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## Abstract

The first generically valid proof of uniqueness for the surface solution to shape from shading is presented. Also, it is proven that the local surface solutions around a singular point have at most a four-fold ambiguity. These results apply for a reflectance function corresponding to illumination from the viewer direction of a uniform albedo Lambertian object. Generic surfaces are studied, and their properties established. The proof may lead to new, faster algorithms for shape recovery. Questions of existence are also discussed. It is argued that most images are impossible, in the sense that they can not be a depiction of any physical object. The proof is based on ideas of dynamical systems theory and global analysis.

## 1. Introduction

The first generically valid proof of uniqueness for the surface solutions for shape from shading is presented. A fortiori, shape from shading is not an ill-posed problem. Partial results applicable to special cases have been derived previously (Horn 1989a).

The results apply primarily to images generated by a particular, well-studied reflectance function, and containing a light region surrounded by a black background; such images can depict solid smooth objects wholly contained in the field of view. The reflectance function corresponds to illumination from the viewer direction of a uniform albedo Lambertian object. For such an image, and assuming the object depicted is smooth and non-self-occluding, it is shown that there is generically at most one solution for the object surface. This result, and the methods used in the proof, suggest the possibility of a new, fast approach to numerically recovering shape from shading. Some ideas as to what such an algorithm might look like are outlined.

Some questions of existence are also discussed. It is argued that images are *generically impossible*. This would mean both that most images can not possibly depict real smooth objects, and that slightly perturbing any image will probably render it an impossible one, for the given reflectance function. The practical consequence is that discretized and noisy images are likely not to correspond exactly to any real 3D surface. Thus, some of the difficulties encountered by Horn (Horn 1989a) using the classical characteristic strip method of solution are not due merely to the discretization error of numerical integration, but inherent in the inaccuracies of the image-formation process. A solution method for shape from shading must rectify the errors in the image in order to obtain any consistent solution at all. However, the proof of uniqueness also shows that the object is structurally stable under perturbations of the image, and thus surface recovery can probably be carried out even for an image that is strictly impossible due to noise.

The approach is based on the ideas of dynamical systems theory and global analysis. The techniques of dynamical system theory, well-developed by mathematicians, have been applied recently to a variety of vision problems. Saxberg (Saxberg 1989) has noted that the problem of shape from shading can be usefully reinterpreted along these lines, and proposed a new method of solution which appears to have some promise. In this paper, a new viewpoint on the problem is developed based on these techniques, that is simple and intuitive, and offers qualitatively new insights.

## 2. The Characteristic Strip Method as a Hamiltonian Dynamical System

The method of characteristic strips is a classical method of non-linear first-order differential equations, first applied by Horn (Horn 1989a) to solve the shape from shading problem. The method is based on finding curves in the image plane $(x(t), y(t))$ along which the surface depth function z(x, y) can be explicitly integrated. (Here $x$ and $y$ are the image plane coordinates.) If the image plane can be filled out completely with such curves, then a complete solution for the surface has been found.

In this section arbitrary reflectance functions $R(p, q)$ will be considered. The image irradiance equation can be written as:

$$H \equiv I(x, y) - R(p, q) = 0. \tag{1}$$

$p$, $q$ represent as usual the derivatives of the surface depth $z$ with respect to $x$ and $y$, respectively. The characteristic strip equations are:

$$\dot{x} = H_p \quad (\equiv \frac{\partial H}{\partial p}), \quad \dot{y} = H_q, \quad \dot{p} = -H_x, \quad \dot{q} = -H_y. \tag{2}$$

The dot denotes a derivative with respect to 'time', an arbitrarily chosen variable that parameterizes the position along the characteristic strip. As pointed out by Saxberg, these equations constitute a *dynamical system*, that is, they can be thought of as determining the 'motion' of a particle whose 'position' is specified by the four parameters $(x(t), y(t), p(t), q(t))$. This four-dimensional 'position' space will be referred to below as *phase space*. For dynamical systems, the quantities on the right hand side of the dynamical equations are referred to as the components of a *vector field*. However, more can be said. These equations determine a very special type of dynamical system, namely a *Hamiltonian* one.

Hamiltonian dynamical systems are familiar and well studied. They have the defining properties that 1) there exists an *energy* function which is a constant of the motion, 2) the motion parameters can be divided into equal numbers of 'coordinate' and 'momentum' parameters, and 3) the evolution of the system is governed by Hamilton's equations. Many problems of classical Newtonian mechanics, celestial mechanics, etc., fall into this category. Essentially, it comprises all systems without frictional, or dissipative, forces.

As an example that will prove to be relevant later on, consider an electrically charged particle moving on a plane, in response to an electric field. This system has a conserved energy which is the sum of a *kinetic energy* of motion, and a *potential energy*, whose gradient is equal to the negative of the electric field. The momentum vector $\vec{p}$ is equal to the mass of the particle times its velocity. The energy is explicitly:

$$H = \frac{1}{2m}\vec{p}^2 + V(\vec{x})$$

with $\vec{E}(\vec{x}) = -\nabla V(\vec{x})$, and mass $m$.

In general, Hamilton's equations are:

$$\dot{x}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial x_i}. \tag{3}$$

for each coordinate–momentum pair $(x_i, p_i)$. $H$ is the energy. For the example, these equations become:

$$\dot{x}_i = \frac{p_i}{m}, \quad \dot{p}_i = \frac{1}{m}\frac{d^2 x_i}{dt^2} = E_i. \tag{4}$$

The second equation is just the familiar Newton's law, $F = ma$, while the first defines the momentum.

On comparing eq. 3 with the characteristic strip equations, one finds that the latter are equivalent to a Hamiltonian system, with conserved energy $H = 0$, and generalized momenta $p$ (corresponding to $x$) and $q$ (corresponding to $y$). With this insight, one can think of the characteristic strip equations as describing the motion of a 'particle' on the image plane, with 'momentum' $\vec{P} = (p, q)$. This is now a two–dimensional problem, much easier to visualize than the apparently four–dimensional dynamical system of eq. 2. Also, more is known about the special case of Hamiltonian systems than about general systems. For example, they satisfy Liouville's theorem that the flow in phase space preserves volume, with the consequence that all fixed points of the flow must be saddle points. (For more about fixed points and flows see below). This is useful in what follows, since the fixed (i.e. singular) points are important in characterizing the shape from shading solution, as noted by many people. (To avoid later confusion, it should be remembered that the fixed points are saddle points in the full four-dimensional phase space parameterized by $(x, y, p, q)$, but *not* necessarily in the image plane.)

## 3. Illumination from the Viewer Direction: Characteristic Strips as a Gradient Dynamical System

From now on a reflectance function corresponding to a Lambertian surface of known, constant albedo, with no self-occlusion, and illuminated by a light source close to the viewer direction, will be assumed. For convenience, the albedo is taken to be unity, without loss of generality. Under these conditions the equation for the image intensity $I$ is:

$$I = \hat{n} \cdot \hat{L} = \frac{(-p, -q, 1) \cdot (0, 0, 1).}{(1 + p^2 + q^2)^{1/2}} = \frac{1}{(1 + p^2 + q^2)^{1/2}}. \tag{5}$$

where $\hat{n}$ is the unit surface normal, and $\hat{L} = \hat{z}$ is a unit vector giving the light source direction. Note that the intensity $I$ reaches its maximum value $I = 1$ at $p = q = 0$. After some algebra, this can be rewritten as:

$$H \equiv \frac{1}{2}(p^2 + q^2) + V(\vec{x}) = 0, \tag{6}$$

with

$$2V(\vec{x}) \equiv 1 - \frac{1}{I^2}. \tag{7}$$

This definition of the Hamiltonian (i.e. energy) function $H$ differs slightly from the one used in the previous section. It is chosen because of its simplicity: the energy is expressed as a sum of a *quadratic* kinetic energy term which contains all the 'momentum' dependence, and of a potential energy term containing all the 'coordinate' dependence. This is the familiar form of the energy for elementary Newtonian mechanics. In fact, this $H$ is exactly the same as the example from the previous section, with the mass taken to be unity. *The characteristic* *strips for the given reflectance function are equivalent to the motion of a particle on a plane in response to a potential.*

Explicitly, the characteristic equations are:

$$\dot{x} = H_p = p, \quad \dot{y} = H_q = q,$$

and

$$\ddot{x} = \dot{p} = -H_x = -V_x, \quad \ddot{y} = \dot{q} = -H_y = -V_y. \tag{8}$$

The first two state that the surface slopes $(p, q) \equiv \vec{P}$ are just the *velocity* of the moving point whose trajectory gives the characteristic strip.

The equation for the surface depth $z$ has been neglected so far, since it is unnecessary in deriving the trajectory. It is:

$$\dot{z} = p\dot{x} + q\dot{y} = \dot{x}^2 + \dot{y}^2 \geq 0. \tag{9}$$

The direction of time has been defined so that $z$ always increases with time along the trajectory. Actually, since

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix} = \vec{\nabla} z, \tag{10}$$

a characteristic trajectory $(x(t), y(t))$ is also the image plane trace of a gradient–climbing curve on the 3D object, i.e. of a surface curve that climbs as fast as possible in $z$, since its tangent is parallel to the surface gradient everywhere. The two–dimensional dynamical system specified by the above equation is an example of a *gradient* system—a dynamical system that is even simpler than a Hamiltonian one. For instance, gradient systems can not have limit cycles (which are closed periodically traversed trajectories).

The ensemble of dynamical trajectories filling up a space defines a *flow*. Visually, a flow can be rendered by drawing flow–lines (also called 'orbits'), indicating all the possible paths taken by 'particles' starting at arbitrary initial 'positions', as they move in accordance with a dynamical equation. More formally, a *flow* is a family of maps $F_t$, depending on the time, such that $F_t(x_0)$ gives the 'position' of a 'particle' at time $t$ whose starting 'position' at time $t = 0$ is $x_0$. Thus $F_0$ is the identity map, and $F_t(F_s) = F_{s+t}$, since a 'particle' propagating for a time $t$, and then a time $s$, has propagated for a total time of $s+t$. An image plane flow composed of characteristic curves is shown in Figure 14. The circles represent singular points, and are the initial or terminal points of the flow lines.

From now on, flows in the two-dimensional space of the image plane, which can be thought of as generated by the dynamical equation eq. 10, will be the primary focus of the argument. It is crucial that the characteristic strip equations generate a flow not only in the four dimensional space of $(x, y, p, q)$, but also, by virtue of eq. 10, in the two dimensional space of the image plane. The point is that the curves representing the image plane projections of characteristic strips cannot intersect in the image plane if the image is that of a real object, since then at the point of intersection there would be two different tangent directions $(p, q)$—i.e. two different solutions for the surface slope of the 3D object—corresponding to a single image point, which is not possible. Non-intersecting trajectories in the image plane generate a flow wholly in the space of the image plane.

In the remainder of this paper the characteristic strip curves in the full four dimensional phase space $(x, y, p, q)$ will be referred to as *trajectories*, as a reminder that they can be considered the trajectories of particles moving on the image plane in response to forces. The projection of the characteristic trajectories onto the image plane will be referred to as *gradient curves*, as a reminder that they can be associated with the two dimensional gradient dynamical system eq. 10.

## 4. Singular Points and the Winding Number

The importance of singular points in determining and fixing a solution to the shape from shading problem has been stressed by many people, and they are crucial in this work as well. The *singular points* of an image are those at which there is a unique solution for the surface slope given the intensity. For the present reflectance function, this occurs at $p = q = 0$, and $I = 1$, i.e. where the object surface is parallel to the image plane and the image is maximally bright. Singular points also have the property that the derivatives of the reflectance function with respect to $p$ and $q$ vanish there. From the characteristic strip equations, eq. 2, this implies that $\dot{x} = \dot{y} = 0$—a characteristic trajectory initially at a singular point never leaves it. Singular points are thus *fixed points* in the language of dynamical systems theory.

The concept of a hyperbolic fixed (i.e. singular) point will also be important. For the present case, a singular point is ***hyperbolic*** if the second derivative matrix of the intensity $I$ is non-singular. It is easy to show that a singular point is hyperbolic if and only if the principal curvatures values at the corresponding point on the object surface are both non-zero, which is clearly the generic case for a surface, at least for an isolated singular point, as was noted by Saxberg (Saxberg 1989).

The object surface at a hyperbolic singular point may be either convex, concave, or saddle shaped. Correspondingly, there are three types of 2D flows in the neighborhood of this point in the image plane, made up of the ensemble of gradient curves. For the concave case, the fixed point is referred to as a *sink*, since all the gradient-climbing curves converge to the fixed point. (Recall that $z$ is the surface *depth*). Similarly, the fixed point is called a *source* in the convex case, and a *saddle* point when the surface is saddle-shaped. In the saddle-type flow, there are two gradient curves that converge to the fixed point, and two that originate at the fixed point. All other gradient curves, like comets, initially approach the saddle point, but miss it and recede into the distance. See Figures 1, 2 and 3.

(Figures 1, 2 and 3 should go here.)

A more formal proof that all fixed points of a dynamical system fall into one of these three categories follows from the Grobman-Hartman theorem. A brief discussion is given in Appendix A.

A fixed point in two dimensions has an integer-valued *index* associated with it. (Note: the sense of index used here differs from its definition in Morse Theory as the dimension of the outset of a fixed point.) To define the index, consider a small circle containing the fixed point and no others. For the given flow, each point on the circle is intersected by a unique flow curve. Define a function giving for each point on the circle the direction of the flow curve passing through that point, as a unit vector. As one goes around the circle (say in a counter-clockwise direction), eventually returning to the starting point, this unit vector rotates, and must also return to its original direction. The number of complete rotations it makes in the process, in the counter-clockwise direction, gives the value of the index. An index corresponding to a rotation in the opposite sense from the path taken around the circle is defined to be negative. See Figure 4.

(Figure 4 should go here.)

The most important fact about the index is that it is a topological invariant. It does not depend on the particular path chosen around the fixed point, which need not be a circle in general. Its value defined with respect to a path will not change as that path is distorted from its original shape, as long as the interior of the path contains only the single fixed point. In general, the "index", or rather *winding number*, can be defined for an arbitrary closed curve, not necessarily containing exactly one fixed point in its interior, in the obvious way. Clearly, the winding number is always an integer. Also, the winding number of a curve is the additive sum of the indices of all fixed points contained within its interior. These general theorems can be proven fairly simply for the case at hand.

A crucial fact is that the the index for sources and sinks of a two-dimensional flow is $+1$, while for saddles it is $-1$. See for example (Arnold 1973) and (Abraham 1983) for good intuitive expositions of these results.

## 5. Uniqueness

In this section the proof of generic uniqueness is given. It is important that the proof applies in the *generic* case, which essentially means general position. This has one immediate and important consequence.

It is assumed from now on that the image consists of a light blob in a black background, i.e. the intensity is non-zero in a compact region and zero elsewhere. The image intensity is also assumed to be smooth; in particular, it falls off smoothly at the boundary of the light region in the image. This boundary will be referred to as the image boundary. It must correspond to the occluding boundary of the imaged object. For most points along the image boundary, there are two possibilities for the shape of the object: either the boundary is protruding (a point on the occluding boundary is closer to the viewer than nearby visible points on the object) or else it recedes (visible points are closer than the occluding point). The former case violates general position for non-transparent objects—the viewing direction accidentally coincides with a head-on view of the object 'rim'. Also it violates the object smoothness assumption. Therefore, it is assumed below that the *occluding boundary recedes from the viewer at every point.* This will be true for smooth solid non-self-occluding objects.

Under these assumptions, it is shown that there is at most one generic solution to shape from shading. A generic solution is one that is *structurally stable* in the language of dynamical systems theory, i.e. one for which the image does not change drastically for small perturbations of the object surface. Also, structurally stable solutions are generic in the usual sense that any non—structurally stable solution can be converted into a stable one by an arbitrarily small perturbation.

To begin the proof, the behavior of the object at the occluding boundary is considered. The solution for the object surface is studied by considering properties of the flow field of gradient curves in the image plane. First, the simple result is established that for each point on the image boundary there is a gradient curve that converges smoothly to it, with a well-defined tangent pointing outwards, normal to the image boundary line. The detailed proof is consigned to Appendix A. See Figure 5 for an illustration of this result.

(Figure 5 should go here.)

As described in the appendix, the image plane flow generated by:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{\nabla} z, \tag{11}$$

where $z$ is the surface depth, can be extended to a smooth flow over a region including the image boundary as well as the interior light region. These combined results imply that the theorem of Peixoto is applicable to the solutions for shape from shading (see e.g. (Palis 1982) and (Guckenheimer 1983)).

Adapted to the present case, this theorem states the following: Let $v$ be a smooth $C^r$ vector field (with a continuous $r$-th derivative), defined on an open region $W$ of the plane, such that $v$ points outwards on the boundary $\delta D$ of an open set $D$ contained in $W$. $v$ is structurally stable if and only if: 1) There are a finite number of fixed points and each is hyperbolic. 2) There are no orbits connecting saddle points. Moreover, the set of structurally stable vector fields is open and dense in the set of all $C^r$ vector fields that point outwards on $\delta D$.

The theorem implies that gradient vector fields which satisfy the two conditions above (and the corresponding surfaces) are stable under small perturbations, in the sense that the flow associated with the vector field does not change drastically—its topology remains the same. Moreover, every gradient vector field, which means every surface, can be approximated arbitrarly closely by a vector field satisfying the above two conditions. This does not quite say that every surface can be approximated by one that is structurally stable, since the approximating vector field may not be a gradient vector field.

The theorem of Palis and Smale (Palis 1968) states that the above theorem is still true if the words 'vector field' are everywhere replaced by 'gradient vector field'. This is the desired result, however it cannot be applied without a bit of additional argument, since it is not possible to extend the gradient vector field past the image boundary while maintaining its character as a gradient vector field. In Appendix A, it is shown that for any smooth surface, it is possible to find a contour in the image interior such that the gradient curves point outwards on this contour. Then the Palis–Smale theorem applies.

Thus the above two conditions of structural stability are generically valid for surfaces. Henceforth, the arguments will be restricted to the generic surfaces satisfying these two conditions. *This means that the image will have a finite number of singular points, and that the second derivative matrix of the intensity will be non-singular at these points.* The meaning of the second condition of structural stability is illustrated in Figure 6.

(Figure 6 should go here.)

The fact that the gradient curves point outwards at the image boundary has another important consequence. Consider the winding number of the image boundary curve. Because of the above fact, and because the curve does not self-intersect, it has winding number +1. (For instance, compare Figure 5 with Figure 4.) It follows from the additivity property of the winding number that:

$$N_i + N_f - N_s = 1, \tag{12}$$

where $N_i$ is the number of sources, i.e. singular points where the object is convex, $N_f$ is the number of sinks (concave singular points on the object), and $N_s$ counts the number of singular saddle points.

Note as an aside that this means that the total number of singular points in the image must be odd. If there is an even number of hyperbolic singular points, the image cannot correspond to a a smooth non-self-occluding solid object that advances towards the viewer from the occluding boundary. The only possibility is that the object has a sharp rim which is accidentablly being observed head on. More 'impossibility' results will be presented later.

It is now proven that there exists no more than one generic solution to shape from shading. First, it is shown that every gradient curve in the image plane begins and ends at unique points, which are either on the image boundary, or else singular. The existence theorems for differential equations state that the gradient curves either extend to infinite times, or else approach the boundary infinitely closely. The image region $W$ including the image boundary is compact, implying that any curve in $W$ contains a subsequence of points that converges to a point in $W$. Suppose a gradient curve converges in this sense to an image boundary point $b$. As stated above, and as shown in Appendix A, the curve can be smoothly extended through $b$ normal to the image boundary. Therefore $b$ is the unique endpoint of the curve. Of course, this is true going forwards or backwards along the curve.

Next suppose that a subsequence from a gradient curve converges to an interior point as $t \rightarrow \infty$. (An equivalent argument works for convergence at $t \rightarrow -\infty$). In appendix A it is shown that any interior point to which a gradient curve converges in this sense must be a singular point. If the curve converges to more than one interior point as $t \rightarrow \infty$, then it must travel back and forth between them an infinite number of times. Different branches of the curve pass through points infinitesimally close to one of the fixed points, and thus, by continuity, these branches converge to an infinite set of points. But in the present case the image contains only a finite number of singular points, so a gradient curve converging to an interior point as $t \rightarrow \infty$ converges to a unique singular point. Thus, as claimed, every gradient curve has unique initial and terminal points, where these are either on the boundary, or else singular. When a point lies on a gradient curve that originates or terminates at some singular point $s$, it will be called *connected* to $s$.

Since, as shown in Appendix A, the flow is directed outwards on the appendix, *every gradient curve begins at one of the finite number of singular points.* The basic idea of the uniqueness proof can now be explained. It is well known that there exist unique local convex and concave solutions for the object surface in the neighborhood of a singular point. This corresponds to unique solutions for the local image plane flow around fixed point sinks or sources. The surface depth $z$ is determined uniquely along any gradient curve if it is known at any point on the curve. Thus, up to an additive constant (the depth of the singular point itself), $z$ is determined throughout the entire region connected to sources or sinks. This includes the whole image region apart from the finite number of curves originating from saddle points. The image is therefore uniquely determined by continuity apart from some finite ambiguity as to which points correspond to sinks or sources. (Some must). One can eliminate this remaining finite ambiguity by global reasoning about the nature of the possible generic flows and solutions.

For completeness, a proof of the local existence of unique concave and convex solutions for $z$ in the neighbor of a singular point in the image is included in Appendix B. See also (Bruss 1982) and (Saxberg 1989). It is also possible to prove local uniqueness of the solution around saddle singular points, apart from some special cases. The proof is contained in Appendix B. This result will not be needed for the present uniqueness proof, which is now presented in more detail.

As stated above, the surface solutions in the neighborhoods of sources and sinks are unique up to an additive constant in $z$. If the intensity function is $r$ times differentiable, i.e. $C^r$, the local solutions are $C^{r-1}$, from the Stable Manifold Theorem (see Appendix B). Since $z$ and its first derivatives are uniquely determined, the existence and uniqueness theorems for differential equations state that the gradient curves (and the solution for $z$) can be uniquely extended from starting points in the neighborhoods of the singular points. Thus $z$ is unique over the region of the image plane connected to sources or sinks, as stated earlier.

There are only a finite number of gradient curves connected to saddle points. But for every point in the image region, including the boundary, there exists a unique gradient curve that passes through it, which must connect to some singular point. For the singular points themselves, this curve collapses to a point. Thus if the depth at the singular points is known, it is determined everywhere but along those curves connecting saddles to the boundary, since by structural stability saddles do not connect to saddles. By continuity, this gives the solution everywhere.

The next step is to show that all singular points are connected to each other along a path of gradient curves. This uniquely specifies the relative depths of the singular points. First, if a solution exists, then every singular point is connected to at least one other singular point by a gradient curve, assuming there are more than one. This is clearly true for any saddle point or sink. These singular points are terminal points of gradient curves, which must originate at singular points, since they can't originate at the boundary. Therefore a sink must be connected to some other singular point. Also, by the second condition of structural stability, a gradient curve can not begin and end both at a saddle point (see Figure 6). Thus for generic surfaces, saddle points also must be connected to some other singular point.

The only remaining case is that of sources. Suppose there exists a source $O$ that only connects to the boundary. The region $U_o$ consisting of all points in the image interior that lie on a gradient curve originating at $O$ is open, since it is a projection of the unstable manifold of the source point onto the image plane. By assumption, there are points in the image interior not contained in $U_o$, for example all the other singular points. Thus the boundary of $U_o$ must contain points in the image interior. Consider a gradient curve passing through one of these points. Every point on this curve must also be on the boundary by continuity of the flow. One end of this curve must connect to a singular point $s$, which is also on the boundary. First, $s$ cannot be a source. If it were, then all points in some neighborhood must originate from $s$. But since $s$ is on the boundary of $U_o$, there are points infinitesimally close to $s$ that originate from $O$. Similarly, it cannot be a sink, since then all points in some neighborhood would terminate at $s$. But there are points infinitesimally close to $s$ that originate at $O$, implying that $O$ is connected to $s$ contrary to assumption. See Figure 7.

(Figure 7 should go here.)

Suppose then that $s$ is a saddle. It cannot be an isolated boundary point. For, suppose there is a neighborhood of $s$ that excluding $s$ itself contains only points of $U_o$. Then since there exist gradient curves converging to $s$, $s$ is connected to $O$ contrary to assumption. On the other hand, if every neighborhood of $s$ contains both points in $U_o$ and points not in $U_o$, then there are boundary points of $U_o$ infinitesimally close to $s$. Consider the gradient curves through a sequence of these boundary points approaching $s$. Since $s$ is hyperbolic and therefore an isolated singular point, one can assume that none of these points is singular. Suppose first that they constitute an infinite number of different gradient curves. All of them must connect to singular points. But only a finite number can connect to saddle points, so some must connect to either a source or a sink, which by the previous arguments is impossible. But if there is a finite number of gradient curves, then some curve must contain an infinite subsequence of these boundary points converging to $s$. This gradient curve $L$ therefore connects to $s$, and is in the boundary of $U_o$. It is one of the four special curves that connect to $s$. (These curves constitute the stable and unstable manifolds associated with $s$ for the two-dimensional gradient dynamical system in the image plane.) If $L$ is converging to $s$, it must originate from a source singular point $s'$, which is also on the boundary. As argued above, this is impossible. Therefore $L$ is assumed originating at $s$. It will now be argued that one of the two gradient curves terminating at $s$ is also on the boundary, which as just stated leads to a contradiction. This is illustrated in Figure 8.

(Figure 8 should go here.)

The Grobman–Hartman result that there exists a ball $V$ around $s$ such that the flow is homeomorphic to a linear flow on $V$ is used. This homeomorphism can be taken to be a finite distance from the identity map. See (Palis 1982). $V$ is divided into four quadrants by the four special curves mentioned above, one of which is $L$. Let $Q$ be one of these quadrants bordering $L$, which contains points of $U_o$ arbitrarily close to $L$. Let $I$ be the other gradient curve bounding quadrant $Q$. $I$ must converge to $s$. See Figure 9. It will be shown that $I$ is also contained in the boundary of $U_o$, which by previous arguments is a contradiction, finally showing that every singular point connects to some other singular point. All that is necessary is to show that some point $i$ from $I$ is on the boundary.

(Figure 9 should go here.)

Let $l$ be a point from $L$. $l$ is chosen so that there are points of $U_o$ arbitrarily close to $l$. Choose $i$ from $I$, with $i$ not on the boundary of $U_o$. Then there is a neighborhood of $i$ which contains no point of $U_o$. The Grobman–Hartman theorem implies that one can choose new coordinates $u$, $v$ on the region $V$ around $s$ such that: a) the curve $I$ converging to $s$ is on the $u$–axis, b) $L$ is on the $v$–axis, c) other gradient curves, characterized by constants $c_1$ and $c_2$, are given by:

$$(u(t), v(t)) = (c_1 e^{-at}, c_2 e^{bt}).$$

where $a$, $b$ are universal positive constants, valid for every gradient curve, d) these coordinates are valid for all $u$, $v$ with $u^2+v^2 \leq D$ for some distance $D$, and e) $u$, $v$ are continuous bounded functions of $x$, $y$. For concreteness, $Q$ is taken to correspond to the upper left quadrant in the $u$, $v$ plane. See Figure 9.

Let $i$ and $l$ be identified respectively with the points $(i, 0)$ and $(0, l)$. These points are chosen to lie in the region $V$ where the linearizing coordinates are valid. It is easy to see from the above equation that for any $\epsilon$, there exists a $\delta$ such that all points within a distance $\delta$ from $l$ lie on gradient curves that pass within a distance $\epsilon$ of $i$. Since the change of coordinates is continuous, there does exist an $\epsilon$ such that no point in $Q$ within $\epsilon$ of $i$ is contained in $U_o$. All points on the same curve as a point not in $U_o$ are also not in $U_o$. Therefore, there is a $Q$ neighborhood of $l$ that contains no point of $U_o$, contrary to assumption. $I$ is accordingly contained in the boundary of $U_o$. It must originate from a source point, which is also on the boundary. But this is impossible by previous arguments. All possibilities have now been exhausted, and therefore the original assumption of disconnectedness must be false.

It is therefore true as claimed that every singular point is connected to at least one of the other singular points. This in turn implies that there is a path along gradient curves between any two singular points. For, consider the set $C$ of all singular points connected to a particular singular point $O$ along some sequence of paths. Clearly, all the points in $C$ are connected to each other through $O$. Suppose that this set is not the complete set of all singular points. Then the remaining singular points form a set $N$ none of which is connected to any point of $C$. Define the set $\hat{C}$ to consist of all points in the image connected to the singular points $C$. Similarly, define $\hat{N}$ to be the set of all points in the image connected to the singular points $N$. Every point in the image is either in $\hat{C}$ or $\hat{N}$, since all points connect to some singular point. $\hat{C}$ contains the union $U$ of a number of open sets consisting of those points lying on gradient curves connecting to the sources and sinks in $C$. Exactly the same argument as above shows that the boundary of $U$ can contain no point connected to a singular point in $N$. The closure $\bar{U}$ of $U$ therefore contains, besides points on the image boundary, only the saddle points in $C$ and gradient curves connecting to these saddle points. Moreover, it must contain all the curves connecting to saddles points in $C$. For, if it does not contain one of these curves, then there is a neighborhood of a point $s$ on the curve entirely disconnected from the sources and sinks of $C$. Thus, there are points infinitesimally close to $s$ connected to the sources and sinks of $N$, i.e. $s$ is a boundary point for the (un)stable manifold of one of these points. Again, by the arguments above, this is ruled out. Therefore, $\bar{U} = \hat{C}$.

A similar argument applies to $\hat{N}$. Define $W$ to be the union of all points connected to the sources and sinks of $N$. Then $\bar{W} = \hat{N}$. The compact connected image region has thus been shown to be the union of two disjoint closures of open sets. But this is impossible, and our original assumption is contradicted.

## 6. Determining the Nature of the Singular Points

The previous section demonstrated that if the nature of the surface at the singular points is known (whether, concave, convex or saddle–shaped), then the surface solution is uniquely determined up to a overall additive constant, assuming it exists. This is true because all singular points are connected to any one singular point following a path of gradient curves. Since the depth $z$ can be solved for uniquely along these curves, the depth is known at every singular point assuming it is known at one.

In this section it is shown that the nature of the surface at the singular points is also uniquely determined. This implies that the solution, if it exists, is determined completely uniquely.

First, it is shown that the singular saddle points are uniquely identifiable as such. Suppose that there are two solutions in which some number $m$ of the saddle points in solution $A$ become sources or sinks in solution $B$. Because of the topological formula,

$$N_i + N_f - N_s = 1, \tag{13}$$

the total number of singular saddle points in the object is fixed. (Of course the total number of singular points is fixed by the image). Thus $m$ singular sources or sinks in solution $A$ must also become saddles in solution $B$.

Divide the singular points in the image into two classes: $Y$, containing the points that change character between $A$ and $B$, and $N$, containing those which don't. Both classes are non–empty. This is true for $Y$ by assumption. From the above, $Y$ contains an even number of singular points. Since the total number is odd, $N$ is also non–empty.

From the arguments in the previous section, the sets $Y$ and $N$ are connected by gradient curves in both solutions $A$ and $B$. $N$ must contain at least one singular source or sink. Suppose that a singular source $n$ in $N$ connects to a singular point $y$ in $Y$. (The same argument will work if $n$ is a sink). $n$ will connect to $y$ in both solutions $A$ and $B$, since the assumption that it is a source or sink determines the curves connected to $n$ uniquely. Now $y$ is a sink in one of the two solutions (say $A$), and in that case there is an open region around $y$ such that every point contained in it connects to $y$. Consider the intersection of this open region with the open set of all points connected to $n$ and lying on gradient curves. The intersection is non–empty by assumption, and must be open. Thus there are infinitely many gradient curves connecting to $n$ which also connect to $y$, and these gradient curves are determined purely by the image intensity and the stipulation that $n$ is a source. In $B$ $y$ switches to a saddle, but the character of $n$ and the image intensity are unchanged, and therefore there will continue to be an infinite number of gradient curves connecting $y$ and $n$, which violates the assumption that $y$ is now a saddle. See Figure 10.

(Figure 10 should go here.)

The remaining possibility is that only saddle points in $N$ connect to $Y$, again for both $A$ and $B$. The rest of this discussion is valid for either solution. Since there are no connections between saddles in a generic solution, the saddle points in $Y$ can connect to other singular points just in $Y$. Each saddle point is the terminating point of two different gradient curves. These gradient curves cannot originate on the image boundary, since all gradient curves point outwards on the boundary, so they must originate on singular source points in $Y$. However, the number of saddle points is equal to the number of non–saddle points in $Y$, while the number of gradient curves that must connect a saddle to a source is twice this number.

It can be proven by induction that the $Y$ as described above are impossible. Suppose that there are just two points in $Y$, which is the minimal case. These must be a saddle and a source, with the saddle connected twice to the source. It is easy to see that this configuration is impossible. See Figure 11. Moreover, any configuration with an exterior boundary given by the two connecting curves between a saddle and a source has winding number +1. To show this, note that it is possible to replace the actual surface corresponding to the interior of the configuration by one that has a single concave singular point (i.e. a sink) in the interior. This new surface joins smoothly onto the surface corresponding to the external region, which is unchanged. The new configuration has winding number +1 by addition of the indices of the singular points. However, it also must have the same winding number as the original configuration, since the vector field in the external region, which is what enters into the winding number computation, is left unchanged. See Figure 12.

(Figures 11 and 12 should go here.)

Next suppose that the result has been proven for $Y$ containing $2m$ points, and consider $2(m+1)$. Select any saddle. It may be connected twice to the same source $s$, forming as before a closed figure with winding number +1. Label this figure a "reduced source", and smoothly contract it to a "source" point with index +1. This reduced configuration has at least two fewer points than the old, but its topology is unchanged in the region external to the reduced source. Moreover, the "reduced" configuration containing this "reduced source" still has equal numbers of saddles and non-saddles. By induction, it is impossible for all gradient curves terminating on $Y$ saddle points in this new configuration to originate from $Y$ source points. However, this means that the old $2(m+1)$ configuration is also impossible: it has no additional possibilities for connecting saddles and sources.

The second possibility is that the chosen saddle connects to two source points. Once again, this region can be enclosed and shrunk, and the above argument repeated.

These arguments show that the saddle points of the flow are uniquely determined. It remains to demonstrate that the source and sink points are also uniquely assigned. The gradient curves terminating at sinks can not connect to the image boundary, and must therefore emerge from singular points. As already stated, the region $S$ containing all points connected to a sink $s$ and lying on gradient curves is open. Its boundary $\delta S$ contains no points from the image boundary. Therefore, by arguments similar to previous ones, the boundary is composed of gradient curves joining singular points. None of these boundary singular points can be sinks, since otherwise there would be gradient curves terminating at two different singular points. The only possibility for the boundary therefore is an alternating series of singular sources and saddles connected by gradient trajectories. The entire region with boundary has winding number +1, and all external gradient curves connecting to this region originate at the boundary. Thus the entire region is essentially an extended source, the reverse of the sole singular point contained within. See Figure 13.

(Figure 13 should go here.)

The above argument applies equally well for the same flow with the direction of time reversed. In other words, for this type of contained flow disconnected to the image boundary, the interior singular point could equally well be a source. It then follows that the boundary consists of an alternating series of sinks and saddles. However, since sinks (i.e. locally concave singular points) can only appear associated with contained flows, this flow type may be a good heuristic for detecting sinks. Moreover, this type of flow may be fairly robust, and may be detectable with a fairly crude characteristic strip algorithm.

By the same token, any non–saddle point connected to the image boundary must be a source. It is now shown that one can uniquely identify the sources and sinks by induction. For a single source, there is clearly no difficulty of identification. Now suppose that for an image with $2m + 1$ singular points, one can uniquely identify the nature of each point. Consider an image with $2m + 3$ singular points. If it has no contained flows, then every non–saddle point must be a source. See Figure 14. If it has a contained flow, define a reduced image derived by contracting the contained flow to a point. In this reduced image, the nature of every point is uniquely identifiable by induction. However, once the nature of the reduced point is known, the nature of every point in it is also known, since by the arguments above the boundary properties of the region determine those of the interior point, and vice versa.

(Figure 14 should go here.)

This suffices to show, finally, that for an image of a light region contained in a black background, where the reflectance function is known and given by eq. 6, there is a unique solution for a generic surface that is smooth and non–self–occluding.

Note that local uniqueness around the saddle solutions has not been used in order to derive this result, although this uniqueness is established in Appendix B.

It is probably true that these results extend in some measure to a reflectance function corresponding to more general light source directions.

## 7. Algorithmic Implications

The difficulty with pure variational algorithms for solving shape from shading is that being near–sighted, they are slow. As indicated by this proof, there is an ambiguity in patching together the local solutions around different singular points that can be resolved by global reasoning. A variational algorithm that neglects to think globally will spend time investigating the wrong local solutions around fixed points. On the other hand, a variational algorithm is desirable at some stage for the sake of accuracy.

The following approach for at least the initial stages of a shape from shading algorithm is proposed. The idea is to find quickly the global structure of the solution, and a rough globally valid solution. Then the variational algorithm could be applied to refine the solution. First, the characteristic strips, or some more continous equivalent, are integrated outwards from every singular point assuming that it is a source. (The flows for sources and sinks in the image plane are equivalent). This is a highly parallelizable operation; every characteristic strip at every fixed point could be integrated independently. The important thing to note is how many of these trajectories reach the boundary. For contained flows, some trajectories are likely to escape the confinement region due to noise, but many will probably fall back in, retracing their steps in an unphysical way. The image intensity essentially looks like a crater with a wavy edge containing a hilltop. The hilltop corresponds to the interior singular point, and the wavy edge to the trajectories joining the boundary singular points. See Figure 13. Returning to the Hamiltonian picture, one can think of the characteristic trajectories as traced out by marbles rolling down from the interior hilltop; many will be confined within the crater. Possibly, one could ensure this confinement by reducing slightly the initial energy of the trajectories originating at the singular point. This image brightness configuration might also be looked for more directly. Thus a contained flow may be detectable in these ways.

In the next step, points whose flows seemed to be confined would be labelled as such, and the confinement regions collapsed. Then, at the higher level, the process would be repeated. Some of these collapsed points might in their turn become part of new collapsed regions. In this way one would obtain a hierarchy of collapsed regions, eventually terminating in a skeleton diagram consisting of a string of connected sources and saddles. Two of the sources (those at the ends of the string) must connect only to a single other fixed point. (See Figure 14.) This property may be quite robust, and uniquely identifies these singular points as sources. This then allows the nature of every point to be identified. Additionally, these sources could be identifiable at the initial go–round if they are actual singular points rather than collapsed regions. This would allow immediate identification of the nature of all singular points in one step. Note also that because of the structurally stable property of the flow, its character will not be modified by small perturbations. It may even be possible to do some degree of smoothing if this is advantageous to reduce noise, since this might not change the topological character of the solution, which is all that is really wanted at this stage.

Next, one could use the properly interpreted characteristic strip results to derive a rough globally valid solution. This solution could be refined by integrating the characteristic strip solutions anchored by the power–series computed solution at the now–identified saddle points. Finally, a variational algorithm starting with the characteristic strip derived solution would further refine the results.

It is of course unclear whether the characteristic strip method will be sufficiently robust for this method to be effective.

## 8. Existence of Solutions

Recently, Horn (Horn 1989b) has proven that for the reflectance function of eq. 6, and smooth non–self–occluding objects, there exist images that are impossible, in the sense that there exist no objects of which they can be the images. His image examples with this property are rather special. An argument is presented here that indicates that images are *generically* *impossible.* Specifically, it will be shown that a generic image of an object, under arbitrarily small perturbations, turns into one that cannot be interpreted as generic. The type of perturbation that will do this is quite unrelated to the type of perturbation that will convert a non-generic surface solution into a generic one. It will also be argued that *any* image can be converted by a small perturbation into one that is strictly impossible. These arguments are not yet at the level of proofs.

For completeness, a simple example of the type of impossibility result due to Horn is first reviewed. Claim: any image corresponding to the standard reflectance function which containes a smooth closed curve of maximally bright points, and with the interior of the curve never maximally bright, is impossible, i.e. it cannot depict any smooth, non-self-occluding object.

Proof: The maximally bright curve is contained in a single plane. The region comprised of this curve and its interior is compact. Any smooth object must have either a maximal or minimal depth in the interior of this curve. But at such a point the depth $z$ is stationary, the surface normal is in the $\hat{z}$ direction, and the image of this point must be maximally bright, i.e. a singular point. But no such point exists in the image, so it is impossible.

This result can be partially extended to the case of generic images of objects wholly contained in the field of view. For such objects, which protrude towards the viewer at the boundary, the number of singular points in the non-black region must be odd, as was already mentioned earlier.

It is now argued that images can be rendered impossible by an arbitrarily small perturbation. The Hamiltonian viewpoint, as discussed earlier, equates the trajectory of a characteristic strip with the motion of a point particle in a potential, i.e. a particle moving in the image plane acted on by position-dependent but not time-dependent forces. As remarked in Appendix B, the main obstruction to interpreting images as representing physical objects is the crossing of characteristic trajectories. This is because a non-self-occluding object cannot have more than one surface slope corresponding to a single point on the image. Also, the surface solution for an image can be interpreted as the ensemble of characteristic trajectories sweeping out the whole of the imaged region, as was discussed in the section on uniqueness. Thus all trajectories have neighboring, infinitesimally close trajectories with which they are never allowed to intersect. However, it is easy to introduce a small local perturbation in the potential that will cause two neighboring trajectories to move towards each other. A potential $V$ represents nothing more than an encoding of the space-dependent forces acting on the "particles" tracing out these trajectories. One simply adds forces along two neighboring trajectories over some region pushing them towards each other. In terms of the potential, this can be done by introducing a local trough in the potential in between the two neighboring unperturbed trajectories. The particles will be perturbed towards each other and the trajectories will cross.

One can use the results on uniqueness to show that one cannot escape from this crossing of trajectories, at least for generic images. For such images, as argued in the section on uniqueness, every trajectory terminates or originates on a singular point. The only possible non-uniqueness has to do with how the different solutions are chosen at different singular points. But the global reasoning that goes into this decision will certainly not be affected if the perturbation is introduced near the boundary, in an open region where all the trajectories are "rushing downhill" towards the boundary. The ultimate destination or origin of no trajectory will be affected by this perturbation. Thus such an a perturbation, arbitrarily small, will render a generic image non-interpretable as a generic surface. An example is depicted in Figure 15.

(Figure 15 should go here.)

Even if one relaxes the condition on the boundary that the trajectories point outwards, so that the global uniqueness proof of the previous sections does not apply, one still has the local uniqueness results at hyperbolic fixed points found in Appendix B. These local solutions can be extended over larger regions by following the characteristic trajectories. Even if all choices of the local solutions are possible, this is still just a finite ambiguity (it equals the number of singular points times four), while the number of perturbations is infinite. It should be possible to introduce enough local perturbations in different regions of the image so that all the different choices of local solutions at singular points will be ruled out by trajectory crossings.

Finally, it follows from the results of Appendix B for saddle singular points, that the four gradient curves that actually connect to the saddle point are uniquely determined locally. Suppose there is a possible image in which a saddle connects to another singular point (e.g. a source). The gradient curve connecting these two will be affected by a perturbation in the intensity along its path. By continuity, there will still be some gradient curve originating at the source and terminating at the saddle. In general however, in the neighborhood of the saddle this curve will no longer match onto one of the four uniquely determined paths which connect to the saddle, and the perturbed image is therefore impossible. Thus any image with saddle points, i.e. any generic image with more than one singular point, is likely to be impossible.

## 9. Conclusions

The image irradiance equation,

$$I(x,y) = \frac{1}{(1+p^2+q^2)^{1/2}}, \tag{14}$$

where $p \equiv \frac{\partial z}{\partial x}$, and $q \equiv \frac{\partial z}{\partial y}$, has been considered. The image intensity $I$ is assumed to be smooth, and non–zero only in a compact region of the image plane. Then a generic, non–self–occluding solution for the object surface $z(x,y)$ was shown to be uniquely determined if it exists. It was noted that generic surface solutions in general position correspond to structural stability of the image plane flow.

Existence and uniqueness of locally convex and concave solutions in the neighborhood of a singular point were proven. Uniqueness of saddle solutions up to a two–fold ambiguity, for the generic case, was also demonstrated. The condition necessary for the existence of local saddle solutions was given. Finally, a possible approach to an algorithm was suggested, which could determine via global reasoning the nature of the surface solution at individual singular points.

In addition, the existence of solutions to shape from shading was discussed. It was claimed that images are generically impossible, so that generically they correspond to no consistent physical object.

## 10. Acknowledgements

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## A. Technical Results

### A.1 Local Characterization of Fixed Points

Consider eq. 10 in the neighborhood of a fixed point. The matrix of partial derivatives of the right hand side of this equation with respect to $x$, $y$, is just the matrix of second derivatives of $z$:

$$\mathbf{A} \equiv \frac{\partial^2 z}{\partial x_i \partial x_j}. \tag{15}$$

where $(x_1, x_2) \equiv (x, y)$. By the hyperbolicity assumption, $\mathbf{A}$ has no zero eigenvalues. Then the Grobman–Hartman theorem (Palis 1982) states that there exists a homeomorphism defined in a neighbohood of the fixed point, mapping orbits of the flow to those of the linear flow $\exp(t\mathbf{A})$, preserving the sense of the orbits, and also the parameterization by time.

The meaning of the linear flow is that

$$\vec{x} \rightarrow \vec{x}(t) = e^{t\mathbf{A}}\vec{x}. \tag{16}$$

In a coordinate frame in which $\mathbf{A}$ is diagonal,

$$\vec{x}(t) = (\exp(tA_{11})x, \exp(tA_{22})y). \tag{17}$$

Depending on the sign of $A_{11}$, $A_{22}$, the flow will be convergent, divergent, or saddle-shaped. The existence of a homeomorphism between the complete flow governed by eq. 10 and the linear flow means that these flows are topologically equivalent locally. Thus, the complete flow also must have one of these three forms.

### A.2 Gradient Curves Converge to Singular Points

Define the $\omega$-limit of a curve to be the points to which the curve converges as $t \rightarrow \infty$.

More precisely, it is the set of points $q$ for which there exists a sequence of points from the curve that converges to $q$. Similarly, one defines the $\alpha$-limit to be those points from which the curve originates, i.e. the points to which it converges when the direction of time is reversed. Points in these sets may be on the image boundary as well as in the interior of the image.

Claim: Interior points in these sets are singular points. See (Palis 1982). Consider an interior point $w$ that is contained in the $\omega$-set for example, and suppose that $\nabla z$ is non-zero at $w$. The trajectory through $w$ itself goes from $z$ smaller than that of $w$ to larger $z$. By continuity, trajectories through all points close enough to $w$ will also reach $z$ greater than that of $w$. Thus no gradient curve can converge to $w$, since if it approaches too closely it will attain larger $z$ and never be able to return to $w$.

## A.3 Gradient Curves Point Outwards at the Boundary

Since the object advances towards the viewer from the boundary, the gradient curves near the image boundary will tend to point towards it, since depth increases along these curves. Actually, as shown here, for each image boundary point there exists a curve that smoothly converges to it.

The curves are determined on the 3D object by the standard equations:

$$\frac{dx}{dt} = p, \qquad \frac{dy}{dt} = q, \qquad \frac{dz}{dt} = p^2 + q^2.$$

Let $\alpha(x)$ be a real infinitely differentiable function such that $1 \geq \alpha(x) \geq 0$, and $\alpha(x) = 1$ if $|x| \leq .2$, $\alpha(x) = 0$ if $|x| \geq .4$. Such a function can be shown to exist. Consider the modified equations for the gradient curves:

$$\frac{dx}{dt} = pR(\frac{1}{p^2+q^2}), \qquad \frac{dy}{dt} = qR(\frac{1}{p^2+q^2}), \qquad \frac{dz}{dt} = (p^2+q^2)R(\frac{1}{p^2+q^2}), \tag{18}$$

with

$$R(x) \equiv 1 + \alpha(x)(x-1). \tag{19}$$

These equations determine exactly the same gradient curves as the previous ones, since at every point the tangent direction to the curve is the same as before. Moreover, the new vector field, i.e. the right hand sides of these equations, is as smooth as the previous one. However, now the flow described by these equations can be defined smoothly on the object surface *on a region containing the occluding boundary* as well as the visible surface.

Consider a point on the image boundary and a corresponding point on the occluding boundary of the object. (Every point on the image boundary corresponds to at least one point on the object's occluding boundary—just consider a line in the image that converges to the image boundary point, and its projection on the object surface.) For smooth surfaces with no self–occlusion, the image boundary is differentiable (Giblin 1987). Suppose for convenience that the tangent to the image boundary line at the image point is in the $x$ direction. Then the tangent plane to the surface at the object point is given by the $x$ and $z$ directions. Because the surface is smooth, it can be parameterized locally by $y(x, z)$. Also,

$$p = \frac{\partial z}{\partial x}|_y \equiv z_x|_y = -\frac{y_x|_z}{y_z|_x}, \quad q = z_y|_x = \frac{1}{y_z|_x}, \tag{20}$$

and

$$p/q = \frac{z_x|_y}{z_y|_x} = -y_x|_z. \tag{21}$$

The notation $y_z|_x$ signifies a partial derivative with respect to $z$ keeping $x$ fixed. From the above:

$$\frac{1}{p^2 + q^2} = \frac{q^{-2}}{1 + p^2/q^2} = \frac{(y_z|_x)^2}{1 + (y_x|_z)^2}.$$

Near the boundary point, the equation for the 3D gradient curve becomes:

$$\frac{d\vec{r}}{dt} = \begin{pmatrix} -y_x|_z y_z|_x/(1 + (y_x|_z)^2) \\ y_z|_x/(1 + (y_x|_z)^2) \\ 1 \end{pmatrix}, \tag{22}$$

where $\vec{r} \equiv (x, y, z)$. Near this point, $y_z|_x$ and $y_x|_z$ go smoothly to zero. Since the surface is assumed smooth, the right hand side of this equation is a smooth vector field. Thus the image boundary points can be treated essentially as interior points, with the caveat that this vector field is no longer a gradient vector field. For some point on the image boundary, consider the 'gradient' curve that passes through it. (The 'gradient' curve is identical to a gradient curve on the object's visible surface, but extends smoothly to the invisible region as well.) Clearly, this gives the unique gradient curve that converges to the point in question.

## A.4 The Applicability of the Palis–Smale Theorem

It is shown that there is a contour within the image interior such that the vector field is pointing outwards on this contour. First, at each point $b$ on the image boundary, there is an open region $R_b$ centered at this point such that throughout $R_b$ the direction of the vector field does not deviate from the normal to the image boundary at $b$ by more than an angle $d\theta << 1$. This is true because of the smoothness of the vector field direction $p/q$ near the boundary, which was shown in the previous sub-section. Similarly, there is an open interval of the image boundary such that the image boundary tangent direction does not vary by more than $d\theta$ over this interval. This follows from the fact that image boundary is smooth (Giblin 1987). Therefore there is clearly an open interval $I_b$ of the image boundary around $b$, and some distance $\epsilon_b$, such that there is a smooth contour transversal to the vector field joining any two points whose distances from the boundary points of $I_b$ are less than $\epsilon_b$. Since the image boundary is compact, there is a finite subset of such open intervals that covers the boundary. Let $\epsilon_{min}$ be the least of the $\epsilon_b$ associated with these intervals. Then one can clearly patch together contours with endpoints less than $\epsilon_{min}$ from the image boundary, corresponding to the different intervals, to form a smooth closed contour that is everywhere transversal to the vector field.

From the Palis–Smale theorem, it now follows that any gradient field of the type under consideration, and therefore any surface, can be approximated arbitrarily well by a structurally stable gradient vector field over the image interior. It is possible that one might have to extend the definition of the new vector field past the old image boundary in order to attain the occluding boundary of the new surface.

## B. Existence and Uniqueness of Local Solutions

This Appendix deals with the local uniqueness of solutions in a neighborhood around a singular point. Since only hyperbolic fixed points are considered, there are just three classes of solutions: the surface is either convex, concave, or saddle–shaped in the fixed point's neighborhood. The first two cases are easy. It has already been noted by Bruss (Bruss 1982) and Saxberg (Saxberg 1989) that the Stable Manifold Theorem implies that the locally convex and concave solutions in a neighborhood of the singular point are unique; these are given by the stable or unstable manifolds.

A proof is included for completeness, which has possibly some new aspects. The arguments for the convex and concave cases are exactly parallel, so just the second will be discussed, which corresponds to an image flow with all gradient curves converging on the fixed point $f$. The 4D dynamical system is:

$$\dot{x} = p, \quad \dot{y} = q, \quad \dot{p} = -V_x, \quad \dot{q} = -V_y, \tag{23}$$

with $2V \equiv 1 - 1/I^2$. The linearized version of this equation is:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ p \\ q \end{pmatrix} \simeq \begin{pmatrix} 0 & 1 \\ -\mathrm{V_{ij}} & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ p \\ q \end{pmatrix}, \tag{24}$$

where $V_{ij}$ is the matrix of seond derivatives of the potential $V$. Let $E$ denote the matrix in the above equation. It is easy to verify that $E$ has two eigenvectors with negative eigenvalues, and two with positive, assuming a hyperbolic fixed point, with no zero eigenvalues. The saddle nature of the fixed points follows from Liouville's theorem for Hamiltonian dynamical systems.

Consider the set of all trajectories $(x(t), y(t), p(t), q(t))$ that converge to $f$ as $t \to \infty$ in the 4D phase space. Let $W^s(f)$ denote this set, which includes $f$ itself. Then the Stable Manifold Theorem states that $W^s(f)$ is a manifold, and the tangent space to $W^s(f)$ at $f$ is spanned by the eigenvectors of $E$ associated with negative eigenvalues. Since there are two such eigenvectors, $W^s(f)$ is a 2D manifold. Any concave solution to shape from shading can clearly be interpreted as a 2D manifold consisting of the points $(x,\ y,\ p(x,y),\ q(x,y))$ which is contained in this stable manifold, and must equal it. Therefore there is at most one concave solution, which will be given by the stable manifold. Moreover, the stable manifold is as smooth as the quantities on the right-hand sides of eq. 23 (Guckenheimer 1983). Thus, if the intensity function has $r$ continous derivatives, then the stable manifold (i.e. $p$ and $q$) has $r-1$ continuous derivatives, and $z$ is $C^r$.

All points in this solution satisfy $H = 0$ since $H$ is constant along the trajectories and the trajectories have the fixed point as a limit. $z$ can be recovered by integrating $\dot{z} = \dot{x}p + \dot{y}q$ along the trajectories of the stable manifold. One can verify from the known tangent space to the stable manifold at the singular point that the consistency condition $p_y = q_x$ is satisfied at the fixed point. If it is satisfied everywhere, then the solution for $p$, $q$ corresponds to a consistent solution for $z$.

From eq. 23 above,

$$\frac{d}{dy}\dot{p} - \frac{d}{dx}\dot{q} = 0. \tag{25}$$

Since $\dot{p} = p_x p + p_y q$, and $\dot{q} = q_x p + q_y q$, this is:

$$p_{xy}p + p_{yy}q - q_{xx}p - q_{xy}q + (p_x + q_y)(p_y - q_x) = 0. \tag{26}$$

Also,

$$\frac{d}{dt}(p_y - q_x) = \left(p\frac{d}{dx} + q\frac{d}{dy}\right)(p_y - q_x) = p_{xy}p + p_{yy}q - q_{xx}p - q_{xy}q. \tag{27}$$

Thus,

$$\frac{d}{dt}(p_y - q_x) = -(p_x + q_y)(p_y - q_x). \tag{28}$$

For the concave case, $p_x + q_y$, which equals the trace of the matrix of second derivatives of $z$, is negative at and near the fixed point, since the latter is hyperbolic. Also, the trajectories in the stable manifold converge to the fixed point as the time $t \to \infty$. From the above equation, if $p_y - q_x$ is non-zero in the neighborhood of the fixed point, then it evolves to infinity along the gradient curve as it approaches the fixed point. Since this contradicts the smoothness of the $p$, $q$ solution,

$$p_y - q_x = 0, \tag{29}$$

everywhere in the stable manifold. The solution for $p$, $q$ corresponds to a consistent solution for $z$. In particular, $z$ obeys $\dot{x} = \frac{\partial z}{\partial x}$, and $\dot{y} = \frac{\partial z}{\partial y}$. These statements also hold for the locally convex solution at a fixed point.

Lastly, the fact that the image plane $x$ and $y$ directions have non–zero overlap with the tangent plane to the stable manifold means that there is a one–to–one relation between the stable manifold and the image plane locally around the fixed point, asumming it is hyperbolic. The only obstruction to defining the surface solution $z$ over the whole region of the stable manifold is the possibility that the stable manifold will fold over, and project non–uniquely to the image plane.

The more interesting question is how many saddle solutions exist. Saxberg has claimed that there are no more than two saddle solutions. The nature of the probable restrictions on this result should become clear below.

Consider the image irradiance equation in the form:

$$-V(x,y) = \frac{1}{2}p^2 + q^2 \equiv \vec{s} \cdot \vec{s}, \tag{30}$$

where $\vec{s} \equiv (p,q)$. Differentiating both sides once yields zero at a fixed point. Taking second derivatives yields:

$$-V_{ij} = 2(\vec{s}_{ij} \cdot \vec{s} + \vec{s}_i \cdot \vec{s}_j). \tag{31}$$

The subscripts $i$, $j$ denote differentiation with respect to the image coordinates $x$ and $y$. At a fixed point this is:

$$-V_{ij} = \vec{s}_i \cdot \vec{s}_j.$$

In a coordinate system rotated so that the cross terms $V_{xy}$ vanish, this becomes:

$$-V_{xx} = z_{xx}^2 + z_{xy}^2, \quad 0 = (z_{xx} + z_{yy})z_{xy}, \quad -V_{yy} = z_{yy}^2 + z_{xy}^2. \tag{32}$$

As long as $V_{yy} \neq V_{xx}$, these imply that $z_{xy} = 0$ and

$$z_{xx} = \pm V_{xx}^{.5}, \quad z_{yy} = \pm V_{xx}^{.5}. \tag{33}$$

The condition that $V_{yy} \neq V_{xx}$ is the first restriction on the validity of the result. For the saddle solutions, $z_{yy}$ and $z_{xx}$ are of opposite sign. Thus for these solutions the second derivatives of $z$

are uniquely determined by the image up to a two-fold ambiguity, except for the exceptional case where the intensity obeys $I_{xx} = I_{yy}$. Select one of the two solutions, and set $z_{xx} = a > 0$, $z_{yy} = -b < 0$. $a$ and $-b$ are the principal curvature values for the surface at the fixed point.

Now we return briefly to the concave stable solution. In the image plane, the characteristic trajectories corresponding to the concave solution obey the 2D dynamical system:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{\nabla} z, \tag{34}$$

in a *local region* around a fixed point. The two dimensional vector field $\chi_2(x,y) \equiv \vec{\nabla} z$ is smooth (say $C^r$) if the image intensity is, as follows from the Stable Manifold Theorem stated above. The matrix of derivatives $D\chi_2$ has two negative eigenvalues ($-a$ and $-b$) at the fixed point. Then there are theorems which state that, apart from certain restricted cases, there exists a $C^r$ smooth change of coordinates $(x,y) \to (Q_1, Q_2)$ such that eq. 34 transforms into the *linear* equations:

$$\dot{Q}_1 = -aQ_1, \qquad \dot{Q}_2 = -bQ_2.$$

on a neighborhood of the fixed point. The situation is a little complicated because the theorems depend on the value of $r$, the degree of smoothness of the vector field $\chi_2$. Some examples are given below.

A theorem of Poincare (cited by Sternberg (Sternberg 1957), (Sternberg 1957) and Arnold (Arnold 1983)) applies to the case $r = \omega$, that is for $\chi_2$ an analytic vector field. It states that as long as neither $a$ nor $b$ is a multiple of the other, then the above linearizing change of coordinates exists, and is analytic. Another example (Ruelle 1989) applies to $r = (1,1)$, i.e. the case where the first derivative of $\chi_2$ is *Lipschitz* (this is better than once differentiable, but not as good as twice differentiable). Then the change of coordinates always exists and is $C^1$. See also (Nelson 1970). The eigenvalue conditions, of which one example is given above, represent another example of the type of restriction that applies to the result. These restrictions always involve integer linear dependences among the eigenvalues of $D\chi_2$. Henceforth these restrictions, which are certainly non-generic, will be neglected, and it will be assumed that a change of coordinates as smooth as desired can be effected.

The image irradiance equation was defined earlier in terms of the Hamiltonian function:

$$H \equiv \frac{1}{2}(p^2 + q^2) + V(\vec{x}) = 0, \tag{35}$$

In terms of the four dimensional Hamiltonian system, the smooth coordinate transform is a point transform. It thus constitutes a *canonical* transform for the Hamiltonian system. The meaning of this is that one can define new variables $P_1$ and $P_2$ in terms of which the Hamiltonian can be written:

$$H = \frac{1}{2}(P_1^2 + P_2^2) + V'(Q_1, Q_2), \tag{36}$$

such that Hamilton's equations are obeyed for these new variables:

$$\dot{Q}_i = \frac{\partial H}{\partial P_i}, \quad \dot{P}_i = -\frac{\partial H}{\partial Q_i}. \tag{37}$$

However, the above equations imply that:

$$P_1 = \dot{Q}_1 = -aQ_1, \quad P_2 = \dot{Q}_2 = -aQ_2,$$

and

$$-a\dot{Q}_1 = a^2 Q_1 = -\frac{\partial V}{\partial Q_1}, \quad -b\dot{Q}_2 = b^2 Q_2 = -\frac{\partial V}{\partial Q_2}. \tag{38}$$

As this last equation is true for all $Q_1, Q_2$ in a region of the fixed point, it implies that $V = (1/2)(a^2 Q_1^2 + b^2 Q_2^2)$, and:

$$H = \frac{1}{2}(P_1^2 + P_2^2 - a^2 Q_1^2 - b^2 Q_2^2). \tag{39}$$

The solutions to Hamilton's equations are:

$$Q_1 = c_1 e^{-at} + c_3 e^{+at}, \qquad Q_2 = c_2 e^{-bt} + c_4 e^{+bt}, \tag{40}$$

for some constants $c_i$. A solution satisfying the image irradiance equation $H = 0$ obeys:

$$a^2 c_1 c_3 + b^2 c_2 c_4 = 0. \tag{41}$$

This equation immediately implies that for either the odd pair or the even pair, the $c_i$ have different signs, while for the other pair they have the same sign.

We are interested in the solutions of the above equations that correspond to saddle points of the object surface, and saddle points of the two dimensional image plane flow. Thus one must look for two dimensional submanifolds of the four dimensional phase space $(Q_1, Q_2, P_1, P_2)$ which could correspond to two dimensional saddle flows in the image plane. These submanifolds must satisfy the stringent condition that they project uniquely onto the $(Q_1, Q_2)$ plane, or equivalently, onto the image plane $(x, y)$.

Saddle flows in two dimensions contain two gradient curves that converge to the fixed point, and two that diverge from the fixed point. Thus the two dimensional submanifold if it exists must contain two trajectories from the stable manifold of the Hamiltonian system, and two from the unstable manifold.

The stable manifold consists of trajectories:

$$(Q_1, Q_2) = (c_1 e^{-at}, c_2 e^{-bt}), \tag{42}$$

while the unstable trajectories are:

$$(Q_1, Q_2) = (c_3 e^{at}, c_4 e^{bt}). \tag{43}$$

Henceforth, it will be assumed that $b > a$. Then all these curves are tangent to the $Q_1$ axis at the origin, unless their $Q_1$ component exactly vanishes. Suppose one of the four trajectories $S$ is:

$$(Q_1, Q_2) = (c_{1s} e^{-at}, c_{2s} e^{-bt}), \tag{44}$$

with $c_{1s}$, $c_{2s} > 0$. It will now be shown that this leads to a contradiction, and that the stable–unstable trajectories must lie on the $Q_1$ and $Q_2$ axes.

Let a second trajectory $U$ of these four be:

$$(Q_1, Q_2) = (c_{1u} e^{at}, c_{2u} e^{bt}). \tag{45}$$

The other trajectories participating in the solution neither originate nor terminate at the fixed point. For any two points $s$ on $S$ and $u$ on $U$, there exist such trajectories that approach these two points and also the fixed point arbitrarily closely, since this is true for a two dimensional saddle flow. See Figures 3 and 9. For such a trajectory, let the time $t = 0$ at the point of closest approach to $s$. Then $s_1 \sim c_1 >> c_3$ and $s_2 \sim c_2 >> c_4$. The condition eq. 41 implies that $c_3,\ c_4 \neq 0$, and that this trajectory crosses either the $Q_1$ or $Q_2$ axis. For convenience, assume without loss of generality that $c_3 > 0$, so that it crosses the axis $Q_2 = 0$. Since the trajectory cannot cross $U$, $c_{4u} < 0$, and also $c_{3u} \geq 0$. Clearly, since the trajectory approaches $U$ at large times, $U$ is in the quadrant with $Q_1 > 0$, $Q_2 < 0$. The stable and unstable trajectories $S$ and $U$ connecting to the saddle point are shown in Figure 16, in the linearized coordinates $Q_1,\ Q_2$.

(Figure 16 should go here.)

If at time $t = t^* >> 1$ the trajectory attains its minimum $Q_1$, then $c_3 \sim s_1 \exp(-2at^*)$. At large times, the trajectory approaches:

$$(c_3 e^{at}, c_4 e^{bt}), \tag{46}$$

with

$$c_4 \sim -\frac{a^2 s_1^2}{b^2 s_2} e^{-2at^*}. \tag{47}$$

At the time $t = 2t^*$, the trajectory is approximately:

$$(s_1, -\frac{a^2 s_1^2}{b^2 s_2} e^{2(b-a)t^*}). \tag{48}$$

As the trajectory is required to approach $s$ and the fixed point arbitrarily closely, i.e. as $t^* \to \infty$, the magnitude of the second component grows arbitrarily large. However, $U$ is finite at $Q_1 = s_1$, and it follows that the trajectory must cross $U$, which is impossible.

Thus either $c_1$ or $c_2$ must be zero for the stable trajectory, while the other choice among $c_3$ and $c_4$ must be true for the unstable manifold. The stable and unstable directions for a saddle solution around a fixed point are uniquely determined apart from a two-fold ambiguity.

Since no trajectory in the solution can cross the $Q_1$ or $Q_2$ axis, it follows that all trajectories must have one of the two forms:

$$(Q_1, Q_2) = (c_3 e^{at}, c_2 e^{-bt}), \qquad (Q_1, Q_2) = (c_1 e^{-at}, c_4 e^{bt}). \tag{49}$$

Clearly, either one choice or the other must be true for *all* trajectories. Thus there are only two possibilities for saddle flows and saddle solutions around a fixed point. The saddle solutions, like the concave–convex solutions, vary smoothly under perturbations as follows from the Stable Manifold Theorem (Palis 1982).

However, it is important to note that a consistent saddle solution for $p$, $q$, may not be a consistent solution for $z$, if $p_y \neq q_x$. Existence of the saddle solutions for $z$ has not been established.

Under what conditions do the saddle solutions exist? Suppose that the consistency condition is satisfied:

$$\dot{x}_y - \dot{y}_x = 0. \tag{50}$$

In terms of the coordinates $w \equiv Q_1$, $v \equiv Q_2$, this translates into:

$$\left(w_y \frac{d}{dw} + v_y \frac{d}{dv}\right) \dot{x} - \left(w_x \frac{d}{dw} + v_x \frac{d}{dv}\right) \dot{y}, \tag{51}$$

with:

$$\dot{x} = \dot{w} x_w + \dot{v} x_v, \qquad \dot{y} = \dot{w} y_w + \dot{v} y_v. \tag{52}$$

Up to an overall determinant factor:

$$w_x \sim y_v, \qquad w_y \sim -x_v, \qquad v_x \sim -y_w, \qquad v_y \sim x_w. \tag{53}$$

Thus:

$$\begin{aligned}(x_w^2 + y_w^2)(\dot{w})_v - (y_v^2 + x_v^2)(\dot{v})_w - (x_w x_v + y_w y_v)((\dot{w})_w - (\dot{v})_v)) \\ +\dot{w}(\frac{d}{dy}(x_w) - \frac{d}{dx}(y_w)) + \dot{v}(\frac{d}{dy}(x_v) - \frac{d}{dx}(y_v)) = 0.\end{aligned} \tag{54}$$

From the equations of motion, $\dot{w} = \pm aw$ and $\dot{v} = \pm bv$ (corresponding to eqs. 45 and 49), the first two terms vanish, and:

$$(x_w x_v + y_w y_v)(\pm a - (\pm b)) = (\pm aw)(\frac{d}{dy}(x_w) - \frac{d}{dx}(y_w)) + (\pm bv)(\frac{d}{dy}(x_v) - \frac{d}{dx}(y_v)). \tag{55}$$

This is a condition purely on the coordinate transform $(x, y) \rightarrow (w, v)$. By previous arguments, this condition is satisfied for the stable or unstable manifold, when the same sign holds for $a$

and $b$. It will also hold for both saddle solutions if the terms involving $a$, and those involving $b$, separately vanish.

## C. Power Series Expansion Around a Saddle Fixed Point

In this Appendix, power series solutions of the image irradiance equation in the local neighborhood of a singular point, for a locally saddle-shaped surface, are considered. The previous appendix described the expansion through second order. These results are now extended to higher derivatives. Differentiating eq. 30 $n$ times with respect to $x$ and $m$ times with respect to $y$ yields the sum:

$$V^{(n,m)} = \sum_{i=0}^{n}\sum_{j=0}^{m}\binom{n}{i}\binom{m}{j}\ \vec{s}^{(i,j)}\cdot\vec{s}^{(n-i,m-j)}, \tag{56}$$

where $\vec{s}^{(i,j)}$ signifies a differentiation i–fold in $x$ and j–fold in $y$. At the singular point $\vec{s} = 0$ and the highest order of derivative appearing in the sum is $n+m-1$. The relevant terms are:

$$2\left(n\vec{s}_x\cdot\vec{s}^{(n-1,m)} + m\vec{s}_y\cdot\vec{s}^{(n,m-1)}\right). \tag{57}$$

Since $\vec{s}_x = (a,0)$ and $\vec{s}_y = (0,-b)$, this is just:

$$(na-mb)z^{(n,m)}.$$

All other terms in equation 56 involve derivatives of lower order which, by induction, can be assumed already known. Thus, if $a/b$ is irrational, all the $z^{(n,m)}$ are uniquely determined by the equations 56 obtained from eq. 30 by differentiating $n$ times with respect to $x$ and $m$ times with respect to $y$. Furthermore if $z$ is real analytic, then $z$ is determined uniquely by its power series in a neighborhood of the singular point.

On the other hand, what happens if $a/b$ is rational? Then for the smallest $\bar{m}$, $\bar{n}$, such that $\bar{n}a-\bar{m}b$ vanishes, $z^{(\bar{n},\bar{m})}$ is not determined at this order. Instead the equation gives a consistency equation for the intensity. The equation with $(n,m) = 2(\bar{n},\bar{m})$ will be in general a polynomial equation in $z^{(\bar{n},\bar{m})}$, and can determine it, but up to the usual ambiguity due to multiple roots. There could also be no real solution. Similarly, $z^{(2\bar{n},2\bar{m})}$ may be determined by the equation with $(n, m) = 3(\bar{n}, \bar{m})$, with the same type of ambiguity. Therefore, it appears possible that with $a/b$ rational the saddle solution may not exist, or else may be ambiguous (for the non–generic cases mentioned in the previous appendix). Of course this ambiguity would have little practical importance since it will generally appear in high orders of the derivative.

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## Figure Captions

Figure 1. A source. This image plane flow corresponds to a locally convex object surface. The source singular point, denoted by a white disk, corresponds to the point on the object whose depth is a local minimum. The lines represent curves following the gradient of the depth, in the direction of increasing depth. The level contours of the depth run perpendicular to these lines.

Figure 2. A sink. This image plane flow corresponds to a locally concave object surface. The sink singular point, denoted by a black disk, corresponds to the point on the object whose depth is a local maximum.

Figure 3. A saddle. This image plane flow corresponds to a locally saddle-shaped surface. The normal to the surface is in the viewer direction at the saddle point; the latter is denoted by a circle containing a cross. Two characteristic curves originate at the saddle point, and two terminate at it. The other trajectories, which fill up the plane, approach but do not attain the saddle point.

Figure 4. The index of a singular point, here a source. The unit vector representing the flow direction describes a counter-clockwise rotation as one make a counter-clockwise circuit along the circle containing the source.

Figure 5. The gradient curves at the image boundary are normal to the boundary.

Figure 6. Above: Two saddle points connected by a gradient curve, an example of a non-generic flow, i.e. one that is not structurally stable. Below: A small perturbation of the object or viewing conditions above gives rise to a structurally stable flow, in which the saddle points are not connected.

Figure 7. It is not possible for a sink to be on the boundary of the region $U_o$ connected to a source O.

Figure 8. A saddle point, and the gradient curve $L$ originating at the saddle point, are on the boundary of the region $U_o$ connected to a source $O$. Then there must be a gradient curve $I$, terminating at the saddle point, which is also on the boundary of $U_o$.

Figure 9. An image plane flow around a saddle point, in linearized coordinates. $V$ is the region over which the linearization is valid. $Q$ is a quadrant bounded by the gradient curves $L$ and $I$. Gradient curves that pass infinitesimally close to a point $l$ on $L$ also pass infinitesimally close to a point $i$ on $I$.

Figure 10. Above: Suppose there exists a solution for the object shape in which a source connects to a sink. Below: There is then no solution when the sink is converted into a saddle, while the source is unchanged, since a saddle cannot connect to an infinite number of gradient curves.

Figure 11. A configuration of a source and saddle, where the source connects twice to the saddle. This configuration is impossible.

Figure 12. This image plane flow corresponds to a volcano-shaped object, i.e. a mountain with a crater, in a top view. The connections between the source and saddle correspond to

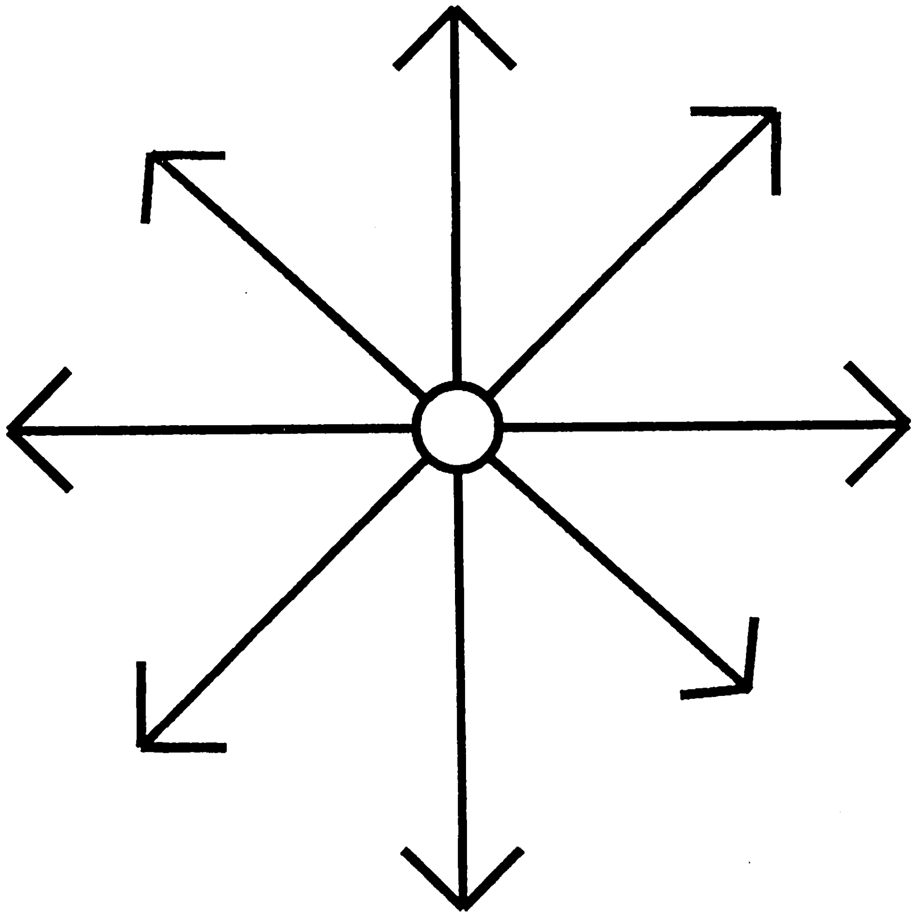
the rim of the crater. The crater is tilted with the source giving the point closest to the viewer, while the point on the rim furthest from the viewer corresponds to the saddle. The sink represents the bottom of the crater. Regardless of the shape of the crater within the rim, the surface exterior to the crater is equivalent to that for a convex mountain with a single peak. In other words, the flow of the gradient curves exterior to the rim is equivalent to that of a single source singular point.

Figure 13. This image plane flow corresponds to a volcano-shaped surface similar to that of Figure 12. The rim of the crater is given by the connections between the source and saddle singular points. In the region exterior to the crater, this configuration looks like an effective source. Sinks can appear only in contained flows of this type, where all lines terminating at the sink originate at singular points on the rim of the region containing the sink.

Figure 14. This image plane flow corresponds to an object with three peaks. The saddle points represent low points dividing the peaks. All generic surfaces without sinks are of this type, with a string of connected sources and saddles. Note that the sources at the ends connect only to one other singular point via a gradient curve, which distinguishes them as sources. In actuality, the gradient curves should flow normal to the image boundary.

Figure 15. A local perturbation of the image intensity near the image boundary has caused two characteristic trajectories to cross, creating an impossible image.

Figure 16. The flow of gradient curves in the neighborhood of a saddle point, in linearized coordinates. The stable and unstable trajectories $s$ and $u$ are both tangent to the $Q_2$ axis at the saddle point. A trajectory which passes infinitesimally close to a point $s$, and also the saddle point, will cross the unstable trajectory $u$, which is a contradiction.



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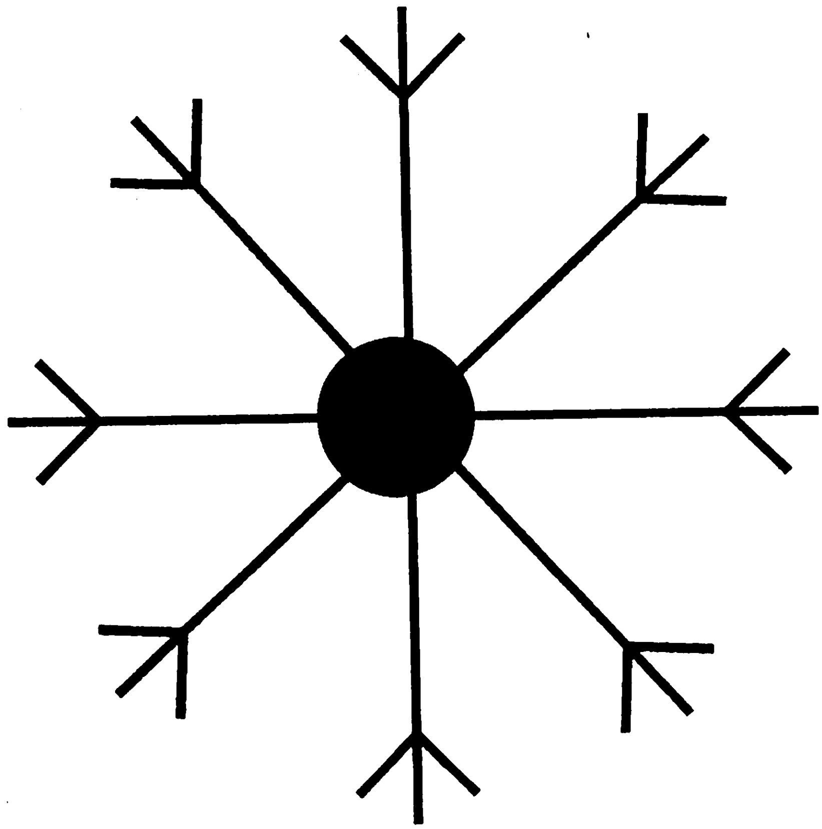

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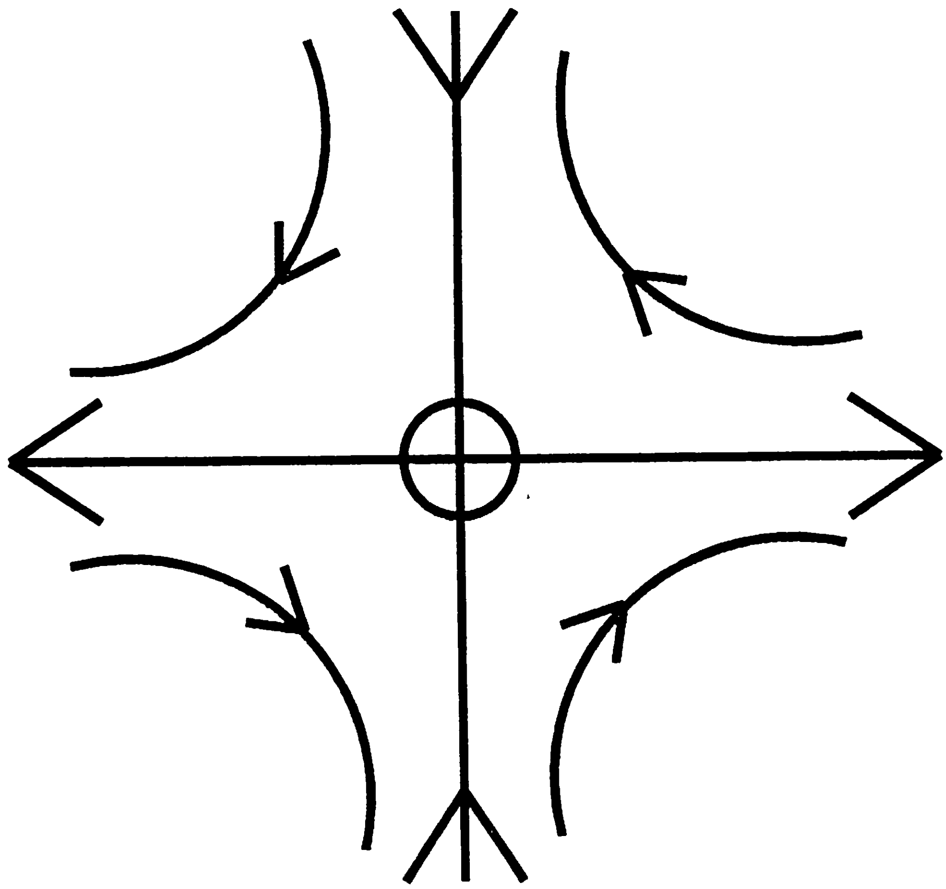

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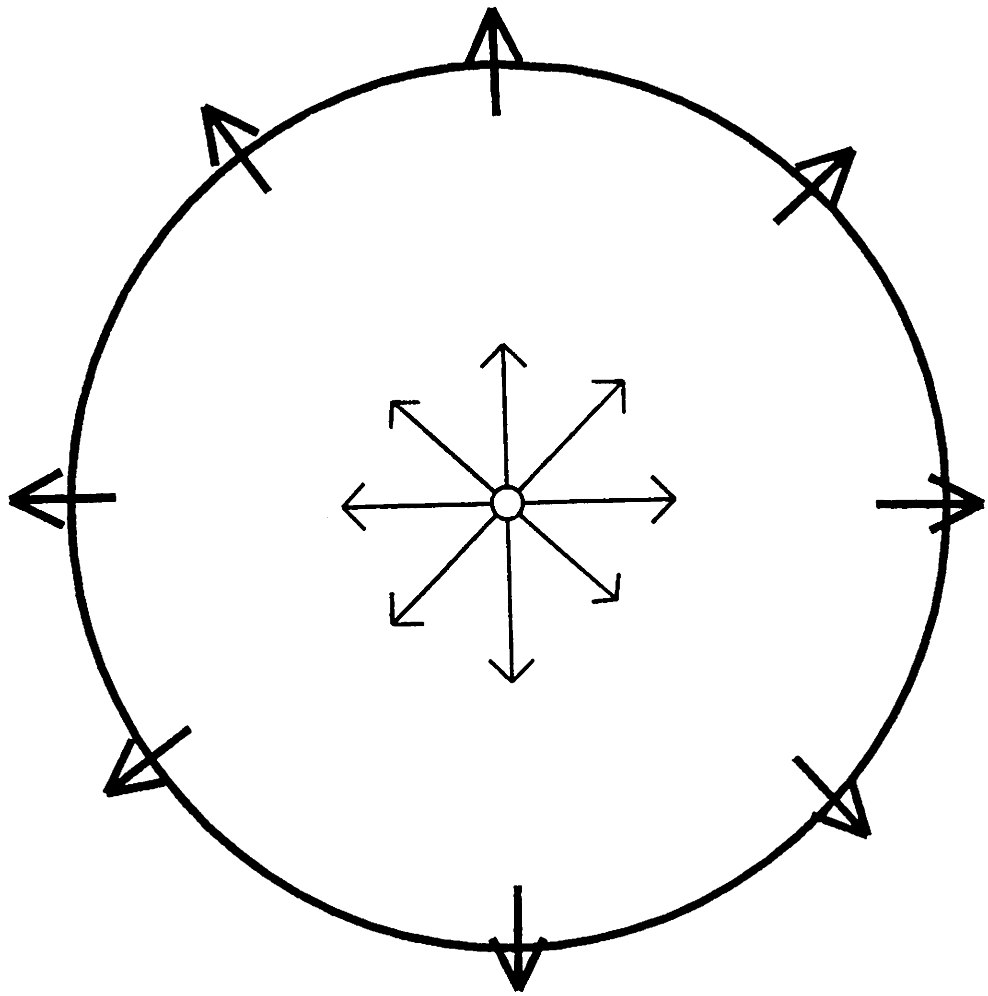

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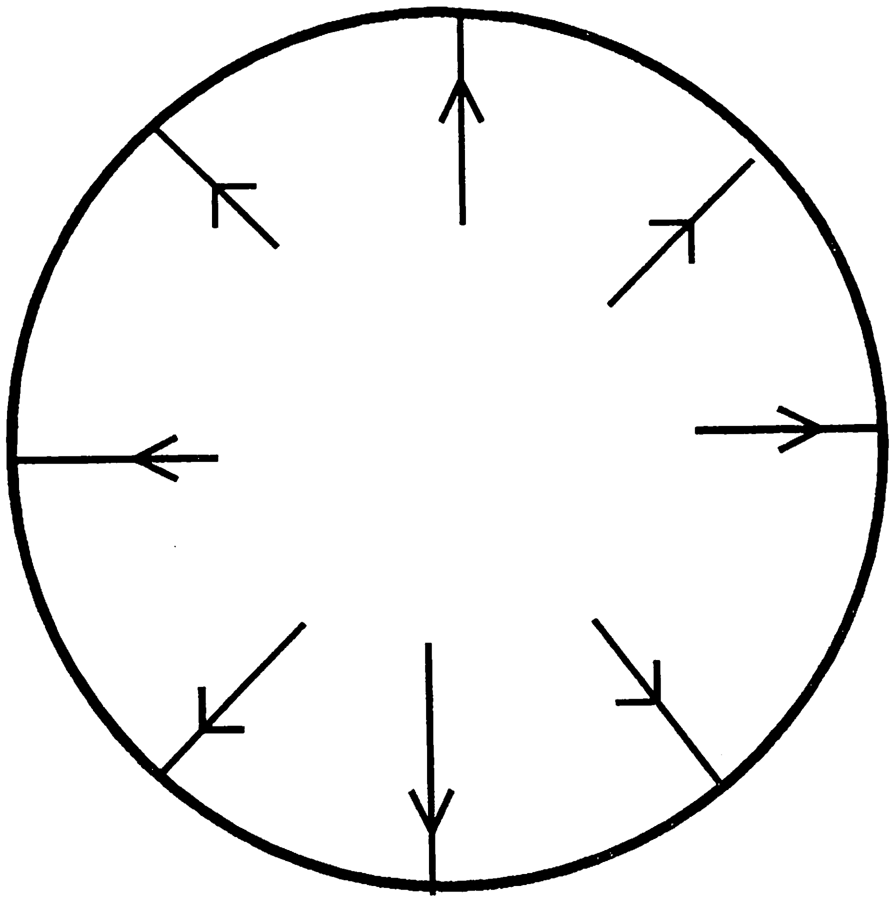

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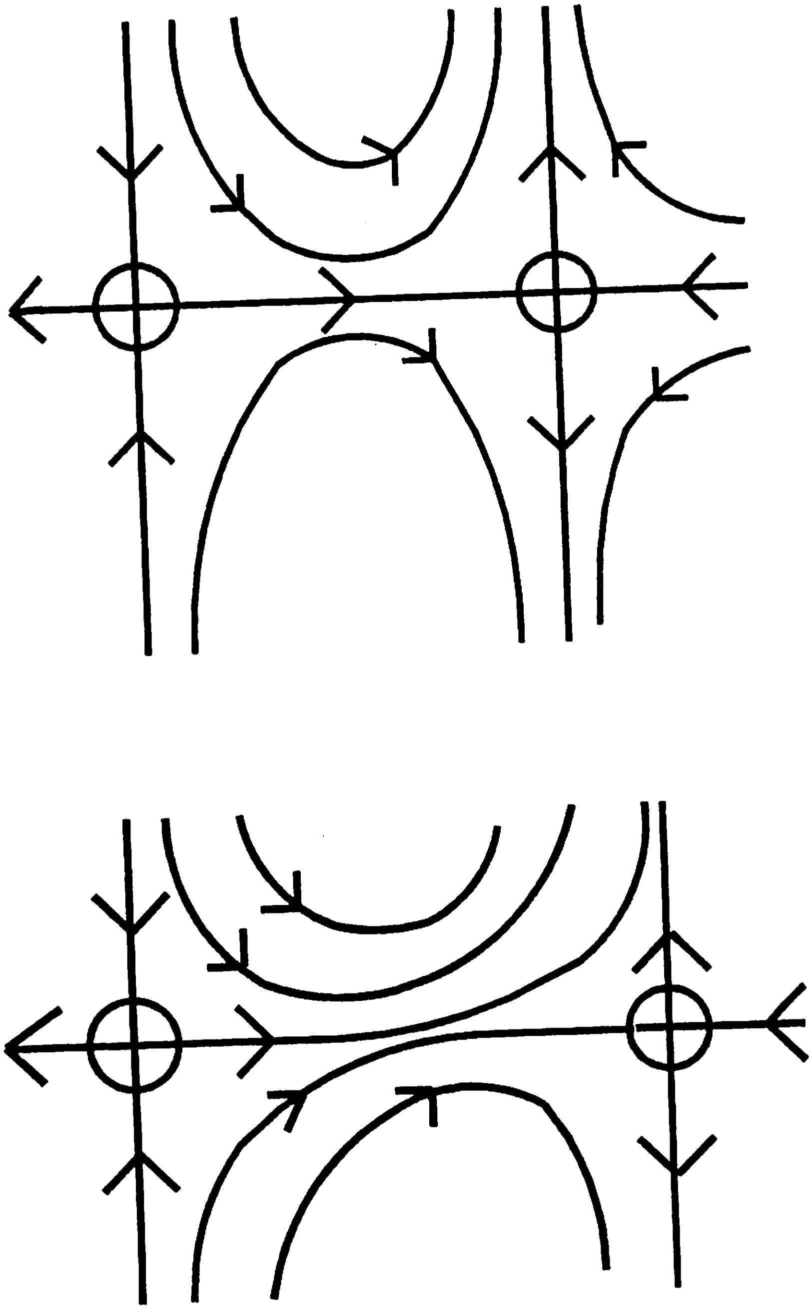

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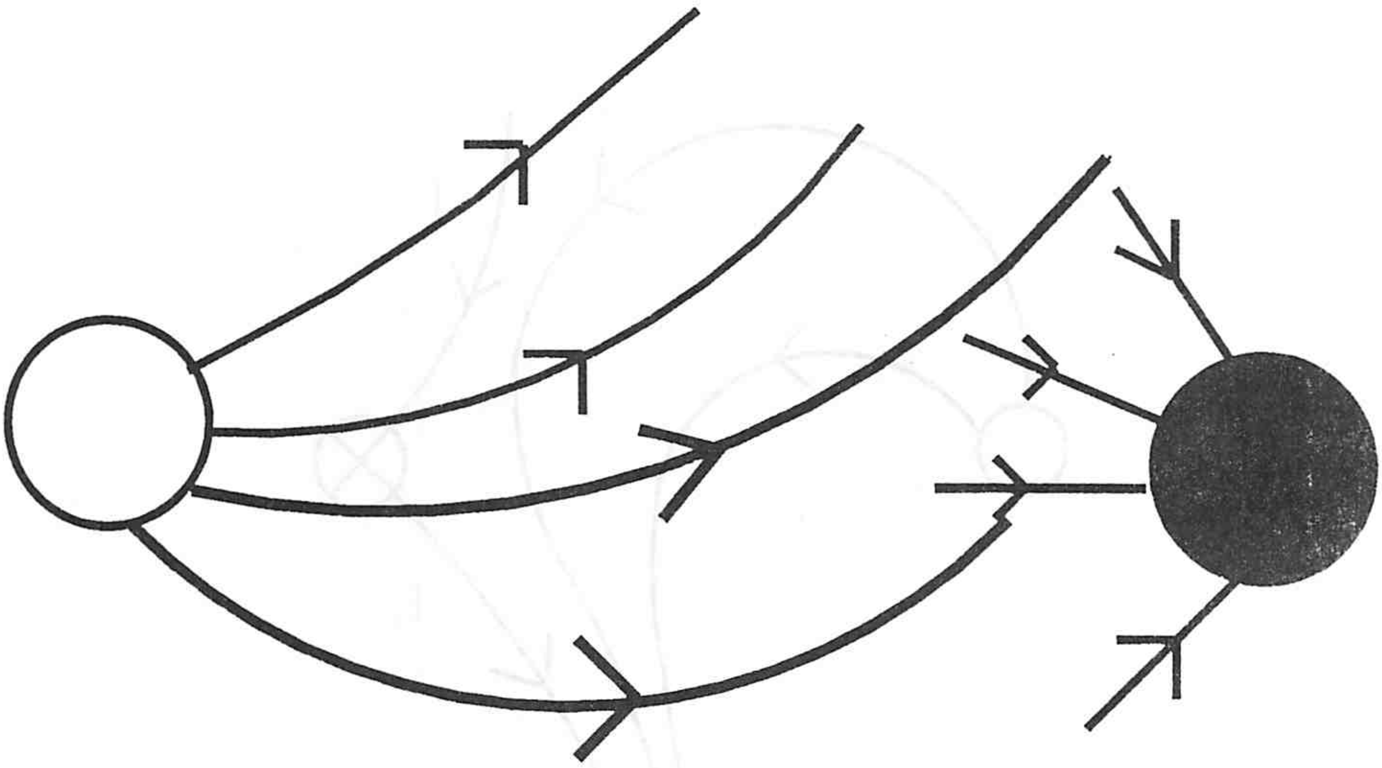

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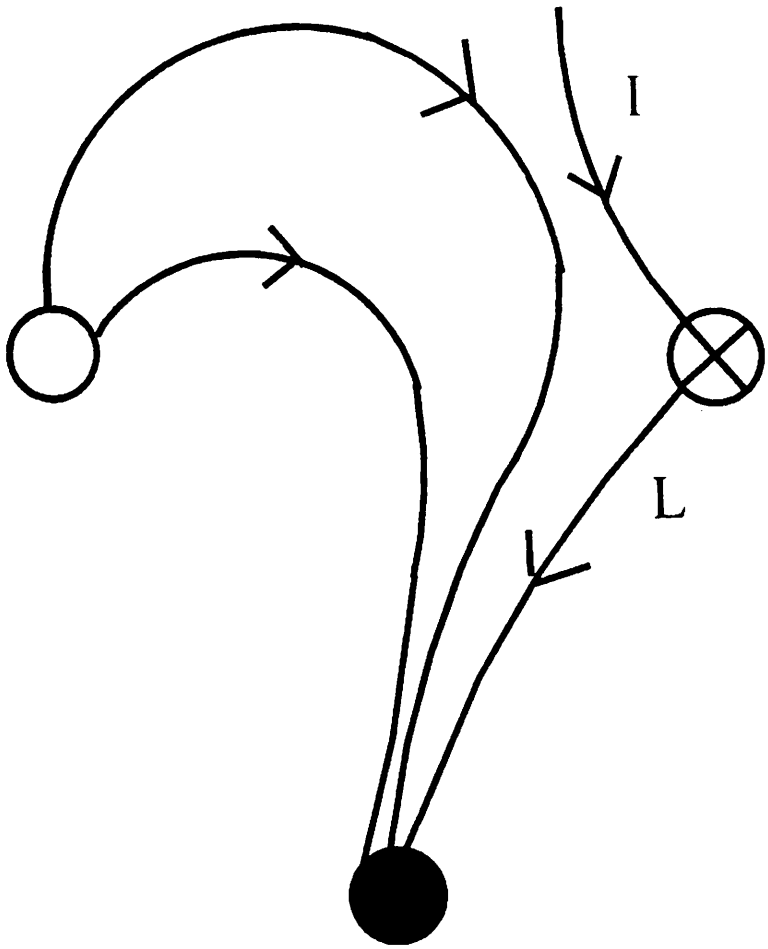


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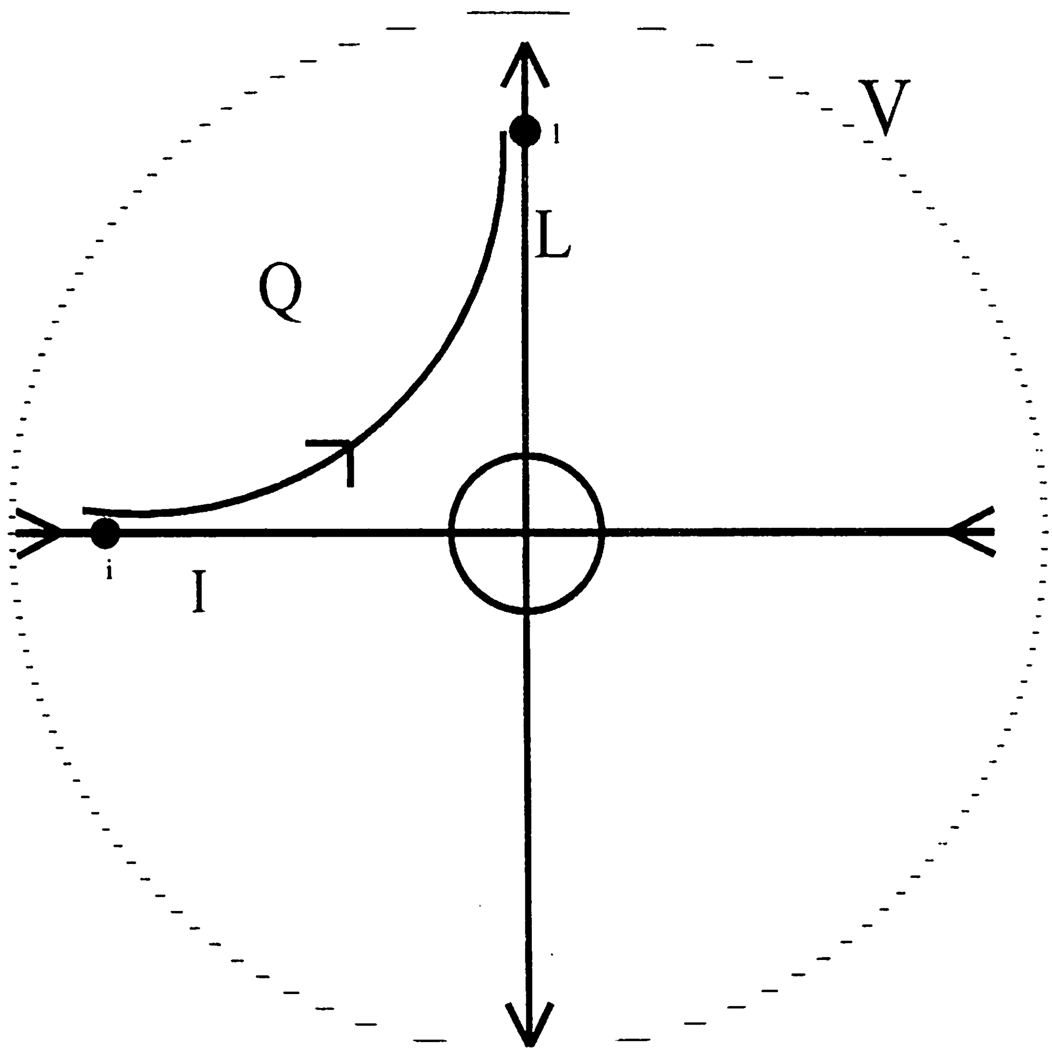


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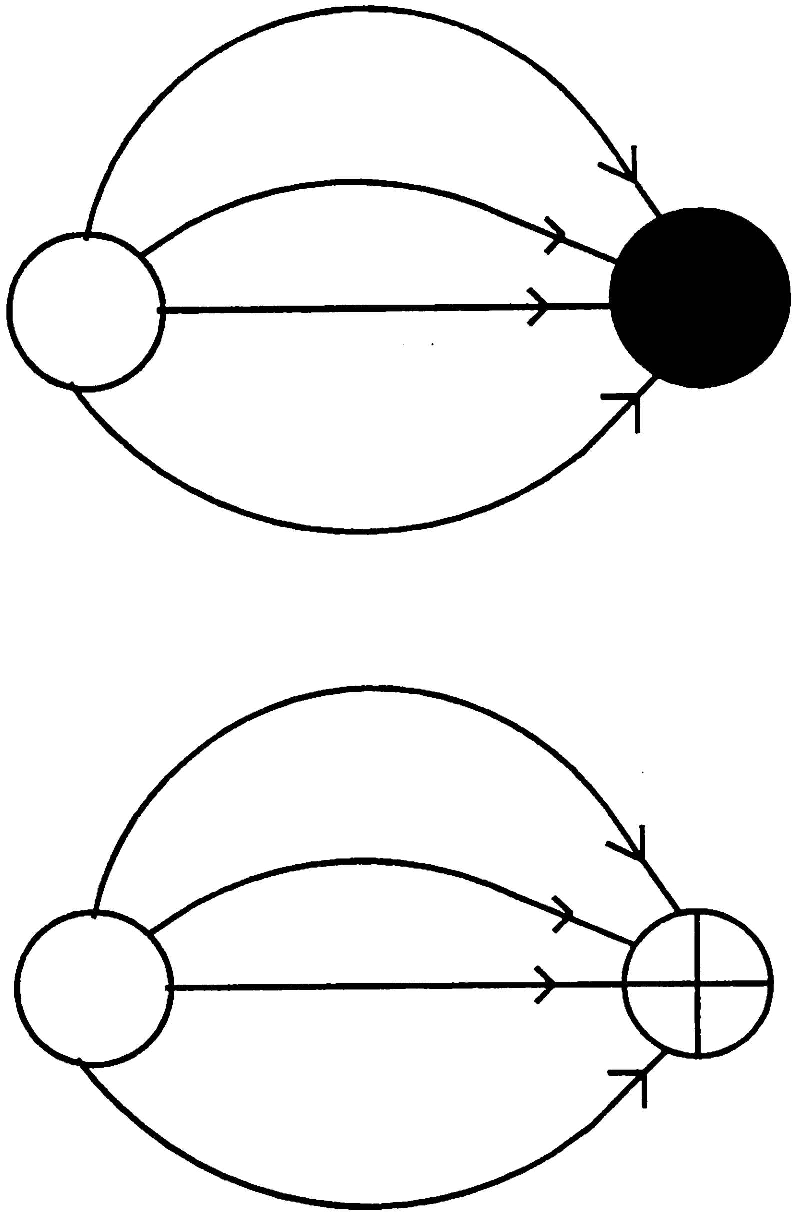

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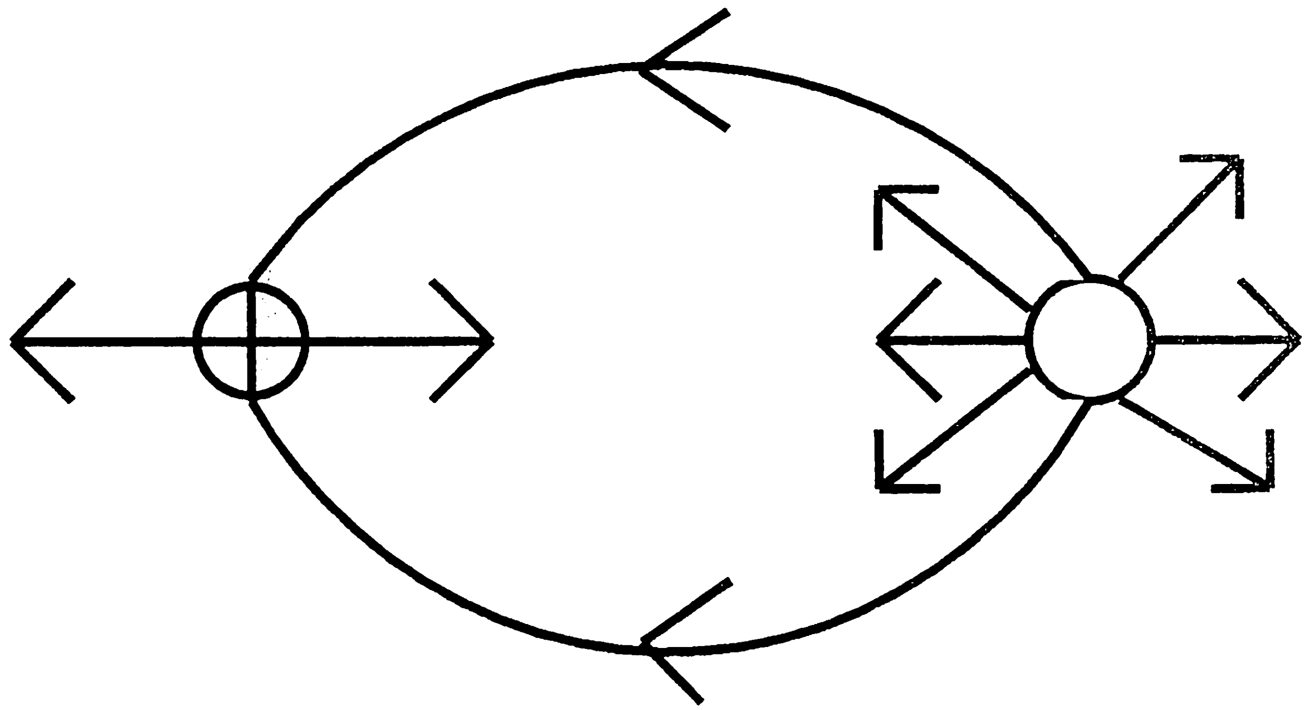

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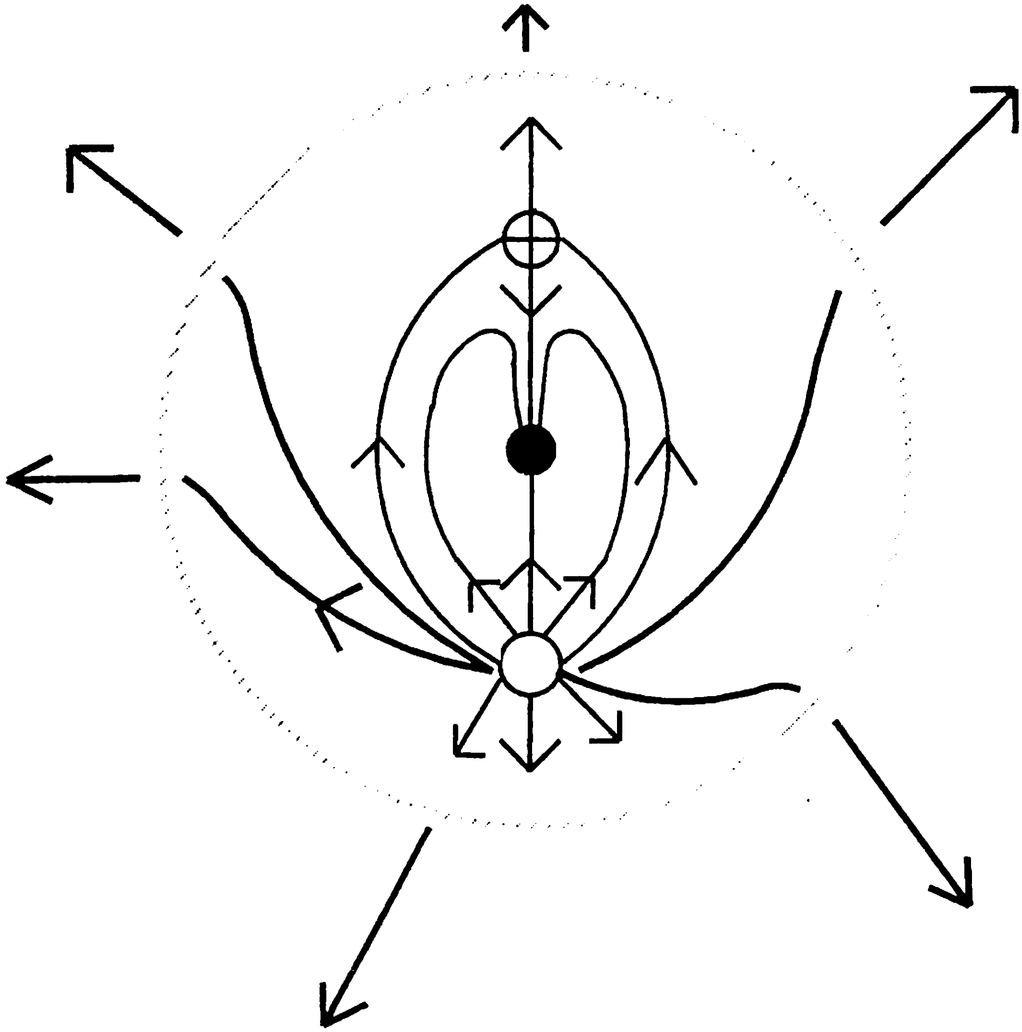

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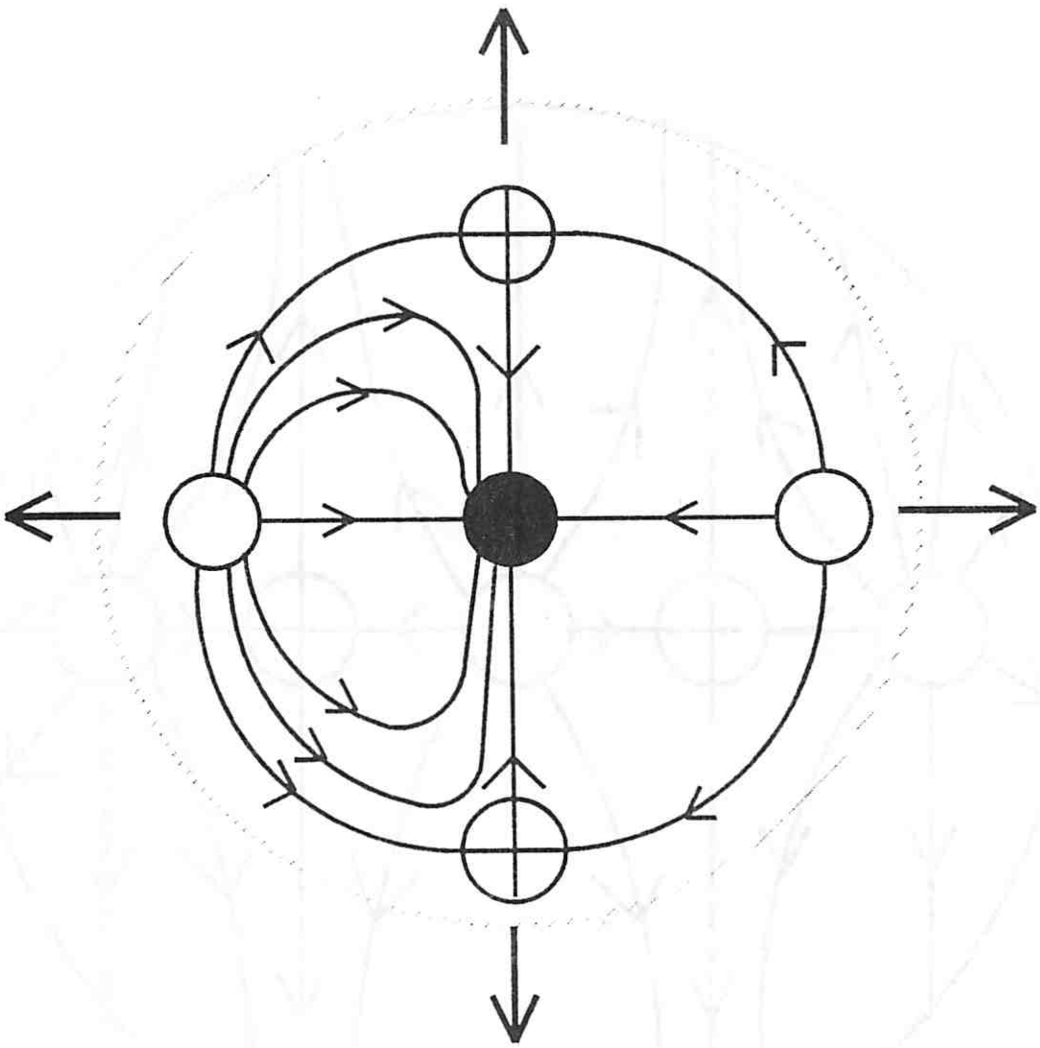

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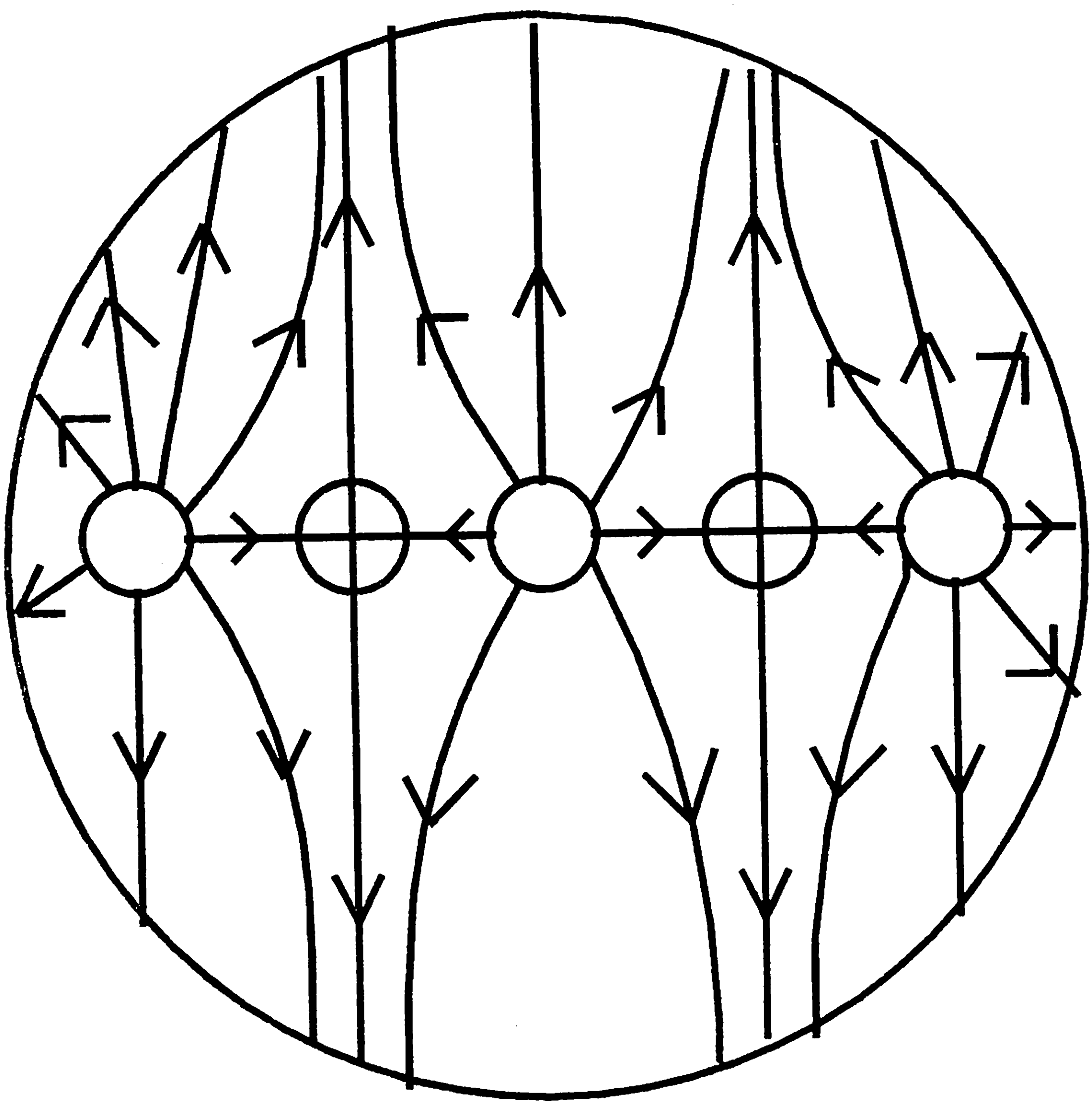

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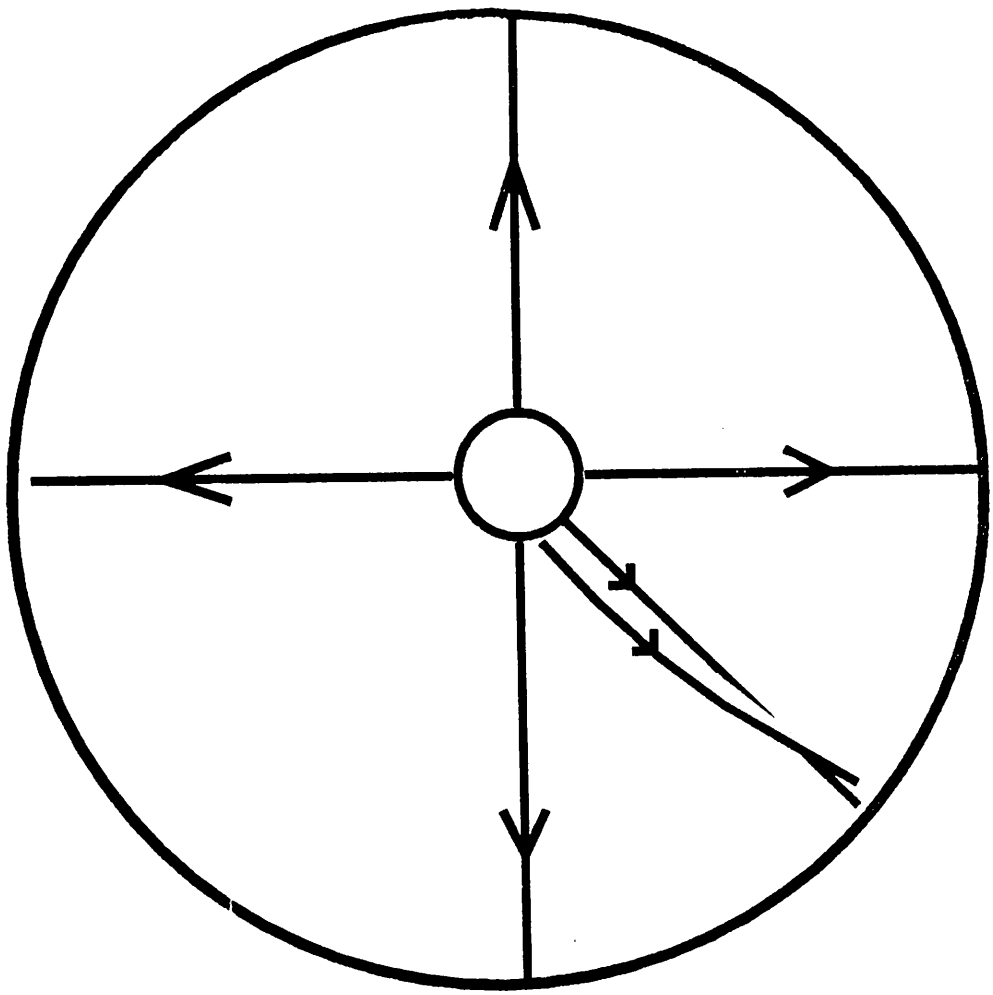

Figure 15.

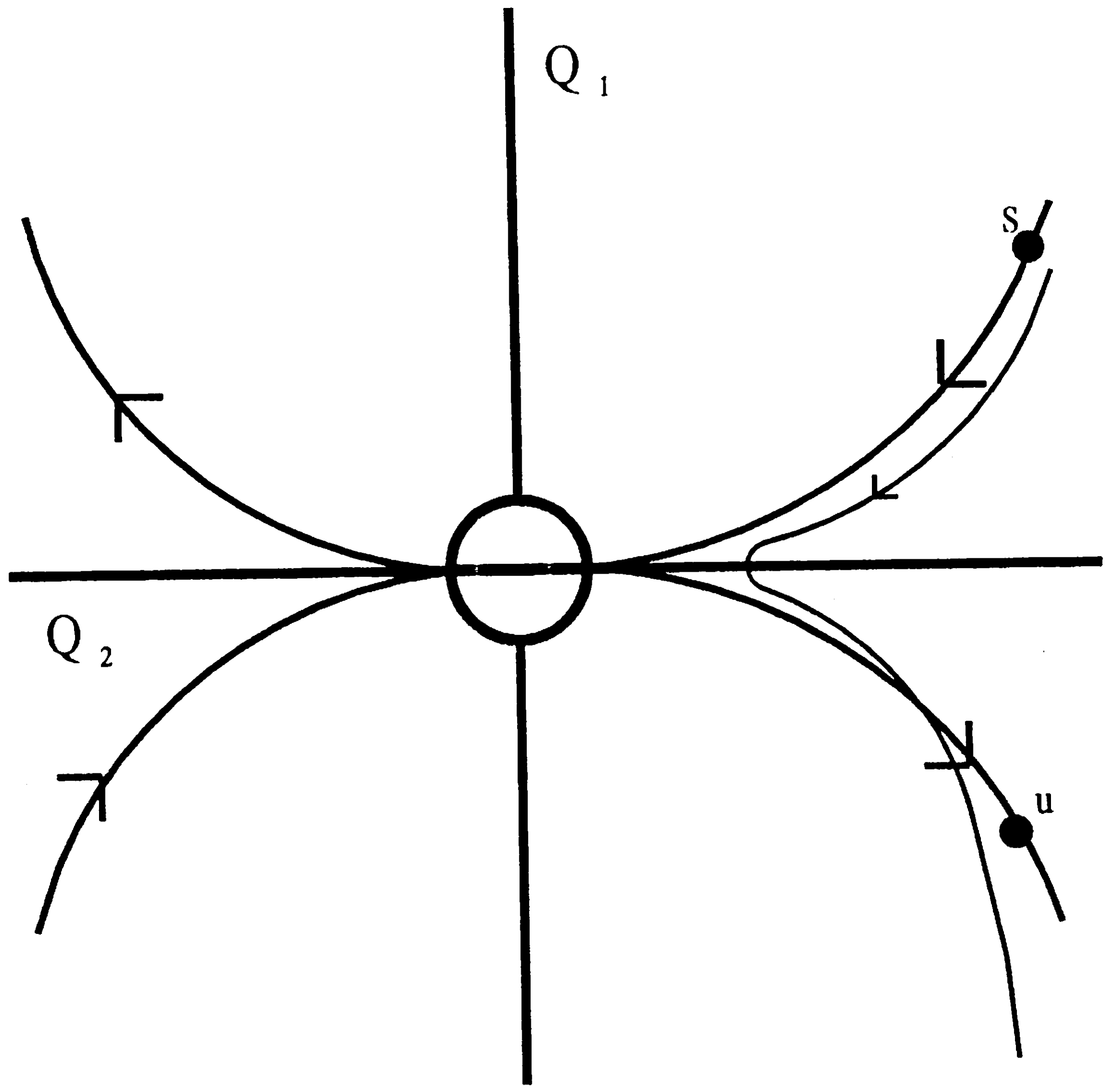


Figure 16.