SYMMETRY PROPERTY OF THE THROUGHPUT IN CLOSED TANDEM QUEUEING NETWORKS WITH FINITE BUFFERS

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Symmetry Property of the Throughput in Closed Tandem Queueing Networks with Finite Buffers

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Abstract

In this paper we consider closed tandem queueing networks with finite buffers and blocking before service. With this type of blocking, a server is allowed to start processing a job only if there is an empty space in the next buffer. It was recently conjectured that the throughput of such networks is symmetrical with respect to the population of the network. That is, the throughput of the network with population N is the same as that with population $C - N$, where C is the total number of buffer spaces in the network. The main purpose of this paper is to prove this result in the case where the service time distributions are of phase type (PH-distribution). The proof is based on the comparison of the sample paths of the network with populations N and $C - N$. Finally, we also show that this symmetry property is related to a reversibility property of this class of networks.

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1. Introduction

We consider a closed tandem queueing network consisting of K service stations and a finite population of N jobs. Each service station, say i, consists of a single server, S_i, and a finite buffer, B_i. Let C_i be the capacity of buffer B_i, including the server space in front of server S_i. Here, C_i is the maximum number of jobs that can be simultaneously present at station i. Let $\mathbf{C} = (C_1, \cdots, C_i, \cdots, C_K)$ be the buffer capacity vector, and let C be the total capacity of the network, i.e.,

$$C = \sum_{i=1}^{K} C_i \tag{1}$$

After completion at station i, $i = 1, \cdots, K-1$, a job must proceed to station $i+1$. After completion at station K, a job goes back to station 1. By convention, station $i-1$ will be station K if $i = 1$, and station $i+1$ will be station 1 if $i = K$. Let $\mathcal{N}$ denote this network. A closed tandem queueing network consisting of $K = 4$ stations is shown in Figure 1. We assume that the network has the following behavior. Server S_i initiates a service period whenever there is at least one job in its upstream buffer, B_i, and at least one hole (empty space) in its downstream buffer, B_{i+1}. The job remains in buffer B_i throughout the service period, i.e., there is no space associated with the servers for storing jobs. At the end of the service period of server S_i, the job is immediately transferred from buffer B_i to buffer B_{i+1}. When buffer B_i is empty, server S_i is said to be ***starved***. When buffer B_{i+1} is full, server S_i is said to be ***blocked***. Note that a server can be both simultaneously starved and blocked. Also, note that the blocking mechanism corresponds to what is referred to as ***blocking before service*** in the literature [7]. The service times of each server are i.i.d. random variables having a common distribution. The service time distributions for each server are assumed to be of ***phase type*** (PH-distribution) [5]. Moreover, we assume that service times at different servers are independent of one another.

The performance parameter of interest to us is the *system throughput*. Note that since we consider a closed tandem queueing network, all servers have the same throughput. The throughput will be denoted by $\theta(\mathcal{N}, N)$ to make explicit that it depends on the population of the network.

It was recently conjectured by Onvural and Perros [6] that the throughput of a closed tandem queueing network with finite buffers and exponential servers is ***symmetrical***

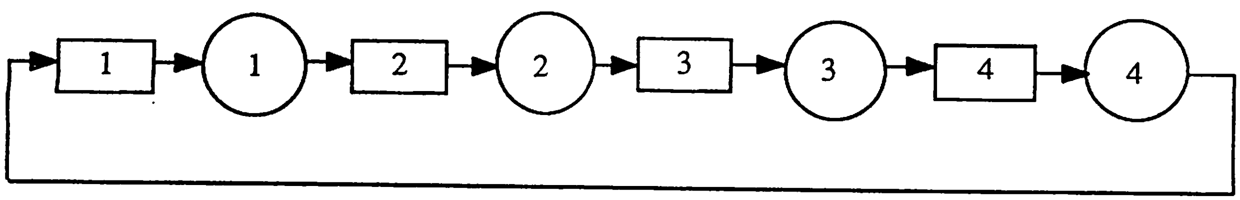

Figure 1: A closed tandem queueing network.

with respect to the population of the network, i.e.,

$$\theta(\mathcal{N}, C - N) = \theta(\mathcal{N}, N), 0 \leq N \leq C. \tag{2}$$

The main purpose of this paper is to provide a proof of this conjecture. Moreover, we show that this result holds not only for exponential distributions but much more generally for phase type (PH) distributions. This result is established in the next section. In the last section, we provide a discussion of this result. In particular, we show that the symmetry property is related to a *reversibility* property of this class of networks.

2. Main Result

In this section we provide a proof of the symmetry property. This proof is based on the comparison of the sample paths of the network with population N and with population $C - N$. We note that the symmetry property is obviously satisfied in the case where either $N = 0$ or $N = C$ since in both cases the throughput is zero. Therefore, we restrict our attention to populations N such that $0 < N < C$.

Let us first consider network $\mathcal{N}$ with population N. The durations of the service periods at server S_i are given by a sequence of non-negative service times, $\{\sigma_{i,n}\}_{n\geq 1}$. The initial conditions are defined as follows. Let $\boldsymbol{m} = (m_1, \cdots, m_i, \cdots, m_K)$ be the *initial marking* of the system where m_i denotes the number of jobs in buffer B_i at time $t = 0$. Since the network has a population of N jobs, we have:

$$\sum_{i=1}^{K} m_i = N \tag{3}$$

We assume that all servers which are neither starved nor blocked for this initial marking initiate a new service period at time $t = 0$. Let $(\mathcal{N}, \boldsymbol{m})$ denote the network coupled with the initial marking. Finally, let $h_i = C_i - m_i$ be the number of holes (empty spaces) in buffer B_i at time $t = 0$. Define the vector $\boldsymbol{h} = (h_1, \cdots, h_i, \cdots, h_K)$; it can be expressed as

$$\boldsymbol{h} = \boldsymbol{C} - \boldsymbol{m} \tag{4}$$

Here $\boldsymbol{m}$ will be referred to as the initial marking of jobs while $\boldsymbol{h}$ will be referred to the initial marking of holes. Note that the total number of holes in the network is

$$\sum_{i=1}^{K} h_i = \sum_{i=1}^{K} (C_i - m_i) = C - N. \tag{5}$$

Throughout this section, the results will be illustrated by means of an example. Consider the network shown in Figure 2. This network consists of $K = 3$ stations and $N = 3$ jobs. The buffer capacity vector is $\boldsymbol{C} = (2, 1, 2)$. Thus, the total buffer capacity of the network is $C = 5$. Finally, the initial marking is $\boldsymbol{m} = (1, 0, 2)$. Note that $\boldsymbol{h} = (1, 1, 0)$.

Let $B_{i,n}$, $n \geq 1$, denote the time of the beginning of the n-th service of server S_i in $(\mathcal{N}, \boldsymbol{m})$. Note that according to the initial timing conditions, $B_{i,1} = 0$ for all servers S_i which are neither starved nor blocked at time $t = 0$, that is $m_i \geq 1$ and $h_{i+1} \geq 1$. Also, let $D_{i,n}$, $n \geq 1$, denote the time of the n-th service completion of server S_i in $(\mathcal{N}, \boldsymbol{m})$. Note that, since we assume that blocking occurs before service, the instant of the service completion is also the instant of the departure of the job. We now establish some relationships between these quantities.

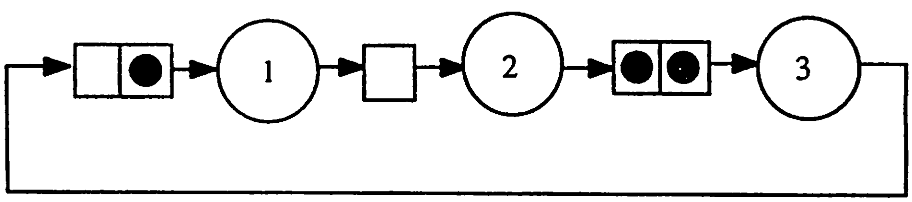

Figure 2: An illustrative example.

Lemma 1 *The evolution equations of* $(\mathcal{N}, m)$ *are:*

$$D_{i,n} = B_{i,n} + \sigma_{i,n}, \quad \forall i, n \geq 1 \tag{6}$$

$$B_{i,n} = \max\left(D_{i,n-1}, D_{i-1,n-m_i}, D_{i+1,n-h_{i+1}}\right), \quad \forall i, n \geq 1 \tag{7}$$

where, by convention, $D_{i,n} = 0$, $\forall i$ *and* $n \leq 0$.

Proof. Equation (6) simply states that the time of the completion of the service is equal to the time of its beginning plus the duration of the service time. Now, let us consider equation (7). There are three conditions that must be satisfied before the n-th service of server S_i can begin:

- the server must be available;
- the upstream buffer, B_i must be non-empty;
- the downstream buffer, B_{i+1} must be non-full.

Now, the first condition is satisfied when server S_i has completed its $(n-1)$-th service, which occurs at time $D_{i,n-1}$. The second condition is satisfied when server S_{i-1} has completed its $(n - m_i)$-th service, which occurs at time $D_{i-1,n-m_i}$. The third condition is satisfied when server S_{i+1} has completed its $(n - h_{i+1})$-th service, which occurs at time $D_{i+1,n-h_{i+1}}$. Since these three conditions must all be satisfied, the instant of beginning of service is thus the maximum of these three times. ∎

Now, let D_n be the time at which n service completions have occurred at all of the servers in the network. It will be referred to as the n-th *overall completion time*. It is given by

$$D_n = \max_i D_{i,n}, \quad \forall n \geq 1. \tag{8}$$

Suppose we are interested in the time at which the last of n_0 service completions at all servers, that is D_{n_0}, for any $n_0 \geq 1$. We now show that this quantity can be expressed as the length of the longest path of a directed graph. This graph will be referred to as the *activity graph* of network $(\mathcal{N}, m)$ and denoted by $\mathcal{G}_{n_0}$. The nodes represent the times of beginning and completion of services and the arcs represent the precedence constraints as expressed in the evolution equations given in Lemma 1. More precisely, node $B_{i,n}$ (resp. $D_{i,n}$) represents the time of beginning (resp. completion) of the n-th service of server S_i, for any $n = 1, \cdots, n_0$. Furthermore, there is one node associated with the instant of the overall completion time, D_{n_0}. Finally, there is one extra node representing the initial timing condition, that is the system starts running at time $t = 0$. This node will be denoted by B_1.

There is one arc oriented from node $B_{i,n}$ to node $D_{i,n}$. This arc represents the service activity and its associated weight is the corresponding service time, that is $\sigma_{i,n}$. This arc expresses the relationship in equation (6). There are, in general, three arcs oriented toward node $B_{i,n}$. The first arc from node $D_{i,n-1}$ (if $n > 1$), the second one from node $D_{i-1,n-m_i}$ (if $n > m_i$), and the third one from node $D_{i+1,n-h_{i+1}}$ (if $n > h_{i+1}$). These three arcs represent the precedence constraints expressed in equations (7). Their associated weights are 0. Also, there is an arc between node B_1 and each node $B_{i,1}$, $1 \leq i \leq K$, with associated weight 0. These arcs ensure that the servers can start working only after time $t = 0$. Finally, there is an arc between each node D_{i,n_0}, $1 \leq i \leq K$, and node D_{n_0} with associated weight 0. These arcs represent the relationship expressed in equation (8). Let $\mathcal{G}_{n_0} = (\mathcal{V}_{n_0}, \mathcal{E}_{n_0})$ where $\mathcal{V}_{n_0}$ is the set of nodes and $\mathcal{E}_{n_0}$ is the set of arcs in the activity graph.

Now, it follows from the construction of $\mathcal{G}_{n_0}$ that the n_0-th *overall completion time* D_{n_0}

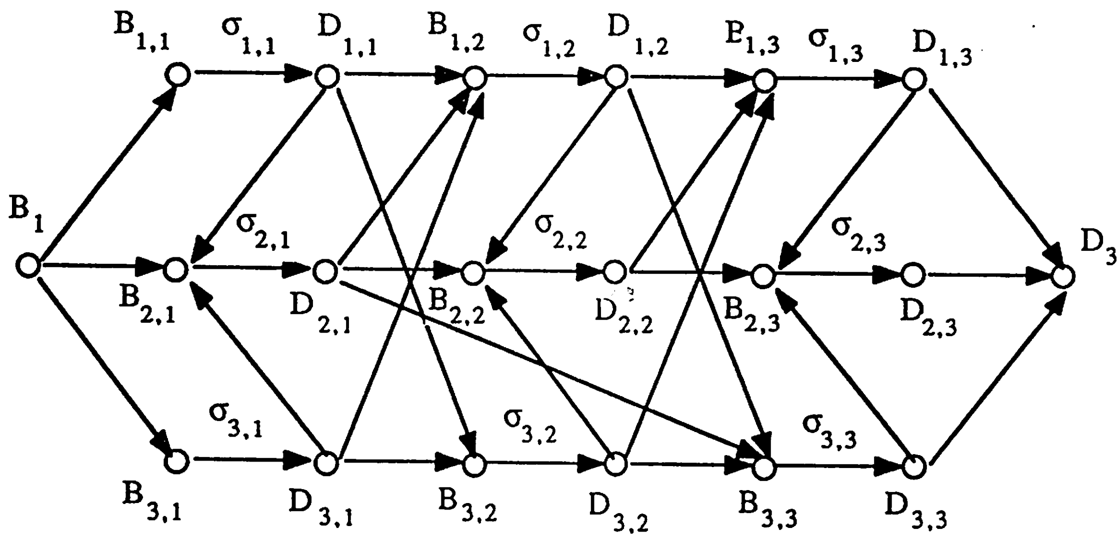

Figure 3: The activity graph associated with the initial marking (1,0,2).

corresponds exactly to the length of the path of maximum length among all possible paths from node B_1 to node D_{n_0}.

The graph $\mathcal{G}_{n_0}$ corresponding to the network shown in Figure 2 is given in Figure 3 in the case $n_0 = 3$. All arcs other than those associated with service activities have weights 0. Consider for instance the third service of server S_3. There is an arc from node $B_{3,3}$ to node $D_{3,3}$ with weight $\sigma_{3,3}$ that corresponds to the third service activity of server S_3. Now, there are three incoming arcs into node $B_{3,3}$, each having weight 0. They indicate that the third service activity of server S_3 cannot start before its second service activity has been completed, the first service activity of server S_2 has been completed, and the second service activity of server S_1 has been completed.

We now consider the same network $\mathcal{N}$ with a *symmetrical initial marking* denoted by m^s, where $m^s = C - m$. Let $(\mathcal{N}, m^s)$ denote the network coupled with this initial marking. Note that the initial marking of jobs in $(\mathcal{N}, m^s)$ corresponds to the initial marking of holes in $(\mathcal{N}, m)$ and vice-versa. That is, $m^s = h$ and $h^s = m$. All quantities that have been defined for $(\mathcal{N}, m)$ can also be defined for $(\mathcal{N}, m^s)$. Quantities pertaining to $(\mathcal{N}, m^s)$ will be denoted by a superscript s. The symmetrical initial marking of the illustrative example is thus $m^s = (1,1,0)$. The network with this

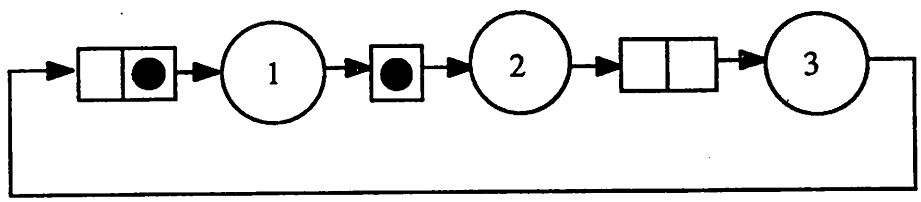

Figure 4: The network with the symmetrical initial marking (1,1,0).

initial marking is illustrated in Figure 4.

The evolution equations of $(\mathcal{N}, m^s)$ are the following:

$$D^s_{i,n} = B^s_{i,n} + \sigma^s_{i,n}, \quad \forall i, n \geq 1 \tag{9}$$

$$B^s_{i,n} = \max\left(D^s_{i,n-1}, D^s_{i-1,n-m^s_i}, D^s_{i+1,n-h^s_{i+1}}\right), \quad \forall i, n \geq 1. \tag{10}$$

The activity graph of this network is shown in Figure 5.

We now establish the fundamental property that relates the activity graphs of the network with the original initial marking and the symmetrical initial marking.

Lemma 2 *Assume the sequences of service times in $(\mathcal{N}, m^s)$ are the reverse of those in $(\mathcal{N}, m)$, that is :*

$$\sigma^s_{i,n} = \sigma_{i,n_0+1-n}, \quad 1 \leq i \leq K; \ 1 \leq n \leq n_0. \tag{11}$$

Then, the activity graph $\mathcal{G}^s_{n_0} = (\mathcal{V}^s_{n_0}, \mathcal{E}^s_{n_0})$ associated with network $(\mathcal{N}, m^s)$ is the reverse of the activity graph $\mathcal{G}_{n_0}$ associated with network $(\mathcal{N}, m)$

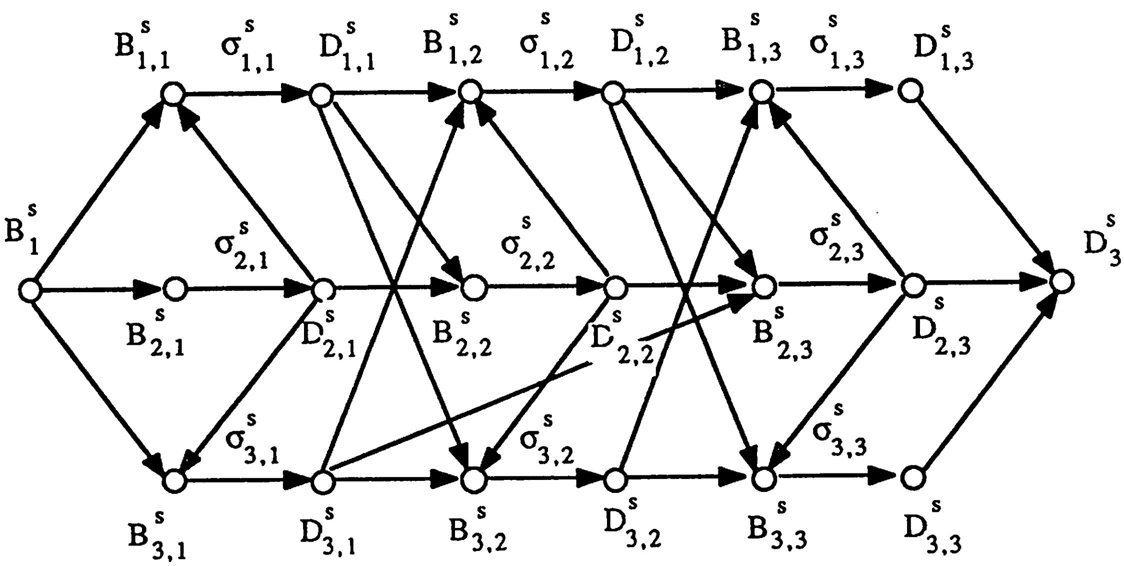

Figure 5: The activity graph associated with the symmetrical initial marking (1,1,0).

Proof. Let us establish a one-to-one correspondence between the nodes of $\mathcal{G}_{n_0}$ and the nodes of $\mathcal{G}^s_{n_0}$. For this purpose define a function $\phi : \mathcal{V}_{n_0} \rightarrow \mathcal{V}^s_{n_0}$ as

$$\phi(v) = \begin{cases} D^s_{i,n_0+1-n}, & v = B_{i,n},\ 1 \le i \le K;\ 1 \le n \le n_0, \\ B^s_{i,n_0+1-n}, & v = D_{i,n},\ 1 \le i \le K;\ 1 \le n \le n_0, \\ D^s_{n_0}, & v = B_1, \\ B^s_1, & v = D_{n_0}. \end{cases} \tag{12}$$

Now, it is easy to check that to any arc connecting two nodes in $\mathcal{G}_{n_0}$, there is an arc connecting the two corresponding nodes in $\mathcal{G}^s_{n_0}$. This arc has the same weight but is oriented in the reverse direction. Consider first the arc connecting node $B_{i,n}$ to node $D_{i,n}$ with weight $\sigma_{i,n}$ in $\mathcal{G}_{n_0}$. Now, according to equation (12), the corresponding nodes are D^s_{i,n_0+1-n} and B^s_{i,n_0+1-n}. Now, there is indeed an arc connecting these two nodes in the reverse direction in $\mathcal{G}^s_{n_0}$ with weight σ^s_{i,n_0+1-n} which by assumption is equal to $\sigma_{i,n}$. Consider now the arc connecting node $D_{i-1,n-m_i}$ to node $B_{i,n}$ with weight 0. Now, according to equations (12), the corresponding nodes are $B^s_{i-1,n_0+1-(n-m_i)}$ and D^s_{i,n_0+1-n}. Now, since $m_i = h^s_i$, there is indeed an arc connecting these two nodes in the reverse direction with weight 0 (see equation (10)). Note that the original arc in $\mathcal{G}_{n_0}$ expresses a non-starvation synchronization constraint while the corresponding arc in $\mathcal{G}^s_{n_0}$ expresses a non-blocking synchronization constraint. Finally, the correspondence for the other arcs can be shown in a similar manner. ∎

We can now state the following result that relates the transient behavior of networks $(\mathcal{N}, m)$ and $(\mathcal{N}, m^s)$.

Theorem 1 *Under the assumption that the service times of all servers are i.i.d. random variables, the overall completion times D_{n_0} in $(\mathcal{N}, m)$ and $D^s_{n_0}$ in $(\mathcal{N}, m^s)$ have the same distribution, for all $n_0 \geq 1$.*

Proof. D_{n_0} is the length of the longest path from node B_1 to node D_{n_0} in $\mathcal{G}_{n_0}$. Similarly, $D^s_{n_0}$ is the length of the longest path from node B^s_1 to node $D^s_{n_0}$ in $\mathcal{G}^s_{n_0}$. Now, under the assumption of Lemma 2, $\mathcal{G}^s$ is the reverse of $\mathcal{G}$. Therefore, the length of the longest path is the same in both graphs and therefore $D^s_{n_0} = D_{n_0}$. Now, under the assumption that the service times are i.i.d. rv's, the sequence $\{\sigma^s_{i,n}\}_{n=1}^{n_0}$ has the same joint distribution as the sequence $\{\sigma_{i,n}\}_{n=1}^{n_0}$, $1 \leq i \leq K$. Therefore, the overall completion times D_{n_0} and $D^s_{n_0}$ have the same distribution. ∎

Let us now restrict our attention to PH-distributions. In this case, since the system can be described as an ergodic Markov chain, we know that the asymptotic throughput exists and depends only on the total population and is otherwise independent of the initial marking. Now, the throughput can be expressed as:

$$\theta(\mathcal{N}, N) = \lim_{n_0 \to \infty} \frac{E[D_{n_0}]}{n_0} \tag{13}$$

The following result establishes the symmetry property of the throughput. It follows from Theorem 1 and the above definition of the throughput.

Corollary 1 *The throughput of a closed tandem queueing network with finite buffers and PH service time distributions is symmetrical with respect to the population of the network, i.e.,*

$$\theta(\mathcal{N}, C - N) = \theta(\mathcal{N}, N), \quad 0 \leq N \leq C. \tag{14}$$

Remark. The transient result on the overall completion time distributions (Theorem 1) holds provided that servers have general independent (GI) distributions. However, for the symmetry property (Corollary 1), we further assumed that the service time distributions are of phase-type. The reason is that, under this assumption, it is trivial to show that the throughput does not depend on the initial marking. Although the symmetry property also holds in the case of GI distributions, it is no longer easy to show. Results pertaining to this more general case is a subject of ongoing research.

3. Discussion

In this section, we discuss a *reversibility property* of closed tandem queueing networks with finite buffers and blocking before service and relate it to the symmetry property established above.

Consider again network $\mathcal{N}$. Let $\mathcal{N}^r$ be the network obtained from $\mathcal{N}$ by reversing the flow of jobs while keeping the same population. Let $\theta(\mathcal{N}^r, N)$ be the throughput of this reverse network. We establish the following reversibility property:

$$\theta(\mathcal{N}^r, N) = \theta(\mathcal{N}, N), \quad 0 \leq N \leq C. \tag{15}$$

Consider network $\mathcal{N}^r$ with the same initial marking as that of $\mathcal{N}$, that is $m^r = m$. The reverse network of the network of Figure 2 is shown in Figure 6.

The evolution equations of $(\mathcal{N}^r, m)$ are:

$$D^r_{i,n} = B^r_{i,n} + \sigma^r_{i,n}, \quad \forall i, n \geq 1 \tag{16}$$

$$B^r_{i,n} = \max\left(D^r_{i,n-1}, D^r_{i+1,n-m_{i+1}}, D^r_{i-1,n-h_i}\right), \quad \forall i, n \geq 1 \tag{17}$$

Note that in the reverse network, the upstream buffer of server S_i is buffer B_{i+1} and the downstream buffer of server S_i is buffer B_i.

Now, since $m^s_i = h_i$ and $h^s_{i+1} = m_{i+1}$, it is easy to check that the evolution equations for $(\mathcal{N}^r, m)$ are the same as those of $(\mathcal{N}, m^s)$ provided that $\sigma^r_{i,n} = \sigma^s_{i,n}$, $1 \leq i \leq K$, $n \geq 1$. Actually, the second (resp. third) term in the right side of equation (17) corresponds to the third (resp. second) term in the right side of equation (10). As a result the activity graph $\mathcal{G}^r_{n_0}$ is the same as the graph $\mathcal{G}^s_{n_0}$. The following results readily follow from this observation.

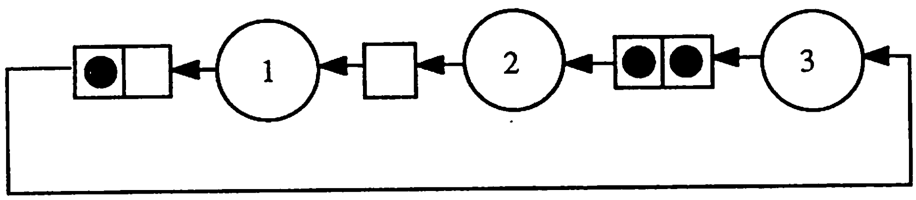

Figure 6: The reverse network with initial marking (1,0,2).

Theorem 2 *Under the assumption that the service times of all servers are i.i.d. random variables, the overall completion times* D_{n_0} *in* $(\mathcal{N}, m)$ *and* $D^r_{n_0}$ *in* $(\mathcal{N}^r, m)$ *have the same distribution, for all* $n_0 \geq 1$.

Corollary 2 *A closed tandem queueing network with finite buffers and PH service time distributions has the same throughput as its reverse network with the same population, i.e.,*

$$\theta(\mathcal{N}^r, N) = \theta(\mathcal{N}, N), \quad 0 \leq N \leq C. \tag{18}$$

Remark. We note that similar reversibility properties have been previously obtained in [3,4,8]. However, these authors considered open tandem queueing networks with finite buffers. Also, the blocking mechanism was different since they assumed blocking after service.

It is interesting to further discuss the equivalence in terms of behavior between networks $(\mathcal{N}^r, m)$ and $(\mathcal{N}, m^s)$. Actually, it can easily be explained using the concept of job/hole duality introduced by Gordon and Newell [2] and more recently developed by Ammar and Gershwin [1]. The idea is that whenever there is a job moving in one direction, there is a hole moving in the other direction. In terms of stochastic processes, it means that the stochastic process characterizing the behavior of holes is equivalent to the stochastic process characterizing the behavior of jobs.

Now, for any network $(\mathcal{N}, m)$, one can define a *dual network*, $(\mathcal{N}^d, m^d)$, which is obtained by reversing the flow of jobs and having as initial marking of jobs the initial marking of holes in the original network, i.e., $\mathcal{N}^d = \mathcal{N}^r$ and $m^d = h$. Now, since we assume blocking before service, it is easy to check that the behavior of jobs in $(\mathcal{N}^d, m^d)$ is the same as the behavior of holes in $(\mathcal{N}, m)$. So finally, combining these observations, we learn that the stochastic process characterizing the behavior of jobs in the dual network is equivalent to the stochastic process characterizing the behavior of jobs in the original network. Now, it is easy to show that network $(\mathcal{N}^r, m)$ is the dual of network $(\mathcal{N}, m^s)$. As a result, their stochastic behavior is equivalent.

Remark. Finally, we note that the reversible properties obtained in this section also hold in the case of closed tandem queueing networks with infinite buffers. This result follows from the observation that all the infinite buffers can equivalently be replaced by finite buffers of capacity N without modifying the behavior of the network.

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