Embeddings de Bruijn and Shuffle-Exchange Graphs in Five Pages

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Abstract

We present algorithms for embedding de Bruijn and shuffle-exchange graphs in books of 5 pages, with cumulative pagewidth $2^n - 1 + (2/3)(2^{n-1} - 2 + (n \bmod 2))$ and $2^{n-1} + (1/3)(2^{n-1} - 2 + (n \bmod 2))$, respectively. These are the first nontrivial bounds on the pagenumber of de Bruijn and shuffle-exchange graphs.

1 Introduction

The *book* of *thickness* p is a set of p half-planes, called *pages*, sharing a common boundary, called the *spine*. A *p-page book-embedding* of a directed graph $G = (V, A)$ is a drawing of G in a book of thickness p so that the nodes of G reside on the spine of the book, while each arc of G is drawn in exactly one page, in such a way that no arcs of G cross. Arc directions are immaterial for book-embedding—the graphs we consider are directed just for clarity of presentation. The *pagenumber* of a graph G is the thickness of the smallest (in number of pages) book into which G can be embedded. The *width* of a page in a book-embedding is its maximum cutwidth. The *cumulative pagewidth* of a book-embedding is the sum of the widths of all pages.

The book-embedding problem appears in several formulations and has various origins (cf. [4]); within the realm of parallel architectures it is relevant for the design of fault-tolerant processor arrays of identical processing elements. The Diogenes approach to the design of such arrays [3, 12] assumes that processing elements are laid in a logical line, while some number of "bundles" of wires runs in parallel with the line. The configuration of the fault-free processors into the desired topology is effected by a network of switches connecting processors to the bundles of wires. The switching mechanism behaves as a *stack*, to which wires are entered or from which they are removed as the linear array of processors is scanned during the configuration process. The most significant cost in a Diogenes layout [3] of an array is the *number* of bundles of wires, organized in hardware stacks, required to configure it. A secondary cost is the total *width* of these bundles. Therefore, a Diogenes design mandates finding a linearization of nodes of the target array such that the edges of its interconnection network can be laid out in few small stacks. This problem, however, is *equivalent to finding an efficient book-embedding* of the graph underlying the interconnection network. The pagenumber of a graph equals the required number of stacks, while the cumulative pagewidth equals the required stackwidth; it is, therefore, desirable to achieve embeddings of important graph families, with optimal pagenumber and pagewidth.

This book-embedding problem is very hard in general: we know [5] that for a given linearization of the nodes of a graph G and a given integer k, the problem of deciding if the linearization admits a k-page book-embedding of G is NP-complete.

At present, book-embeddings of several graph families are known, though it is less often known whether these embeddings are optimal. Exemplifying this fact are the family of complete graphs, whose pagenumber is determined exactly [1], and the family of complete bipartite graphs, where it is not yet established whether the known [11] book-embedding is optimal. Very few efficient algorithms exist for achieving book-embeddings of arbitrary graphs [16] [8].

Optimal (within constant factors, or even absolutely) book-embeddings have been constructed for almost all seriously proposed interconnection networks, including trees [4], grids [4], X-trees [4], butterfly-like networks [6] and hypercubes [4]. Yet, *no efficient book-embeddings have been found so far for shuffle-like networks*, another very popular class of interconnection networks represented by de Bruijn graphs and shuffle-exchange graphs. The very weak upper bound for the much broader class of bounded-degree graphs applies, but is nonconstructive, so it follows from [4] or [10] that there exist book-embeddings of N-node de Bruijn (shuffle-exchange) graphs with pagenumber $O(\sqrt{N})$. This paper presents a construction for embedding these graphs in 5 pages. The best known lower bound on the pagenumber of de Bruijn and shuffle-exchange graphs remains 3, which follows from the graphs' nonplanarity.

It may be interesting to compare our results about shuffle-like networks with the results known about butterfly-like networks. Both families are bounded-degree hypercube-derivative networks; their computational power is a frequent topic of comparative studies (cf. [13]). Both families have small pagenumber: for butterfly-like graphs it is 3, in contrast to hypercubes themselves, whose pagenumber is unbounded (logarithmic) in the size of the network.

We remark that Diogenes can utilize a switching mechanism alternative to stacks of wires; this mechanism consists of *queues* of wires [12], so the success of Diogenes design with queues depends on finding efficient *queue layouts* of graphs. For practically all popular networks the queuenumber is known [9]; for both butterfly-like and shuffle-like graphs it is 2.

In Section 2, we define the bidendral decomposition of de Bruijn graphs, and adduce its relevant properties. In Section 3 we exploit the decomposition for developing the book-embedding of de Bruijn graphs. We start by embedding separately the partial subgraphs produced by the decomposition; then we compose the partial embeddings into an efficient embedding of the whole graph. In Section 4, we adapt the embedding to shuffle-exchange graphs.

Notation: We denote by Z_2 the set $\{0,1\}$ and we use letters from the beginning of Greek alphabet $(\alpha, \beta, \gamma, \ldots)$ as variables ranging over Z_2. For integer $k \geq 0$, we denote by Z_2^k the set of all strings of length k over Z_2, and we let lowercase letters of Roman alphabet $(a, b, \ldots, x, y, z, \ldots)$ range over Z_2^k; $Z_2^0 =_{\text{def}} \{\lambda\}$ is the singleton consisting of the designated *empty string* λ. For $x \in Z_2^k$, $|x| =_{\text{def}} k$ is the length of string x. We define $\alpha^k \in Z_2^k$ as the length-k string whose all elements are equal to α. Let $\overline{\beta} =_{\text{def}} 1-\beta$ and for $y \in Z_2^k$, let $\overline{\beta y} = \overline{\beta}\overline{y}$. The length-$(k-1)$ suffix of a string $y \in Z_2^k$ is denoted by $\sigma(y)$, so $\sigma(\beta y) =_{\text{def}} y$.

2 Bidendral Decomposition of de Bruijn Graphs

We commence by defining the graphs of interest—de Bruijn graphs and complete binary trees.

The *order-n de Bruijn graph* $D(n)$ [2] has node-set Z_2^n; given $y \in Z_2^{n-1}$, two arcs are incident out of each node βy: the *shuffle* arc that leads to $y\beta$ and the *shuffle-exchange* arc that leads to $y\overline{\beta}$. Let $\mathcal{S}(\beta y) =_{\rm def} y\beta$ and $\mathcal{E}(\beta y) =_{\rm def} y\overline{\beta}$. Consequently, $\mathcal{S}(\beta y) = \sigma(\beta y)\beta$ and $\mathcal{E}(\beta y) = \sigma(\beta y)\overline{\beta}$.

The *complete binary tree* $T(h)$ of *height* h has node-set $\bigcup_{0\le k\le h} Z_2^k$ and arcs leading each $y \in Z_2^k$, $0 \le k < h$, to its *children* $y0$ and $y1$. The *root* of the tree $T(h)$ is the empty string λ, the *leaves* of $T(h)$ are all nodes $y \in Z_2^h$.

Levels in the tree $T(h)$ are defined naturally: the 2^k nodes $x \in Z_2^k$, for $0 \le k \le h$, reside at level $h - k$. So, the root is the only node at level h; the leaves are at level 0.

Within the tree $T(h)$, we define *tree-order* on nodes of $T(h)$ as the lexicographic order of nodes as binary strings: node x precedes node y if either $|x| < |y|$, or $|x| = |y|$ and $x < y$, where the latter order is defined on the integers represented in binary by x and y.

We prepare for our embedding of the de Bruijn graph $D(n)$ by identifying two complete binary trees in $D(n)$ and by determining the structure of partial subgraphs induced by the arcs not contained in these trees.

We partition the nodes of $D(n)$ into four sets. The first set is a singleton containing node 0^n; the second set consists of all nodes, other than 0^n, which start with 0. Analogously, the third set is a singleton containing node 1^n; the fourth set consists of all nodes, other than 1^n, which start with 1. So, for each of the two values of $\gamma \in \{0,1\}$, the node-set of $D(n)$ has the following two components:

$$\begin{array}{rcll} S_\gamma & = & \{s_\gamma\} = \{\gamma^n\}, & \gamma \in \{0,1\} \\ V_\gamma & = & \{\gamma y \mid y \in Z_2^{n-1} \setminus \{\gamma^{n-1}\}\}, & \gamma \in \{0,1\} \end{array}$$

Call the elements of V_0 and V_1 *tree nodes*, and call s_0 and s_1 *singular nodes*. This decomposition induces a partition of the arc-set of $D(n)$ into six subsets, so that $D(n)$ is represented as six arc-disjoint partial subgraphs. So, for each of the two values of $\gamma \in \{0,1\}$, $D(n)$ contains the following three subgraphs:

Trees: $T_\gamma = (V_\gamma, A_\gamma)$, where $\gamma \in \{0,1\}$, is the subgraph of $D(n)$ induced on V_γ; it is isomorphic to the complete binary tree $T(n-2)$. The isomorphism Φ_γ of the node-set of $T(n-2)$ to V_γ is defined as follows: For $x \in Z_2^k$, $k \le n-2$,

$$\Phi_0(x) = 0^{n-2-k}01x$$

$$\Phi_1(x) = 1^{n-2-k}10\overline{x}$$

By definition, Φ_γ is injective; it is also surjective by equal cardinality of its domain and its range ($|V_\gamma| = |\bigcup_{0\leq k\leq n-2} Z_2^k| = 2^{n-1}-1$). To show that Φ_γ preserves arcs, note that

$$\Phi_\gamma(x0) = \sigma(\Phi_\gamma(x))\gamma = \mathcal{S}(\Phi_\gamma(x))$$

$$\Phi_\gamma(x1) = \sigma(\Phi_\gamma(x))\overline{\gamma} = \mathcal{E}(\Phi_\gamma(x))$$

whenever $x \in Z_2^k$, $k < n-2$. Call the two arc-sets A_0 and A_1 the *tree arcs* of $D(n)$. Since $T(n-2)$ has $(2^{n-1}-2)$ arcs, there are $2 \times (2^{n-1}-2) = 2^n - 4$ tree arcs in $A_0 \cup A_1$.

Leaf subgraphs: Let V_γ^L, where $\gamma \in \{0,1\}$, be the set of *leaves* of the tree T_γ. Then the graph $L_\gamma = (V_\gamma^L \cup V_{\overline{\gamma}} \cup S_{\overline{\gamma}}, A_\gamma^L)$ has node-set consisting of the leaves of tree T_γ, all nodes of tree $T_{\overline{\gamma}}$, and singular node $s_{\overline{\gamma}}$. For each $x \in V_\gamma^L$, there is $x' \in Z_2^{n-2}$ such that $x = \gamma\overline{\gamma}x'$, so $\mathcal{S}(x) = \overline{\gamma}x'\gamma \in (V_{\overline{\gamma}} \cup S_{\overline{\gamma}})$ and $\mathcal{E}(x) = \overline{\gamma}x'\overline{\gamma} \in (V_{\overline{\gamma}} \cup S_{\overline{\gamma}})$. We define the arc-set A_γ^L as the arcs incident out of leaves of tree T_γ. As just noted, each such arc leads a node in V_γ^L to some node of tree $T_{\overline{\gamma}}$ or to the singular node $s_{\overline{\gamma}}$. Call the two arc-sets A_0^L and A_1^L the *leaf arcs* of $D(n)$. Since two arcs are incident out of each of the 2^{n-2} leaves of T_γ, there are $2 \times 2^{n-2} \times 2 = 2^n$ leaf arcs in $A_0^L \cup A_1^L$.

Singular subgraphs: There is a self-loop incident to node s_γ, where $\gamma \in \{0,1\}$, and there is an arc from s_γ to the root of T_γ. Call these the *singular arcs* of $D(n)$. There are 4 singular arcs altogether.

The two trees T_0 and T_1 are node-disjoint, so embedding one of them does not constrain the embedding of the other; however, each arc in both sets of leaf arcs, A_0^L and A_1^L, connects nodes from different trees. The difficulty in embedding $D(n)$ in a small book is in finding a linearization of nodes of the two trees which both accommodates the leaf arcs and respects the relative ordering of nodes within each tree prescribed by the embedding of the tree arcs. The following lemma clarifies the structure of the partial subgraph of $D(n)$ generated by the leaf arcs, thereby making it possible to define the desired linearization. See Fig. 1.

Lemma 1 *Let $a, b \in V_\gamma^L$, where $\gamma \in \{0,1\}$, be two leaves of T_γ, and let (a,u) and (b,v) be two leaf arcs incident into $T_{\overline{\gamma}}$. Then a precedes b in tree-order of T_γ just when u follows v in tree-order of $T_{\overline{\gamma}}$.*

Proof. We first show that tree-order in T_0 coincides with the lexicographic order on nodes of $D(n)$, while tree-order in T_1 coincides with the reversed lexicographic order on nodes of $D(n)$. Indeed, given nodes x and y of $T(n-2)$ such that x precedes y, we see that

$$\Phi_0(x) = 0^{n-2-|y|}00^{|y|-|x|}1x < 0^{n-2-|y|}01y = \Phi_0(y)$$
$$\Phi_1(x) = 1^{n-2-|y|}11^{|y|-|x|}0\overline{x} > 1^{n-2-|y|}10\overline{y} = \Phi_1(y)$$

We complete the proof for the case $\gamma = 0$, the case $\gamma = 1$ being dual. Because $a, b \in V_0^L$, there are $a', b' \in Z_2^{n-2}$ such that $a = 01a'$, $b = 01b'$, $u \in \{\mathcal{S}(a), \mathcal{E}(a)\}$ and $v \in \{\mathcal{S}(b), \mathcal{E}(b)\}$. We know that a precedes b in tree-order of T_0 just when $a' < b'$, which means that

$$\mathcal{S}(a) = 1a'0 < 1a'1 = \mathcal{E}(a) < 1b'0 = \mathcal{E}(b) < 1b'1 = \mathcal{S}(b).$$

This chain of relations is true just when $u < v$, meaning that u follows v in tree-order of T_1. □

If we extend tree-order in the two trees, consistently with the lexicographic order in $D(n)$, to cover corresponding singular nodes, then Lemma 1 holds also for the case $v = s_{\overline{\gamma}}$, since $s_{\overline{\gamma}} = \overline{\gamma}^n$ becomes the first node of $T_{\overline{\gamma}}$ in tree-order, while the leaf arc incident into it originates at node $\gamma\overline{\gamma}^{n-1}$, which is the last leaf of V_γ in tree-order.

Consider the leaves V_γ^L of T_γ, linearized so that they appear in tree-order. Partition this sequence into $n-1$ successive contiguous *segments* so that the kth segment, $0 \le k \le n-3$, has 2^{n-3-k} nodes, and the $(n-2)$nd segment has 1 node. Let the singular node $s_{\overline{\gamma}}$ be included in level $n-2$, together with the root of $T_{\overline{\gamma}}$. Our picture of the partial subgraphs L_γ, $\gamma \in \{0, 1\}$, is rendered by the following.

Proposition 1 *Let $\gamma \in \{0,1\}$, $0 \le k \le n-2$, $0 \le j < 2^{n-3-k}$. The leaf arcs incident out of the jth node,* in tree-order, *of the kth segment of leaves V_γ^L of T_γ, are incident into the pair of nodes that is the jth pair,* in reversed tree-order, *of the kth level of $T_{\overline{\gamma}}$.*

3 Embedding de Bruijn Graphs in Five Pages

Our main result is stated as follows.

Theorem 1 *The order-n de Bruijn graph $D(n)$ admits a book-embedding in five pages, with cumulative pagewidth* $(5/3)(2^n - 1 - (n \bmod 2)) - (n \bmod 2)$.

The embedding that establishes Theorem 1 is developed in three stages. In the first stage, we specify the subembedding of the two trees T_0 and T_1. The node-sets V_0 and

V_1 are disjoint, so these subembeddings are independent; each requires two pages. So, four pages may be required for the first-stage subembeddings, since we must expect that the node-sets of the trees T_0 and T_1 appear on the spine interleaved in some way dictated by the subembeddings of subsequent stages, thus preventing the pages consumed by one tree from being reused by the other. Four pages are also sufficient, as the tree-subembeddings do not constrain each other. In the second stage, we show how to embed each set of leaf arcs A^L_γ. The resulting leaf-subembeddings are not mutually independent, as each involves the leaf nodes V^L_γ of one tree and the nodes $V_{\bar{\gamma}} \cup S_{\bar{\gamma}}$ of the other. The consideration of interference between the second-stage subembeddings is deferred until the last stage, so these subembeddings are constructed independently; each requires one page. In the last stage, we exhibit a node layout which is consistent with the four subembeddings of the first two stages. Finally, we identify two pages of the first stage that can be combined into a single page in the complete embedding, thus arriving at the total of five pages.

3.1 Embedding the Trees

We present two varieties of what we term a *spiral embedding* of trees, in particular of $T(h)$. In both spiral embeddings, nodes are laid out by an appropriate alternation of tree levels, while each level is contiguous and ordered. In the *inward* spiral embedding, the outermost levels are the *lowest*-numbered levels, while in the *outward* spiral embedding the outermost levels are the *highest*-numbered levels. See Fig. 2 and 3. The following definition makes the layout precise.

Definition 1 *Let* $0 \le k \le \lfloor (h-1)/2 \rfloor$ *and* $0 \le \ell \le \lfloor h/2 \rfloor$.

(a) *In the* **inward spiral embedding** *of* $T(h)$, *the layout of nodes from left to right along the spine is:*

> *nodes at levels* $1, 3, \ldots, 2k+1, \ldots, h-1+(h \bmod 2)$, *in that order, each level in reversed tree-order;*

followed by:

> *nodes at levels* $h-(h \bmod 2), h-2-(h \bmod 2), \ldots, h-2\ell-(h \bmod 2), \ldots, 0$, *in that order, each level in tree-order.*

(b) *In the* **outward spiral embedding** *of* $T(h)$, *the layout of nodes from left to right along the spine is:*

> *nodes at levels* $h-(h \bmod 2), h-2-(h \bmod 2), \ldots, h-2\ell-(h \bmod 2), \ldots, 0$, *in that order, each level in tree-order;*

followed by:

> *nodes at levels* $1, 3, \ldots, 2k+1, \ldots, h-1+(h \bmod 2)$*, in that order, each level in reversed tree-order.*

The spiral embeddings separate odd-numbered tree levels from even-numbered ones, so that all levels of equal parity appear at one side of some point on the spine, while the levels of opposite parity appear at the other side. Call these sides the *even* and the *odd* side of the spine, according to tree levels that occupy them. In the inward spiral embedding the odd side is the left side of the spine, while in the outward spiral embedding the odd side is the right side of the spine. Otherwise, both embeddings place identically the levels of equal parity relative to each other—the odd-numbered ones in the order of increasing level number, the even-numbered ones in the order of decreasing level number. They also place identically the nodes inside each level—in tree-order within even-numbered levels, in reversed tree-order within odd-numbered levels. In summary,

Proposition 2 *The even side of a spiral embedding is laid out in tree-order. The odd side of a spiral embedding is laid out in reversed tree-order.*

We derive now the properties of the arc-assignment.

Lemma 2 *Both outward and inward spiral embeddings of* $T(h)$ *consume two pages, with cumulative pagewidth* $2^{h+1}-2$.

Proof. The arcs of $T(h)$ lead from nodes of one level to nodes at the level below, thus each arc leads either from the even side of the spine to the odd side, or vice versa. We assign the arcs that lead from the odd side to the even side to the *upper* page of a spiral embedding, and the arcs that lead from the even side to the odd side to the *lower* page.

Consider two arcs $(x, x\alpha)$ and $(y, y\beta)$, for distinct x, y with $|x|, |y| < h$. Say that these arcs are assigned to the same page of a spiral embedding. Then x precedes y in the tree-order just when $x\alpha$ precedes $y\beta$. However, the even and the odd sides of the spine are ordered oppositely, so $x\alpha$ and $y\beta$ appear on the spine ordered oppositely to x and y; the two arcs, therefore, nest inside one another.

To verify the cumulative pagewidth, note that the total cutwidth of both pages equals the number of arcs in $T(h)$, the maximum occurring at the pivot point between odd and even side. □

Now we are prepared to specify the first stage of the embedding— the layout of the two trees. At this point, we have a "stand alone" embedding for each of the trees, but we cannot yet specify the positions of the trees on the spine relative to each other.

Node Layout of the Two Trees T_γ:

- T_0 is laid out by *outward* spiral embedding.
- T_1 is laid out by *inward* spiral embedding.

Recalling that the height of each tree is $n-2$, Lemma 2 yields:

Corollary 1 *Each of the two component trees T_γ, where $\gamma \in \{0,1\}$, of $D(n)$ is embedded in two pages, with cumulative pagewidth $2^{n-1}-2$.*

3.2 Embedding the Leaf Arcs

Our next goal is to embed leaf arcs A_0^L and A_1^L, that lead from V_γ^L to $V_{\overline{\gamma}} \cup S_{\overline{\gamma}}$, without violating the *relative* ordering of nodes of $V_{\overline{\gamma}}$, stipulated by the spiral embedding of $T_{\overline{\gamma}}$. See Fig. 4.

In the following, we again coopt the singular node $s_{\overline{\gamma}}$ to level $n-2$ of $T_{\overline{\gamma}}$ and place it beside the root $\overline{\gamma}^{n-1}\gamma$ of $T_{\overline{\gamma}}$, consistently with the ordering described in Proposition 2.

Node Layout of the Partial Subgraph L_γ Generated by Leaf Arcs A_γ^L:
Lay out the nodes $V_{\overline{\gamma}}$ as mandated by the spiral embedding of tree $T_{\overline{\gamma}}$ (inward if $\overline{\gamma}=1$, outward if $\overline{\gamma}=0$). Then *interleave* the leaves V_γ^L of tree T_γ with the *odd* side of the spiral embedding of $T_{\overline{\gamma}}$ so that the following holds:

- Each even-numbered segment of V_γ^L is placed *contiguously, in tree-order*; the kth segment is placed immediately to the *left of the leftmost* node of level $k+1$ of $T_{\overline{\gamma}}$.
- Each node of an odd-numbered segment of V_γ^L is placed *between* the two nodes of $T_{\overline{\gamma}}$ to which it is adjacent via leaf arcs of A_γ^L.

The properties of the second stage embedding are summarized in the following.

Lemma 3 *Each partial subgraph L_γ, where $\gamma \in \{0,1\}$, generated by arcs A_γ^L that lead leaves V_γ^L of T_γ to nodes of $V_{\overline{\gamma}} \cup S_{\overline{\gamma}}$, is embedded in one page of width $(2/3)2^n+(7/3)-(8/3)(n \bmod 2)$. The leaf nodes V_γ^L of T_γ are laid out in tree-order.*

Proof. First, all leaf nodes of odd-numbered segments of V_γ^L are placed immediately beside the corresponding adjacent nodes in the tree $T_{\bar{\gamma}}$, so these arcs do not cross any other arcs on the page; they contribute 1 to the pagewidth.

To complete the proof for leaf arcs incident out of even-numbered segments of V_γ^L, we invoke Propositions 1 and 2. The odd-numbered levels of $T_{\bar{\gamma}}$ are placed in increasing order of level numbers, thus compelling the odd-numbered segments of leaves V_γ^L to appear in order of increasing segment numbers. Further, each even-numbered leaf segment, say the kth, is placed between the levels $k-1$ and $k+1$ of $T_{\bar{\gamma}}$, (assuming both levels exist), thus between the segments $k-1$ and $k+1$ of leaves V_γ^L. This imposes the order of increasing segment numbers on even-numbered segments. So, all nodes of even-numbered segments appear in tree-order and lie within the odd side, while the even-numbered levels of $T_{\bar{\gamma}}$ are also in tree-order and lie within the even side. By Proposition 1, this results in opposite orders of sources and destinations of these leaf arcs; therefore, no two leaf arcs cross.

By appealing to Proposition 1 again, we see that the order of nodes within odd-numbered segments of V_γ^L is opposite to the order within odd-numbered levels of $T_{\bar{\gamma}}$. By Proposition 2, the latter order is reversed tree-order, so the leaves of V_γ^L in odd-numbered segments appear in tree-order. Since leaves in even-numbered segments are in tree-order by construction, and since all segments are laid out in order of increasing segment number, we infer that all leaves V_γ^L are in tree-order.

The contribution of leaf arcs incident out of even-numbered segments of leaves V_γ into even-numbered levels of tree $T_{\bar{\gamma}}$ and, when n is even, into the singular node $s_{\bar{\gamma}}$, is:

$$2 \times \Big(\Big(\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} 2^{2k+(n \bmod 2)} \Big) + 1 - (n \bmod 2) \Big) = \frac{2}{3}\left(2^n + 2 - 4(n \bmod 2)\right)$$

which yields the claimed pagewidth after accounting for the one arc contributed by the leaf arcs incident out of odd-numbered segments. □

3.3 The Complete Embedding

We have to verify now that the partial embeddings defined so far are consistent. The embeddings of the trees in the first stage are trivially so, since they involve disjoint sets of both nodes and arcs. The embedding of each partial subgraph L_γ generated by the leaf arcs is consistent with the corresponding embedding of the tree $T_{\bar{\gamma}}$; it constrains only the leaves V_γ of the other tree T_γ; by Lemma 3, the mandated linearization of the leaves V_γ^L is tree-order; it is, therefore, consistent with the layout of these leaves in the spiral embedding of T_γ. It remains to confirm that each leaf set V_γ^L can be laid out in the odd side of the spiral embedding of the other tree $T_{\bar{\gamma}}$. To that end, we need only recall that the two spiral embeddings have their odd (even) sides in opposite sides of the spine, so the constraint is readily satisfied by identifying the odd side of one spiral embedding with the even side of the other. See Fig. 5.

Finally, the self-loops incident to the singular nodes are embedded easily, and the remaining two singular arcs, each incident out of node s_γ to the root $\gamma^{n-1}\overline{\gamma}$ of T_γ, may be laid out in either of the two pages of the spiral embedding of T_γ, since no other nodes of T_γ are placed between s_γ and $\gamma^{n-1}\overline{\gamma}$.

Corollary 2 *The four partial embeddings of the two trees T_γ, where $\gamma \in \{0,1\}$, and the two partial subgraphs L_γ generated by the leaf arcs, define an embedding of $D(n)$ in six pages.*

Our final task is to show that two of the six pages can be coalesced.

Lemma 4 *Assume that the lower page of a spiral embedding is the one that accommodates the arcs that lead from the even side to the odd side. Then, the two lower pages of the spiral embeddings of the trees T_0 and T_1 can be coalesced.*

Proof. Let (x_0, y_0) be an arc on the lower page of the outward spiral embedding of T_0, and let (x_1, y_1) be an arc on the lower page of the inward spiral embedding of T_1. We prove that the only possible ordering of the endpoints of these arcs on the spine is x_0, y_1, x_1, y_0, in which ordering the two arcs do not cross. Since the sources of the arcs are in the even sides, and the destinations in the odd sides of the corresponding spiral embeddings, we know that both x_0 and y_1 are to the left of x_1 and y_0, as the left side is even for T_0 and odd for T_1.

To prove that x_0 is to the left of y_1, we find a node which is both to the left of y_1 and to the right of x_0. Indeed, x_0 is in some non-leaf even-numbered level of T_0, so it is to the left of all leaves of T_0, by properties of the outward spiral embedding. However, y_1 is in some odd level of T_1, hence is to the right of segment 0 of the leaves of T_0, by properties of the embedding of leaf arcs. Thus, all nodes in segment 0 of V_0^L are to the left of y_1 and to the right of x_0. Analogously, all nodes in segment 0 of V_1^L are to the left of y_0 and to the right of x_1, whence the claimed ordering. □

We complete the proof of Theorem 1 by noting that the cumulative pagewidths of the component embeddings, as established in Corollary 1 and Lemma 3, combine to yield the claimed cumulative pagewidth.

4 Embedding Shuffle-Exchange Graphs in Five Pages

The order-n *shuffle-exchange* graph $S(n)$ [15] has node-set Z_2^n; given $y \in Z_2^{n-1}$ and $\beta \in Z_2$, the *shuffle* arc leads node βy to node $y\beta$ and the *exchange* arc leads node $y\beta$ to node $y\overline{\beta}$.

The book-embedding of de Bruijn graph $D(n)$ almost contains that of the shuffle-exchange graph $S(n)$, as announced by the following. (See Fig. 6.)

Theorem 2 *The order-n shuffle-echange graph $S(n)$ admits a book-embedding in five pages, with cumulative pagewidth* $2^n - (1/3)(2^{n-1} - 1 - (n \bmod 2))$.

Proof. The node layout is identical to that of $D(n)$. All shuffle arcs of $S(n)$ are identified with shuffle arcs of $D(n)$. Each exchange arc of $S(n)$ is incident to nodes $y\gamma$ and $y\overline{\gamma}$, for some $y \in Z_2^{n-1}$. However, one of $\{y\gamma, y\overline{\gamma}\}$ is the immediate successor of the other in the tree-order of one of the trees, say T_γ. There are no nodes of V_γ between $y\gamma$ and $y\overline{\gamma}$, and a new arc between them does not cross any other arc in either of the two pages of the spiral embedding of T_γ.

The claimed cumulative pagewidth is arrived at after removing the shuffle-exchange arcs from the embedding of $D(n)$, and subsequent incrementing by 1 the common width of the pages of the spiral embedding of the two trees. □

5 Conclusion

We have presented algorithms for embedding shuffle-like graphs in books of five pages. It still remains unknown whether five pages are necessary, as the best known lower bound is 3: Fig. 7 presents our embedding of the order-5 de Bruijn graph in four pages.

The *pagewidths* of our book-embeddings are greater than optimal by a factor logarithmic in the size of the graphs. This weakness is found in other book-embeddings of popular interconnection networks (cf. [6, 4]); it would be very interesting to bring the pagewidths of these embeddings closer to optimal, while retaining small pagenumbers, or to find some pagenumber-pagewidth tradeoffs. The general problem of transforming a book-embedding with optimal pagenumber and suboptimal pagewidth into one having the pagewidth in the order of optimal and the pagenumber not much greater than optimal is still open for all but one-page graphs: [7] presents an algorithm which converts one-page book-embeddings into two-page book-embeddings having logarithmic (asymptotically optimal) cumulative pagewidth. Although the general problem for graphs with arbitrary pagenumber is still open, some special cases offer evidence that good solutions are possible: [4] describes a two-page and a three-page graph family whose cumulative pagewidth decreases dramatically (from linear in the number of nodes to a constant) when only one more page is used; [14] constructs families with the same property, but for an arbitrary value of the pagenumber. For the Diogenes approach to fault-tolerant design of processor arrays, simultaneous optimization of both cost measures in the book-embeddings of the prevailing interconnection networks would greatly reduce the price of fault-tolerance.

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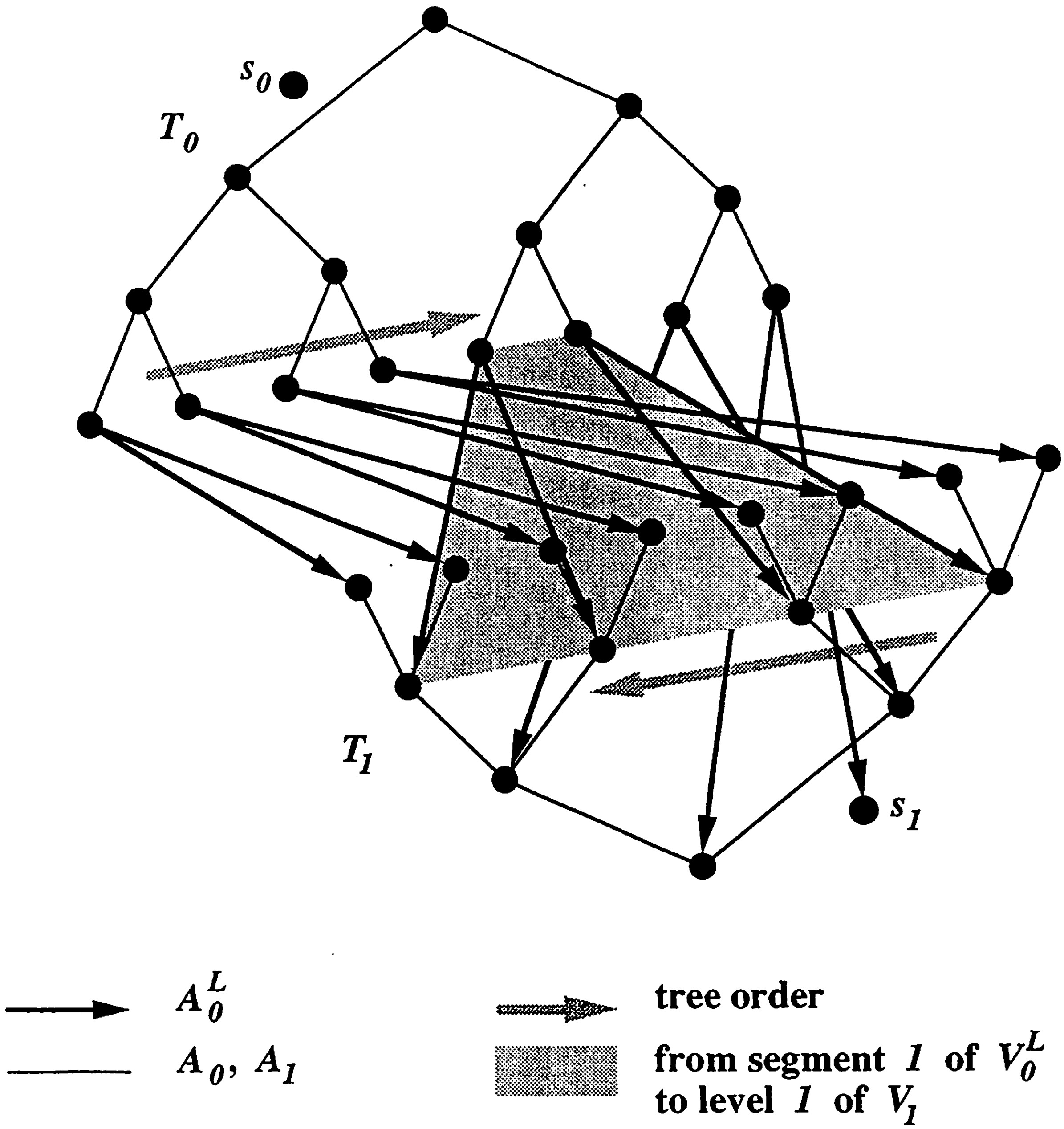

Figure 1: Bidendral decomposition of $D(5)$:
the trees and leaf arcs A_0^L

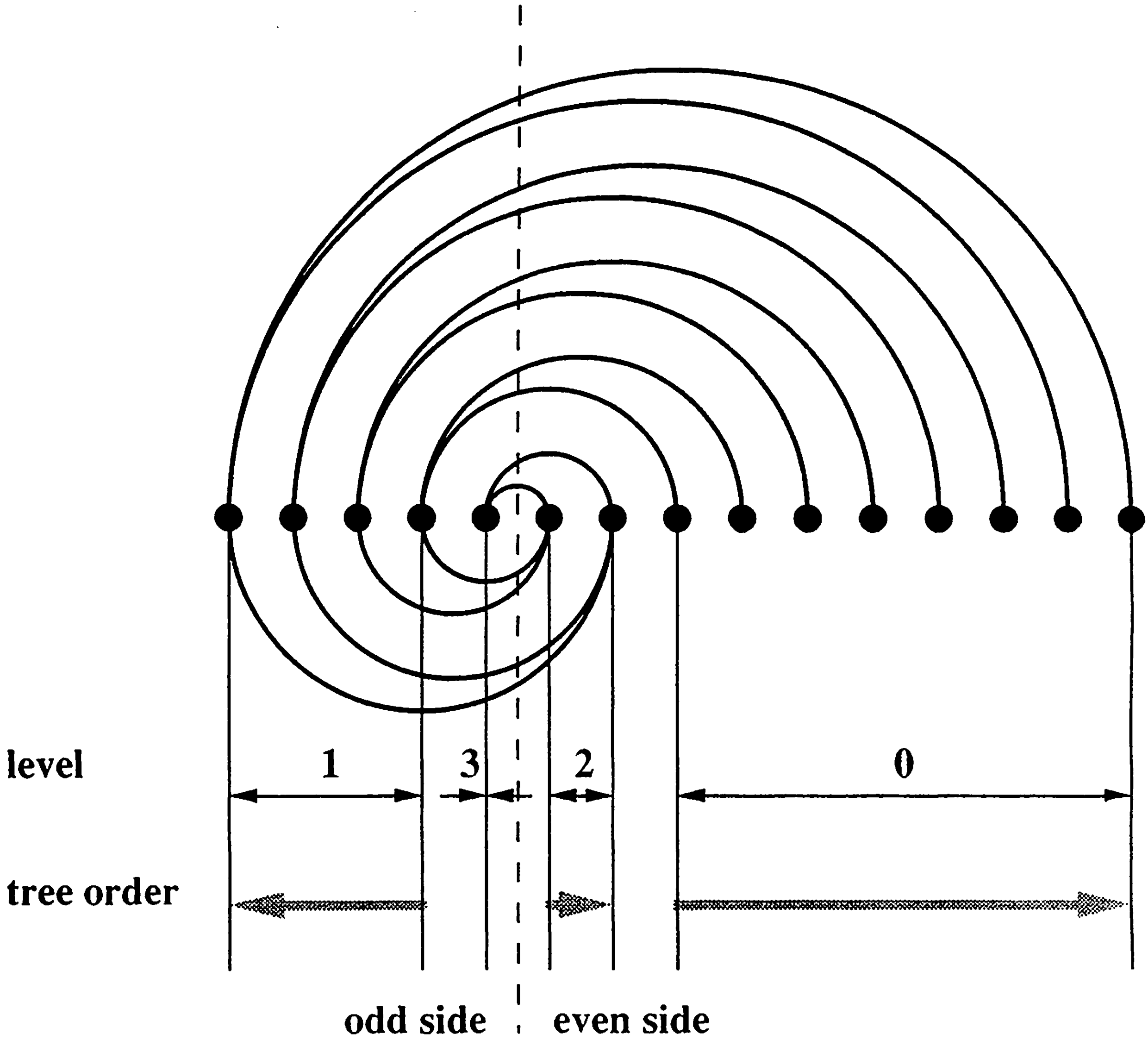

Figure 2: Inward spiral embedding of *T(3)*

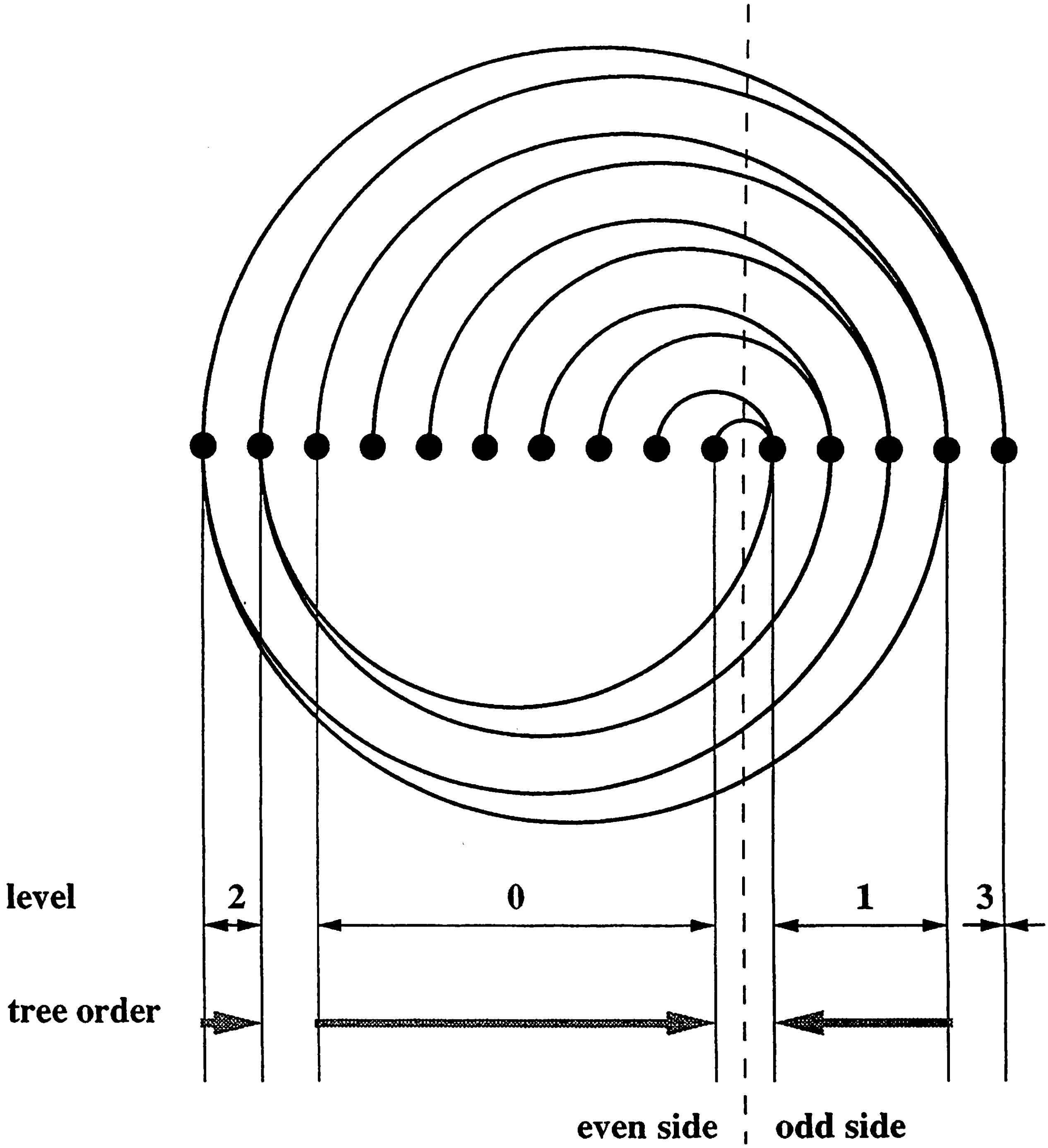

Figure 3: Outward spiral embedding of *T(3)*

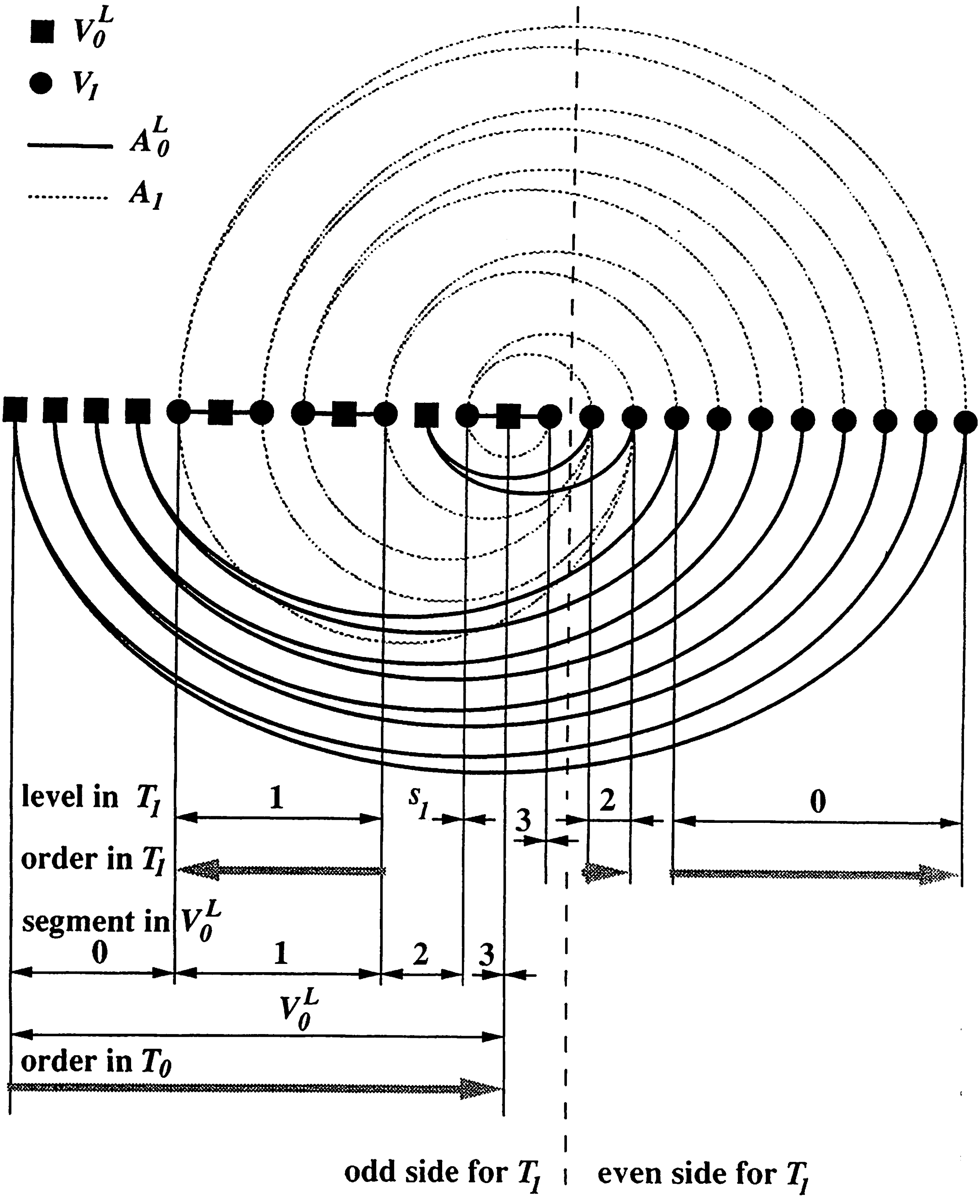

Figure 4: Embedding leaf arcs A_0^L of $D(5)$

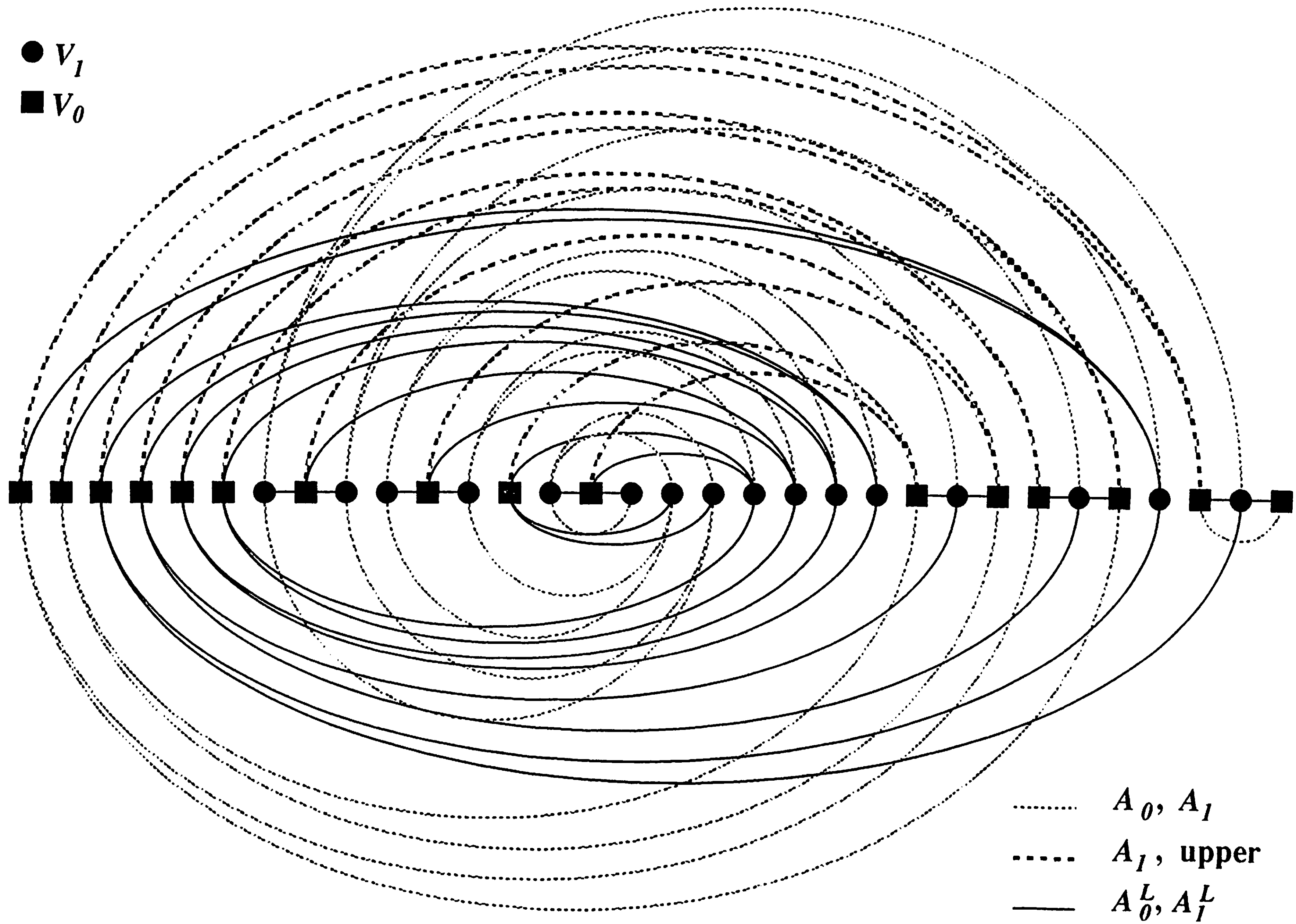

Figure 5: Embedding $D(5)$ in five pages

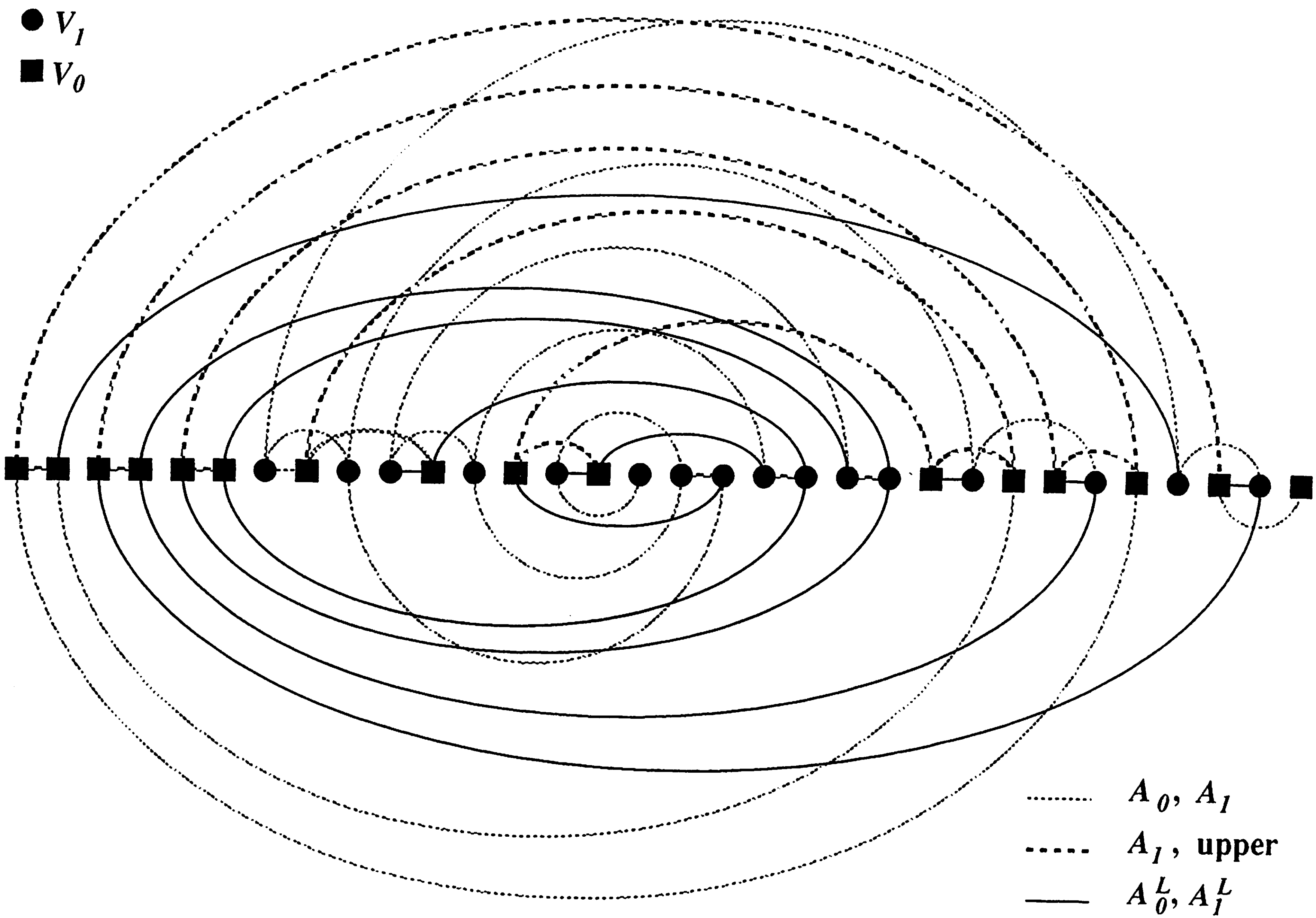

Figure 6: Embedding *S(5)* in five pages

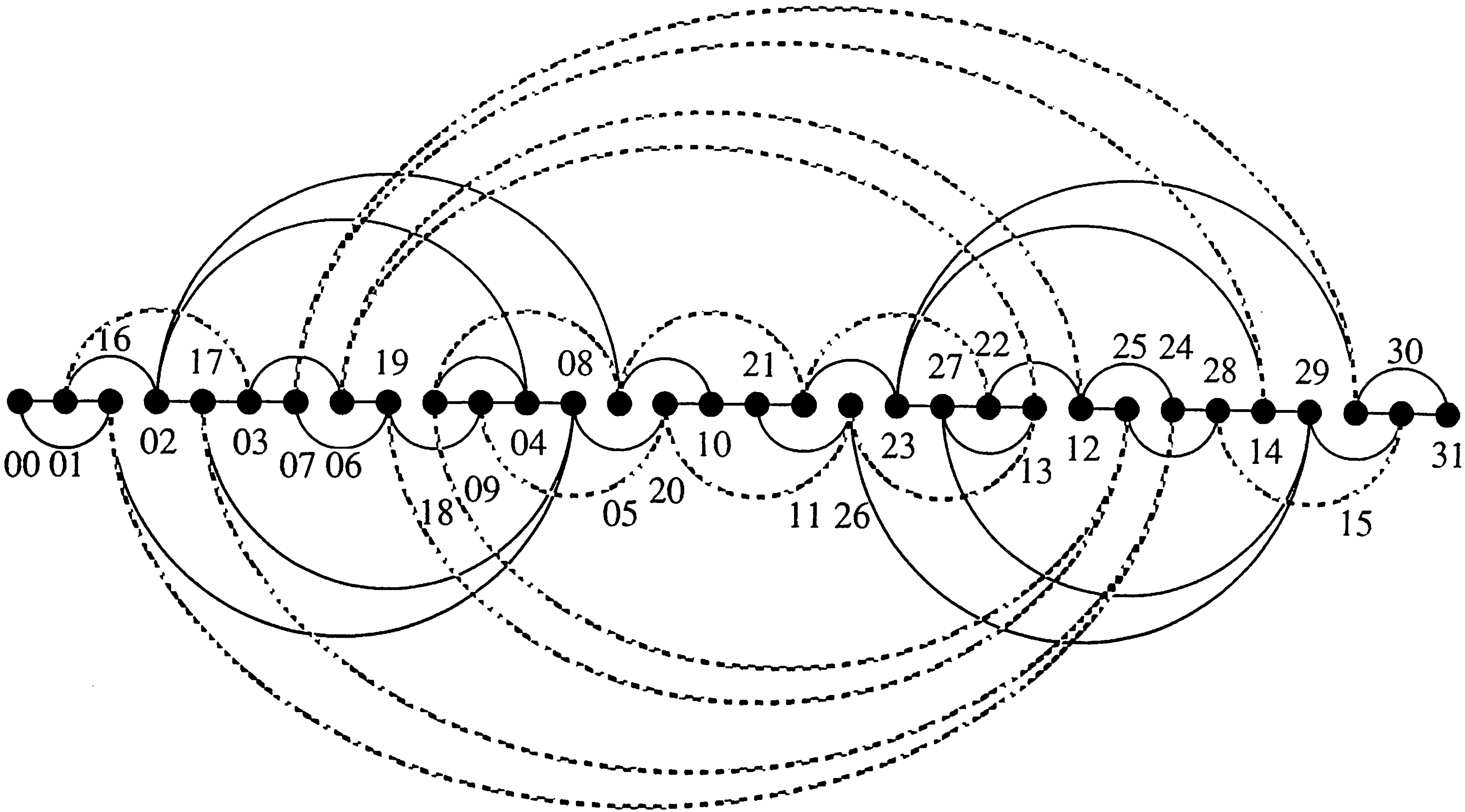

Figure 7: Embedding *D(5)* in four pages