

A Note on “Zippel Denesting”

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Abstract

Radical simplification is an important part of symbolic computation systems. Zippel [7] gave a sufficient condition for a nested radical to be expressed in terms of radicals of lower nesting depth. We fill a lacuna in his proof, and show that his sufficient condition is also necessary. Previous work by Landau and Miller [4] leads to an algorithm for the problem.

Ramanujan observed a number of curiosities amongst nested radicals:

$$\begin{aligned}\sqrt[4]{\frac{3+2\sqrt[4]{5}}{3-2\sqrt[4]{5}}} &= \frac{\sqrt[4]{5}+1}{\sqrt[4]{5}-1} \\ \sqrt{\sqrt[3]{28}-\sqrt[3]{27}} &= 1/3(\sqrt[3]{98}-\sqrt[3]{28}-1) \\ \sqrt[3]{\sqrt[5]{32/5}-\sqrt[5]{27/5}} &= \sqrt[5]{1/25} + \sqrt[5]{3/25} - \sqrt[5]{9/25}\end{aligned}$$

Symbolic computation made such manipulations more than curiosities. For example:

$$\sqrt{5+2\sqrt{6}} = \sqrt{2} + \sqrt{3}$$

means that $\{1, \sqrt{2}, \sqrt{3}, \sqrt{6}\}$ is a basis for $Q(\sqrt{5+2\sqrt{6}})$ over Q . The basis $\{1, \sqrt{2}, \sqrt{3}, \sqrt{6}\}$ is simple to manipulate. An important issue then, is the

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“denesting” of radicals – a term which will be precisely defined in the next section.

In 1985, Borodin, Fagin, Hopcroft and Tompa [1] gave an efficient algorithm for decreasing the nesting depth of a class of expressions involving square roots. More recently Landau showed how to denest radicals by computing in the splitting field of the nested radical [3].

Earlier work by Zippel [7] in 1985 gave a simple sufficient condition under which a radical could denest. Zippel’s theorem omitted a hypothesis. We repair this lacuna, and show that Zippel’s sufficient condition is also necessary. Finally we observe that previous work by Landau and Miller [4] yields an exponential time algorithm for this technique.

1 An Algorithm for a Subclass of Nested Radicals

We begin with the definition of nesting depth. Following [1], a *formula* over a field k and its *depth of nesting* are defined as follows:

- (1) an element of k is a formula of depth 0 over k ,
- (2) an arithmetic combination ($A \pm B$, $A \times B$, A/B) of formulas A and B is a formula whose depth over k is $\max(\text{depth}(A), \text{depth}(B))$, and
- (3) a root $\sqrt[n]{A}$ of a formula A is a formula whose depth over k is $1 + \text{depth}(A)$.

We will say the formula A can be denested over the field k if there is a formula B of lower depth than A such that $\text{value}(A) = \text{value}(B)$. For any α , we define the depth of α over k to be $\text{minimum}\{\text{depth}(A) \mid \text{value}(A) = \alpha\}$. When we are given a formula A of value α such that A can be denested, we will sometimes instead say that α can be denested.

We will be using several classic theorems. Let ξ_n be a primitive n^{th} root of unity.

Theorem 1.1 *Let k be a field, with K a cyclic extension of k of degree n , and suppose ξ_n is in k . Then there is a β in K such that $K = k(\beta)$, and β is a root of $x^n - b$ for some b in k .*

Theorem 1.2 *Let k be a field with ξ_n in k , and suppose β is a root of $x^n - b$. Then $k(\beta)$ is cyclic over k of degree d , where d divides n , and β^d is an element of k .*

Consider the following tower of fields:

$$\begin{array}{ccc} & L = KF & \\ K & & F \\ & K \cap F & \\ & k & \end{array}$$

Then the following is well-known:

Theorem 1.3 (Lang[5], pp. 196-7) *Let the fields be as above, and assume that F is a Galois extension of k . Then KF is Galois over K , and F is Galois over $K \cap F$. Let G be the Galois group of KF over K , and H the group of F over $K \cap F$. If σ is in G , then the restriction of σ to F is in H , and that restriction map is an isomorphism from G onto H .*

Zippel studied the circumstances under which a radical in L can be denested in a field of lower nesting depth. That is, suppose $L = K(\sqrt[d]{\alpha})$ for some α in K . He sought an element β in k such that $\alpha\beta$ is a d^{th} power of an element in K , say λ . (Assume that ξ_d lies in k .) Then $(\alpha\beta) = \lambda^d$ implies that $\sqrt[d]{\alpha} = \lambda/\sqrt[d]{\beta}$.¹ Thus $\sqrt[d]{\alpha}$ may be expressed in terms of an element of lower nesting depth. We call such a denesting a ‘‘Zippel denesting.’’ Zippel [7] presented the following example: let $k = \mathbb{Q}$, $K = \mathbb{Q}(\sqrt{6})$, and $\alpha = 5 + 2\sqrt{6}$, $d = 2$. Since

$$2(5 + 2\sqrt{6}) = (2 + \sqrt{6})^2,$$

we know that

$$\sqrt{5 + 2\sqrt{6}} = \frac{2 + \sqrt{6}}{\sqrt{2}} = \sqrt{2} + \sqrt{3}.$$

Zippel observed that in some cases, information about the associated fields will tell us enough to compute a denesting. The following theorem which omitted, but implicitly assumed, the hypothesis that F is a Galois extension of k , first appeared in Zippel’s paper.

Theorem 1.4 (Zippel [7]) *Assume K is an extension of k , a field containing a primitive d^{th} root of unity. Let $L = K(\sqrt[d]{\alpha})$ be an extension of degree d , where α is in K . If there is a field F which is a Galois extension of $k = K \cap F$, and $L = KF$, then there is a β in k such that $\alpha\beta$ is a d^{th} power of an element of K . Furthermore, $F = k(\sqrt[d]{\beta})$.*

¹Note that Zippel is using the convention of choosing the root $\sqrt[d]{\alpha}$, rather than some other d^{th} root. A different root $\xi_d^i \sqrt[d]{\alpha}$ may have a higher depth of nesting due to the nesting in the root of unity.

Proof By Theorem 1.3, $L = KF$ is Galois over K . If G is the Galois group of L over K , and H is the group of F over k , then there is a map $\rho : G \rightarrow H$ which sends σ in G to σ restricted to the field F . This map is an isomorphism. Then by Theorem 1.2, the extension L over K is cyclic of degree d , and thus so is F over k . Therefore, by Theorem 1.1, $F = k(\sqrt[d]{\gamma})$ for some γ in k .

Let σ be a generator of G , then $\sigma(\sqrt[d]{\alpha}) = \xi_d \sqrt[d]{\alpha}$. Since there is an isomorphism from G to H , we know $\sigma(\sqrt[d]{\gamma}) = \xi_d^c \sqrt[d]{\gamma}$ for some c relatively prime to d . Let $u \equiv c^{-1} \pmod{d}$, and $m = -u$. Then $\xi_d^{cm} = \xi_d^{-cc^{-1}} = \xi_d^{-1}$. We let $\beta = \gamma^m$. Then we have:

$$\sigma(\sqrt[d]{\alpha} \sqrt[d]{\beta}) = \sigma(\sqrt[d]{\alpha}) \sigma(\sqrt[d]{\beta}) = \xi_d \sqrt[d]{\alpha} \sigma((\sqrt[d]{\gamma})^m) = \xi_d \sqrt[d]{\alpha} \xi_d^{cm} (\sqrt[d]{\gamma})^m = \xi_d \xi_d^{-1} \sqrt[d]{\alpha} \sqrt[d]{\beta} = \sqrt[d]{\alpha} \sqrt[d]{\beta}.$$

Thus $\sqrt[d]{\alpha} \sqrt[d]{\beta}$ is fixed by G , and is therefore in K . So $\alpha\beta = \lambda^d$ for some λ in K . Then $\sqrt[d]{\alpha} = \lambda / \sqrt[d]{\beta}$.

Finally observe that m and d are relatively prime. Thus $k(\sqrt[d]{\beta}) = k(\sqrt[d]{\gamma^m}) = k(\sqrt[d]{\gamma}) = F$. ■

Any denesting of this form will cause certain behavior of associated fields. We have the following converse to Zippel's theorem.

Theorem 1.5 *Let α be an element of a field K . Suppose that $\sqrt[d]{\alpha}$ is of degree d over K , and that $\sqrt[d]{\alpha} = \lambda / \sqrt[d]{\beta}$ with λ in K , and β in $k \subset K$. Assume that the d^{th} roots of unity lie in k . Then the field $F = k(\sqrt[d]{\beta})$ satisfies: (i) F over k is Galois and the Galois group of F over $F \cap K$ is isomorphic to the group of FK over K , (ii) $FK = K(\sqrt[d]{\alpha})$, and (iii) $k = F \cap K$.*

Proof Since $F = k(\sqrt[d]{\beta})$, it is clear that F over k is a Galois extension. Then the fields k, F, K, FK satisfy the hypothesis of Theorem 1.3, and $G =$ the Galois group of FK over K is isomorphic to H , the Galois group of F over $F \cap K$. Note that we have $FK = k(\sqrt[d]{\beta})K = K(\sqrt[d]{\beta}) = K(\sqrt[d]{\alpha})$. That $k \subseteq F \cap K$ is clear. To show that the containment is an equality it suffices to observe that $d = [K(\sqrt[d]{\alpha}) : K] = [F : F \cap K] \leq [F : k] \leq d$. This is possible only if $k = F \cap K$. ■

Note that the hypothesis “ $\sqrt[d]{\alpha}$ is of degree d over K ” is necessary. For example

$$\sqrt[6]{7\sqrt[3]{20} - 19},$$

is of degree 3 over $Q(\sqrt[3]{20})$, because $7\sqrt[3]{20} - 19$ is a square in $Q(\sqrt[3]{20})$. In this case there is more than the usual ambiguity in the denesting: if $\sqrt[d]{\alpha}$ is not of degree d over K , then the roots of $x^d - \alpha$ are not conjugate over K .

By Theorems 1.4 and 1.5, a Zippel denesting exists iff there is a field F , with $FK = K(\sqrt[d]{\alpha})$ and F Galois over $k = F \cap K$. To do so, we search subfields of $L = K(\sqrt[d]{\alpha})$ containing k .

In [4] efficient algorithms were given for computing maximal subfields. It follows immediately from Theorem 2.9 and Algorithm 2.2 of [4] that:

Theorem 1.6 (Landau & Miller [4]) *If $f(x)$ in $Z[x]$ is irreducible of degree m with roots $\alpha, \alpha_2, \dots, \alpha_m$, then there is an algorithm to compute β_0, \dots, β_j , where $Q(\beta_0, \dots, \beta_j)$ is a maximal subfield of $Q(\alpha)$. The running time for the algorithm is the time required to factor $f(x)$ over $Q[z]/f(z)$ plus the time needed to calculate m^3 gcd's of polynomials (of degree less than $\deg(f(x))$), with coefficient length less than $m^2 \log \|f(x)\|$ over a field containing two roots of $f(z)$.*

Theorem 1.7 (Landau & Miller [4]) *Let $f(x), g(x)$ in $Z[x]$ be irreducible and monic of degree m and r respectively, and $h(x)$ of degree n in $Q[z, x]/f(z)$ be an irreducible factor of $g(x)$ in $Q[z]/f(z)$. There is an algorithm to compute $B(x) = b_l x^l + b_{l-1} x^{l-1} + \dots + b_0$, a polynomial in $Q[x, z]/f(x)$ whose coefficients determine the field $Q[x]/f(x) \cap Q[x]/g(x)$, in the sense that $Q(b_l, \dots, b_0) = Q[x]/f(x) \cap Q[x]/g(x)$. The algorithm runs in the time required to factor $N_{Q[z]/f(z)}(h(x - cz))$, ($c < (mn)^2$), over $Q[z]/f(z)$, plus the time needed to calculate $(mn)^3$ gcd's of polynomials of degree less than $\max(m, n)$ with coefficient length less than $(m + n)^2 \log \|h(x)\| \log |f(z)|$.*

As before, it is easy to generalize the algorithms to work over a general field k of characteristic 0. We informally describe how to find the field F , if it exists.

First determine if $d = [K(\sqrt[d]{\alpha}) : K]$. Use the Landau & Miller algorithm to find all maximal subfields F_i of L . If $KF_i \subset L$ for each i , then we are done, since no such F exists.

Otherwise, suppose that F_1, \dots, F_t satisfy $KF_i = L$. If any of the F_i are Galois over $K \cap F_i = k$ for some i , then we are done by Theorem 1.4. Check that $K \cap F_i \supseteq k$ for each i . Let F_1, \dots, F_s satisfy $KF_i = L$ and $K \cap F_i \supseteq k$ for each i . If not, compute the maximal subfields of the F_i and repeat the process. If we do not find a field F such that $K(\sqrt[d]{\alpha}) = KF$ and F is Galois over $K \cap F = k$, then by Theorem 1.5, there is no denesting of $\sqrt[d]{\alpha}$ of the form $\lambda/\sqrt[d]{\beta}$, with λ in K , β in k .

The exponential character of the algorithm comes from searching potentially all the subfields of $K(\sqrt[k]{\alpha})$ in order to find F . We conjecture that there is a faster way to handle the search than the essentially brute force approach we are suggesting here. Despite its exponential character, for α of small degree over k , this algorithm is reasonably efficient.

Observe that this method, even if used repeatedly, is not guaranteed to find a minimal denesting of the radical.

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