Distribution of the Loss Period for some Queues in Continuous and Discrete Time

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Abstract

For soft real-time communication systems, packet loss due to excessive delay rather than average delay becomes the critical performance issue. While most previous studies of real-time systems measure loss as a time-average fraction of excessively delayed packets, this paper characterizes the stochastic properties of time-out loss periods for infinite queues, that is, uninterrupted intervals during which the virtual wait is at or above some fixed threshold. We present analytic expressions and numerical techniques for computing both "time-based" measures such as the distribution of periods during which all arriving packets are lost due to excessive delay as well as "packet-based" measures such as the distribution of the number of consecutively lost packets and the number of successful packets between such periods of loss. Both continuous and discrete-time systems are examined. It is shown that the assumption of random packet loss severely underestimates the number of consecutively lost packets. Also, the loss period is found to be independent of the waiting time threshold for the $G/M/1$ queue and $D^{[\mathrm{Geo}]}/D/1$ queue, with very little influence for other queueing models.

Investigations into successive losses for queues with bounded waiting time and bounded queue length show similar results. Furthermore, it is seen that bounding waiting time and queue length also limits, as expected, the length of loss runs, with only marginal increase as the load approaches one.

It is shown that for finite discrete-time queues, FIFO and LIFO service ordering, combined with either front or rear discarding, engender the same loss correlation.

1 Introduction

For soft *real-time* communication systems, packet loss rates rather than average delay becomes the critical performance measure. Interactive voice, video and distributed measurement and control are examples of such systems that impose delay constraints, but also show a certain tolerance for lost packets. Most previous studies of real-time systems only measure loss as a time-average fraction of missing packets [1–6]. In order to judge the "quality" of loss-prone communication, however, it is important to know not just how *many* packets are being lost, but also whether losses occur in clusters or randomly. The importance of accounting for correlated losses has long been recognized in specifying acceptable performance of data circuits. "An errored second is declared when one or more bits in that second is found in error." [7] This leads to the metric of the percentage of error-free seconds (EFS).

Papers on packet voice reconstruction typically assume random, uncorrelated occurrence of packet loss [8, 9]; as we will show, this assumption might be overly optimistic. The work by Shacham and Kenney [10, 11] provides another example. They observed that loss correlation in a finite-buffer system could completely eliminate the advantage of three orders of magnitude predicted for forward-error correction under the assumption of independent (Bernoulli) losses.

In this paper, we consider a single FCFS queue with infinite buffer in isolation in which an arriving customer[1] which reaches the server after waiting h or longer is considered "lost" (even though it still receives service). We characterize the stochastic properties of the *time-out loss period*, the uninterrupted interval of time during which the virtual work in the queue exceeds the given threshold h. Given this time-based quantity, we also arrive at measures of customer-based quantities such as the distribution of the number of consecutive customers which each spends more than an allotted time waiting for service. We show that the assumption that each customer is lost independently from the previous one leads to a significant underestimation of the duration of such loss periods. Using elementary methods of probability, we also prove that for certain classes of queues the duration of the loss period is independent of the value of the threshold, while we show through numerical examples that for other important classes the influence of the threshold is minimal for interesting probabilities of loss. Our numerical results also indicate that the expected number of consecutively lost customers (for the same loss probability) varies by as much as 50% depending on the batch arrival distribution used. We also derive measures of the time between loss periods. Throughout the paper, emphasis is placed on providing results that can be readily numerically evaluated.

A number of authors have investigated related aspects of the problem. Kamitake and Suda [12] consider a discrete-time queue in which traffic is generated by a changing number of active callers, with each active caller generating a packet in a slot according to a Bernoulli process. They compute the steady state loss rate for a given number of active callers and then consider the variation in the number of active callers in computing the amount of uninterrupted time from when this loss rate first exceeds a value ζ until it drops below another value η, with $\eta < \zeta$. Our work differs from [12] in that we directly characterize those periods of time in which arriving customers are lost, rather than characterizing loss as being "quasi-stationary" during periods of times during which the number of active sources remains constant.

Leland [13] mentions, but does not elaborate on measuring consecutive losses per connection in an ATM simulation experiment. Woodruff and Kositpaiboon [14] mention the desirability of specifying the probability and duration of periods of high cell loss rates. Ferrandiz and Lazar [15] investigate the distribution of gaps, that is, consecutive losses, due to blocking and clipping (see

[1]The terms "customer" and "packet" will be used interchangeably.

below) in a multiclass $G/G/m/B$ queueing system; we discuss similarities and differences between our work and [15] in the following sections. Van Doorn [16] and Meier-Hellstern [17,18] characterize the overflow process from a finite Markovian queueing system. As pointed out earlier, [11] underlines the importance of taking loss correlations into account, but investigates their effect on forward-error correction only through simulation. A large body of literature analyzes the overflow process of blocked-calls-cleared systems, but the results do not seem directly applicable to our problem.

The report is organized as follows. After defining more precisely the systems and measures of interest in the section below, continuous-time queues are investigated in Section 2. In Section 3 we then apply similar methods to derive the corresponding measures for a discrete-time queue of interest in packet switching. We conclude by summarizing the work presented and pointing out some issues to be investigated.

2 Clip Loss in Continuous Time ($G/M/1$)

2.1 Performance Measures

This paper focuses on a single-server queue, where customers are processed in the order of arrival. Customers that spend more than a deterministic, fixed amount of time h waiting for service are tagged as lost on leaving the queue, but are still served. (Ferrandiz and Lazar [15] refer to this as clipping loss.) This definition of loss is motivated by considerations of traffic with soft real-time constraints, where packets that are excessively delayed are worthless to the receiver. The loss as defined here differs from that studied in our earlier work [2], where excessively delayed customers depart before occupying the server.

A *loss period* (LP) is an uninterrupted interval during which all arriving customers would experience a waiting time exceeding h. For infinite queues with FCFS service, the loss period equals the interval during which the virtual work in the queue is greater than the threshold h. Loss periods and busy periods are related in that a busy period is a special case of a loss period, with threshold value $h = 0$. Also, both busy periods and loss periods start with the arrival of a customer. They differ, however, in that a busy period ends with a departure of a customer, while the end of a loss period is not connected with a customer departure. A *noloss period* is the interval between two loss periods. For $h = 0$, noloss periods become idle periods of the queue.

While the loss period is of independent interest, we are particularly concerned with measuring the number of consecutively lost customers, called *loss run* for brevity. Note that the number of customers arriving in a loss period is not identical to the number of consecutively lost customers. In particular, the first customer triggering a loss period, i.e., the customer that pushes the virtual work above h, is itself not lost. Thus, loss periods consisting of a single arrival do not contribute to customer loss. Note that there may also be several noloss periods interspersed between two customer losses if each of the loss periods separating the noloss periods consists of only the arrival triggering a loss period. Similar to loss runs, *success runs* denote the number of consecutive customer without any loss.

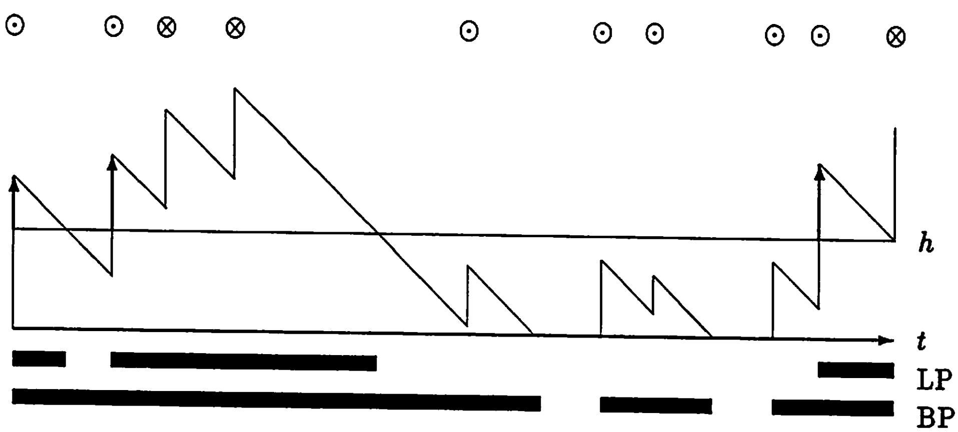

Figure 1: Virtual work sample path

Fig. 1 depicts a virtual work sample path and the measures defined above. Time proceeds along the abscissa from left to right, while the ordinate shows the virtual work at a given instant. In the figure, arrivals that will be tagged as lost on departure are marked with $\otimes$, the others with $\odot$. The extent of loss periods (LP) and busy periods (BP) are indicated by horizontal bars. Bold arrows represent *initial jumps*, the amount of virtual work above h at the beginning of a loss period. The height of the vertical jumps indicates the amount of work brought to the system by an arriving customer. The unfinished work decreases at a unit rate as the server processes jobs.

Let us briefly touch upon other measures of loss behavior for queueing systems. For finite queues with m deterministic servers and bulk arrivals, at least m of the arriving packets are always served. For an individual source, the probability of consecutive losses depends on its position within the bulk, which might be fixed, uniformly distributed or a time-varying stochastic process, depending on the buffer management policy. This system was treated extensively by Li [19]. Li defines as blocking those states where the buffer is full prior to service completion.

We may also look at loss-correlation through a frequency-domain perspective. As an example, the first-order autocorrelation coefficient of intervals between losses was measured experimentally for a five-stage virtual circuit model with bounded waiting times. Using the von-Neumann statistic [20] as a robust estimator, it was found that the intervals between losses were essentially uncorrelated. Autocorrelation information might be useful in comparing different buffer management schemes or service disciplines, but cannot readily be used to predict the performance of packet reconstruction algorithms.

The stochastic properties of the loss period or consecutive customer losses can be quantified in the usual manner, for example through its distribution or its average duration, either in terms of time or the number of customers affected. Details are discussed in the next two sections for continuous and discrete-time queueing systems of interest.

As a first model, we investigate a system with general arrival process of rate λ and exponential service with rate μ. The buffer is infinite, so that only loss due to excessive waiting time occurs. All packets, regardless of whether they exceed the waiting time threshold or not, are served. (A system where late packets are dropped will be treated later.) The special case of Poisson arrivals, i.e., the $M/M/1$ system, will be treated in detail since it yields closed-form expressions for the measures of interest. The $M/M/1/\infty$ system was also investigated by Ferrandiz and Lazar [15] as a special case of their $G/G/m/B$ analysis. Their analysis seems considerably more involved, does not readily yield numerical results and does not make use of the simple connection to the busy period pointed out here. The model is applicable, if only in approximation, to systems with variable packet sizes, for example the PARIS network [21] or a variation of the Knockout switch [22].

Throughout this section, we will verify and illustrate our calculations through a running example consisting of an $M/M/1$ queue with arrival rate $\lambda = 0.8$, service rate $\mu = 1$ (and thus a load of $\rho = \lambda/\mu = 0.8$) and a clipping threshold of $h = 3$ so that $\alpha = 1 - \rho e^{(\lambda-\mu)h} = 43.905\%$ of all arrivals experience a delay of more than 3 (are "lost"). The methods presented below apply equally well at lower loss probabilities; we have chosen this (impractically) high loss to speed simulation verification of our results through simulation.

2.2 Distribution of the $G/M/1$ Initial Jump and Loss Period

As pointed out above, a loss period commences when an arrival causes the virtual work to cross the threshold h from below. In order to establish the distribution of the loss period, the following lemma describes the distribution that governs the initial jump, i.e., the virtual work immediately after the arrival of the first customer in a loss period.

Lemma 1 *For a $G/M/1$ queue, the initial jump has the same distribution as the service time and does not depend on the threshold h.*

PROOF Let the random variable J represent the height of the initial jump and $f_J(j)$ its density. The density $f_J(j)$ can be expressed through conditioning, where we define V as the virtual work just prior to the arrival of the customer whose new work pushes the virtual work above h.

$$f_J(j) = \int_0^h P[V = v | V \leq h] \cdot P[\text{new work} = h + j - v | \text{new work} \geq h - v] \, dv \tag{1}$$

The second conditional probability can be rewritten as

$$\frac{\mu e^{-\mu(h+j-v)}}{e^{-\mu(h-v)}} = \mu e^{-\mu j},$$

which follows immediately from the memorylessness property of the exponential distribution.

Now we can rewrite the jump density as

$$\begin{aligned} f_J(j) &= \mu e^{-\mu j} \int_0^h \frac{P[v = y \cap y \leq h]}{P[v \leq h]} \, dy \\ &= \mu e^{-\mu j} \frac{1}{P[v \leq h]} \int_0^h P[v = y] \, dy \\ &= \mu e^{-\mu j} \frac{1}{P[v \leq h]} P[V \leq h] \\ &= \mu e^{-\mu j} \quad \square \end{aligned}$$

Given this property of the initial jump, the following theorem follows immediately:

Theorem 1 *In a $G/M/1$ queueing system, a loss period is stochastically identical to a busy period. The loss period distribution is independent of the threshold h. For $G/G/1$ queues, a loss period is stochastically identical to a busy period with special first service.*

Busy periods with special (or exceptional) first service are covered by Wolff [23, p. 392-394].

The independence of the loss behavior from the threshold recalls a similar observation made by Li [19] regarding the buffer overflow process in a packet voice system. There, the time spent in the overload state was found to be independent of the buffer size.

Let the random variable L denote the time duration of a loss period. Then, given Theorem 1, busy and loss periods for the $M/M/1$ queue have the density [24, p. 215]

$$f_L(y) = \frac{1}{y\sqrt{\rho}} e^{-(\lambda+\mu)} I_1[2y\sqrt{\lambda\mu}]$$

and mean

$$E[L] = \frac{1}{\mu - \lambda} = \frac{1}{\mu} \frac{1}{1-\rho},$$

where $I_1(y)$ is the modified Bessel function of the first kind of order one. The distribution is best computed by numerical integration.

For non-$G/M/1$ queues, the computation of the initial jump can be difficult. In general, the relationship

$$f_J(j) = \int_0^h \frac{w(v) b(h + j - v)}{W(h)(1 - B(h - v))} \, dv$$

holds, where $b(x)$, $w(x)$, $B(x)$ and $W(x)$ are the densities and distributions of arriving work and the virtual work seen by the arrival, respectively.

However, some stochastic ordering results for busy periods with special first service and loss periods may be of interest.

Lemma 2 *The loss period for a $G/GI/1$ queue is stochastically longer than (shorter than) a busy period if the initial jump is stochastically larger (smaller) than a service duration.*

PROOF We prove the first, non-parenthesized part; the other proceeds similarly. Let the random variables X and Y denote a regular service time and an initial jump, respectively. From the coupling theorem (see [25, Proposition 8.2.2]), we know that if $Y \geq_{st} X$, then there exist random variables $\tilde{X}$ and $\tilde{Y}$, with the same distributions as X and Y, such that $\tilde{Y} \geq \tilde{X}$, i. e., $P(\tilde{Y} \geq \tilde{X}) = 1$. Without changing the duration of the loss period, we preempt the first customer after $\tilde{X}$ and then complete its service in $\tilde{Y} - \tilde{X}$ at the conclusion of the loss period. Denote the sum of the service periods of the remaining customers in the busy or loss period by S and the extension of the loss period caused by arrivals during the $\tilde{Y} - \tilde{X}$ segment as E. Since

$$\tilde{X} + S > a \Rightarrow \tilde{X} + S + (\tilde{Y} - \tilde{X}) + E > a,$$

we write

$$P[\tilde{X} + S + (\tilde{Y} - \tilde{X}) + E > a] \geq P[\tilde{X} + S > a],$$

from which the stochastic inequality for the coupled loss and busy periods follows. Finally, we uncondition to make the result apply to any busy and loss period.

Alternatively, we can argue by using [25, Example 8.2(a)], where it is shown that $f(Y_1, \ldots, Y_n) \geq_{st} f(X_1, \ldots, X_n)$ for any increasing function f if $Y_i \geq_{st} X_i$ and given that $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ are independent. Clearly, the length of the busy period is an increasing function (the sum) of the service periods that constitute it. □

Using this lemma, it is easy to show the following general relation between loss periods and service times.

Theorem 2 *If the service time has decreasing failure rate (DFR), the loss period is stochastically longer than a busy period. Conversely, for service times with inreasing failure rate (IFR), the loss period is stochastically shorter than a busy period.*

PROOF In [25, Proposition 8.1.3], it is shown that iff X is DFR, then $X_t \geq_{st} X$ and iff X is IFR, then $X_t \leq_{st} X$. This is true for any value of t.

Let the conditional random variable X_t denote the initial jump initiating a random loss period, given that the amount of service time needed to reach from the virtual work to h equals t. Let X' be the unconditional initial jump and X the service time. We show the proposition for the IFR case; it follows for the DFR case by reversing the relational operators.

If X is IFR, then $X_t \leq_{st} X$, so that

$$P[X_t < X] \geq P[X < x].$$

X' can be computed from X_t by removing the condition:

$$P[X' < x] = \int_0^\infty P[X_t < x]\, dF_x(t)$$

Thus,

$$P[X' < x] = \int_0^\infty P[X_t < x]\, dF_x(t) > \int_0^\infty P[X < x]\, dF_x(t), = P[X < x].$$

In other words,

$$X' \leq_{st} X,$$

from which Lemma 2 yields the proposition. □

Note that the initial jump does not equal X_t for any one t. As an example, consider the $M/D/1$ queue with unit service time. The survivor function of the residual service time is given by

$$\bar{F}_t(a) = \frac{\bar{F}(t+a)}{\bar{F}(t)} = \begin{cases} 1 & \text{for } t+a<1 \\ 0 & \text{otherwise} \end{cases}$$

for $t \leq 1$ and undefined otherwise. Thus, the density is zero everywhere except at the point $t+a=1$.

The initial jump, on the other hand, is given by

$$P[J=j] = P[V = h-1+j | V \leq h].$$

A closed-form expression for the distribution of the virtual wait for the $M/D/1$ is not available, so that the above equation cannot be further simplified. However, Theorem 2 tells us that the loss period will be stochastically shorter than busy periods. In particular, the mean loss period will be less than $1/(1-\rho)$. The simulation data in Table 1 shows that the expected initial jump and, consequently, the expected loss period depend only weakly on h for "interesting" values of h. It is conjectured that this property holds for general queueing systems. A possible justification can be sought in the exponential form of Kingman's approximation for the tail of the waiting time distribution [26, p. 45].

h	initial jump	loss period
0.0	1.000	5.00
0.2	0.815	4.07
0.5	0.588	2.93
1.0	0.567	2.82
2.0	0.483	2.39
3.0	0.467	2.32
5.0	0.465	2.31

Table 1:

2.3 Consecutive Customers Lost

While the duration of a loss period is distributed like the duration of a busy period, we recall from Section 2.1 that the number of consecutive customers lost does not have the same distribution as the number of customers in a busy period. Defining C_C and C_B as the number of consecutively lost customers and the number of customers in a busy period, respectively, we have

$$P[C_C = n] = \frac{P[C_B = n+1]}{P[C_B > 1]}, \; n > 0$$

where $P[C_B > 1]$ is the probability that the busy period contains more than one customer.

Let us apply these results to the $M/M/1$ queue. With $P[C_B = n]$ given by [24, Eq. (5.157)]

$$P[C_B = n] = \frac{1}{n}\binom{2n-2}{n-1}\rho^{n-1}(1+\rho)^{1-2n}, \; n > 0$$

we compute

$$P[C_B > 1] = 1 - P[C_B = 1] = 1 - \frac{1}{1+\rho} = \frac{\rho}{1+\rho}.$$

Thus,

$$P[C_C = n] = \frac{1}{n+1}\binom{2n}{n}\frac{\rho^{n-1}}{(1+\rho)^{2n}},\ n > 0.$$

Note that this result differs markedly from the geometric distribution postulated by Ferrandiz [15, Corollary 5.3].

Since the average number of customers per busy period is $1/(1-\rho)$, we have that for the $M/M/1$ queue the average number of consecutive customers lost is

$$E[C_C] = \frac{1}{P[C_B > 1]}\left(\frac{1}{1-\rho} - 1\right) = \frac{1+\rho}{1-\rho}. \tag{2}$$

This result differs markedly from that obtained under the assumption that losses occur independently as Bernoulli events with the time-average loss probability α. In that case, the conditional probability mass function (pmf), given one loss, for the number of consecutive losses would be distributed geometrically as

$$P[\hat{C_C} = n] = \alpha^{n-1}(1-\alpha)$$

with an average value of $E[\hat{C_C}] = 1/(1-\alpha)$. For our running example, the independence assumption leads one to conclude that a loss period consists of 1.78 customers on average, while our analysis above shows that the actual number for this system is 9. Thus, customer losses are far more clustered than the assumption of independent losses would suggest.

An additional characterization of loss periods is provided by the *conditional probability of packet loss* given that the previous packet was lost, denoted here by r. It is directly related to the average loss run length, $E[C_C]$, through [15, eq. (5.1)]

$$\begin{aligned} E[C_C] &= \frac{1}{1-r} \approx 1 + r + r^2 + \ldots \\ r &= 1 - \frac{1}{E[C_C]}. \end{aligned}$$

For the $M/M/1$ case,

$$r = \frac{2\rho}{1+\rho}.$$

The clustering of losses in a queueing system is naturally also reflected in this measure. For our $M/M/1$ example, the conditional loss probability equals 0.89, while the assumption of independent losses would result in a conditional loss probability r equal to the loss probability α, which evaluates to 0.44 for our running example.

2.4 Distribution of Noloss Period

The distribution of the time between loss periods is more difficult to determine. This interloss time comprises the interval between the time the virtual work W drops below h from above up to the first time instance it rises above this mark again, i.e., the event $\min\{t : W(t) = h | W(0) = h\}$. The sample path with respect to time t of this stochastic process is continuous in time and right-continuous in state, with drift of rate t and jumps of exponential height at Poisson intervals. The difficulty appears since the process is "sticky" at the zero line, with dwell time corresponding to the interarrival (or queue idle time) distribution. We are interested in the distribution of the time to absorption of this process at the $W = h$ barrier.

To make the problem tractable, a Markovian arrival process has to be assumed; otherwise the duration of the noloss period would depend on the time of the last arrival during the preceding loss period. Thus, the computation of this section will be limited to the $M/G/1$ model.

Aspects of this problem or approximations of it appear in a number of applied stochastic models [27]. In *collective risk theory* [28] the insurance company starts out with some fixed capital, increasing through premiums at a constant rate and decreased (or increased) by claims occurring at Poisson instants. Of interest is the time until the capital reaches zero, that is, the company is ruined. To model noloss periods, the capital would represent the mirror image of the virtual work, with an initial value of zero. However, the model does not allow for the fact that the state cannot exceed h (idle system). Thus, it would tend to overestimate the duration of the noloss period and be most suitable for heavy traffic where the idle period is short. It would give exact results, however, for $t < h$ since the system cannot have reached h by that time.

We select a model involving an approximation that is based on the so-called Moran dam model [29, 30] [31, p. 336f] [32, p. 200]. In this model, the water content of a dam or reservoir is represented by a continuous or discrete-state, discrete-time homogeneous Markov process. For reasons of computability, we choose a discrete-state representation, yielding a discrete-time Markov chain (DTMC). Time is discretized in quantities of τ, a fraction of the service time, and the Markov chain tracks the virtual work W_n in the queue at epochs of integer multiples of τ. Thus, the Markov chain has $k = h/\tau$ states. At the end of each discretization interval, at $(n\tau)^-$, the virtual work, if positive, decreases by one unit, reflecting the fact that the virtual work decreases at unit rate. Arrivals bring in an amount of work X_n, again in multiples of τ, and occur just after the beginning of the discretization interval, at $(n\tau)^+$. The noloss period ends as soon as the virtual work reaches h or state $k-1$. We model this by making state $k-1$ an absorbing state and compute the duration of the noloss period as the time to absorption into $k-1$. Given this description of the DTMC, we can write the state evolution recursively as

$$W_{n+1} = \min(k, W_n + X_n) - \min(1, W_n + X_n).$$

Let us define a_k as the probability that k units of work of size τ arrive. Also, denote the complementary cumulative distribution function g_j as

$$g_j = \sum_{i=j}^{\infty} a_j = 1 - \sum_{i=0}^{i-1} a_j.$$

The state transition matrix follows readily:

$$\mathbf{P} = \left[\begin{array}{c|ccccc} & 0 & 1 & \dots & k-2 & k-1 \\ \hline 0 & a_0 + a_1 & a_2 & & a_{k-1} & g_k \\ 1 & a_0 & a_1 & & a_{k-2} & g_{k-1} \\ \dots & & & & & \\ k-2 & 0 & 0 & & a_1 & g_2 \\ k-1 & 0 & 0 & & 0 & 1 \end{array}\right] \quad (3)$$

The last row stems from the fact that state $k-1$ is absorbing. The state transition probabilities are computed as

$$a_j = P[\tau j \le X \le \tau(j+1)], \quad (4)$$

where X is the amount of work arriving in a slot. We know that the accumulated work from n arrivals in a $G/M/c$ system is Erlang-n distributed with density

$$f(x) = \frac{\mu n(\mu n x)^{n-1}}{\Gamma(n)} e^{-\mu n x}.$$

and cumulative distribution function $F(x) = P(n, \mu x)$, where $P(a, x)$ is the normalized incomplete gamma function

$$P(a,x) \triangleq \frac{\gamma(a,x)}{\Gamma(a)} = \frac{1}{\Gamma(a)} \int_0^x e^{-t} t^{a-1}\, dt.$$

The distribution of arriving work needed in evaluating Eq. (4) is hence given by

$$\begin{aligned} P[X < x] &= \sum_{n=0}^{\infty} P[X < x | n \text{ arrivals}] P[n \text{ arrivals}] \\ &= e^{-\lambda\tau} + \sum_{n=1}^{\infty} P(n, \mu x) e^{-\lambda\tau} \frac{(\lambda\tau)^n}{n!}. \end{aligned}$$

The distribution of the time to absorption can be computed in two basic ways. First, since the probability of having been absorbed by the nth transition is simply the probability that the DTMC is in state $k-1$ after n steps, we can use the basic state probability equation in its recursive or matrix-power form,

$$\pi^{(n)} = \pi^{(n-1)}\mathbf{P} = \pi^{(0)}\mathbf{P}^n$$

where $\pi^{(0)} = (0, 0 \ldots 0, 1, 0)$, i.e., $k-2$ is the initial state. The matrix power form can be evaluated as a special case of the general relationship for any functional f of a matrix, given by $f(\mathbf{P}) = \mathbf{V} f(\Lambda) \mathbf{V}^{-1}$, where $\mathbf{V}$ is the matrix of eigenvectors of $\mathbf{P}$ and the function f is applied element-by-element to the diagonal matrix of eigenvalues Λ [33, p. 8]. The eigenvalue approach may be more accurate for large values of n.

The other alternative defines $f_{il}^{(n)}$ as the probability that the system *first* enters state l after n steps, given the initial state is i. To use the first-passage formulation, the matrix P has to be returned to its recurrent form by replacing the last row with $(0, \ldots, 0, a_0, g_1)$. It is readily seen that this first-passage pmf is recursively defined for all transition matrices as

$$f_{il}^{(1)} = P_{il} \text{ for } i = 0, 1, \ldots \tag{5}$$

$$f_{il}^{(n)} = \sum_{j, j \neq l} P_{ij} f_{jl}^{(n-1)} \tag{6}$$

We obtain a lower bound on the cumulative probability by using $k-2$ as the initial state, an upper bound by using $k-1$.

Sample calculations showed that state equations, matrix computations and the approach using f_{il} yield the same numerical result to within four significant figures, indicating that roundoff errors are not a serious problem here. Also, the computational effort is about the same.

The discretization error incurred by using a particular value of τ can be estimated by computing the expected duration of the noloss period. Since the fraction of packets lost, α, is related to the expected loss period $E[L]$ and the expected noloss period $E[N]$ by[2]

$$\alpha = \frac{E[L]}{E[L] + E[N]} \tag{7}$$

the expected noloss period can be computed as

$$E[N] = E[L]\left(\frac{1}{\alpha} - 1\right).$$

[2]Replacing α by the load, ρ, noloss and loss periods by idle and busy periods yields the well-known relation for busy cycles, again underlining the strong connection between loss and busy periods.

Given the DTMC approximating the virtual work process, the expected noloss period (equivalent to the time to absorption) can be computed as

$$E[N] = \left(\frac{1}{\pi'_{k-1}} - 1 \right) \tau \tag{8}$$

where π'_{k-1} is the *steady-state* probability that the return process corresponding to the DTMC is in state $k-1$. The transition probability matrix of the return process is derived from P by replacing the last row with all zeros, except for a one in column $k-2$. This relationship is derived in [34, p. 112, Problem 3] for the case of two absorbing states, but the result generalizes readily to any number of absorbing states (see also [35, p. 103]).

For our example, the exact value of $E[N]$ is 6.388. For the discretization with $\tau = 0.1$, Eq. (8) yields a value of 6.109, which improves to 6.237 and 6.327 for $\tau = 0.05$ and $\tau = 0.02$, respectively.

2.5 Customers per Noloss Period

It appears difficult to derive an expression for the distribution of the number of customers in a noloss period. The expected value, however, is readily available since the average number of customer arrivals during loss periods, $E[C_L]$, and noloss periods, $E[C_N]$, are related in a similar fashion as the respective duration measures, Eq. (7), yielding

$$\begin{aligned} \alpha &= \frac{E[C_L]}{E[C_L] + E[C_N]} \\ E[C_N] &= E[C_L] \left(\frac{1}{\alpha} - 1 \right). \end{aligned}$$

The difficulty in determining the distribution arises due to the fact that noloss periods do not "see" the same Poisson arrival process with rate λ as a random observer, just as the the arrival rate measured during busy periods is higher than the arrival rate measured over all customers. Thus, the conditional probability of the number of arrivals given a noloss period duration cannot be readily computed.

2.6 Simulation Experiments

A simulation experiment for our running example was performed to lend credence to the performance measures computed in the sections above. The simulation extended over 500000 customers and encompassed 55936 loss and noloss periods.

The comparison between simulation and analysis is carried out in Table 2 for expected values, in Table 3 for the cumulative distributions (of initial jump, loss period duration and customers per loss period) and in Table 4 for the distribution of the noloss period. The results in the latter two tables are also depicted in Fig. 2 and Fig. 3.

The numerical values for the duration of the noloss period were computed using Eq. (6). The lower and upper value of the range indicated for each value of τ were computed with starting states $k-2$ and $k-1$, respectively. In all cases, the range falls within the 90% confidence interval.

For all parameters, the tables and figures show that numerical and simulation results agree quite closely, in the case of noloss measures even for the relatively coarse value of $\tau = 0.1$, with agreement improving on the tail of the distribution.

Performance measure	Theory	Simulation
Waiting time	4.00	3.774... 3.942
System time	5.00	4.771... 4.941
Customer loss, %	43.91	43.034
Idle period	1.25	1.241... 1.255
Busy period	5.00	4.825... 5.004
Customers per busy period	5.00	4.844... 5.002
Initial jump	1.00	9.851... 1.004
Loss period	5.00	4.756... 4.883
Consecutive customers lost	9.00	8.618... 8.818
Noloss period	6.39	6.267... 6.470
Customers per noloss period	11.50	11.330...11.750
Conditional loss probability	0.89	0.884... 0.887

Table 2: Expected values of selected performance measures for our running example

x	Initial jump		Length of loss period		Number of customers/LP	
	Anal.	Simul.	Anal.	Simul.	Anal.	Simul.
1	0.6321	0.6338	0.5032	0.5055	0.3086	0.3089
2	0.8647	0.8668	0.6592	0.6628	0.4611	0.4639
3	0.9502	0.9512	0.7354	0.7386	0.5551	0.5580
4	0.9817	0.9823	0.7818	0.7851	0.6202	0.6242
5	0.9933	0.9934	0.8136	0.8176	0.6684	0.6725
6	0.9975	0.9976	0.8372	0.8413	0.7058	0.7107
7	0.9991	0.9991	0.8553	0.8587	0.7358	0.7394
8	0.9997	0.9997	0.8698	0.8738	0.7605	0.7649
9	0.9999	0.9998	0.8818	0.8852	0.7812	0.7851
10	1.0000	1.0000	0.8918	0.8947	0.7989	0.8038
11	1.0000	1.0000	0.9003	0.9026	0.8142	0.8184
12	1.0000	1.0000	0.9077	0.9104	0.8275	0.8318
13	1.0000	1.0000	0.9143	0.9173	0.8393	0.8428
14	1.0000	1.0000	0.9199	0.9231	0.8497	0.8545
15	1.0000	1.0000	0.9250	0.9283	0.8591	

Table 3: Cumulative distributions for selected performance measures; analysis and simulation

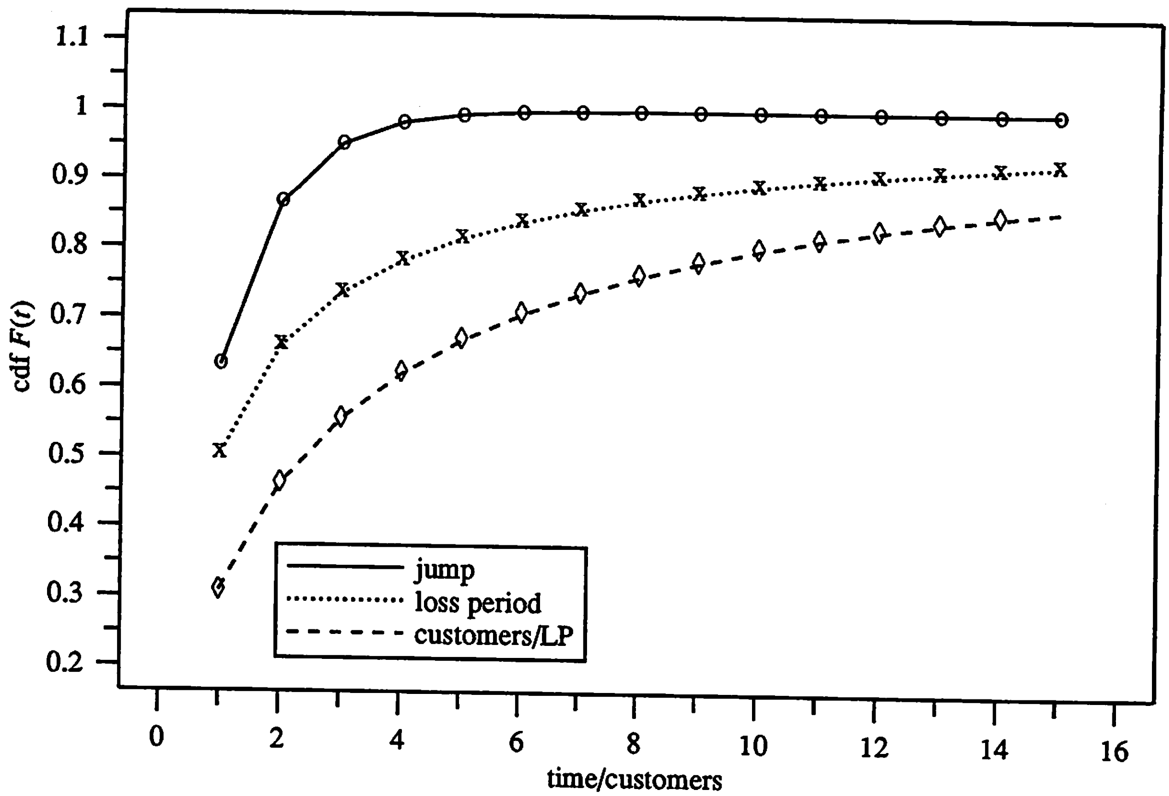

Figure 2: Cumulative distributions for selected performance measures; analysis and simulation

t	DTMC approximation $\tau = 0.1$	DTMC approximation $\tau = 0.05$	DTMC approximation $\tau = 0.02$	Simulation
1	0.3899...0.4180	0.3962...0.4104	0.4000...0.4057	0.4014
2	0.5168...0.5459	0.5221...0.5368	0.5253...0.5312	0.5263
3	0.5807...0.6083	0.5846...0.5986	0.5869...0.5925	0.5872
4	0.6236...0.6490	0.6257...0.6386	0.6269...0.6322	0.6279
5	0.6591...0.6822	0.6595...0.6714	0.6597...0.6645	0.6608
6	0.6905...0.7115	0.6895...0.7003	0.6888...0.6932	0.6889
7	0.7188...0.7379	0.7166...0.7265	0.7152...0.7192	0.7147
8	0.7445...0.7619	0.7413...0.7504	0.7393...0.7430	0.7387
9	0.7678...0.7836	0.7639...0.7721	0.7614...0.7648	0.7607
10	0.7890...0.8034	0.7845...0.7920	0.7816...0.7847	0.7813
11	0.8083...0.8213	0.8033...0.8101	0.8000...0.8029	0.7998
12	0.8258...0.8376	0.8204...0.8267	0.8170...0.8195	0.8162
13	0.8417...0.8524	0.8361...0.8418	0.8324...0.8348	0.8312
14	0.8562...0.8659	0.8503...0.8556	0.8466...0.8488	0.8443
15	0.8693...0.8782	0.8634...0.8681	0.8596...0.8616	0.8566

Table 4: Approximation of distribution of noloss period

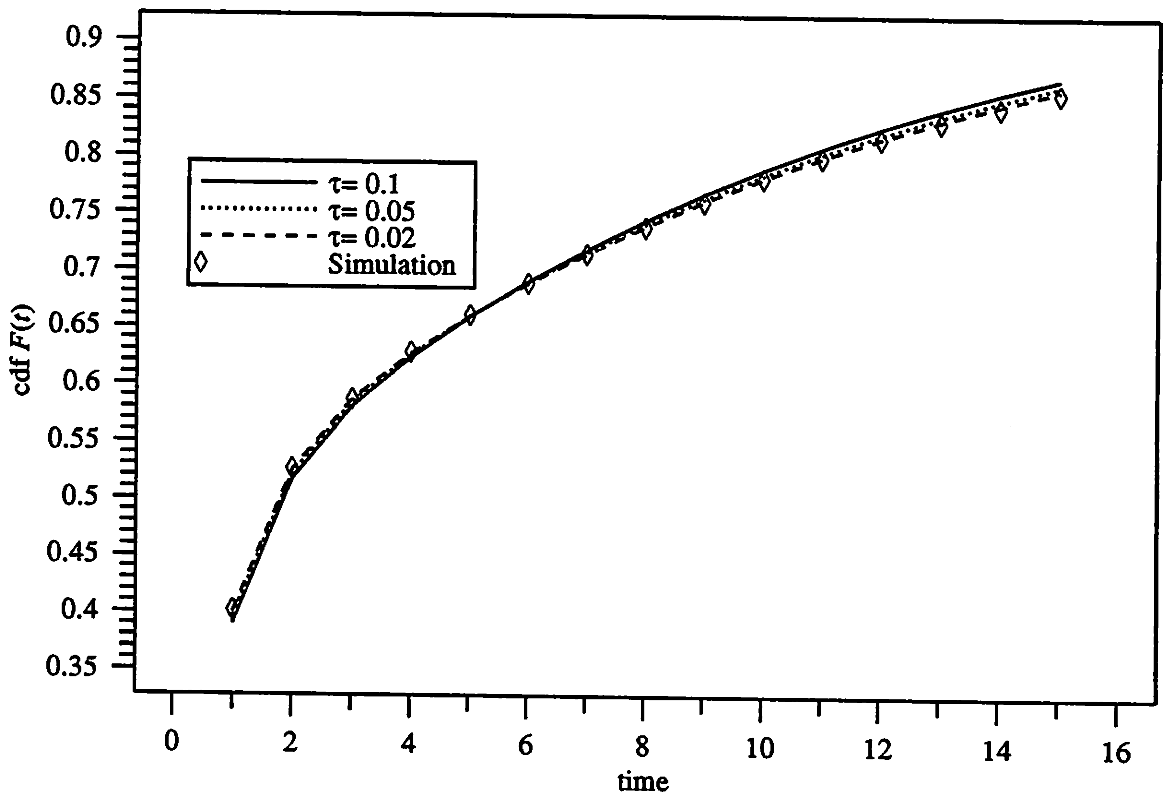

Figure 3: Approximation of distribution of noloss period for $M/M/1$ queue

3 Clip Loss in Discrete-time Systems

We now turn our attention to a queueing model that is commonly used for packet switches and ATM-type networks [36, 37]. In this model, time is discretized, with deterministic service (of duration $\tau = 1$) and batch arrivals, which, in many instances, allow somewhat simpler solutions than their continuous-time counterparts. Batches arrive at the beginning of a slot of unit width, while at most one customer departs at the end of a slot. (Hunter [38, p. 193] refers to this as an early arrival system.) We allow the batch size, A, to have a general distribution, but require the batch sizes to be independent from slot to slot and independent of the state of the queue itself.

We will say that batch sizes are geometrically distributed if their probability mass function (pmf) is $a_n = pq^n$, with $q = 1 - p$ and an average batch size of $\rho = q/p$. We will also cover the Poisson distribution with mean ρ,

$$a_n = \frac{e^{-\rho}\rho^n}{n!} \text{ for } n \in [0, \infty),$$

and the binomial distribution with mean $\rho = \nu p$,

$$a_n = \begin{cases} \binom{\nu}{n} p^n q^{\nu-n} & \text{for } n \in [0, \nu] \\ 0 & \text{otherwise} \end{cases}.$$

While numeric solutions are possible for general batch size distributions, a queue with geometric batch sizes, i.e., the system $D^{[\text{Geo}]}/D/1$, will be shown to share the identity between busy period and loss period described in Section 2 for the continuous-time $GI/M/1$ system. Also, restricting batch sizes to be geometrically distributed will yield closed-form expressions for many distributions of interest. Due to the close relationship between busy and loss periods, we will investigate busy periods in some detail in Section 3.1, followed by derivations of the properties of loss and noloss periods in sections 3.2 and 3.3, respectively.

3.1 The Busy and Idle Period

In discrete-time systems, we define that a *busy period* begins when the first customer in a batch experiences no waiting, i.e., finds the server free. Note that, unlike in continuous time, a server may be continuously occupied for more than one busy period. This occurs if the last customer in a busy period, departing in (n^-, n), is immediately followed by one or more new arrivals in (n, n^+). Thus, the first customer in that batch enters service immediately, starting a new busy period, while the server experiences no idle slot. Later, we will discuss the composite busy period which encompasses time intervals without server idling, consisting of one or more busy periods.

Let us return now to the busy period and compute its distribution. Because each customer occupies the server for one slot, duration and number of customers served are equal in the discrete-time case. For geometric batches[3], we can compute the number of customers in a busy period by making use of Takács combinatorial arguments [39, p. 102f], [24, p. 225f]. Let B be the number served in a busy period and $\tilde{A}_n$ the number of arrivals during the service times of customers 1 through n, where no assumption is made as to whether these n customers belong to the same busy period(s).

[3]The ensuing development requires that the work arriving during the service of each customer be i.i.d.. For batch arrivals, the arrivals that occur while the first customer is being serviced consist of the batch starting the busy period minus one. The distribution of this "shortened" batch has the same distribution as a regular batch only if batches are geometrically distributed. For other batch distributions, the following calculations can serve as an approximation.

The probability mass function of the number served in a busy period is given by [24, Eq. (5.166)]

$$P[B = n] = \frac{1}{n} P[\tilde{A}_n = n - 1].$$

For the case of deterministic service and batch size distribution a_n, the probability on the right-hand side can be readily derived:

$$P[\tilde{A}_n = n - 1] = P[n - 1 \text{ arrivals in } n \text{ slots}] = a_{n-1}^{n*}.$$

Here, a_n^{n*} denotes the n-fold convolution of a_n with itself. For the case of geometrically distributed batches, the convolution becomes the negative binomial or Pascal distribution with probability mass function

$$a_n^{r*} = \binom{r + n - 1}{n} p^r q^n.$$

Thus, the busy period for geometric batches is distributed according to

$$P[B = n] = \frac{1}{qn} \binom{2n - 2}{n - 1} (qp)^n \tag{9}$$

$$= \frac{1}{n} \binom{2n - 2}{n - 1} \rho^{n-1} (1 + \rho)^{1-2n}. \tag{10}$$

The last transformation makes use of the fact that the system load ρ is related to the batch distribution parameter p through $p = 1/(1+\rho)$. We recognize the last expression as the distribution of the number of customers served in an $M/M/1$ busy period [24, p. 218].

The z-transform of the number served in a busy period, $B(z)$, follows from the $M/M/1$-derivation [24, p. 218]:

$$B(z) = \frac{1 + \rho}{2\rho} \left[1 - \sqrt{1 - \frac{4\rho z}{(1 + \rho)^2}} \right] = \frac{1 - \sqrt{1 - 4pqz}}{2q} \tag{11}$$

The expected number served (and arriving) in a busy period can be computed by evaluating an infinite series using Eq. (11) or directly copying the $M/M/1$ result:

$$E[B] = \sum_{n=1}^{\infty} n P[B = n] = \frac{p}{\sqrt{1 - 4pq}} = \frac{1}{1 - \rho}.$$

The *idle period* I for general batch sizes is geometrically distributed with density $P[I = n] = a_0^n (1 - a_0), n \geq 0$. Recall that a_0 is the probability of a batch having zero members. Thus, the average idle period is given by $a_0/(1 - a_0)$. For geometric batch sizes, $a_0 = p$ and thus an idle period last an average of $1/\rho$ slots.

In contrast to the continuous-time case, an idle period may have zero duration. This occurs if a new batch arrives immediately after the last customer of a busy period departs. We will call a period of continuous server occupancy spanning several busy periods a *composite busy period* and identify random variables associated with it by a tilde. A composite busy period consists of ϕ busy periods with the geometric probability $P[\phi = k] = (1 - a_0)^{k-1} a_0$. In the z-transform domain, the number of customers in the composite busy period, $\tilde{B}$, is determined as $\tilde{B}(z) = \phi(B(z))$, where

$$\phi(z) = \frac{pz}{1 - qz}$$

is the probability generating function of the number of busy period constituting a composite busy period.

For geometric batches, $\tilde{B}(z)$ can be expanded using Eq. (11):

$$\begin{aligned}\tilde{B}(z) &= \frac{p}{q}\frac{1-\sqrt{1-4pqz}}{1+\sqrt{1-4pqz}} \\ &= \frac{p}{q}\frac{1-2\sqrt{1-4pqz}+1-4pqz}{4pqz} \\ &= \frac{1}{zq}\left[\frac{1-\sqrt{1-4pqz}}{2q}\right]-\frac{p}{q}\end{aligned}$$

The bracketed fraction is recognized as $B(z)$ and thus $\tilde{B}(z)$ can be inverted easily, yielding

$$\begin{aligned}P[\tilde{B}=n] &= \frac{1}{q}\frac{1}{n+1}\binom{2n}{n}\frac{\rho^n}{(1+\rho)^{2n+1}} \\ &= \frac{1}{n+1}\frac{(2n)!}{(n!)^2}\frac{\rho^{n-1}}{(1+\rho)^{2n}} \qquad (n \geq 1).\end{aligned}$$

The expected number of customers in a composite busy period (and its expected duration in slots) is seen to be

$$E[\tilde{B}] = E[B]\cdot E[\phi] = \frac{E[B]}{a_0},$$

in general or

$$E[\tilde{B}] = \frac{1+\rho}{1-\rho}$$

for geometric batches.

3.2 The Loss Period

A *loss period* begins when one or more customers arriving in a batch see h or more customers already in the system (in other words, if their wait is equal to or greater than h.) Thus, a busy period is (again) a special case of a loss period with h having the value zero. A loss period ends when there are h or fewer customers left after a departure. Just as discussed above for the case of busy periods, a loss period may be followed immediately by another loss period. This occurs if the number of customers reaches h at some point n, i.e., the loss period ends, and a batch arrives in (n, n^+), starting a new loss period. An uninterrupted interval where the number of customers in the system just prior to an arrival never drops below h (or, equivalently, where the occupancy after the arrival instants remains above h) will be referred to as a *composite loss period*. Clearly, it consists of an integral number of loss periods.[4] The random variables L and $\tilde{L}$ represent the loss period and composite loss period, respectively, while the random variable V represents the occupancy on the slot boundary, equivalent to the work in the queue seen by an arriving batch (virtual work). Because of the deterministic service time and the slotted arrivals, duration and customer count properties are the same, i.e., a loss period of l slots leads to l consecutive losses.

Fig. 4 depicts an example of a sample path showing a composite loss period made up of two loss periods (LPs) for a threshold of $h = 3$.

Just like for the continuous-time case, we are interested in determining the distribution of the initial jump J, that is, the work load or, equivalently, the number of customers that begins a loss

[4] However, unlike in the $M/M/1$-case, we do not have to factor out the single-customer busy periods.

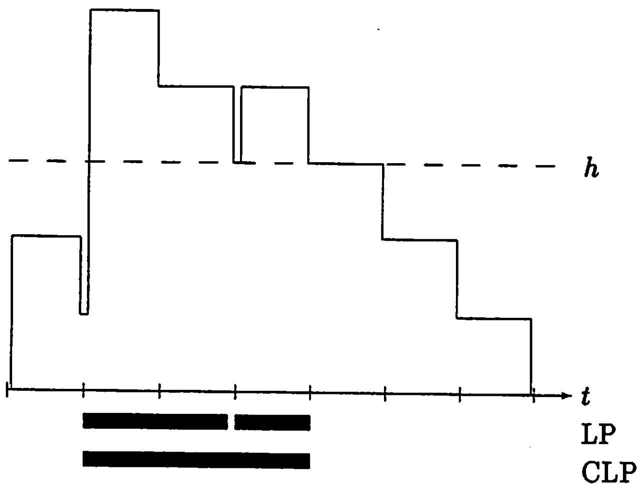

Figure 4: Loss periods in discrete time ($h = 3$)

period. Fig. 4, for example, shows two initial jumps, one occurring at the arrival of batch $\{c \ldots f\}$, with a height of two, and the second at the arrival of $\{g\}$, with a height of one.

Lemma 3 *The initial jump J into a loss period has the distribution*

$$P[J = j] = \frac{\sum_{a=j}^{\min(h+j,\nu)} P[V_h = h + j - a] P[A = a]}{\sum_{a=1}^{\min(h,\nu)} P[V_h > h - a] P[A = a] + P[A > h]},$$

where the distribution of the batch size random variable A is zero outside the range $[0, \nu]$ and the conditional system occupancy distribution seen by an arriving batch is defined as

$$P[V_h = v] \triangleq \frac{P[V = v]}{P[V \le h]}.$$

Proof We sum the probabilities of all events leading to a jump of size j (given that a jump occurred), noting that the random variables A and V are independent. Thus, we have for jumps into loss periods,

$$\begin{aligned}
& P[J = j] \\
&= \sum_{v+a=h+j} P[V_h = v \cap A = a | A + V_h > h] \\
&= \frac{\sum_{v+a=h+w} P[V_h = v \cap A = a \cap A + V_h > h]}{P[A + V_h > h]} \\
&= \frac{\sum_{v+a=h+w} P[V_h = v] P[A = a]}{\sum_{v+a>h} P[V_h = v] P[A = a]} \\
&= \frac{\sum_{a=1}^{\min(h+j,\nu)} P[V_h = h + j - a] P[A = a]}{\sum_{a=1}^{\min(h,\nu)} P[V_h > h - a] P[A = a] + P[A > h]}
\end{aligned}$$

Among loss periods, we have to distinguish those that form the first loss period in a composite loss period. While an arriving batch that starts a loss period may see between 0 and h in the system, a batch that initiates a composite loss period, i.e., the initial jump conditioned on the fact that the loss period is the first in a composite loss period, can see at most $h-1$ customers. Thus, the following lemma provides a separate expression for the jump into a composite loss period, $\tilde{J}$.

Lemma 4 *The initial jump into a composite loss period $\tilde{J}$ has the distribution*

$$P[\tilde{J}=j] = \frac{\sum_{a=j+1}^{\min(h+j,\nu)} P[V_{h-1}=h+j-a]P[A=a]}{\sum_{a=2}^{h} P[V_{h-1}>h-a]P[A=a]+P[A>h]}.$$

The derivation of $P[\tilde{J}]$ proceeds as in the proof of the previous lemma.

Finally, initial jumps of all but the first loss period within a composite loss period always start at h. Since the queue occupancy seen by an arriving batch and the batch size itself are independent, the jump into these "interior" loss periods is distributed like a regular non-zero batch.

In close parallel to the continuous-time case, the memorylessness property of the geometric distribution makes the distribution of both types of initial jump a shifted version of the batch distribution, independent of h. We formulate more precisely in a lemma:

Lemma 5 *For the $D^{[\mathrm{Geo}]}/D/1$ queue, the initial jump J and the initial jump into composite loss periods $\tilde{J}$ are distributed like non-zero batches, with pmf $P[J=j]=P[\tilde{J}=j]=pq^{j-1}=\rho^{j-1}/(\rho+1)^j$.*

The derivation of this invariance property is algebraically rather tedious and relegated to the appendix.

Given the characteristics of the initial jump, the (composite) loss period are seen to be stochastically identical to (composite) busy periods with a special first service given by the initial jump distribution. By the previous lemma, loss periods in a system with geometric batch arrivals have the same distribution as regular busy periods of the same system. Since the members of a batch that initiate a loss period also experience delays of at least h, the number of customers lost in a loss period and the number served in a busy period with the above-mentioned special first service are stochastically identical as well.

For general batch-size distributions, the probabilities for loss periods of length one can easily be written down exactly:

$$\begin{aligned} P[L=1] &= P[J=1] \\ P[\tilde{L}=1] &= P[J=1]a_0 \end{aligned}$$

Closed-form expressions for measures of L and $\tilde{L}$ for longer durations seem difficult to obtain, however. We therefore model the queue state during loss periods as a discrete-time Markov chain with an absorbing state. Since the composite loss period is of greater practical significance, indicating the number of consecutively lost customers, we will focus on this random variable for the remainder of this section. The states of the chain indicate the amount of unfinished work above h just after a batch has arrived. For computational reasons, we truncate the transition matrix to $K+2$ states and write

$$\mathbf{P} = \left[\begin{array}{c|cccccc} & 0 & 1 & 2 & \dots & K & K+1 \\ \hline 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & a_0 & a_1 & a_2 & \dots & a_K & 1-\sum_{j=0}^{K} P_{1j} \\ 2 & 0 & a_0 & a_1 & \dots & a_{K-1} & 1-\sum_{j=0}^{K} P_{2j} \\ \vdots & & & & & & \\ K+1 & 0 & 0 & 0 & \dots & a_0 & 1-a_0 \end{array}\right]$$

Note that the zero state is absorbing since the loss period ends when the amount of unfinished work above h reaches zero. We therefore obtain the duration of a composite loss period by evaluating the chain's absorption probabilities over time, given the initial state probabilities determined by the distribution of the initial jump, also truncated to $K + 2$ values.[5]

Since the distribution of the composite loss probability typically has very long tails, computing its expected value through summation of the weighted probabilities was found to be numerically inaccurate and computationally expensive. However, an alternative approach exists [41, p. 425]. Let d_j be the expected time to absorption into state 0, starting at state j. By the law of total probability, we can write a system of linear equations

$$d_j = \sum_{k=1}^{K+1} \mathbf{P}_{jk} d_k + 1.$$

In matrix from, the linear system consists of the $\mathbf{P}$ matrix with its first row and column removed.

It is instructive to compare the computational costs of the first method, based on the probability density function, to those of the second, based on solving the linear system. The first method requires a matrix multiplication for each time step. If we assume that the time horizon is of the same magnitude as K (judging from examples, a value of $10K$ seems more appropriate for comparative accuracy), the computational cost is approximately $O(K^4)$, while the solution of the linear system is a $O(K^3)$ operation.

The expected value is also of interest in estimating the effect of truncating the transition probability matrix $\mathbf{P}$ and the time horizon in computing the distribution of $\tilde{L}$. Typically, little change is observed for values of K above 30 to 50.

Note that the methods presented in this section can also be used to compute the composite busy period of queues of the $D^{[G]}/D/1$ type. Finally, the conditional loss probability may be evaluated as detailed in Section 2.3.

3.3 The Noloss Period

The state evolution during a noloss period can also be modeled by a discrete-time transient Markov chain with initial state h. Unlike in the continuous-time case, this model is exact. Since the number of possible states is limited to $h + 1$ (including the absorbing state $h + 1$ representing a new loss period), no truncation error is incurred.

We track the number of customers in the system just after arrivals, at n^+. Since we cannot easily accommodate zero first-passage times, we compute the conditional pmf of noloss periods lasting at least one slot. The pmf of the unconditional loss period is then simply the conditional pmf scaled by a_0. We denote the respective random variables by N' and N.

The transition probability matrix is similar to the one used to approximate the noloss period for the continuous-time case, Eq. (3), but since there are no departures when the system is empty,

[5] For the tail of the loss-period, this random walk with drift and one absorbing barrier may be represented by the corresponding Brownian motion [40, p. 437].

the first and second row are identical.

$$P = \left[\begin{array}{c|cccccc} & 0 & 1 & \dots & h-1 & h & h+1 \\ \hline 0 & a_0 & a_1 & & a_{h-1} & a_h & g_{h+1} \\ 1 & a_0 & a_1 & & a_{h-1} & a_h & g_{h+1} \\ \dots & & & & & & \\ h-1 & 0 & 0 & & a_1 & a_2 & g_3 \\ h & 0 & 0 & & a_0 & a_1 & g_2 \\ h+1 & 0 & 0 & & 0 & 0 & 1 \end{array}\right]$$

where

$$g_k = \sum_{j=k}^{\infty} a_j = 1 - \sum_{j=0}^{k-1} a_j.$$

The expected length of the noloss period can be obtained as in the continuous-time case by evaluating the steady-state probabilities of the return process. The state transition matrix of the return process is derived from the matrix **P** by replacing the last row $(h+1)$ with zeros except for a one in column h (see Eq. (8)).

Possibly due to batch effects, the relation

$$\alpha = \frac{1}{\rho} \frac{E[\tilde{L}]}{E[\tilde{L}] + E[\tilde{N}]} = \frac{1}{\rho} \frac{E[L]}{E[L] + E[N]}$$

between α, the loss probability and the expected composite loss period, $E[\tilde{L}]$, and expected conditional noloss period, $E[\tilde{N}]$, holds exactly for geometric batches (due to their memorylessness property), but only approximately otherwise. The correction factor $1/\rho$ accounts for the fact that $\alpha\rho$ packets with excessive waiting time complete service.

The expected number of consecutive successful customers can be computed as discussed in Section 2.5.

Table 5 lists expectations of the performance measures computed for the standard discrete time queues for a load of $\rho = 0.8$ and a threshold $h = 3$. The values were verified by simulation.

Batch size	a_0	α	$E[J]$	$E[\tilde{J}]$	$E[L]$	$E[\tilde{L}]$	$E[N]$	$E[\tilde{N}]$
Geometric	0.5556	0.51200	1.800	1.800	5.000	9.000	7.2071	12.9727
Poisson	0.4493	0.29738	1.371	1.272	2.857	6.358	9.1509	20.3670
Binomial	0.4182	0.22947	1.286	1.162	2.431	5.812	10.8095	25.8477

Table 5: Measures of the loss period and related quantities for $h = 3$, $\rho = 0.8$, $\nu = 5$ and $K = 50$

3.4 Numerical Examples

For a first impression of the behavior of the loss period, we plot the mean value of the composite loss period as a function of the system load ρ in Fig. 5. As expected, the curves follow the characteristic pattern of loss probability and delay curves for queueing systems, with a gradual rise up to a "knee" point, followed by a region of high sensitivity to ρ for high loads. Exhibiting a pattern that will be seen in the following curves as well, the geometric case is clearly separated from the other distributions, which track each other closely in mean composite loss period.

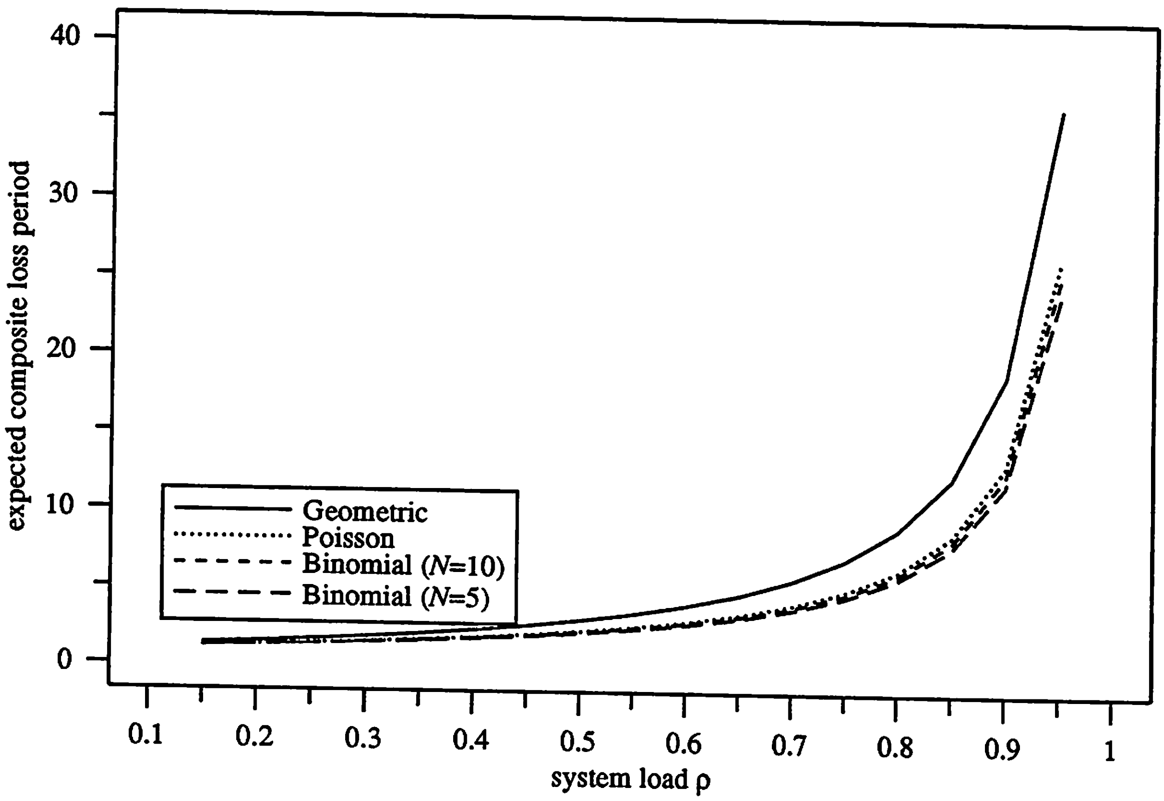

Figure 5: Expected composite loss period as a function of system load for $h = 5$

We stated as Lemma 5 that the distribution of the initial jump and (composite) loss period of the $D^{[Geo]}/D/1$ queue are independent of the threshold h. It seems therefore natural to investigate to what extent these quantities depend on h for other batch distributions commonly used for modeling data communication systems. As an example, consider the Poisson distribution and binomial distribution[6] with an average batch size of $\rho = 0.8$. For values of h ranging from 3 on up (corresponding to losses of about 30% and less), Table 6 shows that h plays no significant role in $E[\tilde{J}]$ and consequently $E[\tilde{L}]$. (The same observation also holds for the distribution, not shown here.) It other words, for loss probabilities of practical interest, the loss period is basically independent of the threshold.

	Poisson			Binomial ($\nu = 2$)		
h	α	$E[\tilde{J}]$	$E[\tilde{L}]$	α	$E[\tilde{J}]$	$E[\tilde{L}]$
0	1.0000	1.453	7.264	1.0000	1.25	6.25
1	0.6936	1.304	6.520	0.5556	1.00	5.00
2	0.4569	1.275	6.376	0.2469	1.00	5.00
3	0.2974	1.272	6.358	0.1097	1.00	5.00
4	0.1933	1.272	6.358	0.0488	1.00	5.00
5	0.1256	1.272	6.358	0.0217	1.00	5.00
6	0.0816	1.272	6.358	0.0096	1.00	5.00
7	0.0531	1.272	6.358	0.0043	1.00	5.00
8	0.0345	1.272	6.358	0.0019	1.00	5.00
10	0.0146	1.272	6.358	0.0004	1.00	5.00

Table 6: Probability of loss, expected composite loss period and jump for Poisson batches as a function of h for $\rho = 0.8$

Batch	$\alpha = 0.01$			$\alpha = 0.1$			$\alpha = 0.6$		
distribution	ρ	$E[\tilde{L}]$	ratio	ρ	$E[\tilde{L}]$	ratio	ρ	$E[\tilde{L}]$	ratio
Geometric	0.398	2.323	1.00	0.631	4.42	1.00	0.904	19.72	1.00
Poisson	0.595	2.937	1.26	0.780	5.74	1.30	0.948	25.57	1.30
Binomial ($N = 10$)	0.655	3.075	1.32	0.798	6.03	1.36	0.953	26.96	1.37
Binomial ($N = 5$)	0.624	3.254	1.40	0.817	6.39	1.45	0.958	28.63	1.45

Table 7: Expected composite loss period for fixed probability of loss; $h = 5$

As a final comparison, we adjust the load ρ so that the loss probability α remains constant over all distributions, as shown in Tab. 7. The column labeled "ratio" depicts the ratio of the expected composite loss period for each distribution with respect to the geometric distribution. It can be seen that the difference is relatively small, on the order of less than 40%, and, surprisingly, increases only slightly with increasing loss probability.

The distribution of the composite loss period is shown in Fig. 6. It is seen here that the distributions differ little for small composite loss periods, with most of the difference in expectation

[6]The value of $\nu = 2$ used here is the smallest non-trivial value. As ν increases, the behavior should approach that of the Poisson distribution.

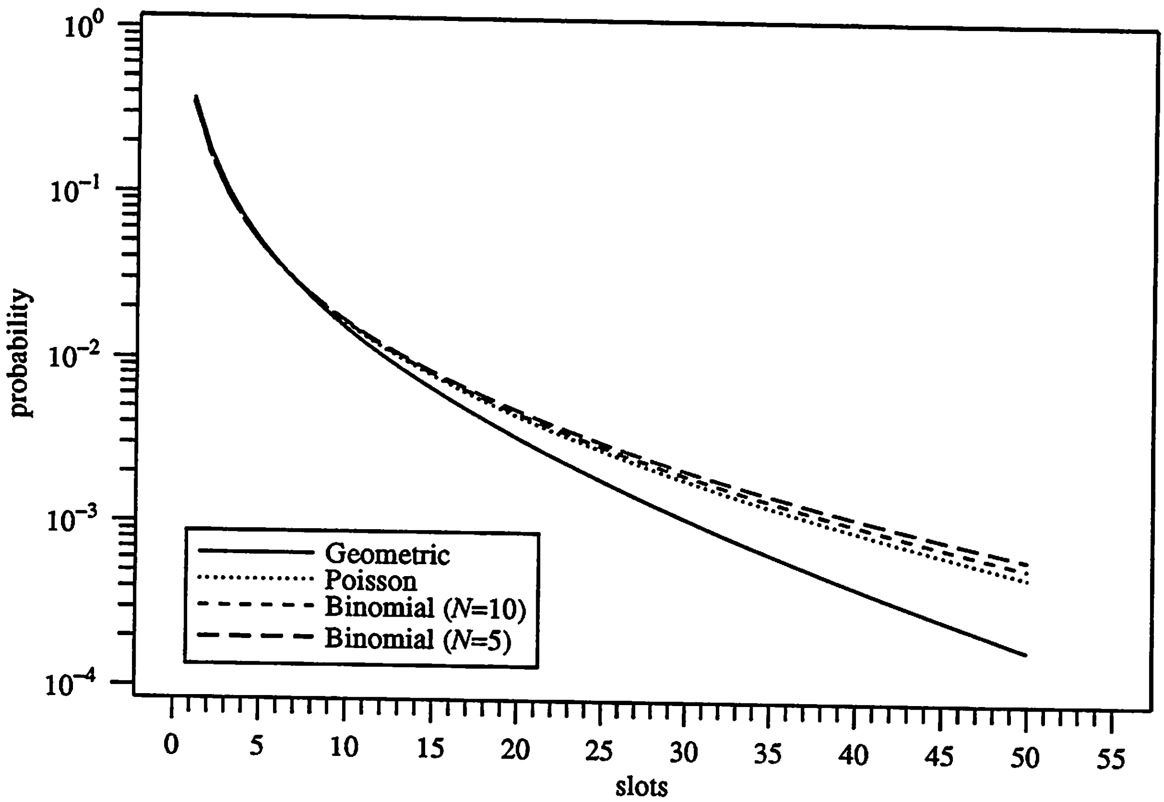

Figure 6: Probability mass function of the composite loss period for $\alpha = 0.1$ and $h = 5$

caused by the divergence in the tail of the distribution. The loss period for geometric batches tails off significantly faster than those for the either the Poisson or the binomial distribution.

4 Queues with Bounded Waiting Time

4.1 Continuous Time

In our earlier analysis in this report, we had assumed that all customers, including those exceeding the waiting time threshold h, are served. In this section, we consider the case when excessively delayed customers are dropped by the server. The queueing model with Poisson arrivals and exponential service was considered in [43] (called FIFO-TO there) and [2] (labelled FIFO-BW).

For exponential service, multiple servers can be handled without difficulty as the c-server system acts like a single, faster server with rate $c\mu$ during loss periods. (Departures from the servers occur as the minimum of exponential service times and interdeparture times are therefore exponentially distributed with rate $c\mu$.)

Let us briefly review the terminology established earlier in this report. A *loss period* commences when an arrival causes the virtual work to cross a given threshold h from below. The amount of work that exceeds h at that instant is called the *initial jump*. It was shown in Theorem 1 that the height of the initial jump is independent of h for $G/M/c$ systems.

Since customers that arrive during a loss period do not enter the system, the duration of the loss period is completely determined by the height of the initial jump J. To determine the distribution of the number of consecutively lost customers (the *loss run*), we compute the number of arrivals during a random interval of length J. For non-Poisson arrivals, it needs to be taken into account that the interval J has an arrival at its beginning. Thus, while the relationship is general, closed-form computation is most likely limited to $G/M/c$ queues due to the independence property noted above. The pmf of the number of arrivals with interarrival density $a(t)$ in an interval $(0, t)$, given that an arrival occurred at 0^-, is labeled $P[C_L = n|t]$ and computed from standard renewal theory [23, p. 57]

$$P[C_L \geq n|t] = \mathcal{L}^{-1}\left\{\frac{1}{s}[A^*(s)]^n\right\} = \int_0^t \mathcal{L}^{-1}\left\{[A^*(s)]^n\right\}\, dt \tag{12}$$

where $A^*(s)$ is the Laplace transform of the interarrival density $a(t)$ and $\mathcal{L}^{-1}\{\}$ denotes the inverse Laplace transform. $E[C_L|t]$ can be obtained similarly:

$$E[C_L|t] = \mathcal{L}^{-1}\left\{\frac{1}{s}\sum_{n=1}^{\infty}[A^*(s)]^n\right\}$$

Given the conditional distribution, we can compute the inverse cumulative distribution and expectation of the number of arrivals during an exponentially distributed loss period, C_L, as

$$\begin{aligned} P[C_L \geq n] &= c\mu \int_0^\infty P[C_L \geq n|t] e^{-c\mu t}\, dt,\ n \geq 0, \\ E[C_L] &= c\mu \int_0^\infty E[C_L|t] e^{-c\mu t}\, dt \end{aligned}$$

which can also be expressed in terms of the A^* evaluated at $s = c\mu$,

$$P[C_L \geq n] = [A^*(c\mu)]^n, \text{ for } n \geq 0 \tag{13}$$

$$E[C_L] = \sum_{n=1}^{\infty}[A^*(c\mu)]^n. \tag{14}$$

The distribution and expectation of the loss run, C_C, follows by appropriate conditioning:

$$\begin{aligned} P[C_C = n] &= \frac{1}{1 - P[C_L = 0]} P[C_L = n], \; n \geq 1 \\ E[C_C] &= \frac{1}{1 - P[C_L = 0]} E[C_L] \end{aligned}$$

Finally, the noloss period is the same as when all customers are served.

4.1.1 Erlangian Arrivals: $E_r/M/c$

Closed-form solutions can be obtained for the $E_r/M/c$ system, i.e., where the interarrival time with mean $1/\lambda$ is distributed as an r-state Erlang distribution with parameter $\theta = r\lambda$. The service time is exponentially distributed with parameter μ. The system load is given by $\rho = \lambda/(c\mu)$.

First, we compute $P[C_L \geq n|t]$ as

$$\begin{aligned} P[C_L \geq n|t] &= \mathcal{L}^{-1}\left\{\frac{1}{s}\left(\frac{\theta}{\theta+s}\right)^{rn}\right\} \\ &= \int_0^t \mathcal{L}^{-1}\left\{\left(\frac{\theta}{\theta+s}\right)^{rn}\right\} dt \\ &= 1 - \int_t^\infty \frac{\theta(\theta t)^{rn-1} e^{-\theta t}}{(rn-1)!} dt \\ &= 1 - \sum_{k=0}^{rn-1} \frac{(\theta t)^k}{k!} e^{-\theta t}, \end{aligned}$$

from which the unconditional survivor function follows readily as

$$\begin{aligned} P[C_L \geq n] &= c\mu \int_0^\infty \left[1 - \sum_{k=0}^{rn-1} \frac{(\theta t)^k}{k!}\right] e^{-c\mu t} \, dt \\ &= c\mu \int_0^\infty e^{-c\mu t} \, dt - c\mu \sum_{k=0}^{rn-1} \int_0^\infty \frac{(\theta t)^k}{k!} e^{-(\theta + c\mu)t} \, dt \\ &= 1 - c\mu \sum_{k=0}^{rn-1} \frac{\theta^k}{k!} \frac{k!}{(\theta + c\mu)^{k+1}} \\ &= 1 - \frac{c\mu}{\theta + c\mu} \sum_{k=0}^{rn-1} \left(\frac{\theta}{\theta + c\mu}\right)^k \\ &= 1 - \frac{c\mu}{\theta + c\mu} \frac{1 - \left(\frac{\theta}{\theta + c\mu}\right)^{rn}}{1 - \left(\frac{\theta}{\theta + c\mu}\right)} \\ &= \left(\frac{\theta}{\theta + c\mu}\right)^{rn} = \left(\frac{r}{r + c\mu/\lambda}\right)^{rn} \\ &= \left(\frac{r}{r + 1/\rho}\right)^{rn} \doteq \alpha^n \end{aligned}$$

where we introduce the definition in the last line as shorthand for the remainder of this section. The result also follows directly from Eq. (13).

Sometimes, it is convenient to have the probability mass function available as well:

$$\begin{aligned} P[C_L = n] &= P[C_L < n+1] - P[C_L < n] \\ &= 1 - \alpha^{n+1} - (1 - \alpha^n) \\ &= \alpha^n (1 - \alpha) \end{aligned}$$

In particular, $P[C_L = 0] = 1 - \alpha$.

$E[C_L]$ and $E[C_C]$ are computed next:

$$\begin{aligned} E[C_L] &= \sum_{n=1}^{\infty} P[C_L \geq n] = \sum_{n=1}^{\infty} \alpha^n = \frac{\alpha}{1-\alpha} \\ E[C_C] &= \frac{1}{\alpha} \frac{\alpha}{1-\alpha} = \frac{1}{1-\alpha} \end{aligned}$$

For Poisson arrivals, i.e., $r = 1$:

$$\begin{aligned} P[W < h] &= \frac{1 - \rho e^{-\nu h}}{1 - \rho^2 e^{-\nu h}} \quad c = 1 \\ P[C_L \geq n] &= \left(\frac{\lambda}{\lambda + c\mu} \right)^n \\ P[C_L = n] &= \frac{c\mu}{\lambda + c\mu} \left(\frac{\lambda}{\lambda + c\mu} \right)^n \\ P[C_C = n] &= \frac{c\mu}{\lambda + c\mu} \left(\frac{\lambda}{\lambda + c\mu} \right)^{n-1} \\ E[C_L] &= \rho \\ E[C_C] &= \rho + 1 \\ r &= 1 - \frac{1}{E[C_C]} = \frac{\rho}{\rho + 1} = \frac{\lambda}{\lambda + \mu} \end{aligned}$$

Note that the loss probability result [43] is only valid for the single-server case. The results for the loss period also follow naturally from the well-known relation [24] $V(z) = B^*(\lambda - \lambda z)$, where $V(z)$ is the z-transform of the probability distribution of the number of Poisson arrivals with rate λ during a time interval with Laplace transform $B^*(\cdot)$. Also note the geometric form of the loss run distributions.

It is instructive to compare $E[C_C]$ for the $M/M/1$ queue with ($= 1 + \rho$) and without discarding ($= (1 + \rho)/(1 - \rho)$). Particularly for high values of h, the loss probabilities for the systems with and without discarding (given by $1 - \rho e^{-\nu h}$) are virtually identical, while the queue without discarding will have significantly more losses without interruption than the queue with discarding. For example, for a load of $\rho = 0.8$, the value of $E[C_L]$ for the $M/M/1$ queue without discarding is five times higher than for the same queue with discarding.

The case of deterministic (periodic) arrivals, i.e., the $D/M/c$ system, emerges as r tends to infinity, where

$$\lim_{r \to \infty} \alpha = \lim_{r \to \infty} \left[\left(1 + \frac{\mu/\lambda}{r} \right)^r \right]^{-1} = e^{-\mu/\lambda}$$

This result follows also directly from Eq. (13) since $P[C_L = n|t] = 1$ for $n/\lambda \leq x < (n+1)\lambda$, zero otherwise. Then,

$$\begin{aligned} P[C_L = n] &= \mu \int_{n/\lambda}^{(n+1)/\lambda} e^{-\mu t}\, dt \\ &= e^{-n\mu/\lambda} \left(1 - e^{-\mu/\lambda} \right), \; n \geq 0. \end{aligned}$$

4.2 Discrete Time

As in continuous time, discarding excessively delayed customers before they enter service simplifies the performance evaluation. Regardless of the batch arrival at some instant n, at $(n+1)$, no more than $h-1$ customers will be in the system, allowing the first arrival at $(n+1)^+$ to complete on time. Using the terminology of Section 3.2, composite and simple loss periods are identical. Furthermore, loss periods have the same length as the initial jump J.

Again, the noloss period is unaffected by whether or not customers are discarded.

5 Buffer Overflow in $G/M/c/K$ Queues

After having investigated systems with constraints on waiting time in some detail, let us now consider Markovian continuous-time queues with finite buffer (of size K, including the customer in service). For $G/M/c/K$ systems, it turns out to be relatively easy to derive measures for loss correlation and noloss periods.

For the $G/M/c/K$ queue, a Markov chain can be embedded at (just before) arrival instants. The conditional loss probability r can be read directly from the transition probability matrix P as

$$r = P_{K,K}.$$

For the $G/M/1/K$ queue, P is of dimension $K+1$ by $K+1$ [44, p. 305]:

$$P = \begin{bmatrix} 1-b_0 & b_0 & 0 & \dots & 0 \\ 1-\sum_{i=0}^{1} b_i & b_1 & b_0 & \dots & 0 \\ 1-\sum_{i=0}^{2} b_i & b_2 & b_1 & \dots & 0 \\ \dots & & & & \\ 1-\sum_{i=0}^{K-1} b_i & b_{K-1} & b_{K-2} & \dots & b_0 \\ 1-\sum_{i=0}^{K-1} b_i & b_{K-1} & b_{K-2} & \dots & b_0 \end{bmatrix}.$$

Note that the two last rows are identical. Here, b_n represents the probability of n service completions during an interarrival time.

$$b_n = \int_0^\infty \frac{e^{-\mu t}(\mu t)^n}{n!}\, dA(t)$$

Thus, with $A(t)$ as the interarrival time distribution, the conditional loss probability becomes

$$r = b_0 = \int_0^\infty e^{-\mu t}\, dA(t) = a^*(\mu), \tag{15}$$

where $a^*(\mu)$ is the Laplace transform of the interarrival time density evaluated at $s = \mu$. For the multi-server case where $K > c$, $P_{K,K}$ is given by [44, p. 313] [24, p. 246]. Eq. (6.12) in [24] shows that the conditional loss probability

$$r = \beta_0 = \int_0^\infty e^{-c\mu t}\, dA(t) = a^*(c\mu),$$

where β_n is the probability of serving n customers during an interarrival time given that all c servers remain busy during this interval.

By now, the similarity to the results in Section 4 should have become apparent. The stochastic process is exactly the same as for a queue with bounded wait and exponentially distributed initial jump. Thus, all the results in Section 4 carry over directly.

For the $M/M/1/K$ queue, the conditional loss probability can also be deduced by noting that it is the probability that the next event is an arrival, given that the queue is full. That probability is given by $\lambda/(\lambda+\mu)$ and is independent of K.

The computation of the distribution of the number of consecutive customers without buffer overflow (called *success runs*) is slightly more involved. Using Eq. (6) to compute the first passage times of the embedded DTMC with matrix P, the pmf of the success runs, s_n, is given by

$$s_n = \sum_{i=0}^{K-1} f_{i,K}^{(n)} \frac{P_{K,i}}{1-P_{K,K}} \quad \text{for } n \geq 1.$$

The fraction describes the probability that the first admitted customer after a loss run sees state i. Note that the distribution is in general not geometrically distributed as assumed in the Gilbert error model.

5.1 Effect of Service Order and Buffer Management on Buffer Loss Correlation

The effect of buffer management on loss probabilities on delay and blocking probability of marked and unmarked traffic was considered by [45,46].

Both papers hint at the issue of loss correlation, but do not elaborate further. Lucantoni and Parekh state that I/O discarding may "mitigate the 'hysteresis effect', i.e., the possibility that the time spent on serving the marked traffic can adversely affect the performance of the unmarked traffic that might follow in a burst." Yin and Hluchyj state that "since the loss at the destination tends to occur to successive packets, which may seriously degrade voice fidelity, we need to avoid or reduce this loss."

For the $M/D/1/1$ system, all customers that arrive during a service period are lost. Thus,

$$P[C_C = n] = \frac{1}{1-a_0} a_n = \frac{1}{1-e^{-\rho}} \frac{\rho^n}{n!} e^{-\rho}$$

where a_n designates the probability of n arrivals during a service period. The expectation follows from the definition:

$$\begin{aligned} E[C_C = n] &= \frac{1}{e^{\rho}-1} \sum_{n=0}^{\infty} n \frac{\rho^n}{n!} \\ &= \frac{\rho}{1-e^{-\rho}} \end{aligned}$$

As for the discrete-time case investigated in Section 6, the influence of the scheduling and buffer management policy on loss runs was studied, with similar results (and, unfortunately, again no proofs for the general case). The distribution of the loss run lengths appears to be independent of the scheduling discipline and on whether packets are dropped from the front or the rear of the queue.

6 Buffer Overflow in Single-Stream Discrete-Time Queues

6.1 *First-Come*, First-Served

In this section, we will derive properties of the loss correlation for a class of discrete-time queues with restricted buffer size. As our queueing model, we consider a FIFO single-server discrete-time queue where arrivals occur in i.i.d. batches of general distribution with mean batch size λ. Each arrival requires exactly one unit of service. For short, we will refer to this system as $D^{[G]}/D/1/K$ [47]. Let K denote the system size, that is, the buffer capacity plus one. Arrivals that do not find space are rejected, but once a customer enters the system, it will be served. This buffer policy will be referred to as rear dropping in section 6.2. Arbitrarily, arrivals are fixed to occur at the beginning of a time slot and departures at the end, creating, in Hunter's terminology [38], an early arrival system. This model is used to represent the output queue of a fast packet switch, for example [48].

The waiting time and loss probability for this model have been analyzed by a number of authors [49, 47, 36, 48, 50, 51, 1]. For Poisson-distributed batches, Birdsall *et al.* [49, p. 392] computes the conditional probability of a run of exactly n slots in which one or more arrivals are rejected given that an arrival was rejected in the preceding slot. We will call it $P[C_R = n]$. The quantity is seen to be the product of the probability that two or more arrivals occur during the next $n-1$ slots and the probability of zero or one arrivals occurs in the terminating interval.

$$P[C_R = n] = e^{-\lambda}(1+\lambda)\left[1-(1+\lambda)e^{-\lambda}\right]^{n-1}.$$

Birdsall *et al.* [49, Eq. (11)] also compute the probability that exactly d arrivals are rejected in the next slot, provided that one or more was rejected in the previous slot. Their result is related to a relation we will derive later (Eq. (18)).

We define Q_k to be the event that the first customer in an arriving batch sees k customers already in the system and q_k to be the probability of that event. For general batch size probability mass function (pmf) a_k, the q_k's are described by the following recursive equations [48]:

$$\begin{aligned}
q_1 &= \frac{q_0}{a_0}(1-a_0-a_1) \\
q_n &= \frac{1}{a_0}\left[q_{n-1} - \sum_{k=1}^{n} a_k q_{n-k}\right], \quad 2 \le n < K \\
q_0 &= 1 - \sum_{n=1}^{K-1} q_n = \left[1 + \sum_{n=1}^{K-1} q_n/q_0\right]^{-1}
\end{aligned}$$

The probability that a packet joins the queue, $P[J]$, is given by

$$P[J] = \frac{1-q_0 a_0}{\lambda},$$

since $1 - a_0 q_0$ is the normalized throughput. Note that q_n, $n = 1, 2, \ldots$, depends only through the factor q_0 on the buffer size, i.e., q_n/q_0 is independent of the buffer size [38, p. 236].

For later use, let us compute the probability $P[S]$ that one or more losses occurs during a randomly selected time slot. By conditioning on the system state probability q_k, we can write

$$P[S] = \sum_{k=0}^{K-1} q_k P[S|Q_k] = \sum_{k=0}^{K-1} q_k \sum_{j=K-k+1}^{\infty} a_j = \sum_{k=0}^{K-1} q_k \left(1 - \sum_{j=0}^{K-k} a_j\right)$$

Let the random variable C_C be the number of consecutively lost customers. The distribution of loss run lengths follows readily,

$$\begin{aligned} P[C_C = n] &= \sum_{s=1}^{K} P[s \text{ spaces available} \mid \text{loss occurs in slot}] \cdot a_{n+s}, \\ &= \frac{1}{P[S]} \sum_{s=1}^{K} q_{K-s} a_{n+s} \end{aligned} \qquad (16)$$

as the number of arrivals in a slot is independent of the system state.

The expected number of consecutive losses can be computed from Eq. (16) or directly by observing that loss runs are limited to a single slot since the first customer in a batch will always be admitted. The expected loss run length is simply the expected number of customers lost per slot, given that a loss did occur in that slot. The expected number of customers lost in a slot is given by $\lambda(1 - P[J])$, so that

$$E[C_C] = E[\text{losses per slot} \mid \text{loss occurs in slot}] = \frac{\lambda(1 - P[J])}{P[S]} = \frac{\lambda - 1 + q_0 a_0}{P[S]}. \qquad (17)$$

Numerical computations show that influence of K on the distribution of C_C is very small (see Table 8). The table also shows that $E[C_C]$ is roughly a linear function of λ.

Losses that occur when a batch arrives to a full system, i.e., a system with only one available buffer space, are independent of the system size and can thus be used to approximate the distribution of C_C quite accurately. For $K = 1$, n consecutive losses occur if and only if $n + 1$ packets arrive during the slot duration, conditioned on the fact that two or more packets arrived. Thus,

$$P[C_C = n] \approx P[C_C = n | K = 1] = \frac{a_{n+1}}{1 - a_0 - a_1} \qquad (18)$$

$$\begin{aligned} E[C_C] &\approx \frac{1}{1 - a_0 - a_1} \sum_{n=1}^{\infty} n a_{n+1} \\ &= \frac{1}{1 - a_0 - a_1} \left[\sum_{n=2}^{\infty} n a_n - \sum_{n=2}^{\infty} a_n \right] \\ &= \frac{1}{1 - a_0 - a_1} [\lambda - a_1 - (1 - a_0 - a_1)] \\ &= \frac{1}{1 - a_0 - a_1} [\lambda - 1 + a_0]. \end{aligned} \qquad (19)$$

As can be seen readily, the above agrees with Eq. (16) and Eq. (17) for $K = 1$.

Also, by the memorylessness property of the geometric distribution, Eq. (18) and Eq. (19) hold exactly for geometrically distributed batches with parameter p and evaluates to

$$\begin{aligned} P[C_C = n] &= \frac{p(1-p)^{n+1}}{1 - p - p(1-p)} = p(1-p)^{n-1} \\ E[C_C] &= \frac{1}{p} = 1 + \lambda \end{aligned}$$

(Regardless of what system occupancy an arriving batch sees, the packets left over after the system is filled are still geometrically distributed.)

a_k	K	$\lambda = 0.5$	$\lambda = 0.8$	$\lambda = 1$	$\lambda = 1.5$
Poisson	1	1.18100	1.30397	1.39221	1.63540
	2	1.15707	1.27511	1.36201	1.60574
	3	1.15707	1.27158	1.35911	1.60403
	4	1.15226	1.27153	1.35910	1.60403
	5	1.15238	1.27159	1.35914	1.60404
	6	1.15242	1.27160	1.35914	1.60404
	∞	1.15242	1.27160	1.35914	1.60404
Geo	any	1.5	1.8	2.0	2.5

Table 8: Expected loss run length ($E[C_C]$) for $D^{[G]}/D/1/K$ system

6.2 Influence of Service and Buffer Policies

It is natural to ask how the burstiness of losses is affected by different scheduling and buffer management policies. As scheduling policies, FIFO (first-in, first-out) and non-preemptive LIFO (last-in, first-out) are investigated. For either policy, we can either discard arriving packets if the buffer is full (*rear discarding*) or push out those packets that have been in the buffer the longest (*front discarding*). Note that this dropping policy is independent of the service policy. From our viewpoint, LIFO serves the packet at the rear of the queue. Obviously, only systems with K greater than one show any difference in behavior.

The analysis of all but FIFO with rear discarding appears to be difficult. Let us briefly discuss the behavior of FIFO and LIFO, each either with front or rear discarding.

FIFO with rear discarding: The first customer in an arriving batch always enters the buffer and will be served eventually. Thus, a loss run never crosses batch boundaries.

FIFO with front discarding: Here, a batch can be completely lost if it partially fills the buffer and gets pushed out by the next arriving batch. However, if a batch was completely lost, the succeeding batch will have at least one of its members transmitted since it must have "pushed through" until the head of the buffer.

LIFO with rear discarding: The first packet in a batch will always occupy the one empty buffer space and be served in the next slot. Again, loss runs are interrupted by packet boundaries.

LIFO with front discarding: A run of losses can consist of at most than one less than arrive in a single batch since the last customer in the batch will be served during the next slot. A loss run never straddles batch boundaries.

For all four systems, indeed over all work-conserving disciplines, the queue state distribution (and, thus, the loss probability) are the same [52, 53, 46]. The mean waiting time results favoring front dropping agree with those of [46] for general queueing systems. Clare and Rubin [53] show that the minimum mean waiting time for non-lost packets is obtained using LCFS with front dropping (referred to as preemptive buffering in [53]).

For all systems, a batch arrival causes the same number of lost packets. If there are q packets in the buffer ($0 \geq q < K$) and a arrive in a batch, $[(q+a)-K]^+$ will be lost.

For rear dropping, at least the first packet in the batch will always enter the system, interrupting any loss run in progress. Thus, we have:

Lemma 6 *The same packets (as identified by their order of generation) will be dropped for all work-conserving service policies and rear dropping.*

Here, we visualize packets within the same batch generated sequentially in the interval $(t, t+0)$.

Lemma 7 *The distributions of loss runs for FIFO with rear and front dropping are the same.*

PROOF We number packets in the order of generation and arbitrarily within a batch so that they are served in order of increasing sequence number. We note first that the buffer for front dropping always contains an uninterrupted sequence of packets. Assume that the buffer contains an uninterrupted sequence. A service completion removes the first element of the sequence, without creating an interruption. A batch arrival that is accepted completely or will also not create a gap. A batch that pushes out some of the customers likewise will continue the sequence. Finally, a batch that pushes out all customers certainly does not create a gap. Note that this property does not hold for rear dropping, as the example of two successive batch arrivals with overflows demonstrates.

As pointed out before, the loss runs for rear dropping are confined to a single arriving batch and comprise $[q+a-K]^+$ packets. For front dropping, the losses are made up of packets already in the buffer and possibly the first part of the arriving batch. By the sequence property shown in the preceding paragraph, all these form a single loss run (again of length $[q+a-K]^+$), which is terminated by serving the next customer. Thus, while the identity of packets dropped may differ, the lengths of the loss runs are indeed the same for both policies. □

The issue is more complicated for front dropping and general service disciplines. A number of simulation experiments were performed to investigate the behavior of the four combinations of service and buffer policies, with results collected in Table 9. (The rows labeled "theory" correspond to the values for FIFO and rear dropping, computed as in discussed in the previous section.) These experiments suggest the following conjecture:

Conjecture 1 *The distribution of loss runs is the same for all combinations of FIFO, LIFO and rear and front dropping. The sample path of loss run lengths is the same for all systems except for LIFO with front dropping.*

arrivals	service	dropping	$E[W]$	$1-P[J]$	$E[C_C]$
geometric	theory		1.985	0.3839	2.500
	FIFO	rear	1.984...1.987	0.3833...0.3840	2.496...2.502
		front	1.349...1.351	0.3833...0.3840	2.496...2.502
	LIFO	rear	1.984...1.987	0.3833...0.3840	2.496...2.502
		front	0.867...0.869	0.3833...0.3840	2.495...2.500
Poisson	theory		2.417	0.341	1.604
	FIFO	rear	2.416...2.418	0.340... 0.341	1.602...1.604
		front	1.582...1.584	0.340... 0.341	1.602...1.604
	LIFO	rear	2.416...2.418	0.340... 0.341	1.602...1.604
		front	0.549...0.553	0.340... 0.341	1.603...1.604

Table 9: Performance measures for geometric and Poisson arrivals, $\lambda = 1.5$, $K = 4$, 90% confidence intervals

Let us briefly outline a possible approach to a more formal analysis. We focus on one batch and construct a discrete-time chain with the state

$$(i,j) = (\text{left in buffer, consecutive losses in batch}) \qquad i \in [1, K-1];\ j \in [0, \infty)$$

The initial state, that is, the state immediately after the arrival of the batch of interest, is determined by the batch size and the system state and should be computable. The states $(0,j)$ are absorbing.

The transition matrix is given by:

$$\begin{array}{lcll}
(i,j) \rightarrow (i,j) & = & a_1 & \forall i,j; \\
(i,j) \rightarrow (i-1,j) & = & a_0 & j > 0; \\
(i,j) \rightarrow (i-1,j+1) & = & a_2 & i,j > 0; \\
(i,j) \rightarrow (i-2,j+2) & = & a_3 & i > 1, j > 0; \\
\ldots & & & \\
(i,j) \rightarrow (i-k,j+k) & = & a_{k+1} & i > k, j > 0; \\
(i,j) \rightarrow (0,i+j) & = & \sum_{k=i+1}^{\infty} a_k & i,j > 0; \\
(0,j) \rightarrow (0,j) & = & 1 &
\end{array}$$

All other entries are zero.

The probability of j losses given initial state (i_0, j_0) is the probability distribution of being absorbed in state $(0,j)$.

Intuitively, random dropping, i.e., selecting a random packet from among those already in the buffer, should reduce the loss run lengths, particularly for large buffers. However, this policy appears to be difficult to implement for high-speed networks. Simulation results for geometric arrivals support this result, as shown in Table 10 as a front or rear dropping would result in an average loss run length of 2.5 for $\lambda = 1.5$ and 1.8 for $\lambda = 0.8$.

	$\lambda = 1.5$			$\lambda = 0.8$		
K	$E[W]$	$1 - P[J]$	$E[C_C]$	$E[W]$	$1 - P[J]$	$E[C_C]$
3	0.913	0.4148	2.304	0.750	0.1731	1.669
4	1.509	0.3833	2.175	1.120	0.1213	1.573
6	2.842	0.3534	2.008	1.771	0.0659	1.451
8	4.304	0.3415	1.915	2.300	0.0386	1.382
10	5.851	0.3365	1.849	2.721	0.0234	1.328
15	9.884	0.3331	1.755	3.399	0.0071	1.238

Table 10: Effect of random discarding for system with geometrically distributed batch arrivals

It should be noted that average run lengths exceed a value of two only for extremely heavy load. Thus, on average, reconstruction algorithms that can cope with two lost packets in a row should be sufficient.

7 Summary and Future Work

In the preceding sections, we have developed probabilistic measures for the behavior of losses in single-server queues commonly used in analyzing high-speed networks and fast packet switches. These measures should be useful in the design and test of packet recovery systems, which face a much harder task than predicted by the optimistic assumption of independent losses.

We found that for certain important queues, the distribution of the loss period is independent of the threshold value used, while for all discrete-time batch distributions investigated the threshold value has very little influence on the loss period. It remains to be seen whether there exists a certain value of h above which the initial jump (and, hence, the loss period) changes no further with increases in h.

It seems natural to extend these results to finite queues in order to analyze periods of buffer overflow, as was done in [19] for a packet voice arrival process.

For $G/M/1$ and $D^{[\mathrm{Geo}]}/D/1$ queues, a busy period can be regarded as a special case of a loss period. Thus, computation of the distribution of busy periods is of particular interest in studying loss phenomena. Our computation of the busy period using combinatorial arguments applied strictly only to geometric batches, but it might provide a readily computable approximation for other distributions.

It is anticipated that the study of correlated arrival processes and non-FIFO service disciplines will require significantly more sophisticated techniques than those found sufficient here.

8 Notation

α	loss probability
ρ	system load
ϕ	number of busy (loss) periods per composite busy (loss) period
ν	population parameter of the binomial distribution
a_k	P[batch size or work per slot = k]
c	number of servers
h	waiting time threshold
$p = 1 - q$	parameters of the geometric and binomial distribution
A	batch sizes
B	busy period
$\tilde{B}$	composite busy period
C_B	customers per busy period
C_L	customers in a loss period
C_C	number of customers lost consecutively
C_N	customers per noloss period
I	idle period
L	loss period
$\tilde{L}$	composite loss period
N	noloss period
N'	noloss period of one or more slots
$\mathbf{P}$	state transition matrix for DTMC
$f_X(x)$	pdf of random variable X
$X(z)$	z-transform of random variable X
$E[X$	] expectation of random variable X
$P[\cdot$	] probability of event

A Proof of Lemma 5

Proof Note that since the geometric distribution has infinite support, $\nu = \infty$. We first need to establish the pmf of the system occupancy at batch arrival instants, V. A simple partial fraction expansion of the generating function $V(z)$ (see [38, p. 278]), using $A(z) = p/(1-qz)$,

$$V(z) = \frac{(1-\rho)(1-z)}{A(z)-z} = 1 + \frac{(1-\rho)\rho}{1-\rho z},$$

yields the pmf

$$P[V = v] = (1-\rho)[\delta(v) + \rho^{v+1}],$$

where $\delta(\cdot)$ denotes the Kronecker delta. Since the cumulative distribution function evaluates to

$$P[V \le h] = 1 - \rho^{h+2},$$

the conditional (scaled) density can be written down as

$$P[V_h = v] = \begin{cases} \frac{1-\rho^2}{1-\rho^{h+2}} & \text{for } v = 0 \\ \frac{(1-\rho)\rho^{v+1}}{1-\rho^{h+2}} & \text{for } v \in [1,h] \\ 0 & \text{otherwise} \end{cases}$$

Since

$$P[V_h > h-a] = 1 - \frac{P[V \le h-a]}{P[V \le h]} = \frac{\rho^{h+2}(\rho^{-a}-1)}{1-\rho^{h+2}},$$

the denominator of the expression for $P[J = j]$ becomes

$$\sum_{a=1}^{h} \frac{\rho^{h+2}}{1-\rho^{h+2}} \left(\frac{1}{\rho^a} - 1\right) \frac{\rho^a}{(\rho+1)^{a+1}} + \left(\frac{\rho}{1+\rho}\right)^{h+1} = \frac{\rho^{h+1}(1-\rho)}{1-\rho^{h+2}}.$$

The numerator is expanded into

$$\frac{1-\rho^2}{1-\rho^{h+2}} \frac{\rho^{h+j}}{(1+\rho)^{h+j+1}} + \frac{1-\rho}{1-\rho^{h+2}} \sum_{a=1}^{h} \rho^{a+1} \frac{\rho^{h+j-a}}{(\rho+1)^{h+j-a+1}},$$

which simplifies to

$$\frac{\rho^{j-1}}{(1+\rho)^j} \frac{(1-\rho)\rho^{h+1}}{1-\rho^{h+2}},$$

from which the result follows immediately. The derivation for $\tilde{J}$ proceeds in a similar fashion. □

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