrix $\mathbf{E}$ is given by

$$\mathbf{E} = \mathbf{U}\mathbf{\Lambda}\mathbf{V}^{\mathrm{T}},$$

then the solutions for the rotation matrix are (using our notation):

$$\mathbf{R}_{\pm} = \mathbf{U}\begin{pmatrix} \pm\mathbf{J} & 0 \\ 0 & s \end{pmatrix}\mathbf{V}^{\mathrm{T}}, \tag{4.32}$$

where $s = \det \mathbf{U} \det \mathbf{V}$. They quote a theorem from matrix algebra which (in our language) states that the matrix $\mathbf{G}$ can be expressed as

$$\mathbf{G} = \mathbf{Q}^{\mathrm{T}}\begin{pmatrix} \mathbf{J} & 0 \\ 0 & 0 \end{pmatrix}\mathbf{Q}, \tag{4.33}$$

for some $\mathbf{Q} \in \mathrm{SO}(3)$. Comparing this with our expression (3.12), we see that while the matrix $\mathbf{Q}$ in [Tsai84] is known only implicitly, it is just the same as the transpose of our matrix $\mathbf{Q}$, which, as we showed, has an explicit physical interpretation as the rotation matrix which rotates the translation vector $\hat{\mathbf{t}}$ onto the $+Z$–axis, and is given explicitly by expression (3.8).

Tsai and Huang go on to make the following argument to show that only one of the expressions (4.32) is physically reasonable. Let (in our notation) the motion of the 3D points P_i (for $1 = 1, 2, \ldots, n$) be given by

$$M_1 \ : \ \mathbf{r}_i \longrightarrow \mathbf{r}'_i \, . \tag{4.34}$$

Then the image motion corresponding to M_1 will be the same as it is for the image motion M_2:

$$M_2 \ : \ \mathbf{r}_i \longrightarrow -\mathbf{r}'_i \, . \tag{4.35}$$

They then make the following statement (p.18 of [Tsai84]):

> Also, the rigidity constraint[s] obviously are not violated [for M_2] since this [$\mathbf{r}'_i \longrightarrow -\mathbf{r}'_i$] is just like rescaling the x, y, z axes by a common factor -1 without altering the right hand rule.

This is wrong. The transformation of an (arbitrary 3D) vector into its opposite is an inversion in the origin. *This is not a rigid motion* unless the points all lie in a plane containing the origin. If you "rescale" the xyz axes by a common factor of -1, then you end up with a left–handed coordinate system, which violates the "right hand rule":

$$\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}} \Longrightarrow (-\hat{\mathbf{x}}) \times (-\hat{\mathbf{y}}) = -(-\hat{\mathbf{z}}) \neq +(-\hat{\mathbf{z}}).$$

They continue with (*op. cit.*)

> Note that for $n \geq 2$, it takes more than just translational motion to move object points from (x'_i, y'_i, z'_i) to $(-x'_i, -y'_i, -z'_i)$ ...

This is true, but not, I think, in the way intended. If $n = 2$, then a rotation by 180° about the perpendicular to the plane containing the origin and the two points will be the indicated transformation. If $n \geq 3$, however, no rigid motion can invert the coordinates in the origin unless all the points lie in a plane containing the origin. The only operation which can do this is the inversion I discussed earlier, and in three dimensions, this inversion is not a rigid motion.

Continuing (*op. cit.*),

> Therefore there are at least two distinct solutions for $\mathbf{R}$, one for the motion $(\mathbf{r}_i \longrightarrow \mathbf{r}'_i)$ [i.e., M_1] and [an]other [for the motion] $(\mathbf{r}_i \longrightarrow -\mathbf{r}'_i)$ [i.e., M_2]. Since (35) [our equation (4.32)] says there are at most two possible solutions, it can be concluded that exactly one of the two solutions in [(4.32)] must correspond to the case when the object moves from the front to the back of the camera or vice–versa.

The point to be made here is that if M_1 is a rigid motion, then M_2 *cannot* be a rigid motion, and vice–versa. Furthermore, the two solutions (4.32) for $\mathbf{R}$ both have determinant $+1$. Consequently, neither can represent the motion M_2. As a consequence, the completeness argument of [Tsai84] is faulty: the two ambiguous solutions given by them cannot be discriminated on the basis that one satisfies the positivity condition while the other does not.

In [Long81], Longuet–Higgins finds a similar two–fold ambiguity in the expression for the rotation matrix, then states without proof that which is the "correct" motion can be determined because one of the ambiguous solutions will satisfy the positivity condition, while the other will not. As we show in Appendix B, this is wrong. We note that he did not actually write down the ambiguous solution, but its existence is implicit in his analysis. We can in fact show that the ambiguity found by Longuet–Higgins is exactly the "duality" ambiguity described in the present work. We suspect that the same is true of the ambiguity found by Tsai and Huang, though we have as yet no proof of this.

We now outline the technique developed by Longuet–Higgins. We have that $\mathbf{E} = \mathbf{GR}$. Let us represent the matrices $\mathbf{E}$ and $\mathbf{R}$ in terms of their columns as:

$$\mathbf{E} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) \; ; \quad \mathbf{R} = (\mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3),$$

where the $\mathbf{e}_i$ and the $\mathbf{R}_i$ are 3×1 column matrices. Then

$$\begin{aligned} (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = \mathbf{G}(\mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3) &= (\mathbf{GR}_1, \mathbf{GR}_2, \mathbf{GR}_3) \\ &= (\hat{\mathbf{t}} \times \mathbf{R}_1, \hat{\mathbf{t}} \times \mathbf{R}_2, \hat{\mathbf{t}} \times \mathbf{R}_3), \end{aligned}$$

i.e.,

$$\mathbf{e}_i = \hat{\mathbf{t}} \times \mathbf{R}_i. \tag{4.36}$$

This expression, incidentally, makes clear the physical origin of the singularity of $\mathbf{E}$: all the column vectors $\mathbf{e}_i$ which compose $\mathbf{E}$ are perpendicular to $\hat{\mathbf{t}}$, and hence must lie in the plane perpendicular to $\hat{\mathbf{t}}$. Since there are three such vectors, they are not linearly independent, and hence $\det \mathbf{E} = 0$. It follows from (4.36) that $\mathbf{R}_i$ and $\mathbf{e}_i$ are orthogonal. Hence, $\mathbf{R}_i$ must be in the plane spanned by $\hat{\mathbf{t}}$ and $\mathbf{e}_i \times \hat{\mathbf{t}}$:

$$\mathbf{R}_i = a_i\hat{\mathbf{t}} + b_i\mathbf{e}_i \times \hat{\mathbf{t}} \qquad \text{(no sum)}$$

(we recall that writing "no sum" means that repeated indices are *not* to be summed over). Substitution of this equation into (4.36) then gives

$$\begin{aligned} \mathbf{e}_i = \hat{\mathbf{t}} \times (a_i\hat{\mathbf{t}} + b_i\mathbf{e}_i \times \hat{\mathbf{t}}) &= b_i\hat{\mathbf{t}} \times (\mathbf{e}_i \times \hat{\mathbf{t}}) = b_i\mathbf{e}_i \quad \text{(no sum)} \\ \Longrightarrow \quad & b_i = 1, \quad i = 1,2,3. \end{aligned}$$

We then define (after Longuet–Higgins)

$$\mathbf{W}_i = \mathbf{e}_i \times \hat{\mathbf{t}},$$

so that

$$\mathbf{R}_i = a_i\hat{\mathbf{t}} + \mathbf{W}_i.$$

It is easy to see that the column vector $\mathbf{R}_i$ is just the rotation (by $\mathbf{R}$) of the i^{th} unit basis vector of $\mathcal{R}^3$. Hence, the set $\{\mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3\}$ must be an orthonormal triad of basis vectors (with positive orientation), i.e.,

$$\mathbf{R}_i = \mathbf{R}_j \times \mathbf{R}_k, \qquad (ijk) \text{ a cyclic permutation of } (123).$$

Hence,

$$\begin{aligned} a_i\hat{\mathbf{t}} + \mathbf{W}_i &= (a_j\hat{\mathbf{t}} + \mathbf{W}_j) \times (a_k\hat{\mathbf{t}} + \mathbf{W}_k) \\ &= a_k\mathbf{W}_j \times \hat{\mathbf{t}} - a_j\mathbf{W}_k \times \hat{\mathbf{t}} + \mathbf{W}_j \times \mathbf{W}_k. \end{aligned}$$

But the $\mathbf{W}_m$ $(m = 1,2,3)$ are all perpendicular to $\hat{\mathbf{t}}$, so that $\mathbf{W}_j \times \mathbf{W}_k$ is parallel to $\hat{\mathbf{t}}$, and $\mathbf{W}_m \times \hat{\mathbf{t}}$ is (of course) perpendicular to $\hat{\mathbf{t}}$. Thus, equating components of the above equation parallel and perpendicular to $\hat{\mathbf{t}}$, we find

$$\begin{aligned} \parallel \hat{\mathbf{t}} \ :& \quad a_i \hat{\mathbf{t}} &= \mathbf{W}_j \times \mathbf{W}_k \\ \perp \hat{\mathbf{t}} \ :& \quad \mathbf{W}_i &= a_k \mathbf{W}_j \times \hat{\mathbf{t}} - a_j \mathbf{W}_k \times \hat{\mathbf{t}} \end{aligned}$$

Hence, we obtain Longuet–Higgins's expression for the columns of the rotation matrix $\mathbf{R}$:

$$\mathbf{R}_i = \mathbf{W}_i + \mathbf{W}_j \times \mathbf{W}_k,$$

where (ijk) is a cyclic permutation of (123).

Now this has been obtained for a particular value of $\hat{\mathbf{t}}$. As noted previously, only the axis of $\hat{\mathbf{t}}$ can be determined from $\mathbf{E}$, so there is an ambiguity (the duality ambiguity) in the direction of $\hat{\mathbf{t}}$ along this axis. If we let $\hat{\mathbf{t}}' = -\hat{\mathbf{t}}$, then the corresponding expression for the columns $\mathbf{R}'_i$ of $\mathbf{R}'$ will be just

$$\mathbf{R}'_i = \mathbf{W}'_i + \mathbf{W}'_j \times \mathbf{W}'_k,$$

where

$$\begin{aligned} \mathbf{W}'_i &= \mathbf{e}_i \times \hat{\mathbf{t}}' \equiv \mathbf{e}_i \times (-\hat{\mathbf{t}}) = -\mathbf{e}_i \times \hat{\mathbf{t}} \\ &\equiv -\mathbf{W}_i. \end{aligned}$$

Therefore,

$$\mathbf{R}'_i = -\mathbf{W}_i + \mathbf{W}_j \times \mathbf{W}_k,$$

We therefore, in Longuet–Higgins's approach, find the two ambiguous solutions:

$$\left\{ \ \hat{\mathbf{t}}, \ \mathbf{R} = (\mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3) \ \ | \ \ \mathbf{R}_i = \mathbf{W}_i + \mathbf{W}_j \times \mathbf{W}_k \ \right\}$$

and

$$\left\{-\hat{\mathbf{t}},\ \mathbf{R}' = (\mathbf{R}'_1, \mathbf{R}'_2, \mathbf{R}'_3) \quad | \quad \mathbf{R}'_i = -\mathbf{W}_i + \mathbf{W}_j \times \mathbf{W}_k \right\},$$

where $\mathbf{W}_i = \mathbf{e}_i \times \hat{\mathbf{t}}$. We want to show that the latter is just the dual of the former. This is easy to do: we need to show that $\mathbf{R}' = \mathbf{R}_0\mathbf{R}$, where, as before, $\mathbf{R}_0 = \exp(\pi\hat{\mathbf{t}} \cdot \mathbf{J})$ represents a rotation by π around the direction of $\hat{\mathbf{t}}$. But since each of the $\mathbf{W}_m$ is perpendicular to $\hat{\mathbf{t}}$, it follows that $\mathbf{R}_0\mathbf{W}_m = -\mathbf{W}_m$. Hence,

$$\begin{aligned} \mathbf{R}_0\mathbf{R}_i &= (-\mathbf{W}_i) + (-\mathbf{W}_j) \times (-\mathbf{W}_k) \\ &= -\mathbf{W}_i + \mathbf{W}_j \times \mathbf{W}_k \\ &= \mathbf{R}'_i \qquad \text{Q.E.D.} \end{aligned}$$

Hence, the ambiguity implicit in Longuet–Higgins's paper is just the same as the one we discuss in the present work.

After the completion of this work, I became aware that what I have called the "duality ambiguity" has been previously discussed (using quite different methods) by Horn [Horn87]. I thank Harpreet Sawhney for bringing this paper to my attention. We note, however, that Horn does not present a proof that this is the only possible ambiguity, as we have done.

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References

[Faug87] O. D. Faugeras, F. Lustmann, and G. Toscani, "Motion and Structure from

Motion from Point and Line Matches," Proc. 1st Int. Conf. Comp. Vis., pp. 25–34, London, England (June 1987).

[Faug89] O. D. Faugeras and S. Maybank, "Motion from Point Matches: Multiplicity of Solutions," Proc. IEEE Workshop on Visual Motion, pp. 248–255, Irvine, Calif. (March 1989).

[Gilm74] R. Gilmore, *Lie Groups, Lie Algebras, and some of their Applications*, Wiley and Sons, N. Y. (1974).

[Gel'f63] I. M. Gel'fand, R. A. Minlos, and Z. Y. Shapiro, *Representations of the Rotation and Lorentz Groups and some of their Applications*, Pergamon, N. Y. (1963).

[Horn87] B. K. P. Horn, "Relative Orientation," MIT Tech. Report (1987).

[Long81] H. C. Longuet–Higgins, "A Computer Algorithm for Reconstructing a Scene from Two Projections," *Nature* **293**, pp. 133–135 (Sept. 1981).

[Snyd89] M. A. Snyder, "On the Calculation of Rigid Motion Parameters from the Essential Matrix," COINS Tech. Rep.89–102, Univ. Mass., Amherst (Oct. 1989).

[Tsai84] R. Y. Tsai and T. S. Huang, "Uniqueness and Estimation of Three–Dimensional Motion Parameters of Rigid Objects with Curved Surfaces," IEEE Trans. Patt. Anal. Mach. Intel., **PAMI–6**, No. 1, pp. 13–27 (Jan. 1984).

A The Lie Group SO(3)

In this Appendix, we briefly discuss the mathematics of the rotation group. We do not aspire to the empyrean heights of mathematical rigor, but rather give an informal exposition of the relevant facts. A general reference for this section is the book of Gilmore [Gilm74]. The reader interested in a rigorous mathematical treatment can refer to a manifold of books on the subject, e.g., the monograph of Gel'fand, et al. [Gel'f63].

The set of all linear transformations of three–dimensional real space $\mathcal{R}^3$ which leaves both the lengths of vectors and the position of the origin fixed can be represented as follows. Let $\mathbf{r} \in \mathcal{R}^3$. Then the most general linear transformation of $\mathcal{R}^3$ which leaves the origin invariant is given by

$$\mathbf{r} \longrightarrow \mathbf{r}' = \mathbf{R}\mathbf{r},$$

where $\mathbf{R}$ is a 3×3 matrix. Clearly, since $\mathbf{r}$ is real, so must be $\mathbf{R}$. If this transformation also leaves the length of $\mathbf{r}$ invariant, then

$$\mathbf{r}^{\mathrm{T}}\mathbf{r} \equiv (\mathbf{r}')^{\mathrm{T}}\mathbf{r}' = \mathbf{r}^{\mathrm{T}}\mathbf{R}^{\mathrm{T}}\mathbf{R}\mathbf{r}.$$

For this to hold for all vectors $\mathbf{r}$, it is necessary and sufficient that

$$\mathbf{R}^{\mathrm{T}}\mathbf{R} = \mathbf{1}_3 \qquad \text{(A.37)}$$

That is, $\mathbf{R}$ is an *orthogonal* matrix. The set of all such real 3×3 matrices evidently has the structure of a group, which is called O(3), for O(rthogonal) group in 3 dimensions.

If we take the determinant of both sides of the relation (A.37), we see that

$$1 = \det \mathbf{R}^{\mathrm{T}} \ \det \mathbf{R} = (\det \mathbf{R})^2 \Longrightarrow \det \mathbf{R} = \pm 1.$$

It therefore follows that O(3) is not a connected group, but consists of two disconnected pieces: $O(3) = O^{+}(3) \bigcup O^{-}(3)$, where

$$O^{\pm}(3) = \{\mathbf{R} \in O(3) \mid \det \mathbf{R} = \pm 1\}.$$

We see immediately that $O^{-}(3)$ is not a group, since it neither possesses an identity element, nor is it closed under the group composition law. The piece $O^{+}(3)$ of O(3), however, is a group, called the (proper) rotation group, denoted by the symbol SO(3), which stands for S(pecial) O(rthogonal) group in 3 dimensions. It is easy to convince oneself that the elements of SO(3) behave in the way we would expect rotations to behave, but we will have to wait until we prove the relation (A.45) to make an explicit identification of SO(3) with the group of rotations in three dimensional space.

We note that an arbitrary element $g \in O^{-}(3)$ can be written as

$$g = \mathbf{IR},$$

where $\mathbf{R} \in SO(3)$. Here **I** is the *inversion* (or *parity*) operation which reverses the direction of each vector in $\mathcal{R}^3$:

$$\mathbf{I} \; : \; \mathbf{r} \longrightarrow \mathbf{r}' \equiv \mathbf{Ir} \equiv -\mathbf{r}.$$

That is, **I** is represented by the matrix $-\mathbf{1}_3$. The elements of $O^{-}(3)$ are sometimes called *improper* rotations, which is an oxymoron: an improper rotation is not a rotation. We note, therefore, that an element of O(3) is in $O^{+}(3)$ ($O^{-}(3)$) if it doesn't (does) reverse the orientation of the coordinate axes. The group SO(3) can therefore be defined as the group of orientation–preserving orthogonal transformations of $\mathcal{R}^3$.

A Lie group has both algebraic structure as a group, and topological structure as a manifold. The group manifold of SO(3), for instance, is the (solid) three–dimensional sphere of radius π, with antipodal points identified. This group manifold is therefore doubly–connected. Considering a Lie group as a manifold means that it makes sense to talk about the tangent space to the group at some element of the group (more precisely, the tangent bundle of the group). This tangent space is called the Lie Algebra of the corresponding group. Speaking loosely, a Lie algebra is a vector space V, together with an algebraic structure which is inherited from the associated group. This structure is reflected in the existence of a binary antisymmetric composition law for associating any two elements of the Lie algebra so as to produce a third element of the Lie algebra. If we let v_1 and v_2 be in V, then the composition law is defined as the *Lie product*

$$v_1 \circ v_2 \equiv [v_1, v_2] = -[v_2, v_1] = -v_2 \circ v_1. \tag{A.38}$$

The product is also required to satisfy the Jacobi Identity:

$$[v_1, [v_2, v_3]] + [v_2, [v_3, v_1]] + [v_3, [v_1, v_2]] = 0, \tag{A.39}$$

where $v_i \in V$. A Lie algebra is thus a generalization of the mathematical structure imposed on $\mathcal{R}^3$ by associating with any two vectors their cross–product (the skeptical reader can easily check that defining the Lie product to be $v_1 \circ v_2 = [v_1, v_2] \equiv v_1 \times v_2$, where v_1 and v_2 are vectors in $\mathcal{R}^3$, satisfies (A.38) and (A.39)).

The relationship between a Lie group and its corresponding Lie algebra is most easily explained by assuming that the Lie algebra is represented as a vector space of matrices (this is always possible). The Lie product is then just the ordinary commutator of the two matrices:

$$\mathbf{M} \circ \mathbf{N} = [\mathbf{M}, \mathbf{N}] \equiv \mathbf{MN} - \mathbf{NM}.$$

We can then find the 3×3 real matrices which represent the Lie algebra *so(3)* of SO(3). Since the Lie algebra of SO(3) is just the tangent space to SO(3) at, say, the identity, it follows that an element $\mathbf{K}$ of SO(3) infinitesimally close to the identity can be written as

$$\mathbf{K} \equiv \mathbf{1}_3 + \lambda \mathbf{F} + O(\lambda^2),$$

where λ is infinitesimal, and $\lambda \mathbf{F}$ (and hence $\mathbf{F}$) is an element of *so(3)*. Consequently,

$$\begin{aligned} \mathbf{1}_3 \equiv \mathbf{K}\mathbf{K}^{\mathrm{T}} &= \left(\mathbf{1}_3 + \lambda \mathbf{F} + O(\lambda^2)\right)\left(\mathbf{1}_3 + \lambda \mathbf{F}^{\mathrm{T}} + O(\lambda^2)\right) \\ &= \mathbf{1}_3 + \lambda\left(\mathbf{F} + \mathbf{F}^{\mathrm{T}}\right) + O(\lambda^2). \end{aligned}$$

This implies that

$$\mathbf{F} + \mathbf{F}^{\mathrm{T}} = 0 \Longrightarrow \mathbf{F}^{\mathrm{T}} = -\mathbf{F}.$$

That is, the Lie algebra of SO(3) consists of real 3×3 antisymmetric matrices.

For compact groups such as SO(3) the converse of the result that the Lie algebra is the tangent space to the group is that the Lie group can be obtained by "exponentiating" the Lie algebra. That is, an arbitrary element g of a (compact) Lie group can be obtained by exponentiating some element $\mathbf{M}$ of the Lie algebra:

$$g = \exp \mathbf{M} \equiv \sum_{n=0}^{\infty} \frac{\mathbf{M}^n}{n!} = \mathbf{1} + \mathbf{M} + \frac{\mathbf{M}^2}{2!} + \dots \quad .$$

This means that an arbitrary element $\mathbf{R}$ of SO(3) can be written as the exponential of an antisymmetric matrix:

$$\mathbf{R} \in \mathrm{SO}(3) \Longrightarrow \mathbf{R} = \exp \mathbf{F} \quad ; \mathbf{F}^{\mathrm{T}} = -\mathbf{F}. \tag{A.40}$$

It is easy to show, using the relation

$$\det(\exp \mathbf{F}) = \exp(\operatorname{tr} \mathbf{F}),$$

that a matrix of the forn. (A.40) satisfies $\det \mathbf{R} = \det(\exp \mathbf{F}) = 1$, since an antisymmetric matrix has 0's on its diagonal.

It is clear that the algebraic structure of the Lie algebra is determined by the Lie product of the basis vectors of the Lie algebra (considered as a vector space). Since for SO(3) the Lie algebra consists of all real 3×3 antisymmetric matrices, a basis for the Lie algebra can be chosen to be the three real antisymmetric matrices $\mathbf{J}_i\ ; i = 1, 2, 3$:

$$\mathbf{J}_1 \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\ ; \mathbf{J}_2 \equiv \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}\ ; \mathbf{J}_3 \equiv \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

i.e., the (j,k) matrix element of the matrix j_i is given by

$$(\mathbf{J}_i)_{jk} = -\epsilon_{ijk}.$$

The $\mathbf{J}_i$'s are called the *generators* of SO(3). Here the *permutation symbol* ϵ_{ijk} is the completely antisymmetric third rank tensor. That is, it is equal to $+1$ if (ijk) is a cyclic permutation of (123), it is odd under the interchange of any two indices, and (hence) it vanishes unless i, j, and k are all different. A very useful property of the permutation symbol which follows immediately from its definition is that

$$\sum_{m=1}^{3} \epsilon_{mjk}\epsilon_{mrs} \equiv \epsilon_{mjk}\epsilon_{mrs} = \delta_{jr}\delta_{ks} - \delta_{js}\delta_{kr}. \tag{A.41}$$

Here δ_{ij} is the Kronecker delta function, which equals 1 if $i = j$, and 0 otherwise, and we use "summation convention"—whenever an index is repeated in a product (such as m above), it is understood to be summed over.

It is easy to check that the $\mathbf{J}_i$'s satisfy the following *commutation relations*:

$$[\mathbf{J}_i, \mathbf{J}_j] = \epsilon_{ijk}\mathbf{J}_k \quad \left(\equiv \sum_{k=1}^{3} \epsilon_{ijk}\mathbf{J}_k\right)$$

The ϵ_{ijk} are called the *structure constants* of *so(3)*.

Since *so(3)* is a vector space, a general element Θ can be written as a linear combination of the generators $\mathbf{J}_i$:

$$\Theta = \boldsymbol{\theta} \cdot \mathbf{J} = \theta_k \mathbf{J}_k = \theta_1 \mathbf{J}_1 + \theta_2 \mathbf{J}_2 + \theta_3 \mathbf{J}_3 = \begin{pmatrix} 0 & -\theta_3 & \theta_2 \\ \theta_3 & 0 & -\theta_1 \\ -\theta_2 & \theta_1 & 0 \end{pmatrix}.$$

We will also use the notation

$$\Theta = \theta\, \mathbf{Z},$$

where

$$\begin{aligned} \mathbf{Z} &= \hat{\boldsymbol{\theta}} \cdot \mathbf{J} \\ \theta &= \|\boldsymbol{\theta}\| \\ \hat{\boldsymbol{\theta}} &= \frac{\boldsymbol{\theta}}{\theta} \end{aligned}$$

It is easy to show from (A.41) that $\mathbf{Z}$ satisfies

$$\mathbf{Z}^3 = -\mathbf{Z}, \tag{A.42}$$

and hence that the general element of SO(3) can be written as

$$\mathbf{R}(\theta) \;=\; \exp(\boldsymbol{\theta} \cdot \mathbf{J}) = \exp(\theta \mathbf{Z})$$

$$= \mathbf{1}_3 + \sin\theta\, \mathbf{Z} + (1-\cos\theta)\, \mathbf{Z}^2. \tag{A.43}$$

This is easily derived by using the Taylor series for the exponential, sine, and cosine functions. We will see next that this expression for an element of SO(3) is equivalent to the well known Rodrigues formula; however, the expression above is more meaningful from a group–theoretical standpoint.

We note that the effect of $\mathbf{Z}$ on a vector $\mathbf{A}$ in $\mathcal{R}^3$ is to produce a vector perpendicular to both $\widehat{\boldsymbol{\theta}}$ and $\mathbf{A}$. To whit:

$$\begin{aligned} (\mathbf{ZA})_i &= \mathbf{Z}_{ir}\mathbf{A}_r = \left(\mathbf{J}_k\widehat{\boldsymbol{\theta}}_k\right)_{ir}\mathbf{A}_r \\ &= -\widehat{\boldsymbol{\theta}}_k\epsilon_{kir}\mathbf{A}_r = \epsilon_{ikr}\widehat{\boldsymbol{\theta}}_k\mathbf{A}_r \\ &= \left(\widehat{\boldsymbol{\theta}}\times\mathbf{A}\right)_i. \end{aligned}$$

That is,

$$\begin{aligned} \mathbf{ZA} &= \widehat{\boldsymbol{\theta}}\times\mathbf{A} \qquad (A.44) \\ \Longrightarrow \mathbf{Z}^2\mathbf{A} &= \widehat{\boldsymbol{\theta}}\times\left(\widehat{\boldsymbol{\theta}}\times\mathbf{A}\right). \end{aligned}$$

Hence,

$$\mathbf{R}(\boldsymbol{\theta})\,\mathbf{A} = \mathbf{A} + \sin\theta\,\widehat{\boldsymbol{\theta}}\times\mathbf{A} + (1-\cos\theta)\,\widehat{\boldsymbol{\theta}}\times\left(\widehat{\boldsymbol{\theta}}\times\mathbf{A}\right).$$

By using

$$\widehat{\boldsymbol{\theta}}\times\left(\widehat{\boldsymbol{\theta}}\times\mathbf{A}\right) = \widehat{\boldsymbol{\theta}}\left(\widehat{\boldsymbol{\theta}}\cdot\mathbf{A}\right) - \mathbf{A},$$

it is easy to show that this is equivalent to Rodrigues' formula for the effect on a vector $\mathbf{A}$ of a rotation by $\boldsymbol{\theta}$:

$$\mathbf{R}(\boldsymbol{\theta})\,\mathbf{A} = \mathbf{A}\cos\theta + \sin\theta\,\widehat{\boldsymbol{\theta}}\times\mathbf{A} + (1-\cos\theta)\,\widehat{\boldsymbol{\theta}}\left(\widehat{\boldsymbol{\theta}}\cdot\mathbf{A}\right). \tag{A.45}$$

This establishes explicitly the equivalence of SO(3) and the rotation group. Therefore, the expression (A.43) for $\mathbf{R}(\boldsymbol{\theta})$ describes a rotation by an angle θ around the axis given by $\boldsymbol{\theta}$.

It is then easy to verify the following identities:

$$\begin{aligned} (\mathbf{A}\cdot\mathbf{J})(\mathbf{B}\cdot\mathbf{J}) &= \mathbf{B}\mathbf{A}^{\mathrm{T}} - \left(\mathbf{A}^{\mathrm{T}}\mathbf{B}\right)\mathbf{1}_3, &\text{(A.46)}\\ [\mathbf{A}\cdot\mathbf{J},\mathbf{B}\cdot\mathbf{J}] &= \mathbf{B}\mathbf{A}^{\mathrm{T}} - \mathbf{A}\mathbf{B}^{\mathrm{T}} \\ &= (\mathbf{A}\times\mathbf{B})\cdot\mathbf{J}, &\text{(A.47)}\end{aligned}$$

where $\mathbf{A}$ and $\mathbf{B}$ are any vectors in $\mathcal{R}^3$.

Since SO(3) is the group of rotations in three–dimensional space, and since the Lie algebra of SO(3) is a three–dimensional space (albeit with addtional structure), it is clear that we can take SO(3) to act on its own Lie algebra. Such an action is called the *adjoint* or *regular* representation of SO(3) (on *so(3)*). We take an element of *so(3)* to be the "dot product" $\mathbf{A}\cdot\mathbf{J}$. The vector $\mathbf{A}$ must transform as follows under the action of an element $\mathbf{R}$ of SO(3):

$$\mathbf{R} \,:\, \mathbf{A} \longrightarrow \mathbf{R}\mathbf{A}.$$

But the action of $\mathbf{R}$ on the Lie algebra element $\mathbf{A}\cdot\mathbf{J}$ must be consistent with this action, i.e.,

$$\mathbf{R} \,:\, \mathbf{A}\cdot\mathbf{J} \longrightarrow (\mathbf{R}\mathbf{A})\cdot\mathbf{J}.$$

We will now show that

$$\mathbf{R}(\mathbf{A}\cdot\mathbf{J})\mathbf{R}^{\mathrm{T}} = (\mathbf{R}\mathbf{A})\cdot\mathbf{J}. \tag{A.48}$$

That is, the adjoint representation of SO(3) on *so(3)* is given by

$$\mathbf{R} \,:\, \mathbf{A}\cdot\mathbf{J} \longrightarrow \mathbf{R}(\mathbf{A}\cdot\mathbf{J})\mathbf{R}^{\mathrm{T}}.$$

We have that $\mathbf{R}$ can be expressed as $\exp(\theta\mathbf{Z})$, for some angle $\boldsymbol{\theta}$:

$$\mathbf{R}\,(\mathbf{A}\cdot\mathbf{J})\,\mathbf{R}^{\mathrm{T}} = \exp(\theta\mathbf{Z})\,(\mathbf{A}\cdot\mathbf{J})\exp(-\theta\mathbf{Z})\,.$$

But it can be shown by induction that

$$e^{\mathbf{M}}\,\mathbf{N}\,e^{-\mathbf{M}} = \sum_{n=0}^{\infty}\frac{[\mathbf{M},\mathbf{N}]_n}{n!},$$

where the Pochhammer symbol $[\mathbf{M},\mathbf{N}]_n$ is defined recursively as

$$\begin{aligned}[\mathbf{M},\mathbf{N}]_0 &\equiv \mathbf{N},\\ [\mathbf{M},\mathbf{N}]_{n+1} &= [\mathbf{M},[\mathbf{M},\mathbf{N}]_n]\end{aligned}$$

i.e.,

$$\begin{aligned}[\mathbf{M},\mathbf{N}]_1 &\equiv [\mathbf{M},\mathbf{N}] = \mathbf{MN}-\mathbf{NM}\\ [\mathbf{M},\mathbf{N}]_2 &= [\mathbf{M},[\mathbf{M},\mathbf{N}]]\\ [\mathbf{M},\mathbf{N}]_3 &= [\mathbf{M},[\mathbf{M},[\mathbf{M},\mathbf{N}]]]\,,\ \text{etc.}\end{aligned}$$

Now, let $\mathbf{M} = \theta\,\mathbf{Z}$ and $\mathbf{N} = \mathbf{A}\cdot\mathbf{J}$. Then, clearly,

$$[\mathbf{M},\mathbf{N}]_n = \theta^n\,[\mathbf{Z},\mathbf{A}\cdot\mathbf{J}]_n\,.$$

Noting that

$$[\mathbf{Z},\mathbf{B}\cdot\mathbf{J}] = \left[\hat{\boldsymbol{\theta}}\cdot\mathbf{J},\mathbf{B}\cdot\mathbf{J}\right] = \left(\hat{\boldsymbol{\theta}}\times\mathbf{B}\right)\cdot\mathbf{J} = (\mathbf{ZB})\cdot\mathbf{J}, \tag{A.49}$$

we see that

$$[\mathbf{Z},\mathbf{A}\cdot\mathbf{J}]_0 \equiv \mathbf{A}\cdot\mathbf{J}$$

$$
\begin{aligned}
[\mathbf{Z},\mathbf{A}\cdot\mathbf{J}]_1 &= (\mathbf{ZA})\cdot\mathbf{J} \\
[\mathbf{Z},\mathbf{A}\cdot\mathbf{J}]_2 &= [\mathbf{Z},[\mathbf{Z},\mathbf{A}\cdot\mathbf{J}]_1] = [\mathbf{Z},(\mathbf{ZA})\cdot\mathbf{J}] \\
&= \left(\mathbf{Z}^2\mathbf{A}\right)\cdot\mathbf{J},
\end{aligned}
$$

where we have used (A.49) in the last equality. Consequently, by induction:

$$
[\mathbf{Z},\mathbf{A}\cdot\mathbf{J}]_m = (\mathbf{Z}^m\mathbf{A})\cdot\mathbf{J} \quad (m \neq 0).
$$

But $\mathbf{Z}^3 = -\mathbf{Z}$, so that

$$
\begin{aligned}
\mathbf{Z}^{2m} &= (-1)^{m+1}\mathbf{Z}^2 \quad (m > 0) \\
\mathbf{Z}^{2m+1} &= (-1)^m\,\mathbf{Z} \qquad (m \geq 0).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
[\mathbf{Z},\mathbf{A}\cdot\mathbf{J}]_{2m} &= (-1)^{m+1}\left(\mathbf{Z}^2\mathbf{A}\right)\cdot\mathbf{J} \quad (m > 0) \\
[\mathbf{Z},\mathbf{A}\cdot\mathbf{J}]_{2m+1} &= (-1)^m\,(\mathbf{ZA})\cdot\mathbf{J} \qquad (m \geq 0).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
e^{\theta\mathbf{Z}}\,(\mathbf{A}\cdot\mathbf{J})\,e^{-\theta\mathbf{Z}} &= \sum_{m=0}^{\infty}\theta^m\,\frac{[\mathbf{Z},\mathbf{A}\cdot\mathbf{J}]}{m!} \\
&= \mathbf{A}\cdot\mathbf{J} + \sum_{m=0}^{\infty}\theta^{2m+1}\frac{[\mathbf{Z},\mathbf{A}\cdot\mathbf{J}]_{2m+1}}{(2m+1)!} + \sum_{m=1}^{\infty}\theta^{2m}\frac{[\mathbf{Z},\mathbf{A}\cdot\mathbf{J}]_{2m}}{(2m)!} \\
&= \mathbf{A}\cdot\mathbf{J} + \sum_{m=0}^{\infty}\theta^{2m+1}(-1)^m\frac{(\mathbf{ZA})\cdot\mathbf{J}}{(2m+1)!} + \sum_{m=1}^{\infty}\theta^{2m}(-1)^{m+1}\frac{(\mathbf{Z}^2\mathbf{A})\cdot\mathbf{J}}{(2m)!} \\
&= \mathbf{A}\cdot\mathbf{J} + (\mathbf{ZA})\cdot\mathbf{J}\sum_{m=0}^{\infty}(-1)^m\frac{\theta^{2m+1}}{(2m+1)!} - \left(\mathbf{Z}^2\mathbf{A}\right)\cdot\mathbf{J}\sum_{m=1}^{\infty}(-1)^m\frac{\theta^{2m}}{(2m)!}
\end{aligned}
$$

$$\begin{aligned} &= \mathbf{A}\cdot\mathbf{J} + ((\mathbf{Z}\mathbf{A})\cdot\mathbf{J})\sin\theta + \left(\left(\mathbf{Z}^2\mathbf{A}\right)\cdot\mathbf{J}\right)(1-\cos\theta) \\ &= \left[\left(\mathbf{1}_3 + \sin\theta\,\mathbf{Z} + (1-\cos\theta)\mathbf{Z}^2\right)\mathbf{A}\right]\cdot\mathbf{J} \\ &\equiv (\mathbf{R}\mathbf{A})\cdot\mathbf{J}, \end{aligned}$$

where we used the definition (A.43). **Q. E. D.**

B The Ubiquity of the Ambiguity

The purpose of this appendix is to show that the ambiguity between $(\mathbf{R}, \hat{\mathbf{t}})$ and $(\mathbf{R}_0\mathbf{R}, -\hat{\mathbf{t}})$ is generally present.

We begin with a simple example, which serves as a counterexample (only one is necessary) to the arguments of Tsai and Huang [Tsai84] and Longuet–Higgins [Long81] for the uniqueness of the motion parameters associated with $\mathbf{E}$.

Let the rotation and translation be given by

$$\begin{aligned} \mathbf{R} &= \{\text{rotation by } \theta \text{ around } +X\text{–axis}\} = \exp(\theta\mathbf{J}_1)\,; \\ \hat{\mathbf{t}} &= \{\text{translation along } +Z\text{–axis}\} = (0,0,1)^{\mathrm{T}} \end{aligned}$$

We assume that $\mathbf{E}$ is given by $(\hat{\mathbf{t}}\cdot\mathbf{J})\mathbf{R}$ using the above, and that the "correct" $\hat{\mathbf{t}}$ has been chosen. Then $\mathbf{Q}\equiv\mathbf{1}_3$, and

$$\mathbf{E} = \begin{pmatrix} \mathbf{E}_0 & \mathbf{e}_1 \\ 0 & 0 \end{pmatrix}\ ;\ \mathbf{E}_0 = \begin{pmatrix} 0 & -\cos\theta \\ 1 & 0 \end{pmatrix}\ ;\ \mathbf{e}_1 = (\sin\theta, 0)^{\mathrm{T}}\,.$$

One then finds that

$$\mathbf{R}_{\pm} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \pm\sin\theta & \pm\cos\theta \end{pmatrix}.$$

Since for this matrix $\det \mathbf{R}_{\pm} = \pm 1$, we must choose the upper sign: $\mathbf{R} = \mathbf{R}_{+}$. The matrix $\mathbf{R}_0$ is just

$$\mathbf{R}_0 = \exp(\pi \mathbf{G}) = \begin{pmatrix} -\mathbf{1}_2 & 0 \\ 0 & 1 \end{pmatrix},$$

so that

$$\mathbf{R}_0\mathbf{R} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -\cos\theta & \sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}.$$

The reader may easily check that this is a rotation by π around the axis $\left(0, \sin\frac{\theta}{2}, \cos\frac{\theta}{2}\right)$.

The ambiguous motion pairs are therefore:

$$\left(\mathbf{R}, \hat{\mathbf{t}}\right) = \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}, (0,0,1) \right) \quad ; \tag{B.50}$$

and the dual motion

$$\left(\mathbf{R}_0\mathbf{R}, -\hat{\mathbf{t}}\right) = \left(\begin{pmatrix} -1 & 0 & 0 \\ 0 & -\cos\theta & \sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}, (0,0,-1) \right) \quad . \tag{B.51}$$

We now let

$$\begin{aligned} \mathbf{r}_1' &= (X_1', Y_1', Z_1')^{\mathrm{T}} \equiv \mathbf{R}\mathbf{r} + \hat{\mathbf{t}}, \\ \mathbf{r}_2' &= (X_2', Y_2', Z_2')^{\mathrm{T}} \equiv \mathbf{R}_0\mathbf{R}\mathbf{r} - \hat{\mathbf{t}}, \end{aligned}$$

where $\mathbf{r} = (X, Y, Z)^{\mathrm{T}}$. We find that

$$X_1' = X = -X_2'$$

$$Y_1' = Y\cos\theta - Z\sin\theta = -Y_2'$$
$$Z_1' = Y\sin\theta + Z\cos\theta + 1$$
$$Z_2' = Y\sin\theta + Z\cos\theta - 1.$$

It is clearly possible for Z, Z_1', and Z_2' to all be positive, i.e., the points $\mathbf{r}$, $\mathbf{r}_1'$, and $\mathbf{r}_2'$ are all in the forward hemisphere. This contradicts the assertion of [Tsai84] and [Long81] that such a situation is impossible.

We now show that the ambiguity is generally present. We do this by showing that except for one case, when the translation is in the image plane (which includes the case of stereo), the set of points which can lead to the discovered ambiguity is "large," in the sense that it is not a set of measure zero in $\mathcal{R}^3$.

Consider the effect of the two ambiguous rigid motions on a point $\rho \in \mathcal{R}^3$:

$$\left(\mathbf{R}, \hat{\mathbf{t}}\right) \; : \; \rho \longrightarrow \mathbf{r} \; \equiv \mathbf{R}\rho + \hat{\mathbf{t}} \tag{B.52}$$

$$\left(\mathbf{R}_0\mathbf{R}, -\hat{\mathbf{t}}\right) \; : \; \rho \longrightarrow \mathbf{r}' \; \equiv \mathbf{R}_0\mathbf{R}\rho - \hat{\mathbf{t}} \tag{B.53}$$

(note that $\mathbf{r}$ no longer denotes the initial point!). From (B.52), it follows that

$$\mathbf{R}\rho = \mathbf{r} - \hat{\mathbf{t}}, \tag{B.54}$$

and hence

$$\mathbf{r}' = \mathbf{R}_0\mathbf{r} - 2\hat{\mathbf{t}}, \tag{B.55}$$

where we have used the fact that $\mathbf{R}_0$ is in the little group of $\hat{\mathbf{t}}$. We want to find the conditions under which ρ, $\mathbf{r}$, and $\mathbf{r}'$ can be imaged by the camera, i.e., they are all in the forward hemisphere. Since at least one of these rigid motions must have the initial and final points in the forward hemisphere, we can assume[2] that it is $\left(\mathbf{R}, \hat{\mathbf{t}}\right)$. We want to find the conditions under which $\mathbf{r}'$ is in the forward hemisphere.

We address this issue as follows. We have that

$$\mathbf{R}_0 = \mathbf{1}_3 + 2\mathbf{G}^2 = 2\hat{\mathbf{t}}\hat{\mathbf{t}}^{\mathrm{T}} - \mathbf{1}_3.$$

Hence,

$$\mathbf{R}_0\mathbf{r} = 2\hat{\mathbf{t}}\left(\mathbf{r}\cdot\hat{\mathbf{t}}\right) - \mathbf{r},$$

so that (using (B.55))

$$\mathbf{r}' = 2\hat{\mathbf{t}}\left(\mathbf{r}\cdot\hat{\mathbf{t}}\right) - \mathbf{r} - 2\hat{\mathbf{t}}.$$

Taking the Z–component of this vector equation, we find that

$$\begin{aligned}(\mathbf{r}')_3 \equiv Z' &= 2t_3\left(\mathbf{r}\cdot\hat{\mathbf{t}}\right) - (\mathbf{r})_3 - 2t_3\\ &= 2t_3\left(\mathbf{r}\cdot\hat{\mathbf{t}}\right) - \hat{\boldsymbol{\zeta}}\cdot\mathbf{r} - 2t_3,\end{aligned}$$

where we recall that $\hat{\boldsymbol{\zeta}} = (0,0,1)^{\mathrm{T}}$. We can therefore write

$$Z' = \hat{\mathbf{k}}\cdot\mathbf{r} - 2t_3,$$

where if $\hat{\mathbf{t}}$ is parametrized by polar angles (θ, φ),

$$\hat{\mathbf{k}} \equiv 2t_3\hat{\mathbf{t}} - \hat{\boldsymbol{\zeta}} = \left(\sin 2\theta\cos\varphi, \sin 2\theta\sin\varphi, \cos 2\theta\right),$$

i.e., $\hat{\mathbf{k}}$ is a unit vector (as is suggested by the notation) parametrized by the polar angles $(2\theta, \varphi)$.

[2] This is a consequence of the fact that the two motions are related by duality.

Now $\mathbf{r}'$ is in the forward hemisphere if and only if Z' is positive. From (B) this occurs when $\mathbf{r}$ is such that

$$\hat{\mathbf{k}} \cdot \mathbf{r} > 2t_3. \tag{B.56}$$

Define the plane $\mathcal{S}$ to be

$$\mathcal{S} \equiv \left\{ \mathbf{r} \in \mathcal{R}^3 \mid \hat{\mathbf{k}} \cdot \mathbf{r} = 2t_3 \right\}. \tag{B.57}$$

The set of points $\mathcal{L} \equiv \{\mathbf{r} \in \mathcal{R}^3 \mid Z' > 0\} = \left\{ \mathbf{r} \in \mathcal{R}^3 \mid \hat{\mathbf{k}} \cdot \mathbf{r} > 2t_3 \right\}$ for which Z' is positive will lie on one side or the other of the plane $\mathcal{S}$ (which side depends on the sign of t_3 and on the direction of the vector $\hat{\mathbf{k}}$). If we require that Z be positive, then the set $\mathcal{K}$ of points $\mathbf{r} \in \mathcal{R}^3$ for which both Z and the transformed point's depth Z' are positive will be the intersection of $\mathcal{L}$ and the forward hemisphere:

$$\mathcal{K} \equiv \left\{ \mathbf{r} \in \mathcal{R}^3 \mid Z > 0 \right\} \bigcap \mathcal{L}$$

In general this will be a wedge–shaped region in $\mathcal{R}^3$, infinite in extent. The ambiguity is, therefore, ubiquitous. There will be no ambiguity only when $\mathcal{K}$ is the null set ϕ, when $\mathcal{L}$ consists only of points for which Z (*sic*) is negative. It is easy to see that this will occur only when $t_3 = 0$, for which $\hat{\mathbf{k}} = -\hat{\zeta}$:

$$\begin{aligned}
\mathcal{L} &= \left\{ \mathbf{r} \in \mathcal{R}^3 | (-\hat{\zeta}) \cdot \mathbf{r} > 0 \right\} \\
&= \left\{ \mathbf{r} \in \mathcal{R}^3 | \hat{\zeta} \cdot \mathbf{r} < 0 \right\} \\
&= \left\{ \mathbf{r} \in \mathcal{R}^3 | Z < 0 \right\}.
\end{aligned}$$

Therefore the only case in which there is no duality ambiguity is when the translation vector is in the X–Y plane. This includes stereo as a special case. The physical reason why the ambiguity vanishes in this case is clear: if $\hat{\mathbf{t}}$ lies in the X–Y plane, then a rotation by π around the translation axis reverses the sign of the Z–coordinate of every point, and subsequent translation by $\hat{\mathbf{t}}$ will not change the value of Z.

When is there an ambiguity in point correspondences?

In general, the two ambiguous solutions for rotation and translation which give the same $\mathbf{E}$ will give different point correspondences. In this section we find the conditions under which they produce the *same* point correspondences.

Let $(\mathbf{R}, \hat{\mathbf{t}})$ give rise to the correspondence (in three–dimensional space) $\mathbf{r} \leftrightarrow \mathbf{r}_1$, and the dual solution $(\mathbf{R}_0\mathbf{R}, -\hat{\mathbf{t}})$ give rise to the correspondence $(\mathbf{r} \leftrightarrow \mathbf{r}_2)$. That is,

$$\begin{aligned} \mathbf{r}_1 = & \quad \mathbf{R}\mathbf{r} + \hat{\mathbf{t}} & = (X_1, Y_1, Z_1) \\ \mathbf{r}_2 = & \quad \mathbf{R}_0\mathbf{R}\mathbf{r} - \hat{\mathbf{t}} & = (X_2, Y_2, Z_2). \end{aligned}$$

The image point correspondences will be the same if and only if $\mathbf{r}_2 = \lambda \mathbf{r}_1$ for some real number λ. Furthermore, we must have $\lambda > 0$ if both points can be imaged (i.e., they appear in front of the image plane). We then find that

$$\begin{aligned} \lambda \mathbf{r}_1 \equiv \mathbf{r}_2 &= \mathbf{R}_0\mathbf{R}\mathbf{r} - \hat{\mathbf{t}} \\ &= \mathbf{R}_0\left(\mathbf{R}\mathbf{r} + \hat{\mathbf{t}} - \hat{\mathbf{t}}\right) - \hat{\mathbf{t}} \\ &= \mathbf{R}_0\mathbf{r}_1 - \mathbf{R}_0\hat{\mathbf{t}} - \hat{\mathbf{t}} \\ &= \mathbf{R}_0\mathbf{r}_1 - 2\hat{\mathbf{t}}, \end{aligned}$$

where we have used the fact that $\mathbf{R}_0$ is in the little group of $\hat{\mathbf{t}}$. We conclude that the solution and its dual will give the same image point correspondences only when there is a non–trivial solution to the equation

$$(\mathbf{R}_0 - \lambda \mathbf{1}_3)\,\mathbf{r}_1 = 2\hat{\mathbf{t}}.$$

This equation can be solved for $\mathbf{r}_1$ only when the matrix on the left hand side of the equation is invertible, i.e., when it is NOT true that

$$\det(\mathbf{R}_0 - \lambda \mathbf{1}_3) = 0.$$

However this is just the secular equation for the rotation matrix $\mathbf{R}_0$. We know that the secular equation is satisfied only when λ is an eigenvalue of $\mathbf{R}_0$. Since the only real such eigenvalue is $\lambda = 1$, we conclude that the matrix $\mathbf{R}_0 - \lambda \mathbf{1}_3$ is invertible unless $\lambda = 1$.

In summary, then, the image point correspondences will be the same for the solution and its dual when

$$\mathbf{r}_1 = (\mathbf{R}_0 - \lambda \mathbf{1}_3)^{-1}\; 2\hat{\mathbf{t}}, \tag{B.58}$$

with $\lambda \neq 1$. The indicated inverse matrix may be easily found as follows: since $\mathbf{R}_0$ is an idempotent matrix, i.e., $\mathbf{R}_0^2 = \mathbf{1}_3$, we can conclude that the inverse is a linear combination of $\mathbf{1}_3$ and $\mathbf{R}_0$:

$$(\mathbf{R}_0 - \lambda \mathbf{1}_3)^{-1} \equiv f(\lambda)\,[\mathbf{1}_3 + a\mathbf{R}_0]\,. \tag{B.59}$$

Multiplying this equation on the left (or right: $\mathbf{1}_3$ and $\mathbf{R}_0$ commute) by $\mathbf{R}_0 - \lambda \mathbf{1}_3$, we find:

$$\begin{aligned}
\mathbf{1}_3 &\equiv f(\lambda)\,(\mathbf{R}_0 - \lambda \mathbf{1}_3)\,[\mathbf{1}_3 + a\mathbf{R}_0] \\
&= f(\lambda)\left[\mathbf{R}_0 - \lambda \mathbf{1}_3 + a\mathbf{R}_0^2 - a\lambda \mathbf{R}_0\right] \\
&= f(\lambda)\,[(a-\lambda)\mathbf{1}_3 + (1-a\lambda)\mathbf{R}_0] \\
&= \{f(\lambda)(a-\lambda)\}\,\mathbf{1}_3 + \{f(\lambda)(1-a\lambda)\}\,\mathbf{R}_0.
\end{aligned}$$

This implies that (since $f(\lambda) = 0$ cannot solve the equation)

$$1 - a\lambda = 0 \Longrightarrow a = \frac{1}{\lambda}.$$

Hence, using $1 = f(\lambda)(a - \lambda)$, we conclude that

$$f(\lambda) = \frac{\lambda}{1-\lambda^2}, \tag{B.60}$$

which then gives, from (B.58):

$$(\mathbf{R}_0 - \lambda \mathbf{1}_3)^{-1} = \frac{1}{1-\lambda^2}\{\lambda \mathbf{1}_3 + \mathbf{R}_0\}. \tag{B.61}$$

The expression for $\mathbf{r}_1$ then becomes

$$\begin{aligned} \mathbf{r}_1 &= \frac{2}{1-\lambda^2}\{\mathbf{R}_0 + \lambda \mathbf{1}_3\}\hat{\mathbf{t}} = 2\frac{1+\lambda}{1-\lambda^2}\hat{\mathbf{t}} \\ \Longrightarrow \mathbf{r}_1 &= \frac{2}{1-\lambda}\hat{\mathbf{t}}. \end{aligned}$$

We conclude that the two transformed points must be

$$\begin{aligned} \mathbf{r}_1 &= \frac{2}{1-\lambda}\hat{\mathbf{t}} \\ \mathbf{r}_2 &= \frac{2\lambda}{1-\lambda}\hat{\mathbf{t}}. \end{aligned} \tag{B.62}$$

This means that

$$\begin{aligned} \mathbf{R}\mathbf{r} + \hat{\mathbf{t}} &= \mathbf{r}_1 = \frac{2}{1-\lambda}\hat{\mathbf{t}} \\ \Longrightarrow \mathbf{R}\mathbf{r} &= \frac{1+\lambda}{1-\lambda}\hat{\mathbf{t}} \\ \Longrightarrow \mathbf{r} &= \frac{1+\lambda}{1-\lambda}\mathbf{R}^{-1}\hat{\mathbf{t}}. \end{aligned} \tag{B.63}$$

We conclude that an environmental point given by (B.63) will be transformed by the rigid motion $(\mathbf{R}, \hat{\mathbf{t}})$ into the point

$$\mathbf{r}_1 = \frac{2}{1-\lambda}\,\hat{\mathbf{t}},$$

whereas it will be transformed by the dual rigid motion $(\mathbf{R}_0\mathbf{R}, -\hat{\mathbf{t}})$ into the point

$$\mathbf{r}_2 = \frac{2\lambda}{1-\lambda}\,\hat{\mathbf{t}},$$

where both $\mathbf{r}_1$ and $\mathbf{r}_2$ correspond to the same *image* point. For points $\mathbf{r} \in \mathcal{R}^3$ satisfying (B.63), then, both the rigid motion and its dual give the same image point correspondence.

We can express the correspondences in a slightly more perspicacious form by defining the quantity Γ to be

$$\Gamma \equiv \frac{1+\lambda}{1-\lambda}.$$

This then gives the expressions:

$$\begin{aligned} \mathbf{r} &= \Gamma\,\mathbf{R}^{\mathrm{T}}\hat{\mathbf{t}} \\ \mathbf{r}_1 &= (\Gamma+1)\,\hat{\mathbf{t}} \\ \mathbf{r}_2 &= (\Gamma-1)\,\hat{\mathbf{t}}. \end{aligned} \qquad \text{(B.64)}$$

There is, therefore, a real ambiguity (at the level of both the $\mathbf{E}$ matrix and the correspondences) only when the initial environmental point lies along the line defined by the vector $\mathbf{R}^{\mathrm{T}}\hat{\mathbf{t}}$. All such environmental points clearly project to the same image point. As a consequence, there is only one pair of corresponding points for which the actual motion and its dual are consistent with this correspondence. Since many more than one pair of corresponding points must be given to find the essential matrix in the first place, this (real) ambiguity would seem to be of only theoretical interest.

C When is $E = GR$?

We showed in Section 3.1 that if $\mathbf{E} = \mathbf{GR}$, then $\mathbf{EE}^\mathrm{T}$ is a projection operator which projects any vector onto its component perpendicular to the vector $\hat{\mathbf{t}}$. It then followed immediately that $\mathbf{EE}^\mathrm{T}$ had eigenvalues 0 and 1, and that the multiplicity of the latter eigenvalue was 2. That is, the matrix $\mathbf{E}$ has singular values 0, 1, and 1. In the paper of Faugeras and Maybank [Faug89], it is shown that this property of $\mathbf{E}$ is not only a necessary condition for $\mathbf{E}$ to be of the form $\mathbf{GR}$, but also a sufficient condition. However, their proof of sufficiency is by explicit construction, and only for the case of $\mathbf{E}$ being a 3×3 matrix. It is the purpose of this section to show that their result is more general, and also to give a more elegant (but less constructive) proof.

The main theorem that we want to prove in this section is this:

THEOREM *(Faugeras and Maybank) A* 3×3 *real matrix* $\mathbf{E}$ *can be written as* $\mathbf{E} = \mathbf{GR}$, *where* $\mathbf{G}$ *is real antisymmetric, and* $\mathbf{R} \in \mathrm{O}(3)$ *if and only if the singular values of* $\mathbf{E}$ *are* $0, T^2, T^2$, *where* $T \in \mathcal{R}$.

The "only if" portion of this theorem was shown in the text to be an immediate consequence of the fact that $\mathbf{EE}^\mathrm{T}$ is a projection operator. The "if" portion was proved in [Faug89] by explicit construction. We give here a more abstract proof.

We first note (as was done by Faugeras and Maybank) that if the singular values of $\mathbf{E}$

(i.e., the eigenvalues of $\mathbf{EE}^{\mathrm{T}}$) are $0, T^2, T^2$, then $\mathbf{E}$ satisfies the nonlinear equations

$$0 = \frac{1}{2}\left[\mathrm{tr}\,(\mathbf{EE}^{\mathrm{T}})\right]^2 - \mathrm{tr}\left[(\mathbf{EE}^{\mathrm{T}})^2\right] \tag{C.65}$$

$$0 = \det \mathbf{E}. \tag{C.66}$$

In order to show that (C.65) and (C.66) imply that $\mathbf{E} = \mathbf{GR}$, where $\mathbf{G}$ is antisymmetric and $\mathbf{R} \in \mathrm{O}(3)$, we will need the following lemma:

LEMMA *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be* $m \times m$ *real matrices satisfying* $\mathbf{AA}^{\mathrm{T}} = \mathbf{BB}^{\mathrm{T}}$. *Then* $\mathbf{A} = \mathbf{BR}$, *where* $\mathbf{R} \in \mathrm{O(n)}$.

Proof of the Theorem

The matrix $\mathbf{EE}^{\mathrm{T}} \equiv \mathbf{F}$ is real and symmetric, and hence may be diagonalized by an orthogonal transformation $\mathbf{U} \in \mathrm{O}(3)$:

$$\mathbf{F} = \mathbf{U\Delta U}^{\mathrm{T}} \;, \qquad \mathbf{\Delta} = \mathrm{diag}\,(\lambda_1, \lambda_2, \lambda_3) \;, \quad \lambda_i \in \mathcal{R}$$

Since $\lambda_1\lambda_2\lambda_3 = \det \mathbf{F} \equiv 0$, it follows that one of the λ_i is zero. We can, without loss of generality, assume that it is λ_1 that vanishes. Then

$$\begin{aligned} \mathrm{tr}\,\mathbf{F} &= \lambda_2 + \lambda_3 \\ \mathrm{tr}\,\mathbf{F}^2 &= \lambda_2^2 + \lambda_3^2. \end{aligned}$$

Since $\mathbf{F}$ satisfies (C.65),

$$\begin{aligned} 0 = \frac{1}{2}(\mathrm{tr}\,\mathbf{F})^2 - \mathrm{tr}\,\mathbf{F}^2 &= \frac{1}{2}(\lambda_2 + \lambda_3)^2 - \left(\lambda_2^2 + \lambda_3^2\right) \\ &= -\frac{1}{2}(\lambda_2 - \lambda_3)^2 . \end{aligned}$$

Hence, $\lambda_2 = \lambda_3$. Now, assume that $\lambda_2 = \lambda_3 = 0$. If this were true, then it would imply that $\mathbf{E}\mathbf{E}^{\mathrm{T}} = 0$. Therefore, $0 = \mathrm{tr}\left(\mathbf{E}\mathbf{E}^{\mathrm{T}}\right) = \sum_{i,j} \mathbf{E}_{ij}^2$. Since each of the terms of this sum is non–negative, this would imply that each of the terms in the sum is itself zero. Hence, the matrix $\mathbf{E}$ would be equal to the zero matrix. Such a matrix clearly satisfies the theorem. Hence, we may assume that $\lambda_2 = \lambda_3 \neq 0$. Using the same procedure as above, this means that $\mathbf{E} \neq 0$, and hence that $0 < \mathrm{tr}\,\mathbf{E}\mathbf{E}^{\mathrm{T}} = \lambda_2 + \lambda_3 = 2\lambda_2$. Therefore we can define $\lambda_2 = \lambda_3 \equiv T^2 > 0$.

Now, since 0 is an eigenvalue of $\mathbf{E}\mathbf{E}^{\mathrm{T}}$, it follows that there exists an eigenvector $\hat{\mathbf{t}}$ of $\mathbf{E}\mathbf{E}^{\mathrm{T}}$ corresponding to that eigenvalue. Since $T^2 \neq 0$, the two eigenvectors of $\mathbf{E}\mathbf{E}^{\mathrm{T}}$ corresponding to the eigenvalue T^2 must be perpendicular to $\hat{\mathbf{t}}$. Therefore, $\mathbf{E}\mathbf{E}^{\mathrm{T}}$ must be a projection operator onto the plane perpendicular to eigenvector $\hat{\mathbf{t}}$ corresponding to the zero eigenvalue:

$$\mathbf{E}\mathbf{E}^{\mathrm{T}} = T^2\left(\mathbf{1}_3 - \hat{\mathbf{t}}\hat{\mathbf{t}}^{\mathrm{T}}\right)$$

Letting $\mathbf{T} = T\hat{\mathbf{t}}$,

$$\begin{aligned} \mathbf{E}\mathbf{E}^{\mathrm{T}} = T^2\mathbf{1}_3 - \mathbf{T}\mathbf{T}^{\mathrm{T}} &= -(\mathbf{T}\cdot\mathbf{J})(\mathbf{T}\cdot\mathbf{J}) \\ &= (\mathbf{T}\cdot\mathbf{J})(\mathbf{T}\cdot\mathbf{J})^{\mathrm{T}}. \end{aligned}$$

Therefore $\mathbf{E}\mathbf{E}^{\mathrm{T}} = \mathbf{B}\mathbf{B}^{\mathrm{T}}$, where $\mathbf{B} \equiv \mathbf{T}\cdot\mathbf{J}$ is manifestly antisymmetric. We then use the Lemma (with $m = 3$) to conclude that $\mathbf{E} = \mathbf{B}\mathbf{R}$, where $\mathbf{R} \in \mathrm{O}(3)$, which proves the theorem.

We now prove the Lemma.

PROOF OF THE LEMMA

We can write the singular value decomposition of $\mathbf{A}$ and $\mathbf{B}$ as

$$\mathbf{A} = \mathbf{U}_1\mathbf{\Sigma}_1\mathbf{W}_1^{\mathrm{T}}$$
$$\mathbf{B} = \mathbf{U}_2\mathbf{\Sigma}_2\mathbf{W}_2^{\mathrm{T}}$$

where $\mathbf{U}_i, \mathbf{W}_i \in \mathrm{O(m)}$, and $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$ are $m \times m$ diagonal matrices, with non-negative entries on the diagonal. We then note that

$$\mathbf{A}\mathbf{A}^{\mathrm{T}} = \mathbf{U}_1\mathbf{\Sigma}_1\mathbf{W}_1^{\mathrm{T}}\mathbf{W}_1\mathbf{\Sigma}_1\mathbf{U}_1^{\mathrm{T}} = \mathbf{U}_1\mathbf{\Sigma}_1^2\mathbf{U}_1^{\mathrm{T}}$$
$$\mathbf{B}\mathbf{B}^{\mathrm{T}} = \mathbf{U}_2\mathbf{\Sigma}_2\mathbf{W}_2^{\mathrm{T}}\mathbf{W}_2\mathbf{\Sigma}_2\mathbf{U}_2^{\mathrm{T}} = \mathbf{U}_2\mathbf{\Sigma}_2^2\mathbf{U}_2^{\mathrm{T}}.$$

Since these two matrices are the same, we see that $\mathbf{\Sigma}_1^2$ and $\mathbf{\Sigma}_2^2$ must have the same set of diagonal entries, although the identical entries need not appear at the same place on the diagonal. That is, we can write

$$\mathbf{\Sigma}_2^2 = \mathbf{P}\mathbf{\Sigma}_1^2\mathbf{P}^{\mathrm{T}}, \tag{C.67}$$

where $\mathbf{P}$ is some permutation matrix, i.e., a matrix each of whose rows, and whose columns, has a single non-zero entry, and that non-zero entry is 1. Such a matrix, acting on a vector in $\mathcal{R}^m$, just permutes the components of the vector. Since this leaves the length of the vector invariant, it follows that $\mathbf{P}$ must be an element of O(m) (but not necessarily an element of SO(m)!). Since $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$ are known to be diagonal and non-negative, it follows from (C.67) that

$$\mathbf{\Sigma}_2 = \mathbf{P}\mathbf{\Sigma}_1\mathbf{P}^{\mathrm{T}}.$$

Furthermore, we can find another permutation matrix $\mathbf{P}' \in \mathrm{O(m)}$ such that the matrix $\mathbf{\Sigma}_1$ is given by

$$\mathbf{\Sigma}_1 = \mathbf{P}'\mathbf{\Sigma}\mathbf{P}'^{\mathrm{T}},$$

where $\boldsymbol{\Sigma}$ is a matrix having the same diagonal entries as $\boldsymbol{\Sigma}_1$, but in non-decreasing order as read from upper-left to lower-right.

Hence,

$$\begin{aligned} \mathbf{A} &= \mathbf{U}_1\boldsymbol{\Sigma}_1\mathbf{W}_1^{\mathrm{T}} = \mathbf{U}_1\mathbf{P}'\boldsymbol{\Sigma}\mathbf{P}'^{\mathrm{T}}\mathbf{W}_1^{\mathrm{T}} = (\mathbf{U}_1\mathbf{P}')\boldsymbol{\Sigma}(\mathbf{W}_1\mathbf{P}')^{\mathrm{T}}, \\ \mathbf{B} &= \mathbf{U}_2\boldsymbol{\Sigma}_2\mathbf{W}_2^{\mathrm{T}} = \mathbf{U}_2\mathbf{P}\mathbf{P}'\boldsymbol{\Sigma}\mathbf{P}'^{\mathrm{T}}\mathbf{P}^{\mathrm{T}}\mathbf{W}_2^{\mathrm{T}} = (\mathbf{U}_2\mathbf{P}\mathbf{P}')\boldsymbol{\Sigma}(\mathbf{W}_2\mathbf{P}\mathbf{P}')^{\mathrm{T}}. \end{aligned}$$

But $\mathbf{U}_1, \mathbf{U}_2, \mathbf{W}_1, \mathbf{W}_2, \mathbf{P}$, and $\mathbf{P}'$ are all in $\mathrm{O(m)}$, and hence so is any product of them. Consequently, we can write

$$\begin{aligned} \mathbf{A} &= \mathbf{Y}_1\boldsymbol{\Sigma}\mathbf{Z}_1^{\mathrm{T}} \\ \mathbf{B} &= \mathbf{Y}_2\boldsymbol{\Sigma}\mathbf{Z}_2^{\mathrm{T}}, \end{aligned}$$

where $\mathbf{Y}_i$ and $\mathbf{Z}_i$ are in $\mathrm{O(m)}$. We then have that

$$\begin{aligned} \mathbf{A}\mathbf{A}^{\mathrm{T}} &= \mathbf{Y}_1\boldsymbol{\Sigma}^2\mathbf{Y}_1^{\mathrm{T}} \\ \mathbf{B}\mathbf{B}^{\mathrm{T}} &= \mathbf{Y}_2\boldsymbol{\Sigma}^2\mathbf{Y}_2^{\mathrm{T}}. \end{aligned}$$

But by assumption, $\mathbf{A}\mathbf{A}^{\mathrm{T}} = \mathbf{B}\mathbf{B}^{\mathrm{T}} \equiv \mathbf{K}$. Hence, $\boldsymbol{\Sigma}^2$ must be the matrix of eigenvalues of $\mathbf{K}$, arranged in non-decreasing order, and $\mathbf{Y}_1$ and $\mathbf{Y}_2$ must be $\mathrm{O(m)}$ matrices which effect the diagonalization of $\mathbf{K}$. This allows us to find the relationship between $\mathbf{Y}_1$ and $\mathbf{Y}_2$, as follows.

Since the diagonal entries of $\boldsymbol{\Sigma}^2$ are in non-decreasing order, we can write $\boldsymbol{\Sigma}^2$ in the block-diagonal form

$$\boldsymbol{\Sigma}^2 = \begin{bmatrix} \overline{\lambda}_1^2\mathbf{1}_{m_1} & & & \\ & \overline{\lambda}_2^2\mathbf{1}_{m_2} & & \\ & & \ddots & \\ & & & \overline{\lambda}_k^2\mathbf{1}_{m_k} \end{bmatrix},$$

where the eigenvalues $\overline{\lambda}_i^2$ appear in *increasing* order. Here $\mathbf{1}_p$ is the $p \times p$ identity matrix, where p is the multiplicity of the eigenvalue $\overline{\lambda}_p$. Finally, the m_i's must satisfy

$$\sum_{i=1}^{k} m_i \equiv m.$$

As a result of this, the first m_1 columns of $\mathbf{Y}_1$ and of $\mathbf{Y}_2$ must separately form an orthonormal basis for $\mathcal{R}^{m_1}$, since they are eigenvectors of $\mathbf{K}$ corresponding to the eigenvalue $\overline{\lambda}_1^2$, and $\mathbf{Y}_1$, $\mathbf{Y}_2 \in O(m_1)$. Let these columns be denoted by

$$\begin{aligned} \text{first } m_1 \text{ columns of } \mathbf{Y}_2 &\equiv (\mathbf{u}_1, \ldots, \mathbf{u}_{m_1}) \\ \text{first } m_1 \text{ columns of } \mathbf{Y}_1 &\equiv (\mathbf{v}_1, \ldots, \mathbf{v}_{m_1}), \end{aligned}$$

where $\mathbf{u}_i$ and $\mathbf{v}_i$ are $m \times 1$ column vectors. Since each set of columns separately forms a basis for $\mathcal{R}^{m_1}$, we can write each in terms of the others:

$$\mathbf{u}_i = a_{ij}\mathbf{v}_j,$$

where the understood summation goes from 1 to m_1. But the orthonormality of each set of columns means that

$$\begin{aligned} \delta_{ik} \equiv \mathbf{u}_i^T\mathbf{u}_k &= a_{ij}\mathbf{v}_j^T\, a_{kr}\mathbf{v}_r^T \\ &= a_{ij}a_{kr}\, \underbrace{\mathbf{v}_j^T\mathbf{v}_r}_{\delta_{jr}} \\ &= a_{ij}a_{kj}. \end{aligned} \tag{C.68}$$

Defining the matrix $\mathbf{R}^{(m_1)}$ to have matrix elements

$$\left(\mathbf{R}^{(m_1)}\right)_{ik} \equiv a_{ki},$$

we see that $\mathbf{R}^{(m_1)}$ must be an element of $O(m_1)$. But then

$$\begin{aligned} \mathbf{u}_i &= \left(\mathbf{R}^{(m_1)}\right)_{ji} \mathbf{v}_j = \mathbf{v}_j \left(\mathbf{R}^{(m_1)}\right)_{ji} \\ &\Longrightarrow (\mathbf{u}_1, \ldots, \mathbf{u}_{m_1}) = (\mathbf{v}_1, \ldots, \mathbf{v}_{m_1})\mathbf{R}^{(m_1)}. \end{aligned}$$

The same argument clearly holds for the $(m_1+1)^{\text{th}}$ through $(m_1+m_2)^{\text{th}}$ columns, and so on. Hence, we conclude that the relationship between $\mathbf{Y}_2$ and $\mathbf{Y}_1$ is simply:

$$\mathbf{Y}_2 = \mathbf{Y}_1 \Re,$$

where $\Re$ is a block diagonal matrix of the form

$$\Re = \begin{bmatrix} \mathbf{R}^{(m_1)} & & & \\ & \mathbf{R}^{(m_2)} & & \\ & & \ddots & \\ & & & \mathbf{R}^{(m_k)} \end{bmatrix},$$

where $\mathbf{R}^{(m_i)} \in O(m_i)$. But since

$$O(m_1) \otimes O(m_2) \otimes \cdots \otimes O(m_k) \subset O(m),$$

it follows that $\Re \in O(m)$.

We are now approaching the end of our proof. Given the form of $\mathbf{\Sigma}^2$, it follows that $\mathbf{\Sigma}$ must have the form

$$\mathbf{\Sigma} = \begin{bmatrix} \overline{\lambda}_1 \mathbf{1}_{m_1} & & & \\ & \overline{\lambda}_2 \mathbf{1}_{m_2} & & \\ & & \ddots & \\ & & & \overline{\lambda}_k \mathbf{1}_{m_k} \end{bmatrix}.$$

But since the matrix $\mathbf{R}^{(m_i)}$ clearly commutes with the identity matrix $\mathbf{1}_{m_i}$, it follows that $\Re$ and $\mathbf{\Sigma}$ commute:

$$[\Re, \mathbf{\Sigma}] = 0.$$

Consequently, the matrix $\mathbf{B}$ is given by

$$\begin{aligned}\mathbf{B} = \mathbf{Y}_2\boldsymbol{\Sigma}\mathbf{Z}_2^{\mathrm{T}} &= \mathbf{Y}_1\left(\Re\boldsymbol{\Sigma}\right)\mathbf{Z}_2^{\mathrm{T}} \\ &= \mathbf{Y}_1\left(\boldsymbol{\Sigma}\Re\right)\mathbf{Z}_2^{\mathrm{T}} \\ &= \left(\mathbf{Y}_1\boldsymbol{\Sigma}\mathbf{Z}_1^{\mathrm{T}}\right)\mathbf{Z}_1\Re\mathbf{Z}_2^{\mathrm{T}} \\ &= \mathbf{A}\left(\mathbf{Z}_1\Re\mathbf{Z}_2^{\mathrm{T}}\right) \\ &\equiv \mathbf{AR}.\end{aligned}$$

Since $\mathbf{Z}_1$, $\Re$, and $\mathbf{Z}_2^{\mathrm{T}}$ are all in $\mathrm{O}(m)$, so is the matrix $\mathbf{R}$ defined above. Consequently, the lemma is proven, and so is the theorem.

Figure