The Mathematical Foundations of Smoothness Constraints: a Complete List of a New Class of Coupled Constraints

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Abstract

Gradient–based approaches to the computation of optical flow often use a minimization technique incorporating a smoothness constraint on the optical flow field. Smoothness constraints are also of interest in surface interpolation, where they are known as "performance functions." All known smoothness constraints used to compute optical flow have a subtle property, namely that they do not mix derivatives of different components of the optical flow field. We present an analysis of smoothness constraints which do not satisfy this "decoupled" property, but rather in which derivatives of different components of the flow can interact. By using the representation theory of the group of Euclidean motions in the image plane, we show that the the single assumption that the smoothness constraint is invariant under

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this group of transformations allows us to write down a *complete* list of all possible invariant smoothness constraints of type (p, q), by which it is meant that they are quadratic in p^{th} derivatives of the optical flow field, and in q^{th} derivatives of the grey level image intensity function. This is done explicitly for the values $0 \leq p, q \leq 2$. So far as the author is aware, all of these smoothness constraints, excepting those linear combinations which are decoupled, are new. We find that there are 4 of type $(1,0)$, 5 of type $(2,0)$, 8 of type $(1,1)$, 14 of type $(1,2)$, 15 of type $(2,1)$, and 24 of type $(2,2)$. In addition, we find using our method all invariant "performance measures," used in surface interpolation, when the performance measure is quadratic in no higher than fourth derivatives of the objective function.

1 Introduction

In this work we consider smoothness constraints which can be written as an integral, over the image plane, of a quantity called the smoothness density Σ (see [Snyd89] for a fuller discussion). We consider only densities which are quadratic in both p^{th} derivatives of the optical flow field $\mathbf{U} = (U_1, U_2)^{\text{T}} \equiv (u, v)$, and in q^{th} derivatives of the grey–level image intensity function I. These smoothness densities are therefore of the form:

$$\Sigma = f^{a_1,\ldots,a_p;b_1,\ldots,b_p;r,s}_{c_1,\ldots,c_q;d_1,\ldots,d_q}(\partial_{a_1}\ldots\partial_{a_p}U_r)(\partial_{b_1}\ldots\partial_{b_p}U_s)\; I_{c_1\ldots c_q} I_{d_1\ldots d_q}, \tag{1}$$

where the quantity $f^{\cdots}_{\cdots}$ does not involve derivatives of either $\mathbf{U}$ or of I. As in our previous work [Snyd89], we are using the Einstein summation convention, in which all repeated indices are understood to be summed over from 1 to 2. We are using the notation that $\partial_k \equiv \partial/\partial x_k$ ($k = 1, 2$), and a subscript of I denotes differentiation with respect to that coordinate, e.g.,

$$I_{d_1\ldots d_q} = \frac{\partial^q I}{\partial x_{d_1}\cdots\partial x_{d_q}}. \tag{2}$$

(Note that the subscripts of U denote components, not derivatives.) We will call a smoothness density of the form (1) a *smoothness density of type* (p, q). The well known smoothness constraint of Horn and Schunck [Horn81], for instance, is a particular smoothness constraint of type $(1, 0)$, while that of Nagel and Enkelmann [Nage86] is of type $(1, 2)$.

In our previous work [Snyd89], we asserted that a smoothness density should satisfy three physically reasonable conditions:

1. The smoothness density should be invariant under a change in the Cartesian coordinate system of the image, i.e., it should be invariant under the Euclidean group of the plane ISO(2). We called this the 0^{th} *Law of Computer Vision.*

2. The smoothness density should be positive definite.

3. The smoothness density should be decoupled, by which is meant that no terms involving the product of derivatives of different components of **U** (e.g., $u_x v_y$) should appear in (1).

We used these three constraints in [Snyd89] to give a completeness proof for decoupled smoothness constraints of type $(1,1)$ and $(1,2)$, i.e., we found all the independent such smoothness constraints.

It is the purpose of the present work to drop the third assumption, and hence to find *all* possible smoothness densities which satisfy the first two assumptions, namely Euclidean invariance and positive definitenss. We recall that the "decoupling" assumption in [Snyd89] had no obvious physical or mathematical motivation, but was characteristic of all known smoothness densities.

We cannot yet justify the *experimental* relevance of considering coupled smoothness constraints, but it is possible that coupled smoothness constraints may find future application in the analysis of visual motion. Our own interest in this problem, however, is from a purely theoretical standpoint.

We will use the notation and ideas developed in [Snyd89], to which the interested reader is referred for mathematical background and the physical justification of our approach.

We recall that an element of the Euclidean group ISO(2) of the plane is specified by a rotation matrix $\mathbf{R} \in SO(2)$ and a translation vector $\mathbf{t}$, and that this element $(\mathbf{R},\mathbf{t})$ of the Euclidean group has the following effect on the position vector $\mathbf{r}$ of a point with respect to some origin:

$$(\mathbf{R},\mathbf{t}) \ : \ \mathbf{r} \longrightarrow \mathbf{r}' = \mathbf{R}\mathbf{r} + \mathbf{t}. \tag{3}$$

Furthermore, an object $F_{k_1 \ldots k_r}$ which transforms under this Euclidean transformation according to the rule:

$$F_{k_1 \ldots k_r}(\mathbf{r}) \longrightarrow F'_{k_1 \ldots k_r}(\mathbf{r}') = \mathbf{R}_{k_1 k'_1} \cdots \mathbf{R}_{k_r k'_r} F_{k'_1 \ldots k'_r}(\mathbf{r}), \tag{4}$$

where $\mathbf{r}$ and $\mathbf{r}'$ are related by (3), is called an r^{th} *rank tensor* (under ISO(2)). An "ordinary" vector $\mathbf{A}$ (for example, the optical flow vector $\mathbf{U}$) is just a 1^{st} rank tensor, i.e., $\mathbf{A} \longrightarrow \mathbf{A}' = \mathbf{R}\mathbf{A}$.

It is easy to show from the fact that the p^{th} derivative of a scalar quantity is a p^{th} rank tensor, and that the p^{th} derivative of the (vector) quantity U_a is a $(p+1)^{\text{th}}$ rank tensor that the quantity containing all the derivatives in (1) is a tensor of rank $2p + 2q + 2$. It then follows easily from the tensor transformation law that if Σ is to be an invariant under the Euclidean group, the quantity $f^{\cdots}_{\cdots}$ in (1) must also be a tensor of rank $2p + 2q + 2$.

It follows immediately from the definition of a tensor product of two–dimensional spaces that a tensor of rank $2p + 2q + 2$ must be a linear combination of all possible $p + q + 1$–fold products of the elementary second–rank tensors of the underlying two–dimensional space. In two dimensions, there are only two such tensors, the symmetric *Kronecker Delta function* δ_{ab} and the antisymmetric *permutation symbol* ϵ_{ab}, where a, $b = 1, 2$. These two tensors are defined as follows:

$$\delta_{ab} = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if } a \neq b \end{cases} \tag{5}$$

and

$$\epsilon_{ab} = \begin{cases} 1 & \text{if } (ab) = (12) \\ -1 & \text{if } (ab) = (21) \\ 0 & \text{if } a = b \end{cases} . \tag{6}$$

It follows from the definition of ϵ that it obeys the constraint

$$\epsilon_{ab}\epsilon_{cd} = \delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc} \ . \tag{7}$$

The quantity $f^{\cdots}_{\cdots}$ in equation (1) must be a linear combination of terms of the form

$$\lambda_{k_1 k_2}\lambda_{k_3 k_4} \cdots \lambda_{k_{2p+2q+1} k_{2p+2q+2}} \ , \tag{8}$$

where λ is either a δ or an ϵ, and where $\{k_1, k_2, \ldots, k_{2p+2q+2}\}$ is a permutation of the set consisting of the indices of $f^{\cdots}_{\cdots}$.

We first address the question of how many possible smoothness interactions there are of type (p, q). This is easily found by a combinatoric argument. We first note that because of the relation (7), we can restrict our considerations to products of the form (8) where either all of the λ's are δ's, or where only one of the λ's is an ϵ, the remainder being δ's. Without losing generality, we can assume that if there is an ϵ, it is in the first position.

If we think of the subscripts in (8) as slots into which the $2p + 2q + 2$ indices of $f^{\cdots}_{\cdots}$ must be inserted, there are clearly $(2p + 2q + 2)!$ ways of fitting the indices into the slots. Owing to the symmetry (or antisymmetry, depending on whether a particular λ is a δ or an ϵ) of λ, however, we can always permute the two subscripts of each λ without changing the character of the tensor. This means we have overcounted by a factor of 2^{p+q+1}. Furthermore, for the case where all the λ's are δ's, we can permute the set of $(p + q + 1)$ δ's without affecting the set of all possible such tensors, so that in this case we have overcounted by a factor of $(p + q + 1)!$. Similarly, for the case where the first λ is an ϵ, we can permute the set of remaining $(p + q)$ δ's without affecting the set of all possible such tensors, so we have overcounted this set by a factor of $(p + q)!$. Putting all of this together, we find that the number of independent smoothness constraints of type (p, q) is at most

$$\frac{(2p+2q+2)!}{2^{p+q+1}\,(p+q+1)!} + \frac{(2p+2q+2)!}{2^{p+q+1}\,(p+q)!} = (p+q+2)(2p+2q+1)!!\ , \tag{9}$$

where $(2n+1)!! = (2n+1)(2n-1)(2n-3)\cdots(3)(1)$. Of course, not all of these tensors can be independent. In particular, the maximum number of different indices in the tensor $f^{\cdots}_{\cdots}$ is clearly $2^{2p+2q+2}$, since each index can take only the two values 1 and 2. Furthermore, owing to the commutativity of differentiation, there are (if $p \neq 0$) only $(p+1)(2p+3)$ different products of p^{th}–order derivatives of U, and $(q+1)(q+2)/2$ different q^{th} derivatives of I. Hence the maximum number of independent quantities of the form $(\partial^p \mathrm{U})^2(\partial^q I)^2$ is

$$\frac{(p+1)(2p+3)(q+1)(q+2)}{2}.$$

Therefore, the maximum number of independent constraints of type (p, q) is the smaller of this number, $2^{2p+2q+2}$, and (9). In the table which follows, we give the number of tensors (9), and the maximum number of such tensors which can be independent. In the last column, we give the actual number of such independent tensors found in the present work.

Clearly, the calculation of all the possible invariant smoothness constraints rapidly becomes a daunting task if one proceeds by trying to write down all possible tensors of the form (8), even though various symmetries lead to relations between the tensors. We will not pursue this train of thought farther.

In the next section, we develop the representation theory of ISO(2) (or, since all our quantities are manifestly translation invariant, the representation theory of SO(2)). In Section 3, we use these results to find simply and quickly *all* possible ISO(2)–invariant smoothness constraints of type (1,0), (2,0), (0,1), and (0,2). We reproduce the well known decoupled smoothness constraints of these four types, and exhibit new coupled smoothness constraints. In Section 4, we use the results of Section 3 to find all invariant smoothness constraints of type $(1,1)$ and $(1,2)$. We reproduce the well known decoupled constraints of Nagel and Enkelmann [Nage83, Nage86] and give new coupled constraints. In Section 5, we find all the constraints of type $(2,1)$ and $(2,2)$, all of which, so far as we are aware, are new. In Section 6, we consider the case of performance functions for surface interpolation, and find all the invariant smoothness functions quadratic in 3$^{\text{rd}}$ and 4$^{\text{th}}$ derivatives. These are, so far as we are aware, new. We conclude with some remarks on further investigations of these ideas.

2 The Representation Theory of ISO(2)

Because all the quantities which appear in our smoothness density are translationally invariant, we need only consider the SO(2) subgroup of ISO(2). The reader unfamiliar with SO(2) can consult [Snyd89] for the necessary mathematical background. A general reference for this section is the book by Murnaghan [Murn38].

It is well known that the irreducible representations of SO(2) are labelled by an integer

Type	Example	Number of Tensors	Maximum # Independent	Actual # Independent
(1,0)	$(\partial U)^2$	9	9	4
(2,0)	$(\partial^2 U)^2$	60	21	5
(1,1)	$(\partial U)^2(\partial I)^2$	60	30	8
(1,2)	$(\partial U)^2(\partial^2 I)^2$	525	60	14
(2,1)	$(\partial^2 U)^2(\partial I)^2$	525	64	15
(2,2)	$(\partial^2 U)^2(\partial^2 I)^2$	5,670	126	24
(0,1)	$(\partial I)^2$	9	3	1
(0,2)	$(\partial^2 I)^2$	60	6	2
(0,3)	$(\partial^3 I)^2$	525	10	2
(0,4)	$(\partial^4 I)^2$	5,670	15	3

n, called the *weight* of the corresponding irreducible representation. We will denote this irreducible representation by $\mathbf{n}$. In what follows we will use superscripts and subscripts to label where a particular representation comes from, but these indices have no other significance. The carrier space for such an irreducible representation can be taken to be the image plane, so that we say that a "vector" (not to be confused with an ordinary vector) $\mathbf{A} = (A_1, A_2)^{\mathrm{T}}$ transforms according to the irreducible representation (irrep) $\mathbf{n}$ if, under a rotation (of the coordinate system) by θ, $\mathbf{A}$ transforms like

$$\mathbf{A} \longrightarrow \mathbf{A}' = \mathrm{R}(n\theta)\mathbf{A}, \tag{10}$$

where

$$\mathrm{R}(n\theta) = \begin{pmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{pmatrix}. \tag{11}$$

We write the fact that $\mathbf{A}$ transforms according to (10) as $\mathbf{A} \sim \mathbf{n}$. We note that if $n = 1$, then $\mathbf{A}$ is an "ordinary" vector (corresponding to the irrep $\mathbf{1}$).

There is also a minor point to be noted here. In (10), n can be any integer, but we can show that we can restrict our attention to the non–negative integers, as follows. Suppose that $n = -|n| < 0$. Then a vector of weight n must transform like

$$\mathbf{A} \equiv \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \longrightarrow \begin{pmatrix} A'_1 \\ A'_2 \end{pmatrix} = \begin{pmatrix} \cos|n|\theta & -\sin|n|\theta \\ \sin|n|\theta & \cos|n|\theta \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}.$$

However, if we define the associated vector

$$\mathbf{A}^\circ \equiv \begin{pmatrix} A^\circ_1 \\ A^\circ_2 \end{pmatrix} \equiv \begin{pmatrix} A_1 \\ -A_2 \end{pmatrix},$$

then it is easy to see that

$$\mathbf{A}^\circ \longrightarrow \begin{pmatrix} A_1^{\circ\prime} \\ A_2^{\circ\prime} \end{pmatrix} = \begin{pmatrix} \cos|n|\theta & -\sin|n|\theta \\ \sin|n|\theta & \cos|n|\theta \end{pmatrix} \begin{pmatrix} A_1^\circ \\ A_2^\circ \end{pmatrix},$$

i.e., if $\mathbf{A} \sim -|\mathbf{n}|$, then $\mathbf{A}^\circ \sim \mathbf{n}$. The representations of weight n and weight $-n$ are called *equivalent* representations. As a consequence, we can restrict our considerations to non-negative weights. Later, in our discussion of direct product representations, we will assume that the second component of a vector has been multiplied by -1, if necessary, in order to have the vector transform with a non-negative weight. We note that an alternative, and somewhat more powerful, approach to the representation theory of SO(2) is to associate with each vector $\mathbf{A} = (A_1, A_2)^{\mathrm{T}}$ the complex number $A = A_1 + iA_2$. The transformation law for a vector of weight n then assumes the simple form $A \longrightarrow e^{in\theta}A$. We do not pursue this avenue here[1], although the complex approach is used extensively in the sequel to this work [Snyd90].

In our work, we are interested in quantities of the form

$$\mathbf{A}^{(1)}_{a_1}\mathbf{A}^{(2)}_{a_2}\cdots\mathbf{A}^{(k)}_{a_k}, \tag{12}$$

where $\mathbf{A}^{(i)}$ is a vector of weight n_i (normally, $n_i = 1$). Formally, (12) is a tensor. Such tensors are not, in general, irreps of SO(2) (i.e., they do not transform like (10) for some n) but rather have "pieces" which do transform irreducibly under SO(2). That is, a quantity like (12) can be written as a sum of quantities which transform irreducibly under SO(2). In the language of group representation theory, we say that the representation of SO(2) given by (12) is *reducible* under the action of SO(2) into a direct sum of irreps of SO(2).

[1]The reader interested in such matters can easily show that if $\mathbf{A} = (A_1, A_2)$ is a vector of weight 1, then the complex number $A = A_1 + iA_2$ transforms like $A \longrightarrow A' = \exp i\theta\, A$, and hence, $(\Re A^n, \Im A^n)^{\mathrm{T}}$ is a vector of weight n, where $\Re Z$ and $\Im Z$ are the real and complex parts of the complex number Z, respectively.

These comments can be made more precise. A quantity of the form (12) is an element of the direct product space $V_2 \otimes V_2 \otimes \cdots \otimes V_2$ (p factors of V_2), where V_2 is the image plane. It therefore transforms (following (12) according to the direct product representation

$$\mathbf{n_1} \otimes \mathbf{n_2} \otimes \cdots \otimes \mathbf{n_k} \tag{13}$$

of SO(2). This transformation is not, in general, irreducible, but can be written as the direct sum of irreps of SO(2). Our interest here is in the specific combinations of the $\mathrm{A}^{(i)}_{a_i}$ that transform as *scalars* (invariants) under SO(2), since these will correspond to ISO(2)–invariant smoothness constraints. That is, we are interested in the irreps $\mathbf{0}$ contained in the direct sum decomposition of the direct product (13). This can be done as follows.

We first note that the direct product and direct sum of representations $\mathbf{m}$, $\mathbf{n}$, and $\mathbf{p}$ satisfy the following two rules:

$$\begin{aligned} \mathbf{m} \otimes (\mathbf{n} \oplus \mathbf{p}) &= (\mathbf{m} \otimes \mathbf{n}) \oplus (\mathbf{m} \otimes \mathbf{p}) \\ \mathbf{m} \otimes (\mathbf{n} \otimes \mathbf{p}) &= (\mathbf{m} \otimes \mathbf{n}) \otimes \mathbf{p}. \end{aligned}$$

Consequently the reduction of (13) into a direct sum of irreps can be made if we can reduce the direct product $\mathbf{m_1} \otimes \mathbf{m_2}$ of *two* irreps $\mathbf{m_1}$ and $\mathbf{m_2}$.

To see how this is done, consider a vector $\mathbf{A}$ of weight $m_1 \neq 0$, and a vector $\mathbf{B}$ of weight $m_2 \neq m_1 \neq 0$. Then the direct product representation $\mathbf{m_1} \otimes \mathbf{m_2}$ consists of the four quantities

$$\{A_a B_b \; ; \; a, b = 1, 2\} = \{A_1 B_1, A_1 B_2, A_2 B_1, A_2 B_2\}.$$

Since

$$\mathbf{A} \longrightarrow \mathbf{A}' = \mathrm{R}(m_1 \theta) \mathbf{A} \tag{14}$$

$$\mathbf{B} \longrightarrow \mathbf{B}' = \mathrm{R}(m_2 \theta) \mathbf{B}, \tag{15}$$

it is easily seen by explicit calculation that if the following basis is taken for the direct product space:

$$\begin{aligned} \Psi_1 &= A_1B_1 + A_2B_2 = \mathbf{B}\cdot\mathbf{A} \\ \Psi_2 &= A_2B_1 - A_1B_2 = \mathbf{B}\times\mathbf{A} \qquad (16) \\ \Phi_1 &= A_1B_1 - A_2B_2 \\ \Phi_2 &= A_2B_1 + A_1B_2, \qquad (17) \end{aligned}$$

then we have that under a rotation of the coordinate system by θ

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} \longrightarrow \mathbf{R}((m_1 - m_2))\Psi \qquad (18)$$

$$\Phi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} \longrightarrow \mathbf{R}((m_1 + m_2))\Psi. \qquad (19)$$

That is, Ψ is a vector of weight $m_1 - m_2$, so that $\Psi \sim \boldsymbol{m_1 - m_2}$ and Φ is a vector of weight $m_1 + m_2$, so that $\Phi \sim \boldsymbol{m_1 + m_2}$. (Note that $\boldsymbol{m \pm n}$ is *not* any sort of "sum" or "difference" of two irreps—it is the representation of weight $m \pm n$.) This is exactly the decomposition of the direct product into a direct sum of irreps:

$$\mathbf{m_1} \otimes \mathbf{m_2} = \boldsymbol{m_1 - m_2} \oplus \boldsymbol{m_1 + m_2} \qquad (m_1 \neq m_2 \neq 0). \qquad (20)$$

If either m_1 or m_2 is zero, then the above derivation is invalid, since there are not four independent quantities of the form $\{A_aB_b\}$. In that case, it is clear that a scalar ($\sim \mathbf{0}$) times a vector of weight m just gives back a vector of weight m, i.e., $\mathbf{m}\otimes\mathbf{0} = \mathbf{m}$. A slight complication does arise, however, if $m_1 = m_2$. In that case (18) says that $\Psi_1 = A_1B_1 + A_2B_2 = \mathbf{B}\cdot\mathbf{A}$ and $\Psi_2 = A_2B_1 - A_1B_2 = \mathbf{B}\times\mathbf{A}$ are *both* invariants (note that in two dimensions, the cross product of two vectors which transform in the same way under SO(2) is a scalar, not a vector). Furthermore, if $m_1 = m_2$, and both of these irreps come from the same quantity (i.e., $\mathbf{A} = \mathbf{B}$), then $\Psi_2 = 0$, so there is only one non–trivial invariant instead of two. In the latter case, we say that the representations m_1 and m_2 are *identical*. We summarize these results as:

$$\mathbf{m} \otimes \mathbf{m} = \mathbf{0}^+ \oplus \mathbf{0}^- \oplus \mathbf{2m} \qquad (\mathbf{A} \neq \mathbf{B},\ m \neq 0) \tag{21}$$

$$\mathbf{m} \otimes \mathbf{m} = \mathbf{0} \oplus \mathbf{2m} \equiv (\mathbf{m} \otimes \mathbf{m})_{\mathrm{id}} \quad (\mathbf{A} = \mathbf{B},\ m \neq 0) \tag{22}$$

$$\mathbf{m} \otimes \mathbf{0} = \mathbf{0} \otimes \mathbf{m} = \mathbf{m}. \tag{23}$$

Here, $\mathbf{0}^+$ and $\mathbf{0}^-$ are the appropriate Ψ_1 and Ψ_2, respectively, and $\mathbf{2m}$ is the representation of weight $2m$. We have introduced the notation $(\mathbf{m} \otimes \mathbf{m})_{\mathrm{id}}$ to denote the direct product of the two representations $\mathbf{m}$ when the representations are in fact identical.

These results will be crucial to the remainder of the analysis.

3 The Direct Sum Decomposition of $\mathbf{1}^{\otimes m}$

The problem of finding all (coupled or decoupled) ISO(2)–invariant smoothness constraints of type (p, q) is just the problem of finding all possible linear combinations of quantities like

$$\left(\partial_{a_1} \ldots \partial_{a_p} U_r\right)\left(\partial_{b_1} \ldots \partial_{b_p} U_s\right)\left(I_{c_1 \ldots c_q} I_{d_1 \ldots d_q}\right),$$

such that the linear combinations transform as scalars under ISO(2) (this is exactly the point of finding all the tensors $f^{\cdots}_{\cdots}$!). But the set of all quantities of the form (3) spans the tensor product space $V_2 \otimes V_2 \otimes \cdots \otimes V_2$ ($2p + 2q + 2$ factors of V_2). Since $\{\partial_a\}$ and $\{U_a\}$ transform according to the $\mathbf{1}$ representation of ISO(2), and I is a scalar, so that $\{\partial_a I\}$ transforms according to the 1 representation as well, it follows that

$$\partial_{a_1}\cdots\partial_{a_p}U_r \;\sim\; \underbrace{1\otimes\cdots\otimes 1}_{p+1} \equiv 1^{\otimes(p+1)}$$

$$I_{c_1\ldots c_q} \;\sim\; \underbrace{1\otimes\cdots\otimes 1}_{q} \equiv 1^{\otimes q}. \tag{24}$$

$$\tag{25}$$

Consequently, (3) must transform according to the (reducible) representation

$$\left[1^{(p,0)}\right]\otimes\left[1^{(0,q)}\right], \tag{26}$$

of ISO(2), where we have defined:

$$\left(\partial_{a_1}\cdots\partial_{a_p}U_r\right)\left(\partial_{b_1}\cdots\partial_{b_p}U_s\right) \sim \left[1^{\otimes(p+1)}\otimes 1^{\otimes(p+1)}\right] \;\equiv\; 1^{(p,0)}$$

$$I_{c_1\ldots c_q}I_{d_1\ldots d_q} \sim \left[1^{\otimes q}\otimes 1^{\otimes q}\right] \;\equiv\; 1^{(0,q)}.$$

Therefore, there is a one–to–one correspondence between invariants of the form (1), and representations of weight 0 which occur in the direct sum decomposition of (26) into irreps. *Therefore, we can find all the invariants of type* (p,q) *by simply making this decomposition.* Indeed, the decomposition will tell us not only how many invariants there are, but also gives us their explicit form, by using (16,17)! **That is, the theory of group representations gives both a complete list of all the invariants and their explicit construction.** (This is the most important sentence in this paper!)

We can therefore find all the invariants of type (p,q) by finding the direct sum decomposition of $1^{(p,0)}$ and $1^{(0,q)}$, taking their direct product, and identifying all the representations of weight 0 which occur in the direct sum decomposition of that direct product.

3.1 Some general remarks on invariant constraints

We note a few general properties of smoothness constraints of type (p, q). Our first remark is that if we denote the invariant constraints of type $(p, 0)$ by $\{\mathcal{F}_i^{(p0)},\ i = 1, \ldots, m\}$ and the invariant constraints of type $(0, q)$ by $\{\mathcal{F}_i^{(0q)},\ i = 1, \ldots, n\}$, then the mn quantities

$$\left\{\mathcal{F}_{i,j}^{(p,q)} \equiv \mathcal{F}_i^{(p0)}\mathcal{F}_j^{(0q)},\ i = 1, \ldots, m; j = 1, \ldots, n\right\}$$

are all smoothness constraints of type (p, q). We call such constraints *composite*.

Our second remark is that there is a relation between the constraints of type $(0, q)$, used for surface interpolation, and the *de-coupled* constraints of type $(q, 0)$ used for optical flow computations. If we show the explicit functional dependence of the former constraints as

$$\left\{\mathcal{F}_i^{(0q)}[I],\ i = 1, \ldots, n\right\},$$

Then it is easy to see that the decoupled constraints given by

$$\left\{\mathcal{F}_i[\mathbf{U}] = \mathcal{F}_i^{(0q)}[u] + \mathcal{F}_i^{(0q)}[v]\ ;\ \ i = 1, \ldots, n\right\}.$$

are decoupled constraints of type $(q, 0)$. This is, for instance, the relation between the smoothness constraint used by Anandan and Weiss [Anan85] for computing optical flow, and the "quadratic variation" smoothness constraint introduced by Brady and Horn [Brad83] and by Grimson [Grim81].

In the next two sections, we find the direct sum decomposition of $\mathbf{1}^{(p,0)}$ and $\mathbf{1}^{(0,q)}$.

3.2 The direct sum decomposition of $\mathbf{1}^{(p,0)}(p = 1, 2)$

According to the remarks in the previous section, we have

$$\partial_a U_r \sim \mathbf{1} \otimes \mathbf{1},$$

and so

$$\partial_a U_r \partial_b U_s \sim (\mathbf{1} \otimes \mathbf{1}) \otimes (\mathbf{1} \otimes \mathbf{1}) = \mathbf{1}^{(1,0)}.$$

Our goal is to find the direct sum decomposition of this direct product.

We will need the results (20), (21), (22), and (23) in order to effect this decomposition. We have

$$\mathbf{1} \otimes \mathbf{1} = \mathbf{0}^+ \oplus \mathbf{0}^- \oplus \mathbf{2}$$

if the two **1**'s are not identical, and

$$\mathbf{1} \otimes \mathbf{1} = (\mathbf{1} \otimes \mathbf{1})_{\text{id}} = \mathbf{0} \oplus \mathbf{2}$$

if the two **1**'s are identical. We note that since ∂_a and U_r are not identical,

$$\partial_a U_r \sim \mathbf{1} \otimes \mathbf{1} = \mathbf{0}_1 \oplus \mathbf{0}_2 \oplus \mathbf{2}_2,$$

where, according to (16) and (17),

$$\mathbf{0}_1 \sim \partial_1 U_1 + \partial_2 U_2 = u_x + v_y \equiv \lambda_1 \tag{27}$$

$$\mathbf{0}_2 \sim \partial_2 U_1 - \partial_1 U_2 = u_y - v_x \equiv \lambda_2 \tag{28}$$

$$\mathbf{2}_1 \sim \begin{pmatrix} \partial_1 U_1 - \partial_2 U_2 \\ \partial_2 U_1 + \partial_1 U_2 \end{pmatrix} = \begin{pmatrix} u_x - v_y \\ u_y + v_x \end{pmatrix} \equiv \mathbf{K}^{(1)} . \tag{29}$$

An identical analysis holds for $\partial_b U_s$. Note that $\lambda_1 \sim \mathbf{0}_1$ is just the divergence of the flow field, and $\lambda_2 \sim \mathbf{0}_2$ is just its curl (recall that in two dimensions, the curl is a scalar, not a vector). Therefore

$$\partial_a U_r \partial_b U_s \sim (\mathbf{1} \otimes \mathbf{1}) \otimes (\mathbf{1} \otimes \mathbf{1}) = (\mathbf{0}_1 \oplus \mathbf{0}_2 \oplus \mathbf{2}_1) \otimes (\mathbf{0}_1 \oplus \mathbf{0}_2 \oplus \mathbf{2}_1)$$

Using the fact that $\otimes$ distributes over $\oplus$, we find that

$$\begin{aligned}(\mathbf{0}_1 \oplus \mathbf{0}_2 \oplus \mathbf{2}_1) \quad \otimes & \quad (\mathbf{0}_1 \oplus \mathbf{0}_2 \oplus \mathbf{2}_1) \\ = & \quad \Big[(\mathbf{0}_1 \otimes \mathbf{0}_1) \oplus (\mathbf{0}_1 \otimes \mathbf{0}_2) \oplus (\mathbf{0}_2 \otimes \mathbf{0}_1) \oplus (\mathbf{0}_2 \otimes \mathbf{0}_2)\Big] \oplus \\ & \oplus \Big[(\mathbf{2}_1 \otimes \mathbf{0}_1) \oplus (\mathbf{2}_1 \otimes \mathbf{0}_2) \oplus (\mathbf{0}_1 \otimes \mathbf{2}_1) \oplus (\mathbf{0}_2 \otimes \mathbf{2}_1)\Big] \oplus \\ & \oplus \Big[\mathbf{2}_1 \otimes \mathbf{2}_1\Big].\end{aligned}$$

But it is easy to see that

$$\begin{aligned}\mathbf{0}_1 \otimes \mathbf{0}_2 &= \mathbf{0}_2 \otimes \mathbf{0}_1, \\ \mathbf{2}_1 \otimes \mathbf{0}_1 &= \mathbf{0}_1 \otimes \mathbf{2}_1, \\ \mathbf{2}_1 \otimes \mathbf{0}_2 &= \mathbf{0}_2 \otimes \mathbf{2}_1,\end{aligned}$$

so that in the direct sum, we are presented with two copies of each of the above direct products (the number of such copies is called the *multiplicity* of the representation). It should be obvious that the multiplicity of a representation is irrelevant to our considerations, and so we define the symbol $\simeq$ as follows: if a reducible representation $\mathbf{Q}$ is a direct sum $\mathbf{Q}'$ of other (not necessarily irreducible) representations with various multiplicities, then $\mathbf{Q} \simeq \mathbf{Q}''$, where $\mathbf{Q}''$ is just $\mathbf{Q}'$ with all the multiplicities set equal to 1. That is, if $\mathbf{Q}'$ gives the direct sum structure of $\mathbf{Q}$, then $\mathbf{Q}''$ gives the representation structure of $\mathbf{Q}$ (we are concerned only with the latter in this work).

Consequently, we can write

$$\begin{aligned}(\mathbf{0}_1 \oplus \mathbf{0}_2 \oplus \mathbf{2}_1) \quad \otimes & \quad (\mathbf{0}_1 \oplus \mathbf{0}_2 \oplus \mathbf{2}_1) \\ \simeq & \quad \Big[(\mathbf{0}_1 \otimes \mathbf{0}_1) \oplus (\mathbf{0}_1 \otimes \mathbf{0}_2) \oplus (\mathbf{0}_2 \otimes \mathbf{0}_2)\Big] \oplus\end{aligned}$$

$$\oplus\Big[(\mathbf{2}_1 \otimes \mathbf{0}_1) \oplus (\mathbf{2}_1 \otimes \mathbf{0}_2)\Big] \oplus$$

$$\oplus\Big[\mathbf{2}_1 \otimes \mathbf{2}_1\Big].$$

We can easily find the quantities in the above direct sum:

$$\begin{aligned}
\mathbf{0}_1 \otimes \mathbf{0}_1 &\equiv \mathbf{0}_3 \sim \lambda_1^2 \\
\mathbf{0}_1 \otimes \mathbf{0}_2 &\equiv \mathbf{0}_4 \sim \lambda_1\lambda_2 \\
\mathbf{0}_2 \otimes \mathbf{0}_2 &\equiv \mathbf{0}_5 \sim \lambda_2^2 \\
\mathbf{2}_1 \otimes \mathbf{0}_1 &\equiv \mathbf{2}_2 \sim \lambda_1\mathbf{K}^{(1)} \equiv \mathbf{K}^{(2)} \\
\mathbf{2}_1 \otimes \mathbf{0}_2 &\equiv \mathbf{2}_3 \sim \lambda_2\mathbf{K}^{(1)} \equiv \mathbf{K}^{(3)}.
\end{aligned}$$

The only remaining one is $\mathbf{2}_1 \otimes \mathbf{2}_1$. Since the two $\mathbf{2}$'s are the same,

$$\mathbf{2}_1 \otimes \mathbf{2}_1 \equiv \Big(\mathbf{2}_1 \otimes \mathbf{2}_1\Big)_{\text{id}} = \mathbf{0}_6 \oplus \mathbf{4}_1, \tag{30}$$

where, using (16,17), we see that

$$\begin{aligned}
\mathbf{0}_6 &\sim \mathbf{K}^{(1)} \cdot \mathbf{K}^{(1)} = (u_x - v_y)^2 + (u_y + v_x)^2 \\
\mathbf{4}_1 &\sim \mathbf{M}^{(1)} = \begin{pmatrix} {K_1^{(1)}}^2 - {K_2^{(1)}}^2 \\ 2K_1^{(1)}K_2^{(1)} \end{pmatrix} = \begin{pmatrix} (u_x - v_y)^2 - (u_y + v_x)^2 \\ 2(u_x - v_y)(u_y + v_x) \end{pmatrix}.
\end{aligned}$$

In summary, then, the representation $\mathbf{1}^{(1,0)}$ has the direct sum decomposition:

$$(\partial \mathbf{U})^2 \sim \mathbf{1}^{(1,0)} \simeq \mathbf{0}_3 \oplus \mathbf{0}_4 \oplus \mathbf{0}_5 \oplus \mathbf{0}_6 \oplus \mathbf{2}_2 \oplus \mathbf{2}_3 \oplus \mathbf{4}_1, \tag{31}$$

where

$$\mathbf{0}_3 = \lambda_1^2 \equiv \mathcal{F}_1^{(10)}$$

$$\begin{aligned}
\mathbf{0}_4 &= \lambda_1\lambda_2 \equiv \mathcal{F}_2^{(10)} \\
\mathbf{0}_5 &= \lambda_2^2 \equiv \mathcal{F}_3^{(10)} \\
\mathbf{0}_6 &= \mathbf{K}^{(1)} \cdot \mathbf{K}^{(1)} \equiv \mathcal{F}_4^{(10)} \\
\mathbf{2}_2 &= \lambda_1 \mathbf{K}^{(1)} \qquad (32) \\
\mathbf{2}_3 &= \lambda_2 \mathbf{K}^{(1)} \qquad (33) \\
\mathbf{4}_1 &= \mathbf{M}^{(1)}
\end{aligned}$$

We note that in the process of finding the direct sum decomposition of $\mathbf{1}^{(1,0)}$, we have also found that the invariant smoothness constraints of type $(1,0)$ are given simply by the four invariants in (31), namely

$$\left\{\mathcal{F}_1^{(10)},\ \mathcal{F}_2^{(10)},\ \mathcal{F}_3^{(10)},\ \mathcal{F}_4^{(10)}\right\}. \qquad (34)$$

It is easy to check that these are linearly independent, and that the only de–coupled invariant smoothness constraint that can be formed by taking linear combinations of these is just the smoothness constraint $\mathcal{F}_{\text{H-S}}$ of Horn and Schunck [Horn81] :

$$u_x^2 + u_y^2 + v_x^2 + v_y^2 = \mathcal{F}_{\text{H-S}} = \mathcal{F}_1^{(10)} + \mathcal{F}_3^{(10)} + \mathcal{F}_4^{(10)}.$$

Looking ahead a little, we note that if the invariant of type $(0,1)$ given by equation (44) is denoted by $F[I]$, then the decoupled invariant of type $(1,0)$ is just of the form

$$F[u] + F[v].$$

This is an example of the general comments made in Section 3.1.

We now perform a similiar analysis for the direct sum decomposition of $\mathbf{1}^{(2,0)}$. The method is identical to that for $\mathbf{1}^{(1,0)}$, so we give only the main details.

The quantity $\partial_a\partial_b$ transforms like $(\mathbf{1}\otimes\mathbf{1})_{\rm id}$, where $\mathbf{1}\sim\partial_a$. From (16,17), we see that

$$(\mathbf{1}\otimes\mathbf{1})_{\rm id}\sim\mathbf{0}\oplus\mathbf{2},$$

where

$$\begin{aligned}\mathbf{0} &\sim \partial_1^2+\partial_2^2\equiv\nabla^2\,,\\ \mathbf{2} &\sim \begin{pmatrix}\partial_1^2-\partial_2^2\\ 2\partial_1\partial_2\end{pmatrix}\equiv\begin{pmatrix}\partial^+\\ \partial^-\end{pmatrix}.\end{aligned}$$

Hence,

$$\begin{aligned}\partial_a\partial_b\mathbf{U}_r &\sim (\mathbf{1}\otimes\mathbf{1})_{\rm id}\otimes\mathbf{1}\\ &= \Big(\mathbf{0}\oplus\mathbf{2}\Big)\otimes\mathbf{1}\\ &= (\mathbf{0}\otimes\mathbf{1})\oplus(\mathbf{2}\otimes\mathbf{1})\\ &\equiv \mathbf{1}_1\oplus\mathbf{1}_2\oplus\mathbf{3}_1,\end{aligned}$$

where, using (16,17),

$$\begin{aligned}\mathbf{0}\otimes\mathbf{1}\equiv\mathbf{1}_1 &= \nabla^2\mathbf{U}=\begin{pmatrix}\nabla^2u\\ \nabla^2v\end{pmatrix}\\ \mathbf{1}_2 &= \begin{pmatrix}\partial^+U_1+\partial^-U_2\\ \partial^-U_1-\partial^+U_2\end{pmatrix}=\begin{pmatrix}\partial^+u+\partial^-v\\ \partial^-u-\partial^+v\end{pmatrix}\\ \mathbf{3}_1 &= \begin{pmatrix}\partial^+U_1-\partial^-U_2\\ \partial^-U_1+\partial^+U_2\end{pmatrix}=\begin{pmatrix}\partial^+u-\partial^-v\\ \partial^-u+\partial^+v\end{pmatrix}.\end{aligned}$$

Consequently,

$$\partial_a\partial_bU_r\partial_c\partial_dU_s \sim \Big(\mathbf{1}_1\oplus\mathbf{1}_2\oplus\mathbf{3}_1\Big)\otimes\Big(\mathbf{1}_1\oplus\mathbf{1}_2\oplus\mathbf{3}_1\Big)$$

$$
\begin{aligned}
&= (\mathbf{1}_1 \otimes \mathbf{1}_1) \oplus (\mathbf{1}_1 \otimes \mathbf{1}_2) \oplus (\mathbf{1}_2 \otimes \mathbf{1}_1) \oplus (\mathbf{1}_2 \otimes \mathbf{1}_2) \oplus \\
&\quad \oplus (\mathbf{3}_1 \otimes \mathbf{1}_1) \oplus (\mathbf{3}_1 \otimes \mathbf{1}_2) \oplus (\mathbf{1}_1 \otimes \mathbf{3}_1) \oplus (\mathbf{1}_2 \otimes \mathbf{3}_1) \oplus (\mathbf{3}_1 \otimes \mathbf{3}_1) \\
&\simeq (\mathbf{1}_1 \otimes \mathbf{1}_1)_{\mathrm{id}} \oplus (\mathbf{1}_1 \otimes \mathbf{1}_2) \oplus (\mathbf{1}_2 \otimes \mathbf{1}_2)_{\mathrm{id}} \oplus \\
&\quad \oplus (\mathbf{3}_1 \otimes \mathbf{1}_1) \oplus (\mathbf{3}_1 \otimes \mathbf{1}_2) \oplus (\mathbf{3}_1 \otimes \mathbf{3}_1)_{\mathrm{id}}.
\end{aligned}
$$

We then use (20 ff) repeatedly to find

$$
\begin{aligned}
(\mathbf{1}_1 \otimes \mathbf{1}_1)_{\mathrm{id}} &= \mathbf{0}_7 \oplus \mathbf{2}_4, \\
(\mathbf{1}_2 \otimes \mathbf{1}_2)_{\mathrm{id}} &= \mathbf{0}_8 \oplus \mathbf{2}_5, \\
(\mathbf{1}_1 \otimes \mathbf{1}_2) &= \mathbf{0}_9 \oplus \mathbf{0}_{10} \oplus \mathbf{2}_6, \\
(\mathbf{3}_1 \otimes \mathbf{1}_1) &= \mathbf{2}_7 \oplus \mathbf{4}_2, \\
(\mathbf{3}_1 \otimes \mathbf{1}_2) &= \mathbf{2}_8 \oplus \mathbf{4}_3, \\
(\mathbf{3}_1 \otimes \mathbf{3}_1)_{\mathrm{id}} &= \mathbf{0}_{11} \oplus \mathbf{6}_1.
\end{aligned}
$$

Here, according to (16,17),

$$
\begin{aligned}
\mathbf{0}_7 &\sim \mathcal{F}_1^{(20)} \equiv (\nabla^2 u)^2 + (\nabla^2 v)^2 \\
\mathbf{0}_8 &\sim \mathcal{F}_2^{(20)} \equiv (\partial^+ u + \partial^- v)^2 + (\partial^- u - \partial^+ v)^2 \\
\mathbf{0}_9 &\sim \mathcal{F}_3^{(20)} \equiv \nabla^2 u\,(\partial^+ u + \partial^- v) + \nabla^2 v\,(\partial^- u - \partial^+ v) \\
\mathbf{0}_{10} &\sim \mathcal{F}_4^{(20)} \equiv \nabla^2 u\,(\partial^- u - \partial^+ v) - \nabla^2 v\,(\partial^+ u + \partial^- v) \\
\mathbf{0}_{11} &\sim \mathcal{F}_5^{(20)} \equiv (\partial^+ u - \partial^- v)^2 + (\partial^- u + \partial^+ v)^2
\end{aligned}
$$

$$
\mathbf{2}_4 \sim \mathbf{K}^{(4)} \equiv \begin{pmatrix} (\nabla^2 u)^2 - (\nabla^2 v)^2 \\ 2\nabla^2 u\,\nabla^2 v \end{pmatrix} \tag{35}
$$

$$
\mathbf{2}_5 \sim \mathbf{K}^{(5)} \equiv \begin{pmatrix} (\partial^+ u + \partial^- v)^2 - (\partial^- u - \partial^+ v)^2 \\ 2(\partial^+ u + \partial^- v)(\partial^- u - \partial^+ v) \end{pmatrix} \tag{36}
$$

$$\mathbf{2}_6 \sim \mathrm{K}^{(6)} \equiv \begin{pmatrix} \nabla^2 u\,(\partial^+ u + \partial^- v) - \nabla^2 v\,(\partial^- u - \partial^+ v) \\ \nabla^2 u\,(\partial^- u - \partial^+ v) + \nabla^2 v\,(\partial^+ u + \partial^- v) \end{pmatrix} \tag{37}$$

$$\mathbf{2}_7 \sim \mathrm{K}^{(7)} \equiv \begin{pmatrix} \nabla^2 u\,(\partial^+ u - \partial^- v) + \nabla^2 v\,(\partial^- u + \partial^+ v) \\ \nabla^2 u\,(\partial^- u + \partial^+ v) - \nabla^2 v\,(\partial^+ u - \partial^- v) \end{pmatrix} \tag{38}$$

$$\mathbf{2}_8 \sim \mathrm{K}^{(8)} \equiv \begin{pmatrix} (\partial^+ u + \partial^- v)(\partial^+ u - \partial^- v) + (\partial^- u - \partial^+ v)(\partial^- u + \partial^+ v) \\ (\partial^+ u + \partial^- v)(\partial^- u + \partial^+ v) - (\partial^- u - \partial^+ v)(\partial^+ u - \partial^- v) \end{pmatrix} \tag{39}$$

$$\mathbf{4}_2 \sim \mathrm{M}^{(2)} \equiv \begin{pmatrix} \nabla^2 u\,(\partial^+ u - \partial^- v) - \nabla^2 v\,(\partial^- u + \partial^+ v) \\ \nabla^2 u\,(\partial^- u + \partial^+ v) + \nabla^2 v\,(\partial^+ u - \partial^- v) \end{pmatrix} \tag{40}$$

$$\mathbf{4}_3 \sim \mathrm{M}^{(3)} \equiv \begin{pmatrix} (\partial^+ u + \partial^- v)(\partial^+ u - \partial^- v) - (\partial^- u - \partial^+ v)(\partial^- u + \partial^+ v) \\ (\partial^+ u + \partial^- v)(\partial^- u + \partial^+ v) + (\partial^- u - \partial^+ v)(\partial^+ u - \partial^- v) \end{pmatrix} \tag{41}$$

$$\mathbf{6}_1 \sim \mathrm{N}_1 \equiv \begin{pmatrix} (\partial^+ u - \partial^- v)^2 - (\partial^- u + \partial^+ v)^2 \\ 2(\partial^+ u - \partial^- v)(\partial^- u + \partial^+ v) \end{pmatrix}.$$

Consequently, the direct sum decomposition of $\mathbf{1}^{(2,0)}$ is (neglecting multiplicities) given by:

$$\left(\partial^2 \mathrm{U}\right)^2 \sim \mathbf{1}^{(2,0)} \simeq \left[\mathbf{0}_7 \oplus \mathbf{0}_8 \oplus \mathbf{0}_9 \oplus \mathbf{0}_{10} \oplus \mathbf{0}_{11}\right] \oplus \left[\mathbf{2}_4 \oplus \mathbf{2}_5 \oplus \mathbf{2}_6 \oplus \mathbf{2}_7 \oplus \mathbf{2}_8\right] \oplus \oplus\left[\mathbf{4}_2 \oplus \mathbf{4}_3\right] \oplus \mathbf{6}_1. \tag{42}$$

We note that there are here five invariants of type (2,0):

$$\left\{\mathcal{F}_1^{(20)}, \mathcal{F}_2^{(20)}, \mathcal{F}_3^{(20)}, \mathcal{F}_4^{(20)}, \mathcal{F}_5^{(20)}\right\}$$

where (expanding out the expressions above):

$$\mathcal{F}_1^{(20)} = (u_{xx} + u_{yy})^2 + (v_{xx} + v_{yy})^2$$

$$\mathcal{F}_2^{(20)} = (u_{xx} - u_{yy} + 2v_{xy})^2 + (v_{xx} - v_{yy} - 2u_{xy})^2$$

$$\mathcal{F}_3^{(20)} = u_{xx}^2 - u_{yy}^2 - v_{xx}^2 + v_{yy}^2 + 2u_{xy}(v_{xx} + v_{yy}) + 2v_{xy}(u_{xx} + u_{yy})$$

$$\mathcal{F}_4^{(20)} = 2\Big[u_{xy}(u_{xx}+u_{yy}) - v_{xy}(v_{xx}+v_{yy}) + u_{yy}v_{yy} - u_{xx}v_{xx}\Big]$$

$$\mathcal{F}_5^{(20)} = \Big(u_{xx}-u_{yy}-2v_{xy}\Big)^2 + \Big(v_{xx}-v_{yy}+2u_{xy}\Big)^2$$

It is easy to show that these five invariant constraints are linearly independent. It is also easy to show that precisely two linear combinations of these invariants are decoupled. $\mathcal{F}_1^{(20)} \equiv \mathcal{G}_1^{(20)}$ is manifestly so, and so is the linear combination

$$\mathcal{G}_2^{(20)} \equiv \frac{2\mathcal{F}_1^{(20)} + \mathcal{F}_2^{(20)} + \mathcal{F}_5^{(20)}}{4} = u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2 + \{u \longrightarrow v\}.$$

It is clear that both of these constraints are positive definite. We note that $\mathcal{G}_2^{(20)}$ is just the smoothness constraint used by Anandan and Weiss [Anan85] for computing optical flow. We note also that the decoupled constraints of type $(2,0)$, $\mathcal{G}_1^{(20)}$ and $\mathcal{G}_2^{(20)}$, are related to those of type $(0,2)$ (cf. the discussion following equation (51)), as was remarked upon in Section 3.1.

In summary, then, there are 5 scalar smoothness constraints of type $(2,0)$. The two decoupled ones are:

$$\mathcal{G}_1^{(20)} = (u_{xx}+u_{yy})^2 + (v_{xx}+v_{yy})^2$$

$$\mathcal{G}_2^{(20)} = \Big[(u_{xx}-u_{yy})^2 + 4u_{xy}^2\Big] + \Big[(v_{xx}-v_{yy})^2 + 4v_{xy}^2\Big]$$

and the three coupled ones can be taken to be:

$$\mathcal{F}_3^{(20)} = u_{xx}^2 - u_{yy}^2 - v_{xx}^2 + v_{yy}^2 + 2u_{xy}(v_{xx}+v_{yy}) + 2v_{xy}(u_{xx}+u_{yy})$$

$$\frac{1}{8}\Big(\mathcal{F}_2^{(20)} - \mathcal{F}_5^{(20)}\Big) = (u_{xx}-u_{yy})v_{xy} - (v_{xx}-v_{yy})u_{xy}$$

$$\frac{1}{2}\mathcal{F}_2^{(20)} = u_{xy}(u_{xx}+u_{yy}) - v_{xy}(v_{xx}+v_{yy}) + u_{yy}v_{yy} - u_{xx}v_{xx}.$$

3.3 The Decomposition of $\mathbf{1}^{(0,1)}$ and $\mathbf{1}^{(0,2)}$

We will also need the direct sum decomposition of quantities quadratic in 1[st] and 2[nd] derivatives of the grey–level intensity function I. We easily see that

$$\begin{aligned} \partial_a I &\sim \mathbf{1} \sim \begin{pmatrix} I_x \\ I_y \end{pmatrix} \equiv \nabla I \\ \partial_a \partial_b I &\sim (\mathbf{1} \otimes \mathbf{1})_{\mathrm{id}} \sim \mathbf{0} \oplus \mathbf{2}, \end{aligned} \tag{43}$$

where

$$\begin{aligned} \mathbf{0} &\sim \nabla^2 I \\ \mathbf{2} &\sim \begin{pmatrix} \partial_1^2 I - \partial_2^2 I \\ 2\partial_1\partial_2 I \end{pmatrix} \equiv \begin{pmatrix} I_{xx} - I_{yy} \\ 2I_{xy} \end{pmatrix}. \end{aligned}$$

It is then easy to show that

$$\partial_a I \partial_b I \sim (\mathbf{1} \otimes \mathbf{1})_{\mathrm{id}} \sim \mathbf{0}_{12} \oplus \mathbf{2}_9,$$

where

$$\mathbf{0}_{12} \sim |\nabla I|^2 \equiv \mathcal{F}_1^{(01)} \tag{44}$$

$$\mathbf{2}_9 \sim \mathbf{K}^{(9)} \equiv \begin{pmatrix} I_x^2 - I_y^2 \\ 2I_xI_y \end{pmatrix}. \tag{45}$$

Similarly,

$$\begin{aligned} \partial_a\partial_b I \partial_c \partial_d I &\sim \Big(\mathbf{0} \oplus \mathbf{2}\Big) \otimes \Big(\mathbf{0} \oplus \mathbf{2}\Big) \\ &= (\mathbf{0} \otimes \mathbf{0}) \oplus (\mathbf{2} \otimes \mathbf{0}) \oplus (\mathbf{0} \otimes \mathbf{2}) \oplus (\mathbf{2} \otimes \mathbf{2}) \\ &\simeq (\mathbf{0} \otimes \mathbf{0}) \oplus (\mathbf{2} \otimes \mathbf{0}) \oplus (\mathbf{2} \otimes \mathbf{2})_{\mathrm{id}}. \end{aligned}$$

Now then:

$$\mathbf{0} \otimes \mathbf{0} \equiv \mathbf{0}_{13} \sim (\nabla^2 I)^2 \equiv \mathcal{F}_1^{(02)} \tag{46}$$

$$\mathbf{2} \otimes \mathbf{0} \equiv \mathbf{2}_{10} \sim \mathbf{K}^{(10)} \equiv (\nabla^2 I) \begin{pmatrix} I_{xx} - I_{yy} \\ 2I_{xy} \end{pmatrix} \tag{47}$$

$$(\mathbf{2} \otimes \mathbf{2})_{\text{id}} = \mathbf{0}_{14} \oplus \mathbf{4}_4,$$

where, according to (16,17),

$$\mathbf{0}_{14} \sim (I_{xx} - I_{yy})^2 + 4I_{xy}^2 \equiv \mathcal{F}_2^{(02)} \tag{48}$$

$$\mathbf{4}_4 \sim \mathbf{M}^{(4)} \equiv \begin{pmatrix} (I_{xx} - I_{yy})^2 - 4I_{xy}^2 \\ 4(I_{xx} - I_{yy})\, I_{xy} \end{pmatrix}. \tag{49}$$

In summary, then,

$$(\partial I)^2 \simeq \mathbf{0}_{12} \oplus \mathbf{2}_9 \tag{50}$$

$$(\partial^2 I)^2 \simeq \mathbf{0}_{13} \oplus \mathbf{0}_{14} \oplus \mathbf{2}_{10} \oplus \mathbf{4}_4, \tag{51}$$

where the quantities in each of the direct sums are defined in (44)—(49).

We note that in finding these direct sum decompositions we have also found all the invariant smoothness constraints of types $(0,1)$ and $(0,2)$. Namely, the smoothness constraint of type $(0,1)$ is just the single quantity $\mathcal{F}_1^{(01)}$ defined in (44), whereas the smoothness constraints of type $(0,2)$ are two in number: $\mathcal{F}_1^{(02)}$ and $\mathcal{F}_2^{(02)}$, defined in (46) and (48), respectively. These constraints are manifestly positive definite, and hence are suitable for "performance functions" for surface interpolation. Our results therefore constitute a generalization of the results of [Brad83] and [Grim81], who also gave completeness proofs for smoothness constraints of this type, but for the more restricted case of "rotational symmetry." Our method is, we believe, more direct than that of the aforementioned authors. We remark also that $\mathcal{F}_1^{(02)}$ has been called by Grimson [Grim81] the "squared Lagrangian," and that the (positive definite) linear combination

$$\frac{1}{2}\left[\mathcal{F}_1^{(02)} + \mathcal{F}_2^{(02)}\right] = I_{xx}^2 + 2I_{xy}^2 + I_{yy}^2 \equiv \mathcal{F}_+^{(02)}$$

has been called by him the "quadratic variation." As is well known, the quantity $\mathcal{F}_1^{(02)} - \mathcal{F}_+^{(02)}$ is a total derivative (the divergence of a vector).

4 Invariant Smoothness Constraints of Type $(1,1)$ and $(1,2)$

Now that we have found the direct sum decompositions of the portion of each smoothness constraint of type (p,q) which depends only on the optical flow, or only on the image intensity, it is a simple matter to find the invariant constraints for arbitrary p and q. We limit ourselves in this section to the cases $(p,q) = (1,1),\ (1,2)$.

Before proceeding, we recall (see Section 3.1) that some of the invariants of type (p,q) can be written down immediately. Namely, the product of an invariant of type $(p,0)$ and one of type $(0,q)$ is clearly an invariant of type (p,q). We shall call such invariants "composite invariants of type (p,q)." Since there are 4 invariants of type $(1,0)$, 2 of type $(2,0)$, 1 of type $(0,1)$, and 2 of type $(0,2)$, it follows that there are $4 \times 1 = 4$ composite invariants of type $(1,1)$, $4 \times 2 = 8$ of type $(1,2)$, $2 \times 1 = 2$ of type $(2,1)$, and $2 \times 2 = 4$ of type $(2,2)$. We will see this explicitly in the remainder of this Section and in Section 5

4.1 Invariant Smoothness Constraints of Type (1,1)

These smoothness constraints are quadratic in 1st derivatives of the optical flow and of the image intensity function:

$$\left(\partial_a U_r \partial_b U_s\right)\partial_c I \partial_d I \sim (\partial U)^2(\partial I)^2 \tag{52}$$

From (31) and (50), we have that (52) transforms like the direct product representation

$$\begin{aligned}
\partial_a U_r \partial_b U_s \partial_c I \partial_d I &\simeq \left\{\left(\sum_{i=3}^{i=6\oplus} \mathbf{0}_i\right) \oplus \mathbf{2}_2 \oplus \mathbf{2}_3 \oplus \mathbf{4}_1\right\} \otimes \left(\mathbf{0}_{12} \oplus \mathbf{2}_9\right) \\
&= \left(\sum_{i=3}^{i=6\oplus} \left(\mathbf{0}_i \otimes \mathbf{0}_{12}\right)\right) \oplus \left(\mathbf{2}_2 \otimes \mathbf{0}_{12}\right) \oplus \left(\mathbf{2}_3 \otimes \mathbf{0}_{12}\right) \oplus \left(\mathbf{4}_1 \otimes \mathbf{0}_{12}\right) \oplus \\
&\quad \oplus \left(\sum_{i=3}^{i=6\oplus} \left(\mathbf{0}_i \otimes \mathbf{2}_9\right)\right) \oplus \left(\mathbf{2}_2 \otimes \mathbf{2}_9\right) \oplus \left(\mathbf{2}_3 \otimes \mathbf{2}_9\right) \oplus \left(\mathbf{4}_1 \otimes \mathbf{2}_9\right).
\end{aligned}$$

Here, we have defined

$$\sum_{i=1}^{i=n\oplus} \mathbf{m}_i \equiv \mathbf{m}_1 \oplus \cdots \oplus \mathbf{m}_n.$$

We are only interested in the terms in the direct sum decomposition that transform like $\mathbf{0}$. These can arise only from the terms of the form $\mathbf{m} \otimes \mathbf{m}$ in the above expression. Thus:

$$\partial_a U_r \partial_b U_s \partial_c I \partial_d I \sim \left(\sum_{i=3}^{i=6\oplus} \left(\mathbf{0}_i \otimes \mathbf{0}_{12}\right)\right) \oplus (\mathbf{2}_2 \otimes \mathbf{2}_9) \oplus (\mathbf{2}_3 \otimes \mathbf{2}_9) \oplus \{\text{O.T.}\},$$

where {O.T.} represents the terms which cannot give rise to scalars. We then find the direct sum decomposition of each of these double products, and identify the resulting scalars.

We have (recalling, once again, the insignificance of indices on representations)

$$\mathbf{2}_i \otimes \mathbf{2}_9 = \mathbf{0}^{+i} \oplus \mathbf{0}^{-i} \oplus \mathbf{4} \quad (i = 2, 3),$$

where, according to (16), (17), (32), and (33),

$$\mathbf{0}^{+2} \sim \lambda_1[\mathbf{K}_1^{(1)}\mathbf{K}_1^{(9)} + \mathbf{K}_2^{(1)}\mathbf{K}_2^{(9)}]$$

$$\mathbf{0}^{-2} \sim \lambda_1[\mathbf{K}_2^{(1)}\mathbf{K}_1^{(9)} - \mathbf{K}_1^{(1)}\mathbf{K}_2^{(9)}]$$

$$\mathbf{0}^{+3} \sim \lambda_2[\mathbf{K}_1^{(1)}\mathbf{K}_1^{(9)} + \mathbf{K}_2^{(1)}\mathbf{K}_2^{(9)}]$$

$$\mathbf{0}^{-2} \sim \lambda_2[\mathbf{K}_2^{(1)}\mathbf{K}_1^{(9)} - \mathbf{K}_1^{(1)}\mathbf{K}_2^{(9)}],$$

or, explicitly, using (29) and (45):

$$\mathbf{0}^{+2} \sim (u_x + v_y)\sigma_1$$

$$\mathbf{0}^{-2} \sim (u_x + v_y)\sigma_2$$

$$\mathbf{0}^{+3} \sim (u_y - v_x)\sigma_1$$

$$\mathbf{0}^{-3} \sim (u_y - v_x)\sigma_2,$$

where we have made the definitions

$$\sigma_1 \equiv \mathbf{K}^{(9)} \cdot \mathbf{K}^{(1)} = (u_x - v_y)(I_x^2 - I_y^2) + (u_y + v_x)(2I_xI_y) \quad (53)$$

$$\sigma_2 \equiv \mathbf{K}^{(9)} \times \mathbf{K}^{(1)} = (u_y + v_x)(I_x^2 - I_y^2) + (v_y - u_x)(2I_xI_y) \quad (54)$$

In addition, we have the four "composite" invariants of the form $\mathbf{0}_i \otimes \mathbf{0}_{12}$, i.e., there are *eight* invariants of type $(1,1)$:

$$\mathcal{F}_1^{(11)} = \mathcal{F}_1^{(01)}\mathcal{F}_1^{(10)} = |\nabla I|^2(u_x + v_y)^2$$

$$\mathcal{F}_2^{(11)} = \mathcal{F}_1^{(01)}\mathcal{F}_2^{(10)} = |\nabla I|^2(u_x + v_y)(u_y - v_x)$$

$$\mathcal{F}_3^{(11)} = \mathcal{F}_1^{(01)}\mathcal{F}_3^{(10)} = |\nabla I|^2(u_y - v_x)^2$$

$$
\begin{aligned}
\mathcal{F}_4^{(11)} &= \mathcal{F}_1^{(01)}\mathcal{F}_4^{(10)} = |\nabla I|^2(u_y+v_x)^2+(u_x-v_y)^2 \\
\mathcal{F}_5^{(11)} &= \lambda_1\sigma_1 = (u_x+v_y)[(u_x-v_y)(I_x^2-I_y^2)+(u_y+v_x)(2I_xI_y)] \\
\mathcal{F}_6^{(11)} &= \lambda_1\sigma_2 = (u_x+v_y)[(u_y+v_x)(I_x^2-I_y^2)+(v_y-u_x)(2I_xI_y)] \\
\mathcal{F}_7^{(11)} &= \lambda_2\sigma_1 = (u_y-v_x)[(u_x-v_y)(I_x^2-I_y^2)+(u_y+v_x)(2I_xI_y)] \\
\mathcal{F}_8^{(11)} &= \lambda_2\sigma_2 = (u_y-v_x)[(u_y+v_x)(I_x^2-I_y^2)+(v_y-u_x)(2I_xI_y)]
\end{aligned}
\tag{55}
$$

where the $\mathcal{F}_i^{(10)}$ are the invariants for constraints of type $(1,0)$, and σ_1 and σ_2 are as defined in (53) and (54), respectively.

It is easy to check that the eight invariants listed above are, in fact, linearly independent. One can further show that there are exactly *three* independent linear combinations of these eight invariants which are decoupled constraints (of type $(1,1)$):

$$
\begin{aligned}
\mathcal{G}_1^{(11)} &= \mathcal{F}_1^{(11)}+\mathcal{F}_3^{(11)}+\mathcal{F}_4^{(11)} = |\nabla I|^2(u_x^2+u_y^2+v_x^2+v_y^2) \equiv |\nabla I|^2\mathcal{F}_{\text{H-S}} \\
\mathcal{G}_2^{(11)} &= \mathcal{F}_5^{(11)}-\mathcal{F}_8^{(11)} = (u_x^2+v_x^2-u_y^2-v_y^2)(I_x^2-I_y^2)+(u_xu_y+v_xv_y)(4I_xI_y) \\
\mathcal{G}_3^{(11)} &= \mathcal{F}_6^{(11)}+\mathcal{F}_7^{(11)} = (u_x^2+v_x^2-u_y^2-v_y^2)(-2I_xI_y)+(u_xu_y+v_xv_y)(2(I_x^2-I_y^2)).
\end{aligned}
$$

These are related to the two decoupled constraints of type $(1,1)$ defined in [Snyd89] (but earlier introduced by Nagel and Enkelmann [Nage86] (see also [Nage83, Nage87])) :

$$
\begin{aligned}
(NE)_1 &= \mathrm{tr}\,[\Omega^{\mathrm{T}}\nabla I\,\nabla^{\mathrm{T}} I\,\Omega] \\
(NE)_2 &= \mathrm{tr}\,[\Omega^{\mathrm{T}}\widetilde{\nabla} I\widetilde{\nabla}^{\mathrm{T}} I\,\Omega],
\end{aligned}
$$

where $\Omega = \nabla \mathbf{U}^{\mathrm{T}}$, $\widetilde{\nabla} I = (I_y, -I_x)^{\mathrm{T}}$, by

$$\begin{aligned} \mathcal{G}_1^{(11)} &= (NE)_1 + (NE)_2 \\ \mathcal{G}_2^{(11)} &= (NE)_1 - (NE)_2 \end{aligned}$$

These were both shown in our earlier work to be positive definite, while we show later that $\mathcal{G}_3^{(11)}$ is not positive definite.

4.2 ISO(2) invariant smoothness constraints of type (1,2)

In this section, we consider smoothness constraints of type $(1,2)$, i.e., combinations of the form

$$\left(\partial_a U_r \partial_b U_s\right) \partial_c \partial_d I \partial_m \partial_n I \tag{56}$$

which are invariant. From (31) and (51) we have

$$(\partial U)^2 \simeq \Big(\sum_{i=3}^{i=6^{\oplus}} \mathbf{0}_i\Big) \oplus \mathbf{2}_2 \oplus \mathbf{2}_3 \oplus \mathbf{4}_1$$

and

$$(\partial^2 I)^2 \simeq \mathbf{0}_{13} \oplus \mathbf{0}_{14} \oplus \mathbf{2}_{10} \oplus \mathbf{4}_4.$$

Hence, a quantity of the form (56) will transform under ISO(2) like the direct product representation

$$\begin{aligned} (\partial U)^2 (\partial^2 I)^2 &\simeq \left(\Big(\sum_{i=3}^{i=6^{\oplus}} \mathbf{0}_i\Big) \oplus \mathbf{2}_2 \oplus \mathbf{2}_3 \oplus \mathbf{4}_1\right) \otimes \left(\mathbf{0}_{13} \oplus \mathbf{0}_{14} \oplus \mathbf{2}_{10} \oplus \mathbf{4}_4\right) \\ &\simeq \left[\Big(\sum_{i=3}^{i=6^{\oplus}} \mathbf{0}_i\Big) \otimes \left(\mathbf{0}_{13} \oplus \mathbf{0}_{14}\right)\right] \oplus \left[\left(\mathbf{2}_2 \oplus \mathbf{2}_3\right) \otimes \mathbf{2}_{10}\right] \oplus \end{aligned}$$

$$\oplus\left[\mathbf{4}_1\otimes\mathbf{4}_4\right]\oplus\left\{\text{O.T.}\right\},$$

where, as before, "O.T." denotes terms which cannot give invariants. We see immediately that there are $4\times 2=8$ "composite" invariants, $2\times 2=4$ invariants coming from the two $\mathbf{0}$'s in each of the two $\mathbf{2}\otimes\mathbf{2}''$s, and $2\times 1=2$ invariants coming from the two $\mathbf{0}$'s in the $\mathbf{4}_1\otimes\mathbf{4}_4$, making a total of $8+4+2=14$ invariants of type $(1,2)$. We easily construct these explicitly.

We have that

$$\begin{aligned}\mathbf{2}_2\otimes\mathbf{2}_{10} &= \mathbf{0}^{+(20)}\oplus\mathbf{0}^{-(20)}\oplus\{\text{O.T.}\}\\ \mathbf{2}_3\otimes\mathbf{2}_{10} &= \mathbf{0}^{+(30)}\oplus\mathbf{0}^{-(30)}\oplus\{\text{O.T.}\},\end{aligned}$$

where

$$\begin{aligned}\mathbf{0}^{+(20)} &\sim \lambda_1\mathbf{K}^{(1)}\cdot\mathbf{K}^{(10)}=\\ &= (u_x+v_y)\Big[(u_x-v_y)(I_{xx}^2-I_{yy}^2)+(u_y+v_x)(2I_{xy}\nabla^2 I)\Big]\\ \mathbf{0}^{-(20)} &\sim \lambda_1\mathbf{K}^{(10)}\times\mathbf{K}^{(1)}=\\ &= (u_x+v_y)\Big[(u_y+v_x)(I_{xx}^2-I_{yy}^2)-(u_x-v_y)(2I_{xy}\nabla^2 I)\Big]\\ \mathbf{0}^{+(30)} &\sim \lambda_2\mathbf{K}^{(1)}\cdot\mathbf{K}^{(10)}=\\ &= (u_y-v_x)\Big[(u_x-v_y)(I_{xx}^2-I_{yy}^2)+(u_y+v_x)(2I_{xy}\nabla^2 I)\Big]\\ \mathbf{0}^{-(30)} &\sim \lambda_2\mathbf{K}^{(10)}\check{\times}\mathbf{K}^{(1)}=\\ &= (u_y-v_x)\Big[(u_y+v_x)(I_{xx}^2-I_{yy}^2)-(u_x-v_y)(2I_{xy}\nabla^2 I)\Big]\end{aligned}$$

We also have

$$\mathbf{4}_1\otimes\mathbf{4}_4=\mathbf{0}^{+(17)}\oplus\mathbf{0}^{-(17)}\oplus\{\text{O.T.}\},$$

where

$$0^{+(17)} \sim \mathbf{M}^{(1)} \cdot \mathbf{M}^{(4)} =$$
$$= \Big((u_x - v_y)^2 - (u_y + v_x)^2\Big)\Big((I_{xx} - I_{yy})^2 - 4I_{xy}^2\Big) + \Big(2(u_x - v_y)(u_y + v_x)\Big)\Big(4I_{xy}(I_{xx} - I_{yy})\Big)$$

$$0^{-(17)} \sim \mathbf{M}^{(4)} \times \mathbf{M}^{(1)} =$$
$$= \Big((I_{xx} - I_{yy})^2 - 4I_{xy}^2\Big)\Big(2(u_x - v_y)(u_y + v_x)\Big) - \Big(4I_{xy}(I_{xx} - I_{yy})\Big)\Big((u_x - v_y)^2 - (u_y + v_x)^2\Big).$$

In summary, then, there are 14 invariant smoothness constraints of type $(1,2)$:

$$0_3 \otimes 0_{13} \sim \mathcal{F}_1^{(12)} \equiv \mathcal{F}_1^{(10)}\mathcal{F}_1^{(02)} = (u_x + v_y)^2(I_{xx} + I_{yy})^2$$

$$0_3 \otimes 0_{14} \sim \mathcal{F}_2^{(12)} \equiv \mathcal{F}_1^{(10)}\mathcal{F}_2^{(02)} = (u_x + v_y)^2\Big[(I_{xx} - I_{yy})^2 + 4I_{xy}^2\Big]$$

$$0_4 \otimes 0_{13} \sim \mathcal{F}_3^{(12)} \equiv \mathcal{F}_2^{(10)}\mathcal{F}_1^{(02)} = (u_x + v_y)(u_y - v_x)(I_{xx} + I_{yy})^2$$

$$0_4 \otimes 0_{14} \sim \mathcal{F}_4^{(12)} \equiv \mathcal{F}_2^{(10)}\mathcal{F}_2^{(02)} = (u_x + v_y)(u_y - v_x)\Big[(I_{xx} - I_{yy})^2 + 4I_{xy}^2\Big]$$

$$0_5 \otimes 0_{13} \sim \mathcal{F}_5^{(12)} \equiv \mathcal{F}_3^{(10)}\mathcal{F}_1^{(02)} = (u_y - v_x)^2(I_{xx} + I_{yy})^2$$

$$0_5 \otimes 0_{14} \sim \mathcal{F}_6^{(12)} \equiv \mathcal{F}_3^{(10)}\mathcal{F}_2^{(02)} = (u_y - v_x)^2\Big[(I_{xx} - I_{yy})^2 + 4I_{xy}^2\Big]$$

$$0_6 \otimes 0_{13} \sim \mathcal{F}_7^{(12)} \equiv \mathcal{F}_4^{(10)}\mathcal{F}_1^{(02)} = \Big[(u_x - v_y)^2 + (u_y + v_x)^2\Big](I_{xx} + I_{yy})^2$$

$$0_6 \otimes 0_{14} \sim \mathcal{F}_8^{(12)} \equiv \mathcal{F}_4^{(10)}\mathcal{F}_2^{(02)} = \Big[(u_x - v_y)^2 + (u_y + v_x)^2\Big]\Big[(I_{xx} - I_{yy})^2 + 4I_{xy}^2\Big]$$

$$0^{+(20)} \sim \mathcal{F}_9^{(12)} = (u_x + v_y)\Big[(u_x - v_y)(I_{xx}^2 - I_{yy}^2) + (u_y + v_x)(2I_{xy}\nabla^2 I)\Big]$$

$$0^{-(20)} \sim \mathcal{F}_{10}^{(12)} = (u_x + v_y)\Big[(u_y + v_x)(I_{xx}^2 - I_{yy}^2) - (u_x - v_y)(2I_{xy}\nabla^2 I)\Big]$$

$$0^{+(30)} \sim \mathcal{F}_{11}^{(12)} = (u_y - v_x)\Big[(u_x - v_y)(I_{xx}^2 - I_{yy}^2) + (u_y + v_x)(2I_{xy}\nabla^2 I)\Big]$$

$$0^{-(30)} \sim \mathcal{F}_{12}^{(12)} = (u_y - v_x)\Big[(u_y + v_x)(I_{xx}^2 - I_{yy}^2) - (u_x - v_y)(2I_{xy}\nabla^2 I)\Big]$$

$$0^{+(17)} \sim \mathcal{F}_{13}^{(12)} = \Big((u_x - v_y)^2 - (u_y + v_x)^2\Big)\Big((I_{xx} - I_{yy})^2 - 4I_{xy}^2\Big) +$$

$$+\Big(2(u_x - v_y)(u_y + v_x)\Big)\Big(4I_{xy}(I_{xx} - I_{yy})\Big)$$

$$0^{-(17)} \sim \mathcal{F}_{14}^{(12)} = \Big((I_{xx} - I_{yy})^2 - 4I_{xy}^2\Big)\Big(2(u_x - v_y)(u_y + v_x)\Big) -$$

$$-\Big(4I_{xy}(I_{xx} - I_{yy})\Big)\Big((u_x - v_y)^2 - (u_y + v_x)^2\Big).$$

One can show that all fourteen of these are linearly independent. Further, one can show that there are exactly *four* linear combinations of the $\mathcal{F}_i^{(12)}$ which are decoupled:

$$\mathcal{G}_1^{(12)} \equiv \frac{1}{2}\Big[\mathcal{F}_1^{(12)} + \mathcal{F}_5^{(12)} + \mathcal{F}_7^{(12)}\Big] = \mathcal{F}_{\text{H-S}}\Big[\nabla^2 I\Big]^2$$

$$\mathcal{G}_2^{(12)} \equiv \frac{1}{2}\Big[\mathcal{F}_2^{(12)} + \mathcal{F}_6^{(12)} + \mathcal{F}_8^{(12)}\Big] = \mathcal{F}_{\text{H-S}}\Big[(I_{xx} - I_{yy})^2 + 4I_{xy}^2\Big]$$

$$\mathcal{G}_3^{(12)} \equiv \frac{1}{2}\Big[\mathcal{F}_{10}^{(12)} + \mathcal{F}_{11}^{(12)}\Big] =$$

$$= (I_{xx}^2 - I_{yy}^2)(u_x u_y + v_x v_y) + 2I_{xy}\,\nabla^2 I(-u_x^2 + u_y^2 - v_x^2 + v_y^2)$$

$$\mathcal{G}_4^{(12)} \equiv \frac{1}{2}\Big[\mathcal{F}_9^{(12)} - \mathcal{F}_{12}^{(12)}\Big] =$$

$$= (I_{xx}^2 - I_{yy}^2)(u_x^2 - u_y^2 + v_x^2 - v_y^2) + 2I_{xy}\,\nabla^2 I(u_x u_y + v_x v_y)$$

The relation between these decoupled invariants and the two positive definite ones found in [Snyd89] (originally proposed by Nagel and Enkelmann [Nage86]):

$$\text{tr}\,(\boldsymbol{\Omega}^{\text{T}}\mathbf{L}^2\boldsymbol{\Omega}) \quad\text{and}\quad \text{tr}\,(\boldsymbol{\Omega}^{\text{T}}\bar{\mathbf{L}}^2\boldsymbol{\Omega}),$$

where

$$\boldsymbol{\Omega} = \nabla\mathbf{U}^{\text{T}}$$

$$\mathbf{L} = \begin{pmatrix} I_{xx} & I_{xy} \\ I_{xy} & I_{yy} \end{pmatrix}$$

$$\tilde{\mathbf{L}} = \begin{pmatrix} I_{yy} & -I_{xy} \\ -I_{xy} & I_{xx} \end{pmatrix},$$

is that

$$\mathrm{tr}\,(\Omega^{\mathrm{T}}\mathbf{L}^2\Omega) = \frac{1}{4}\left[\mathcal{G}_1 + \mathcal{G}_2 + 2\mathcal{G}_4\right]$$

$$\mathrm{tr}\,(\Omega^{\mathrm{T}}\tilde{\mathbf{L}}^2\Omega) = \frac{1}{4}\left[\mathcal{G}_1 + \mathcal{G}_2 - 2\mathcal{G}_4\right].$$

The remaining linear combinations are not positive definite.

5 Invariants of Type $(2,1)$ and $(2,2)$

As far as we are aware, smoothness constraints quadratic in the 2nd derivative of optical flow have been considered only by Anandan and Weiss [Anan85], who used a decoupled constraint of type $(2,0)$. Although there are probably good reasons why such constraints have been neglected (they would probably be sensitive to small errors), this may be an artifact of present algorithms. At any rate, we present the scalar constraints of type $(2,1)$ and $(2,2)$ for completeness.

5.1 Invariants of Type $(2,1)$

The smoothness constraints of type $(2,1)$ are linear combinations of quantities of the form

$$(\partial^2\mathrm{U})^2(\partial I)^2 \sim \left(\partial_a\partial_b U_r \partial_c\partial_d U_s\right)\left(\partial_m I \partial_n I\right).$$

Such objects transform, according to (42) and (50), like the direct product

$$(\partial^2 \mathrm{U})^2(\partial I)^2 \simeq \left\{\Big(\sum_{i=7}^{i=11\oplus} \mathbf{0}_i\Big) \oplus \Big(\sum_{i=4}^{i=8\oplus} \mathbf{2}_i\Big) \oplus \Big(\sum_{i=2}^{i=3\oplus} \mathbf{4}_i\Big) \oplus \mathbf{6}_1\right\} \otimes \left\{\mathbf{0}_{12} \oplus \mathbf{2}_9\right\}$$

$$= \left[\Big(\sum_{i=7}^{i=11\oplus} \mathbf{0}_i\Big) \otimes \mathbf{0}_{12}\right] \oplus \left[\Big(\sum_{i=4}^{i=8\oplus} \mathbf{2}_i\Big) \otimes \mathbf{2}_9\right] \oplus \Big[\mathrm{O.T.}\Big] .$$

We therefore have 5 composite invariant smoothness constraints of the form $\mathbf{0} \otimes \mathbf{0}$, and $5 \times 2 = 10$ invariant constraints from the two scalars contained in each of the $\mathbf{2} \otimes \mathbf{2}$'s, for a total of 15 invariant constraints of type $(2,1)$. It is then an easy matter to find these constraints explicitly:

$$\mathbf{2}_i \otimes \mathbf{2}_9 = \mathbf{0}^{+i} \oplus \mathbf{0}^{-i} \oplus \{\mathrm{O.T.}\} \qquad (i = 4, 5, 6, 7, 8)$$

where

$$\mathbf{0}^{+i} \sim \mathbf{K}^{(9)} \cdot \mathbf{K}^{(i-2)}$$

$$\mathbf{0}^{-i} \sim \mathbf{K}^{(9)} \times \mathbf{K}^{(i-2)} .$$

Therefore we have the 15 invariant smoothness constraints of type $(2,1)$:

$$\mathbf{0}_7 \otimes \mathbf{0}_{12} \sim F_1^{(21)} = F_1^{(20)} F_1^{(01)} = \left[u_{xx}^2 + 2u_{xx}u_{yy} + u_{yy}^2 + v_{xx}^2 + 2v_{xx}v_{yy} + v_{yy}^2\right](I_x^2 + I_y^2)$$

$$\mathbf{0}_8 \otimes \mathbf{0}_{12} \sim F_2^{(21)} = F_2^{(20)} F_1^{(01)} = \left[(u_{xx} - u_{yy} + 2v_{xy})^2 + (v_{xx} - v_{yy} - 2u_{xy})^2\right](I_x^2 + I_y^2)$$

$$\mathbf{0}_9 \otimes \mathbf{0}_{12} \sim F_3^{(21)} = F_3^{(20)} F_1^{(01)} =$$

$$= \left[(u_{xx} + u_{yy})(u_{xx} - u_{yy} + 2v_{xy}) - (v_{xx} + v_{yy})(v_{xx} - v_{yy} - 2u_{xy})\right](I_x^2 + I_y^2)$$

$$\mathbf{0}_{10} \otimes \mathbf{0}_{12} \sim F_4^{(21)} = F_4^{(20)} F_1^{(01)} =$$

$$= \Big[(u_{xx}+u_{yy})(v_{xx}-v_{yy}-2u_{xy})+(v_{xx}+v_{yy})(u_{xx}-u_{yy}+2v_{xy})\Big](I_x^2+I_y^2)$$

$$\mathbf{0}_{11}\otimes\mathbf{0}_{12} \sim F_5^{(21)} = F_5^{(20)}F_1^{(01)} = \Big[(u_{xx}-u_{yy}-2v_{xy})^2+(v_{xx}-v_{yy}+2u_{xy})^2\Big](I_x^2+I_y^2)$$

$$\mathbf{0}^{+4} \sim F_6^{(21)} = \mathbf{K}^{(9)}\cdot\mathbf{K}^{(2)} =$$

$$= (I_x^2-I_y^2)\Big[(u_{xx}+u_{yy})^2-(v_{xx}+v_{yy})^2\Big]+2I_xI_y\Big[2(u_{xx}+u_{yy})(v_{xx}+v_{yy})\Big]$$

$$\mathbf{0}^{-4} \sim F_7^{(21)} = \mathbf{K}^{(9)}\times\mathbf{K}^{(2)} =$$

$$= (I_x^2-I_y^2)\Big[2(u_{xx}+u_{yy})(v_{xx}+v_{yy})\Big]-2I_xI_y\Big[(u_{xx}+u_{yy})^2-(v_{xx}+v_{yy})^2\Big]$$

$$\mathbf{0}^{+5} \sim F_8^{(21)} = \mathbf{K}^{(9)}\cdot\mathbf{K}^{(3)} = (I_x^2-I_y^2)\Big[(u_{xx}-u_{yy}+2v_{xy})^2-(v_{xx}-v_{yy}-2u_{xy})^2\Big]-$$

$$-2I_xI_y\Big[(u_{xx}-u_{yy}+2v_{xy})(v_{xx}-v_{yy}-2u_{xy})\Big]$$

$$\mathbf{0}^{-5} \sim F_9^{(21)} = \mathbf{K}^{(9)}\times\mathbf{K}^{(3)} = (I_x^2-I_y^2)\Big[-2(u_{xx}-u_{yy}+2v_{xy})(v_{xx}-v_{yy}-2u_{xy})\Big]-$$

$$-2I_xI_y\Big[(u_{xx}-u_{yy}+2v_{xy})^2-(v_{xx}-v_{yy}-2u_{xy})^2\Big]$$

$$\mathbf{0}^{+6} \sim F_{10}^{(21)} = \mathbf{K}^{(9)}\cdot\mathbf{K}^{(4)} =$$

$$= (I_x^2-I_y^2)\Big[(u_{xx}-u_{yy}+2v_{xy})(u_{xx}+u_{yy})+(v_{xx}-v_{yy}-2u_{xy})(v_{xx}+v_{yy})\Big]+$$

$$+2I_xI_y\Big[(u_{xx}-u_{yy}+2v_{xy})(v_{xx}+v_{yy})-(v_{xx}-v_{yy}-2u_{xy})(u_{xx}+u_{yy})\Big]$$

$$\mathbf{0}^{-6} \sim F_{11}^{(21)} = \mathbf{K}^{(9)}\times\mathbf{K}^{(4)} =$$

$$= (I_x^2-I_y^2)\Big[(u_{xx}-u_{yy}+2v_{xy})(v_{xx}+v_{yy})-(v_{xx}-v_{yy}-2u_{xy})(u_{xx}+u_{yy})\Big]-$$

$$-2I_xI_y\Big[(u_{xx}-u_{yy}+2v_{xy})(u_{xx}+u_{yy})+(v_{xx}-v_{yy}-2u_{xy})(v_{xx}+v_{yy})\Big]$$

$$\mathbf{0}^{+7} \sim F_{12}^{(21)} = \mathbf{K}^{(9)}\cdot\mathbf{K}^{(5)} =$$

$$= (I_x^2-I_y^2)\Big[(u_{xx}-u_{yy}-2v_{xy})(u_{xx}+u_{yy})+(v_{xx}-v_{yy}+2u_{xy})(v_{xx}+v_{yy})\Big]+$$

$$+2I_xI_y\Big[(u_{xx}-u_{yy}-2v_{xy})(v_{xx}+v_{yy})-(v_{xx}-v_{yy}+2u_{xy})(u_{xx}+u_{yy})\Big]$$

$$0^{-7} \sim F_{13}^{(21)} = \mathbf{K}^{(9)} \times \mathbf{K}^{(5)} =$$

$$= (I_x^2 - I_y^2)\Big[(u_{xx}-u_{yy}-2v_{xy})(v_{xx}+v_{yy})-(v_{xx}-v_{yy}+2u_{xy})(u_{xx}+u_{yy})\Big] -$$

$$-2I_xI_y\Big[(u_{xx}-u_{yy}-2v_{xy})(u_{xx}+u_{yy})+(v_{xx}-v_{yy}+2u_{xy})(v_{xx}+v_{yy})\Big]$$

$$0^{+8} \sim F_{14}^{(21)} = \mathbf{K}^{(9)} \cdot \mathbf{K}^{(6)} = (I_x^2 - I_y^2)\Big[(u_{xx}-u_{yy})^2+4u_{xy}^2-(v_{xx}-v_{yy})^2-4v_{xy}^2\Big] -$$

$$-2I_xI_y\Big[2(u_{xx}-u_{yy})(v_{xx}-v_{yy})+4u_{xy}v_{xy}\Big]$$

$$0^{-8} \sim F_{15}^{(21)} = -\mathbf{K}^{(9)} \times \mathbf{K}^{(6)} = (I_x^2 - I_y^2)\Big[2(u_{xx}-u_{yy})(v_{xx}-v_{yy})+4u_{xy}v_{xy}\Big] +$$

$$+2I_xI_y\Big[(u_{xx}-u_{yy})^2+4u_{xy}^2-(v_{xx}-v_{yy})^2-4v_{xy}^2\Big]$$

It is easy to see that there are only two decoupled constraints of type $(2,1)$, corresponding to a product of the one of the two decoupled constraints of type $(2,0)$, times the single constraint of type $(0,1)$, as noted in Section 3.1. These can be chosen to be

$$\mathcal{G}_1^{(21)} \equiv \mathcal{F}_1^{(21)} = \Big[(u_{xx}+u_{yy})^2+(v_{xx}+v_{yy})^2\Big]|\nabla I|^2$$

$$\mathcal{G}_2^{(21)} \equiv \frac{\mathcal{F}_2^{(21)}+\mathcal{F}_5^{(21)}+2\mathcal{F}_1^{(21)}}{4} = \Big[u_{xx}^2+2u_{xy}^2+u_{yy}^2+\{u \longrightarrow v\}\Big]|\nabla I|^2$$

5.2 Invariant Constraints of Type $(2,2)$

The smoothness constraints of type $(2,2)$ are linear combinations of quantities of the form

$$(\partial^2\mathbf{U})^2(\partial^2 I)^2 \sim \Big(\partial_a\partial_b U_r \partial_c\partial_d U_s\Big)\Big(\partial_m\partial_n I \partial_k\partial_\ell I\Big).$$

Such objects transform, according to (42) and (51), like the direct product

$$(\partial^2\mathrm{U})^2(\partial^2 I)^2 \simeq \left\{\Big(\sum_{i=7}^{i=11\oplus} \mathbf{0}_i\Big) \oplus \Big(\sum_{i=4}^{i=8\oplus} \mathbf{2}_i\Big) \oplus \Big(\sum_{i=2}^{i=3\oplus} \mathbf{4}_i\Big) \oplus \mathbf{6}_1\right\} \otimes \left\{\Big(\sum_{i=13}^{i=14\oplus} \mathbf{0}_i\Big) \oplus \mathbf{2}_{10} \oplus \mathbf{4}_4\right\}.$$

We therefore have $5 \times 2 = 10$ composite invariant smoothness constraints of the form $\mathbf{0} \otimes \mathbf{0}$, $2 \times 5 = 10$ from the two scalars contained in each of the five $\mathbf{2} \otimes \mathbf{2}$'s, and $2 \times 2 = 4$ from the two scalars contained in each of the two $\mathbf{4} \otimes \mathbf{4}$'s, for a total of $10 + 10 + 4 = 24$ invariant smoothness constraints of type $(2, 2)$.

The ten composite constraints are:

$$\mathbf{0}_7 \otimes \mathbf{0}_{13} \;\sim\; \mathcal{F}_1^{(22)} = \mathcal{F}_1^{(20)}\mathcal{F}_1^{(02)} = \Big[(u_{xx} + u_{yy})^2 + (v_{xx} + v_{yy})^2\Big]\Big(I_{xx} + I_{yy}\Big)^2$$

$$\mathbf{0}_8 \otimes \mathbf{0}_{13} \;\sim\; \mathcal{F}_2^{(22)} = \mathcal{F}_2^{(20)}\mathcal{F}_1^{(02)} = \Big[(u_{xx} - u_{yy} + 2v_{xy})^2 + (v_{xx} - v_{yy} - 2u_{xy})^2\Big]\Big(I_{xx} + I_{yy}\Big)^2$$

$$\mathbf{0}_9 \otimes \mathbf{0}_{13} \;\sim\; \mathcal{F}_3^{(22)} = \mathcal{F}_3^{(20)}\mathcal{F}_1^{(02)} = \Big[u_{xx}^2 - u_{yy}^2 - v_{xx}^2 + v_{yy}^2 + 2u_{xy}(v_{xx} + v_{yy}) + 2v_{xy}(u_{xx} + u_{yy})\Big] \times$$
$$\times\Big(I_{xx} + I_{yy}\Big)^2$$

$$\mathbf{0}_{10} \otimes \mathbf{0}_{13} \;\sim\; \mathcal{F}_4^{(22)} = \mathcal{F}_4^{(20)}\mathcal{F}_1^{(02)} = \Big[2\Big(u_{xy}(u_{xx} + u_{yy}) - v_{xy}(v_{xx} + v_{yy}) + u_{yy}v_{yy} - u_{xx}v_{xx}\Big)\Big] \times$$
$$\times\Big(I_{xx} + I_{yy}\Big)^2$$

$$\mathbf{0}_{11} \otimes \mathbf{0}_{13} \;\sim\; \mathcal{F}_5^{(22)} = \mathcal{F}_5^{(20)}\mathcal{F}_1^{(02)} = \Big[(u_{xx} - u_{yy} - 2v_{xy})^2 + (v_{xx} - v_{yy} + 2u_{xy})^2\Big]\Big(I_{xx} + I_{yy}\Big)^2$$

$$\mathbf{0}_7 \otimes \mathbf{0}_{14} \;\sim\; \mathcal{F}_6^{(22)} = \mathcal{F}_1^{(20)}\mathcal{F}_2^{(02)} = \Big[(u_{xx} + u_{yy})^2 + (v_{xx} + v_{yy})^2\Big]\Big((I_{xx} - I_{yy})^2 + 4I_{xy}^2\Big)$$

$$\mathbf{0}_8 \otimes \mathbf{0}_{14} \;\sim\; \mathcal{F}_7^{(22)} = \mathcal{F}_2^{(20)}\mathcal{F}_2^{(02)} =$$
$$= \Big[(u_{xx} - u_{yy} + 2v_{xy})^2 + (v_{xx} - v_{yy} - 2u_{xy})^2\Big]\Big((I_{xx} - I_{yy})^2 + 4I_{xy}^2\Big)$$

$$\mathbf{0}_9 \otimes \mathbf{0}_{14} \;\sim\; \mathcal{F}_8^{(22)} = \mathcal{F}_3^{(20)}\mathcal{F}_2^{(02)} =$$

$$= \left[u_{xx}^2 - u_{yy}^2 - v_{xx}^2 + v_{yy}^2 + 2u_{xy}(v_{xx} + v_{yy}) + 2v_{xy}(u_{xx} + u_{yy})\right]\left((I_{xx} - I_{yy})^2 + 4I_{xy}^2\right)$$

$$\mathbf{0}_{10} \otimes \mathbf{0}_{14} \sim \mathcal{F}_9^{(22)} = \mathcal{F}_4^{(20)}\mathcal{F}_2^{(02)} =$$

$$= \left[2\Big(u_{xy}(u_{xx} + u_{yy}) - v_{xy}(v_{xx} + v_{yy}) + u_{yy}v_{yy} - u_{xx}v_{xx}\Big)\right]\left((I_{xx} - I_{yy})^2 + 4I_{xy}^2\right)$$

$$\mathbf{0}_{11} \otimes \mathbf{0}_{14} \sim \mathcal{F}_{10}^{(22)} = \mathcal{F}_5^{(20)}\mathcal{F}_2^{(02)} =$$

$$= \left[(u_{xx} - u_{yy} - 2v_{xy})^2 + (v_{xx} - v_{yy} + 2u_{xy})^2\right]\left((I_{xx} - I_{yy})^2 + 4I_{xy}^2\right)$$

Clearly, exactly $2 \times 2 = 4$ of these composite constraints are decoupled, corresponding to the two independent decoupled constraints of type $(2,0)$ multiplied by one of the two invariant constraints of type $(0,2)$:

$$\mathcal{G}_1^{(22)} = \mathcal{F}_1^{(22)} = \mathcal{G}_1^{(20)}\mathcal{F}_1^{(02)} = \left[(u_{xx} + u_{yy})^2 + (v_{xx} + v_{yy})^2\right]\left(I_{xx} + I_{yy}\right)^2$$

$$\mathcal{G}_2^{(22)} = \frac{2\mathcal{F}_1^{(22)} + \mathcal{F}_2^{(22)} + \mathcal{F}_5^{(22)}}{4} = \mathcal{G}_2^{(20)}\mathcal{F}_1^{(02)} =$$

$$= \left(u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2 + (u \longrightarrow v)\right)\left(I_{xx} + I_{yy}\right)^2$$

$$\mathcal{G}_3^{(22)} = \mathcal{F}_6^{(22)} = \mathcal{G}_1^{(20)}\mathcal{F}_2^{(02)} = \left[(u_{xx} + u_{yy})^2 + (v_{xx} + v_{yy})^2\right]\left((I_{xx} - I_{yy})^2 + 4I_{xy}^2\right)$$

$$\mathcal{G}_4^{(22)} = \frac{2\mathcal{F}_6^{(22)} + \mathcal{F}_7^{(22)} + \mathcal{F}_{10}^{(22)}}{4} = \mathcal{G}_2^{(20)}\mathcal{F}_2^{(02)} =$$

$$= \left(u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2 + (u \longrightarrow v)\right)\left((I_{xx} - I_{yy})^2 + 4I_{xy}^2\right).$$

The 14 additional invariant constraints of type $(2,2)$ which arise from the $\mathbf{2} \otimes \mathbf{2}$ and $\mathbf{4} \otimes \mathbf{4}$ components of the direct product are defined as:

$$\mathbf{2}_i \otimes \mathbf{2}_{10} = \mathbf{0}^{+(ii)} \oplus \mathbf{0}^{-(ii)} \oplus \{\text{O.T.}\} \quad (i = 4, 5, 6, 7, 8)$$

$$4_j \otimes 4_4 = 0^{+(4j)} \oplus 0^{-(4j)} \oplus \{\mathrm{O.T.}\} \quad (j = 2, 3),$$

where

$$\begin{aligned} 0^{+(ii)} &\sim \mathbf{K}^{(10)} \cdot \mathbf{K}^{(i)} \\ 0^{-(ii)} &\sim \mathbf{K}^{(10)} \times \mathbf{K}^{(i)} \\ 0^{+(4i)} &= \mathbf{M}^{(4)} \cdot \mathbf{M}^{(i)} \\ 0^{-(4i)} &= \mathbf{M}^{(4)} \times \mathbf{M}^{(i)}, \end{aligned}$$

and $\mathbf{K}^{(10)}$ and $\mathbf{M}^{(4)}$ are defined in (47) and (49), respectively. These are given explicitly by

$$\begin{aligned} 0^{+(44)} &\sim \mathcal{F}_{11}^{(22)} = \mathbf{K}^{(10)} \cdot \mathbf{K}^{(4)} \\ &= \left\{I_{xx}^2 - I_{yy}^2\right\}\left[(\nabla^2 u)^2 - (\nabla^2 v)^2\right] + \left\{4I_{xy}\nabla^2 I\right\}\left[\nabla^2 u \nabla^2 v\right] \\ 0^{-(44)} &\sim \mathcal{F}_{12}^{(22)} = \mathbf{K}^{(10)} \times \mathbf{K}^{(4)} \\ &= \left\{I_{xx}^2 - I_{yy}^2\right\}\left[\nabla^2 u \nabla^2 v\right] - \left\{4I_{xy}\nabla^2 I\right\}\left[(\nabla^2 u)^2 - (\nabla^2 v)^2\right] \\ 0^{+(55)} &\sim \mathcal{F}_{13}^{(22)} = \mathbf{K}^{(10)} \cdot \mathbf{K}^{(5)} \\ &= \left\{I_{xx}^2 - I_{yy}^2\right\}\left[(u_{xx} - u_{yy} + 2v_{xy})^2 - (v_{xx} - v_{yy} - 2u_{xy})^2\right] - \\ &\quad -\left\{4I_{xy}\nabla^2 I\right\}\left[(u_{xx} - u_{yy} + 2v_{xy})(v_{xx} - v_{yy} - 2u_{xy})\right] \\ 0^{-(55)} &\sim \mathcal{F}_{14}^{(22)} = \mathbf{K}^{(10)} \times \mathbf{K}^{(5)} \\ &= \left\{I_{xx}^2 - I_{yy}^2\right\}\left[(u_{xx} - u_{yy} + 2v_{xy})(v_{xx} - v_{yy} - 2u_{xy})\right] - \\ &\quad -\frac{1}{4}\left\{4I_{xy}\nabla^2 I\right\}\left[(u_{xx} - u_{yy} + 2v_{xy})^2 - (v_{xx} - v_{yy} - 2u_{xy})^2\right] \\ 0^{+(66)} &\sim \mathcal{F}_{15}^{(22)} = \mathbf{K}^{(10)} \cdot \mathbf{K}^{(6)} \end{aligned}$$

$$= \left\{I_{xx}^2 - I_{yy}^2\right\}\left[u_{xx}^2 - u_{yy}^2 - 2u_{xy}\nabla^2 v + v_{xx}^2 - v_{yy}^2 + 2v_{xy}\nabla^2 u\right] +$$

$$+\left\{4I_{xy}\nabla^2 I\right\}\left[u_{xy}\nabla^2 u + v_{xy}\nabla^2 v + u_{xx}v_{yy} - u_{yy}v_{xx}\right]$$

$$\mathbf{0}^{-(66)} \sim \mathcal{F}_{16}^{(22)} = \mathbf{K}^{(10)} \times \mathbf{K}^{(6)}$$

$$= \left\{I_{xx}^2 - I_{yy}^2\right\}\left[u_{xy}\nabla^2 u + v_{xy}\nabla^2 v + u_{xx}v_{yy} - u_{yy}v_{xx}\right] -$$

$$-\frac{1}{4}\left\{4I_{xy}\nabla^2 I\right\}\left[u_{xx}^2 - u_{yy}^2 - 2u_{xy}\nabla^2 v + v_{xx}^2 - v_{yy}^2 + 2v_{xy}\nabla^2 u\right]$$

$$\mathbf{0}^{+(77)} \sim \mathcal{F}_{17}^{(22)} = \mathbf{K}^{(10)} \cdot \mathbf{K}^{(7)}$$

$$= \left\{I_{xx}^2 - I_{yy}^2\right\}\left[u_{xx}^2 - u_{yy}^2 + 2u_{xy}\nabla^2 v + v_{xx}^2 - v_{yy}^2 - 2v_{xy}\nabla^2 u\right] +$$

$$+\left\{4I_{xy}\nabla^2 I\right\}\left[u_{xy}\nabla^2 u + v_{xy}\nabla^2 v - u_{xx}v_{yy} + u_{yy}v_{xx}\right]$$

$$\mathbf{0}^{-(77)} \sim \mathcal{F}_{18}^{(22)} = \mathbf{K}^{(10)} \times \mathbf{K}^{(7)}$$

$$= \left\{I_{xx}^2 - I_{yy}^2\right\}\left[u_{xy}\nabla^2 u + v_{xy}\nabla^2 v - u_{xx}v_{yy} + u_{yy}v_{xx}\right] -$$

$$-\frac{1}{4}\left\{4I_{xy}\nabla^2 I\right\}\left[u_{xx}^2 - u_{yy}^2 + 2u_{xy}\nabla^2 v + v_{xx}^2 - v_{yy}^2 - 2v_{xy}\nabla^2 u\right]$$

$$\mathbf{0}^{+(88)} \sim \mathcal{F}_{19}^{(22)} = \mathbf{K}^{(10)} \cdot \mathbf{K}^{(8)}$$

$$= \left\{I_{xx}^2 - I_{yy}^2\right\}\left[(u_{xx} - u_{yy})^2 + 4u_{xy}^2 - (v_{xx} - v_{yy})^2 - 4v_{xy}^2\right] +$$

$$+\left\{4I_{xy}\nabla^2 I\right\}\left[(u_{xx} - u_{yy})(v_{xx} - v_{yy}) + 4u_{xy}v_{xy}\right]$$

$$\mathbf{0}^{-(88)} \sim \mathcal{F}_{20}^{(22)} = \mathbf{K}^{(10)} \times \mathbf{K}^{(8)}$$

$$= \left\{I_{xx}^2 - I_{yy}^2\right\}\left[(u_{xx} - u_{yy})(v_{xx} - v_{yy}) + 4u_{xy}v_{xy}\right] -$$

$$-\frac{1}{4}\left\{4I_{xy}\nabla^2 I\right\}\left[(u_{xx} - u_{yy})^2 + 4u_{xy}^2 - (v_{xx} - v_{yy})^2 - 4v_{xy}^2\right]$$

$$
\begin{aligned}
0^{+(42)} &\sim \mathcal{F}_{21}^{(22)} = \mathrm{M}^{(4)} \cdot \mathrm{M}^{(2)} \\
&= \Big\{(I_{xx} - I_{yy})^2 - 4I_{xy}^2\Big\}\Big[u_{xx}^2 - u_{yy}^2 - 2u_{xy}\nabla^2 v - v_{xx}^2 + v_{yy}^2 - 2v_{xy}\nabla^2 u\Big] + \\
&\quad +8\Big\{I_{xy}(I_{xx} - I_{yy})\Big\}\Big[u_{xy}\nabla^2 u - v_{xy}\nabla^2 v + u_{xx}v_{xx} - u_{yy}v_{yy}\Big] \\
0^{-(42)} &\sim \mathcal{F}_{22}^{(22)} = \mathrm{M}^{(4)} \times \mathrm{M}^{(2)} \\
&= \Big\{(I_{xx} - I_{yy})^2 - 4I_{xy}^2\Big\}\Big[u_{xy}\nabla^2 u - v_{xy}\nabla^2 v + u_{xx}v_{xx} - u_{yy}v_{yy}\Big] + \\
&\quad +2\Big\{I_{xy}(I_{xx} - I_{yy})\Big\}\Big[u_{xx}^2 - u_{yy}^2 - 2u_{xy}\nabla^2 v - v_{xx}^2 + v_{yy}^2 - 2v_{xy}\nabla^2 u\Big] \\
0^{+(43)} &\sim \mathcal{F}_{23}^{(22)} = \mathrm{M}^{(4)} \cdot \mathrm{M}^{(3)} \\
&= \Big\{(I_{xx} - I_{yy})^2 - 4I_{xy}^2\Big\}\Big[(u_{xx} - u_{yy})^2 - 4u_{xy}^2 + (v_{xx} - v_{yy})^2 - 4v_{xy}^2\Big] + \\
&\quad +16\Big\{I_{xy}(I_{xx} - I_{yy})\Big\}\Big[u_{xy}(u_{xx} - u_{yy}) + v_{xy}(v_{xx} - v_{yy})\Big] \\
0^{-(43)} &\sim \mathcal{F}_{24}^{(22)} = \mathrm{M}^{(4)} \times \mathrm{M}^{(3)} \\
&= \Big\{(I_{xx} - I_{yy})^2 - 4I_{xy}^2\Big\}\Big[u_{xy}(u_{xx} - u_{yy}) + v_{xy}(v_{xx} - v_{yy})\Big] - \\
&\quad -\Big\{I_{xy}(I_{xx} - I_{yy})\Big\}\Big[(u_{xx} - u_{yy})^2 - 4u_{xy}^2 + (v_{xx} - v_{yy})^2 - 4v_{xy}^2\Big].
\end{aligned}
$$

Somewhat surprisingly, there are two linear combinations of these non–composite invariants which are decoupled (all the previous decoupled invariants were composite). They are:

$$
\begin{aligned}
\mathcal{G}_5^{(22)} &= \frac{\mathcal{F}_{15}^{(22)} + \mathcal{F}_{17}^{(22)}}{2} = (I_{xx}^2 - I_{yy}^2)\Big[u_{xx}^2 - u_{yy}^2 + v_{xx}^2 - v_{yy}^2\Big] + 4I_{xy}\nabla^2 I\Big[u_{xy}\nabla^2 u + v_{xy}\nabla^2 v\Big] \\
\mathcal{G}_6^{(22)} &= \frac{\mathcal{F}_{16}^{(22)} + \mathcal{F}_{18}^{(22)}}{2} = (I_{xx}^2 - I_{yy}^2)\Big[u_{xy}\nabla^2 u + v_{xy}\nabla^2 v\Big] - 4I_{xy}\nabla^2 I\Big[u_{xx}^2 - u_{yy}^2 + v_{xx}^2 - v_{yy}^2\Big]
\end{aligned}
$$

6 Invariant Constraints of Type $(0,3)$ and $(0,4)$

Since the optical flow field is in a sense a field of derivatives, the invariant constraints of type $(2,q)$ are, in a sense, quadratic in third order derivatives of $\mathbf{U}$. For this reason, such invariants are probably numerically unstable; we considered them primarily for reasons of completeness. For the same reasons, it may be considered unwise to look at performance functions for surface interpolation which are quadratic in derivatives higher than the second. Once again, however, for completeness we list the performance functions which are quadratic in these higher–order derivatives, namely the invariant constraints of type $(0,3)$ and $(0,4)$. We do note, however, that a smoothness constraint involving derivatives higher than the 2^{nd} may be of interest in regions of the image where the grey level intensity function varies rapidly, such as in the neighborhood of discontinuities. We note that Geman and Reynolds have recently [Gema90] considered a smoothness constraint which involves 3^{rd} order derivatives in their work on the recovery of image discontinuities. The relation of their work to ours is not at present clear, since their constraint is explicitly a discrete one, while ours are continuous.

6.1 Invariant Constraints of Type $(0,3)$

Invariant constraints of type $(0,3)$ are of the form

$$\left(\partial^3 I\right)^2 \sim \partial_a\partial_b\partial_c I \partial_d\partial_e\partial_g I.$$

We have that

$$\partial^3 \sim \partial(\partial^2) \sim \mathbf{1} \otimes (\mathbf{1} \otimes \mathbf{1}),$$

where the $(\mathbf{1} \otimes \mathbf{1})$ term is given by (43):

$$\partial^2 \sim \mathbf{0} \oplus \mathbf{2},$$

where

$$\mathbf{0} \sim \nabla^2$$

$$\mathbf{2} \sim \begin{pmatrix} \partial_x^2 - \partial_y^2 \\ 2\partial_x\partial_y \end{pmatrix}.$$

Consequently,

$$\partial^3 \sim \mathbf{1} \otimes (\mathbf{0} \oplus \mathbf{2}) = (\mathbf{1} \otimes \mathbf{0}) \oplus (\mathbf{2} \otimes \mathbf{1})\ .$$

But

$$\mathbf{1} \otimes \mathbf{0} = \mathbf{1} \sim \nabla^2 \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} \equiv \mathrm{N},$$

and

$$\mathbf{2} \otimes \mathbf{1} = \mathbf{1}' \oplus \mathbf{3},$$

where, according to (16,17)

$$\mathbf{1}' \sim \begin{pmatrix} \partial_x^3 + \partial_x\partial_y^2 \\ \partial_x^2\partial_y + \partial_y^3 \end{pmatrix} = \begin{pmatrix} \partial_x\nabla^2 \\ \partial_y\nabla^2 \end{pmatrix} \equiv \mathrm{N} \sim \mathbf{1}$$

The equality of $\mathbf{1}$ and $\mathbf{1}'$ is "accidental," in the sense that they would be different if differentiation were not commutative. In addition,

$$\mathbf{3} \sim \begin{pmatrix} (\partial_x^2 - \partial_y^2)\partial_x - (2\partial_x\partial_y)\partial_y \\ \partial_x(2\partial_x\partial_y) + \partial_y(\partial_x^2 - \partial_y^2) \end{pmatrix} = \begin{pmatrix} \partial_x^3 - 3\partial_x\partial_y^2 \\ 3\partial_x^2\partial_y - \partial_y^3 \end{pmatrix}.$$

Therefore, we have the decomposition

$$\partial^3 I \sim \mathbf{1} \oplus \mathbf{3}, \tag{57}$$

where

$$\mathbf{1} \sim \mathbf{N} = \begin{pmatrix} I_{xxx} + I_{xyy} \\ I_{xxy} + I_{yyy} \end{pmatrix}$$

$$\mathbf{3} \sim \mathbf{P} = \begin{pmatrix} I_{xxx} - 3I_{xyy} \\ 3I_{xxy} - I_{yyy} \end{pmatrix}.$$

As a consequence of this decomposition, we find that

$$\begin{aligned} (\partial^3 I)^2 &\simeq (\mathbf{1} \oplus \mathbf{3}) \otimes (\mathbf{1} \oplus \mathbf{3}) \\ &= (\mathbf{1} \oplus \mathbf{1})_{\mathrm{id}} \oplus (\mathbf{3} \oplus \mathbf{3})_{\mathrm{id}} \oplus \{\mathrm{O.T.}\} \\ &= \mathbf{0} \oplus \mathbf{0}' \oplus \{\mathrm{O.T.}\}, \end{aligned}$$

where

$$\begin{aligned} \mathbf{0} &\sim \mathbf{N}^2 \\ \mathbf{0}' &\sim \mathbf{P}^2. \end{aligned}$$

Consequently, there are exactly two invariants of type $(0,3)$:

$$\begin{aligned} \mathcal{F}_1^{(03)} &= \mathbf{N}^2 = (I_{xxx} + I_{xyy})^2 + (I_{xxy} + I_{yyy})^2 \\ \mathcal{F}_2^{(03)} &= \mathbf{P}^2 = (I_{xxx} - 3I_{xyy})^2 + (3I_{xxy} - I_{yyy})^2. \end{aligned}$$

Both of these are manifestly positive definite, and would be suitable 3$^{\mathrm{rd}}$ order smoothness constraints for surface interpolation. We note, however, that similar to the case for constraints of type $(0,2)$, only one of these two is necessary, since a linear combination of the two is a total divergence. It is easy to see that

$$\frac{1}{8}\left[\mathcal{F}_1^{(03)} - \mathcal{F}_2^{(03)}\right] = \nabla \cdot \mathbf{A},$$

where

$$A = (I_{xx} - I_{yy})\widetilde{\nabla} I_{xy}.$$

6.2 Invariant Constraints of Type (0, 4)

Invariant constraints of type (0, 4) are of the form

$$\left(\partial^4 I\right)^2 \sim \partial_a\partial_b\partial_c\partial_d I \partial_e\partial_g\partial_h\partial_j I.$$

We have that

$$\partial^4 \sim \left((\partial^2)\partial^2\right),$$

and so, using (43),

$$\begin{aligned} \partial^4 &\sim (\mathbf{1} \otimes \mathbf{1}) \otimes (\mathbf{1} \otimes \mathbf{1}) = (\mathbf{0} \oplus \mathbf{2}) \otimes (\mathbf{0} \oplus \mathbf{2}) \\ &\simeq (\mathbf{0} \otimes \mathbf{0}) \oplus (\mathbf{2} \otimes \mathbf{0}) \oplus (\mathbf{2} \otimes \mathbf{2})_{\text{id}} = \mathbf{0} \oplus \mathbf{0}' \oplus \mathbf{2} \oplus \mathbf{4}, \end{aligned}$$

where

$$\begin{aligned} \mathbf{0} &\sim (\nabla^2)^2 \\ \mathbf{0}' &\sim (\partial_x^2 - \partial_y^2) + (2\partial_x\partial_y)^2 = \partial_x^4 + \partial_y^4 + 2\partial_x^2\partial_y^2 \equiv (\nabla^2)^2 \sim \mathbf{0}. \end{aligned}$$

Thus, $\mathbf{0}$ and $\mathbf{0}'$ are in fact the same quantity (which equality is "accidental" in the same sense as before). Furthermore, according to (16,17) :

$$\begin{aligned} \mathbf{2} &\sim \nabla^2 \begin{pmatrix} \partial_x^2 - \partial_y^2 \\ 2\partial_x\partial_y \end{pmatrix} \\ \mathbf{4} &\sim \begin{pmatrix} (\partial_x^2 - \partial_y^2)^2 - (2\partial_x\partial_y)^2 \\ 2(\partial_x^2 - \partial_y^2)(2\partial_x\partial_y) \end{pmatrix} = \begin{pmatrix} \partial_x^4 - 4\partial_x^2\partial_y^2 + \partial_y^4 \\ 4\partial_x\partial_y(\partial_x^2 - \partial_y^2) \end{pmatrix}. \end{aligned}$$

Hence,

$$(\partial^4 I) \simeq \mathbf{0} \oplus \mathbf{2} \oplus \mathbf{4}, \tag{58}$$

where

$$\begin{aligned}
\mathbf{0} &\sim (I_{xxxx} + 2I_{xxyy} + I_{yyyy}) \\
\mathbf{2} &\sim \equiv \begin{pmatrix} I_{xxxx} - I_{yyyy} \\ 2I_{xxxy} + 2I_{xyyy} \end{pmatrix} \equiv \mathbf{N}' \\
\mathbf{4} &\sim \begin{pmatrix} I_{xxxx} - 4I_{xxyy} + I_{yyyy} \\ 4I_{xxxy} - 4I_{xyyy} \end{pmatrix} \equiv \mathbf{Q}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
(\partial^4 I)^2 &\simeq (\mathbf{0} \oplus \mathbf{2} \oplus \mathbf{4}) \otimes (\mathbf{0} \oplus \mathbf{2} \oplus \mathbf{4}) \\
&\simeq (\mathbf{0} \otimes \mathbf{0}) \oplus (\mathbf{2} \otimes \mathbf{2})_{\mathrm{id}} \oplus (\mathbf{4} \otimes \mathbf{4})_{\mathrm{id}} \oplus \{\mathrm{O.T.}\} \\
&= \mathbf{0}_1 \oplus \mathbf{0}_2 \oplus \mathbf{0}_3 \oplus \{\mathrm{O.T.}\},
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{0}_1 &\sim \left[(\nabla^2)^2 I\right]^2 \\
\mathbf{0}_2 &\sim \mathbf{N}'^2 \\
\mathbf{0}_3 &\sim \mathbf{Q}^2.
\end{aligned}$$

We therefore find the three invariants of type $(0,4)$, all of which are manifestly positive definite:

$$\begin{aligned}
\mathcal{F}_1^{(04)} &= (I_{xxxx} + 2I_{xxyy} + I_{yyyy})^2 \\
\mathcal{F}_2^{(04)} &= (I_{xxxx} - I_{yyyy})^2 + 4(I_{xxxy} + I_{xyyy})^2 \\
\mathcal{F}_3^{(04)} &= (I_{xxxx} - 4I_{xxyy} + I_{yyyy})^2 + 16(I_{xxxy} - I_{xyyy})^2.
\end{aligned}$$

Although I expect that some linear combination of these is a total divergence, I have not yet been able to show that.

7 Summary and Conclusions

We have used the representation theory of ISO(2) to give a complete list of all possible ISO(2)–invariant smoothness constraints which are quadratic in p^{th} derivatives of the optical flow, and in q^{th} derivatives of the grey–level intensity, for $p, q \leq 2$. We also found all the possible smoothness constraints for surface interpolation which were quadratic in no higher than fourth derivatives of the performance function. In the sequel to this work [Snyd90], we use the complex formulation of the representation theory of SO(2) to find explicit formulas for the performance function quadratic in n^{th} derivatives, for arbitrary n. We show that the invariant constraints of type $(0, n)$ are $\left[(n+3)/2\right]$ in number, where $[m]$ is the integer part of m.

It should also be clear that the methods developed here are applicable to the case of smoothness constraints which are of higher order than quadratic in derivatives of both the image flow and the grey–level intensity function, i.e., constraints of the form $(\partial^p \mathrm{U})^{2m}(\partial^q I)^{2n}$, where m and n are greater than 1. The group–theoretical analysis of smoothness constraints can also be applied to the case of smoothness of the $3D$ velocity fields, in which case the representation theory of the three–dimensional rotation group SO(3) will be relevant. Such smoothness constraints may be of interest when projected down to the image plane. These and other topics will be addressed in future papers.

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