On the Angular Resolution of Planar Graphs

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Abstract

A famous theorem of I. Fàry states that any planar graph can be drawn in the plane so that all edges are straight-line segments and no two edges cross. The *angular resolution* of such a drawing is the minimum angle subtended by any pair of incident edges. The *angular resolution* of a planar graph is the maximum angular resolution over all such planar straight-line drawings of the graph. In a recent paper by Formann et al., *Drawing graphs in the plane with high resolution*, Symp. on Found. of Comp. Sci. (1990), the following question is posed: does there exist a constant $r(d) > 0$ such that every planar graph of maximum degree d has angular resolution $\geq r(d)$? We answer this question in the affirmative by showing that any planar graph of maximum degree d has angular resolution at least α^d radians where $0 < \alpha < 1$ is a constant. In an effort to assess whether or not this lower bound is existentially tight (up to constant α), we analyze a very natural linear program that bounds the angular resolution of any fixed planar graph G from above. The optimal value of this LP is shown to be $\Omega(1/d)$. This suggests that our α^d lower bound might be improved to $\Omega(1/d)$, although currently, we are unable to settle this issue for general planar graphs. For the class of outerplanar graphs with triangulated interior and maximum degree d, we show not only that $\Omega(1/d)$ is lower bound on angular resolution, but in fact, this angular resolution can be achieved in a planar straight-line drawing where all interior faces are *similar* isosceles triangles. Finally, we show that three methods previously described in the literature for generating *convex* planar straight-line drawings of 3-connected planar graphs fail to guarantee angles bounded away from zero, even for the class of 3-connected planar graphs with maximum degree 3.

1 Introduction

Graphs are useful tools for representing many important structures in computer science, including VLSI circuits, parallel computer architectures, networks of terminals, state graphs, data-flow graphs, Petri nets, entity-relationship diagrams, etc. In some applications, the graph representing one of these objects must be displayed visually so that a human can interpret its structure easily. How a graph is best displayed typically depends on the application. If the graph represents, say, a VLSI circuit, it may be most appropriate to represent the vertices of the graph as points in a plane and the edges as rectilinear paths. In other applications, the edges may be more naturally displayed as straight line segments. Say we wish to generate a display of this latter kind. Since aesthetics are an important consideration, there are certain measures of a graph drawing that are worth optimizing. These could include one or some combination of, say, minimizing the number of edge-crossings, spacing the vertices and edges in relatively uniform fashion, obtaining good "angular spread" of the edges incident to each vertex, etc.

It is in fact this latter goal, trying to obtain good angular spread of the edges incident to each vertex, that is the subject of the present paper. The first authors to formalize this concept of "angular spread" and explore some of its properties, were Formann et al. [9] where they called this quantity the *resolution* of the graph. In the present paper, we shall call this quantity the *angular resolution* to distinguish it from other possible notions of resolution that might, for example, be related to distances between vertices. The paper [9] contains many interesting results. For example, they prove general upper and lower bounds on the angular resolution of any graph with maximum *degree* d (the *degree* of a vertex is the number of neighbirs it has.) In addition, they show how to construct straight-line drawings with high angular resolution for many interesting graph families.

Following [9], given an arbitrary graph G, the *angular resolution* of a straight-line drawing for G in the plane is defined to be the minimum angle subtended by any pair of incident edges. The *angular resolution* of G is the maximum angular resolution over all straight-line drawings of G. Thus, for example, if the maximum degree of G is d, the angular resolution is bounded above by $2\pi/d$. In general, however, this bound is not tight, as witnessed by the complete graph K_3 on 3 vertices which has maximum degree 2 and angular resolution $\pi/3$.

Besides being a natural and mathematically interesting concept associated with graphs, and having possible application in the realm of graph display as remarked earlier, there is another motivation for considering angular resolution—it concerns the design of opticalcommunication networks (see [1], [11].) Quoting from [9], "...consider a network in which each node represents a processor that can communicate via optical beams with it's neighbors in the graph. By maximizing the angular resolution of the layout, we simplify the task of designing the processors and the task of recognizing one's neighbors. (It is hard to send or recieve at very tight angles for a unit-size processor.)"

One of the many results demonstrated in [9] is that any planar graph with maximum degree d has angular resolution $\Omega(\frac{1}{d})$, independent of the number of vertices. In these drawings, however, the line segments representing edges of the planar graph are allowed to cross. This lead the authors of [9] to pose the following question: is there a constant $r(d) > 0$ such that any planar graph with maximum degree d has angular resolution $\geq r(d)$, where now angular resolution is computed over all *planar* straight-line drawings, i.e., drawings where the edges do not cross? (That every planar graph has a planar straight-line drawing is a famous theorem of I. Fàry [8].)

The present paper answers this question in the affirmative, and shows that the angular resolution of any planar graph with maximum degree d is at least α^d radians where $0 < \alpha < 1$ is a constant. (The proof utilizes a recent deep result by Andreev and Thurston which says that any triangulated planar graph can be realized as a disc-packing in the plane.) In addition, we show that no stronger lower bound (up to constant α), can be obtained by our methods. Later, we explore another interesting consequence of the Andreev-Thurston result, namely that it leads to a high-resolution drawing for the dual graph of a triangulated planar graph, a drawing in which every interior face is convex.

In an effort to assess whether or not the α^d lower bound is existentially tight (up to constant α) for the class of arbitrary planar graphs, we analyze a very natural linear program related to the angular resolution of a planar graph. The linear program has a collection of variables called "angles". For a fixed planar graph G, the linear program says maximize the minimum "angle" subject to the following constraints: (1) the "angles" at every vertex sum to 2π; (2) the "angles" inside any interior face with k edges sum to $(k-2)\pi$; (3) the values of all "angles" are greater than or equal to zero. Clearly any planar straight-line drawing of G satisfies these constraints. The converse is not necessarily so—not every solution to the constraints corresponds to a planar straight-line drawing. Consider, for instance, the complete graph K_4 on 4 vertices. Figure 1 shows an assignment to the angles that satisfies all the constraints, but which does not correspond to a planar straight-line drawing. So the LP above only partly captures the problem of computing angular resolution. More precisely, the value of this LP is an *upper bound* on the angular resolution of the associated graph. A question we sought to answer was, how small can this upper bound be in terms of d, the maximum degree of the graph? By casting the LP as a maximum flow problem and applying the Max-Flow Min-Cut Theorem, we prove that the optimum value of the LP is always $\Omega(\frac{1}{d})$. (This proof simplifies the earlier proof in Malitz [12] which argues from LP duality.) This suggests that $\Omega(\frac{1}{d})$, and not α^d, is perhaps the true lower bound on angular resolution for arbitrary planar graphs with maximum degree d. We are currently unable to answer this question.

Where we did make progress along these lines, is in the consideration of outerplanar graphs with maximum degree d. We demonstrate not only that $\Omega(\frac{1}{d})$ is a lower bound on angular resolution, but that any outerplanar graph with triangulated interior and maximum degree d admits a planar straight-line drawing in which every angle is $\Omega(\frac{1}{d})$ and all interior faces are similar isosceles triangles.

Finally, we show that three methods described in the literature for generating convex drawings of 3-connected planar graphs fail to guarantee angles bounded away from zero, even for the class of 3-connected planar graphs with maximum degree 3. The methods we consider here are those of Tutte [15], Becker and Hotz [2], and Becker and Osthof [4].

The organization of the paper is as follows. Section 2 derives the α^d lower bound on angular resolution for arbitrary planar graphs with maximum degree d. Section 3 shows that the disc-packing approach used in Section 2 cannot be pushed further to yield a stronger lower bound. Section 4 shows that the dual graph of a triangulated planar graph with maximum degree d admits a planar straight-line drawing in which all angles are $\Omega(1/d^2)$ and all interior faces are convex. Section 5 analyzes a linear program that bounds the angular resolution of a planar graph from above, and shows that its optimal value is always $\Omega(\frac{1}{d})$. Section 6 establishes the lower bound of $\Omega(\frac{1}{d})$ on angular resolution for arbitrary outerplanar graphs with maximum degree d, and describes an interesting planar straight-line drawing for triangulated outerplanar graphs. Section 7 demonstrates that three earlier methods for generating convex drawings of 3-connected planar graphs fail to guarantee angles bounded away from zero, even for the class of planar graphs with maximum degree 3. Finally, Section 8 concludes the paper with some open questions.

2 The Angular Resolution of Planar Graphs

In this section, we show that given any triangulated planar graph G with maximum degree d and specified exterior face f_{ext}, there is a planar straight-line drawing of G that respects f_{ext} and has angular resolution bounded below by α^{d-2} where $\alpha = \frac{1}{3+2\sqrt{3}} \approx .15$.

To begin the argument, we need some definitions. A *disc packing* P is a set $\{D_1, \ldots, D_n\}$ of closed discs (of zero, finite, or infinite radius) in the plane some of which may be *adjacent* (i.e., touching), but none of which overlap on interior points. A disc packing P induces a planar graph G in the obvious way: place a vertex at the center of each disc and for each pair of vertices in adjacent discs create an edge. If the edges are drawn as straight line segments, P determines a planar straight-line drawing of G. Let f_{ext} be the exterior face of G in this planar drawing. In this case, we say P *realizes* the pair (G, f_{ext}).

We start with the following remarkable result:

Theorem 2.1 (Andreev and Thurston) *For every triangulated planar graph G with exterior face f_{ext}, there is a disc packing P that realizes (G, f_{ext}).*

Given any triangulated planar graph G with exterior face f_{ext}, let P be a disc-packing that realizes (G, f_{ext}). Let $\Gamma_P(G)$ denote the planar straight-line drawing of G induced by P. We will now show that P can be selected so that for any two adjacent discs in P, the ratio of the smaller radius to the larger radius is bounded below by a constant depending only on d, the maximum degree of G. Clearly, for such a P, the angular resolution of $\Gamma_P(G)$ is bounded below by a constant depending only on d.

Consider any planar straight-line drawing of the triangulated planar G with exterior face f_{ext}. Consider any vertex v not lying on f_{ext}. If we were to walk around v at close range in say clockwise fashion, we would encounter the edges incident to v in a specifice order: say $(v, u_0), (v, u_2), \ldots, (v, u_{t-1}), (v, u_0)$. Since G is triangulated, u_i is adjacent to $u_{i+1 \bmod t}$ for each i, so that the u_i's form a cycle. The sequence $v, u_0, \ldots, u_{t-1}$ is called a *wheel* of *length t* with v as *hub*. If $u_{i+1}, \ldots, u_{i+j \bmod t}$ is a path in the above cycle, then the sequence $v, u_{i+1}, u_{i+2}, \ldots, u_{i+j}$ is called a *fan* of *length j* with v as *hub*.

By analogy, an ordered disc packing $P = (C, D_0, \ldots, D_{t-1})$ is called a *wheel* of *length t* with C as *hub* if all D_i are adjacent to C and each D_i is adjacent to $D_{i+1 \bmod t}$. Let P' be any subsequence of P that includes C. If P' is itself a wheel with hub C, then P' is called a *subwheel* of P.

An ordered disc packing $P = (C, D_1, \ldots, D_t)$ is called a *fan* of *length t* with C as *hub* if all D_i are adjacent to C and each D_i is adjacent to D_{i+1}. The discs $D_2, \ldots, D_{t-1}$ are called the *intermediate discs* of P. Let P' be any subsequence of P that includes C. If P' is itself a fan with hub C, then P' is called a *subfan* of P.

In the two lemmas that follow, we show that for any wheel $P = (C, D_0, \ldots D_{t-1})$ of length t, the radius of any D_i divided by that of C cannot be smaller than a certain constant depending only on t.

To prove the next lemma, we need a few more preliminaries. If one removes from the plane three mutually adjacent closed discs A, B, and C, then the region left over has two connected components, one bounded, one not. Define the *cranny* of A, B, and C to be the closure of the bounded component.

Given four mutually adjacent discs A, B, C, D, where D lies in the cranny of A, B, C, then say D is *snug* in the cranny of A, B, C. For this arrangement of four discs, there is beautiful formula of Descartes (see Coxeter [6, pp. 13–15]) that relates their radii. Let ϵ_A, ϵ_B, ϵ_C, ϵ_D be the reciprocals of the radii of A, B, C, and D respectively. Descartes' formula says

$$2(\epsilon_A^2 + \epsilon_B^2 + \epsilon_C^2 + \epsilon_D^2) = (\epsilon_A + \epsilon_B + \epsilon_C + \epsilon_D)^2.$$

Solving for ϵ_D from this formula yields

$$\epsilon_D = \epsilon_A + \epsilon_B + \epsilon_C + 2\sqrt{\epsilon_A\epsilon_B + \epsilon_A\epsilon_C + \epsilon_B\epsilon_C}. \tag{1}$$

By merely relabeling the ϵ's, all the formulas relating one ϵ to the other three are obtained.

Lemma 2.1 *Let the ordered disc packing $P = (C, A, D_1, \ldots, D_t, B)$ be a fan of length $t+2$ with hub C. Let r_C, r_A, $r_1, \ldots, r_t$, r_B respectively denote the radii of the discs in P. Suppose further that A is adjacent to B. Let $r = \min\{r_A, r_B, r_C\}$. Then each D_j has radius $r_j \geq \alpha^t r$, where $\alpha = \frac{1}{3+2\sqrt{3}} \approx .15$.*

Proof. The proof is by induction on t, which is the number of intermediate discs in the fan P.

For the base case, we consider $t = 1$. Here P consists of 4 discs C, A, D_1, and B, where all discs are mutually adjacent, and D_1 lies in the cranny of A, B, and C. We now bound r_1 from below. Without loss of generality, assume $r = r_A$. Take the disc packing P, and shrink discs B and C (maintaining adjacencies) until both have radius r. To have maintained adjacencies, the disc D_1 must also have shrunk in radius. This shows that r_1

is smallest, when A, B, and C all have the same radius r. In this case, formula (1) gives $r_1 = \alpha r$.

Suppose the lemma is true for all $s < t$. We now prove the lemma for t.

Fix r_A, r_B, and r_C. Without loss of generality, assume the disc-packing P is such that disc D_i acheives its smallest possible radius. (Such a P is always seen to exist by considering an infinite sequence of drawings where the radius of D_i is decreasing monotonically to its infimum. This sequence has an accumulation point.) Again without loss of generality, assume $D_i \neq D_1$.

We claim that in P, the disc D_1 is adjacent to some other disc besides C, A, and D_2. Suppose, for contradiction, this is not the case. Place the origin of the complex plane at the touching point of B and C and apply the inversive map $1/z$. See Figure 2. Keeping A, B, and C fixed, expand the disc D_1, while sliding the assembly $\{D_2, \ldots, D_t\}$ to the left. At all times, maintain the adjacencies of P, and halt the expansion of D_1 when it first touches some disc besides A, C, and D_2. Take this new diagram and again apply the inversive map $1/z$. The outcome is a new disc packing P' that preserves all the adjacencies of P (and adds some new adjacencies), but for which the disc D_i is now of smaller radius than in P. This contradicts the definition of P. So in P, the disc D_1 is adjacent to some other disc besides A, C and D_2.

So P has a proper sub-fan P' of the form $(C, A, D_1, D_{k+2}, D_{k+3}, \ldots, D_t, B)$ where $1 \leq k \leq t$. Since P' has $t - k < t$ intermediate discs, we can apply the induction hypothesis to conclude that each disc D_j in P' has radius $r_j \geq \alpha^{t-k} r$. But observe that $P'' = (C, D_1, D_2, \ldots, D_{k+1}, D_{k+2})$ is also a proper sub-fan of P where D_1 is adjacent to D_{k+2}. Since P'' has $k < t$ intermediate discs, we can apply the induction hypothesis again to conclude that each disc D_j in P'' has radius $r_j \geq \alpha^k(\alpha^{t-k} r) = \alpha^t r$. ■

Lemma 2.2 *Suppose the ordered disc packing $P = (C, D_0, \ldots, D_{t-1})$ is a wheel of length t with hub C. Let the discs in P have respective radii r_C, $r_0, \ldots, r_{t-1}$. Then each disc D_j has radius $r_j \geq \alpha^{t-3} r_C$.*

Proof. The proof is by induction on t, the length of the wheel P.

The lemma clearly holds for $t = 3$ since every disc in this case has radius at least as large as r_C.

Suppose the lemma holds for all $s < t$. We now prove the lemma for t.

Fix r_C. Without loss of generality, suppose the disc packing P is such that D_0 assumes its smallest possible radius. (Such a P is always seen to exist by considering an infinite sequence of drawings where the radius of D_0 is decreasing monotonically to its infimum. Since we are allowing discs of infinite and zero radius, this sequence has an accumulation point, which we take to be P.)

We claim that P contains a proper subwheel. Clearly this is true, if D_0 is just a single point. So let us suppose that D_0 has some positive finite radius. Suppose for contradiction, that P contains no proper subwheel. Place the origin of the complex plane at the touching point of D_0 and C. Apply the inversive map $1/z$. See Figure 3. Keeping $C, D_2, \ldots, D_{t-2}$ fixed, simultaneously expand the discs D_1 and D_{t-1} while translating D_0 upward as necessary, so as to maintain all the adjacencies of P. Halt the expansion of D_1 and D_{t-1} as soon as either touches a disc other than C and its two neighbors on the cycle $D_0, \ldots, D_{t-1}, D_0$. To the resulting diagram, apply the inversive map $1/z$. This yields a new disc packing P' that preserves all adjacencies in P (and adds some new ones), but where D_0 now has smaller radius than it did in P. We see this contradicts the definition of P.

So, P contains a proper subwheel P' of length $s < t$. Without loss of generality, we may assume that P' is of the form $(C, D_{i+1}, D_{i+2}, \ldots, D_{i+s})$ where and $2 \leq s < t$. By the induction hypothesis, each disc D_j in P' has radius at least $\alpha^{s-3} r_C$. By Lemma 2.1, each of the remaining $t - s$ discs has has radius at least $\alpha^{t-s}(\alpha^{s-3} r_C) = \alpha^{t-3} r_C$. ■

Lemma 2.3 *Every triangulated planar graph G with exterior face f_{ext} is realized by a disc packing in which all three exterior discs have the same radius.*

Proof. Let P be any disc packing for the triangulated planar graph G with exterior face f_{ext}. Label the three exterior discs A, B, C, and let $r_A \leq r_B \leq r_C$ denote the relationship among their respective radii. We now indicate how to obtain a new disc packing P'' for G where r_A and r_B are unchanged, but for which $r_C = r_A$. Place the origin of the complex plane at the touching point of the discs A and B, and apply the inversive map $1/z$. Call the resulting disc packing P'. See Figure 4. Now slide C and the entire assembly of discs between A and B to the right a distance l while maintaining contact with A and B. After performing this slide, apply the map $1/z$ once again to yield a new disc packing P'' for G where r_A and r_B remain unchanged but r_C is decreased. Choose the distance l over which C slides in P' so as to make $r_C = r_A$ in P''.

Now perform a similar sequence of manuevers on P'' to reduce the radius of r_B, while leaving the other two radii untouched, yielding ultimately a disc packing for G in which $r_B = r_A = r_C$. ■

Given any triangulated planar graph G with maximum degree d and exterior face f_{ext}, we know by Theorem 2.1 and Lemma 2.3 that there is a disc packing P for (G, f_{ext}) where the three outer discs all have the same radius, say unit radius. Consider any adjacent pair of discs C,D in P and let D have the smaller radius. If C is not one of the three outer discs, then C and D are part of a wheel of length $\leq d$ with C as hub. By Lemma 2.2, the radius of D divided by that of C is at least α^{d-3}. Now suppose C is an exterior disc and D is an interior disc. In this case, C and D are part of a fan with hub C that includes the other two exterior discs, call them A and B. Since A and B are adjacent, we can use Lemma 2.1 to conclude that the radius of D divided by that of C is at least α^{d-2}. Hence, given any two adjacent discs in P, the radius of the smaller disc divided by that of the larger is at least α^{d-2}.

Consider any triangle in the planar straight-line drawing of G induced by the disc-packing P above. Call the three discs that generate this triangle A, B, and C with radii $r_A \leq r_B \leq r_C$. If we take $r_C = 1$, then by the remarks above, $\alpha^{d-2} \leq r_A \leq r_B$. It is rather easy to see that the smallest angle θ in the triangle is acheived when $r_A = r_B = \alpha^{d-2}$. In this situation, $\sin(\frac{\theta}{2}) = \frac{\alpha^{d-2}}{1+\alpha^{d-2}}$. Since $\frac{\theta}{2} > \sin\frac{\theta}{2}$, we obtain $\theta > \frac{2\alpha^{d-2}}{1+\alpha^{d-2}} > \alpha^{d-2}$. We have proved

Theorem 2.2 *The angular resolution of any triangulated planar graph G of maximum degree d and exterior face f_{ext} is at least α^{d-2} where $\alpha = \frac{1}{3+2\sqrt{3}} \approx .15$.*

As far as algorithms are concerned, we do not know of an efficient way to obtain the disc packing of Theorem 2.1 for arbitrary triangulated planar graphs G. (See the open questions in Section 7.) If we did, we would have an efficient algorithm for generating a planar straight-line drawing of G with angular resolution bounded away from zero (for fixed maximum degree d.) This could be very useful in the display of planar graphs.

3 Tightness of the Above Analysis

We now demonstrate that no better lower bound (up to constant α) for the angular resolution of arbitrary planar graphs is obtainable via the disc-packing method of the last section.

Proposition 3.1 *For every integer $d \geq 2$, there is a triangulated planar graph G_d with specified exterior face f_{ext} and maximum degree d such that for any disc-packing P realizing*

(G_d, f_{ext}), the induced planar straight-line drawing $\Gamma_P(G_d)$ contains an angle less than μ^d where $0 < \mu < 1$ is a constant.

Proof. We start by defining the triangulated planar graph G_d and its exterior face f_{ext}. Let $C, D_0, \ldots, D_{d-1}$ denote the $d+1$ vertices of G_d. Let D_0, D_{d-2}, and D_{d-1} be connected in a triangle, and take this triangle to be the exterior face f_{ext}. Place an edge between C and each of D_0, D_{d-2}, and D_{d-1}. Place an edge between D_1 and each of C, D_0, and D_{d-2}. For all j with $2 \leq j \leq d-3$, place an edge between D_j and each of C, D_{j-2} and D_{j-1}. This concludes the description of G_d. See Figure 5(a).

Let $P = (C, D_0, \ldots, D_{d-1})$ be any disc packing realizing (G_d, F). See Figure 5(b). We see that P is a wheel of length d with hub C. Furthermore, disc C is snug in the cranny of D_0, D_{d-2}, and D_{d-1}, disc D_1 is snug in the cranny of D_0, D_{d-2}, and C, and for each j with $2 \leq j \leq d-3$, disc D_j is snug in the cranny of D_{j-2}, D_{j-1}, and C. In what follows, we show that the radii of D_{d-4} and D_{d-3} are small by comparison with that of C, which means that in $\Gamma_P(G_d)$, the angle at C induced by discs C, D_{d-4}, D_{d-3} is small.

We bound the radii of D_{d-4} and D_{d-3} from above as follows. Without loss of generality, we assume C has radius 1. Let $r_C = 1, r_0, \ldots, r_{d-1}$ denote the radii of the discs $C, D_0, \ldots, D_{d-1}$, respectively. Let $\epsilon_C = 1, \epsilon_0, \ldots, \epsilon_{d-1}$ denote the reciprocals of the above radii. It is clear that that for any disc packing P realizing (G_d, f_{ext}), there holds $\epsilon_2 > \epsilon_1 \geq 1$, and in fact, there are constants $\beta_0, \gamma_0 > 1$ such that for any disc-packing P as above, $\gamma_0\epsilon_1 \geq \epsilon_2 \geq \beta_0\epsilon_1$. Let $\gamma = \max\{\gamma_0, 3+2\sqrt{3}\}$ and $\beta = \min\{\beta_0, 1+\frac{1}{\gamma}\}$. Obviously, $\gamma\epsilon_1 \geq \epsilon_2 \geq \beta\epsilon_1$. For the purpose of induction, suppose we have shown $\gamma\epsilon_{k-1} \geq \epsilon_k \geq \beta\epsilon_{k-1}$, and want to conclude $\gamma\epsilon_k \geq \epsilon_{k+1} \geq \beta\epsilon_k$. Let us see that this conclusion is valid for any k. From (1), we have

$$\epsilon_{k+1} = 1 + \epsilon_{k-1} + \epsilon_k + 2\sqrt{\epsilon_{k-1} + \epsilon_k + \epsilon_{k-1}\epsilon_k}.$$

Applying the induction hypothesis to the equation above, there follows

$$(3+2\sqrt{3})\epsilon_k \geq \epsilon_{k+1} \geq (1+\frac{1}{\gamma})\epsilon_k.$$

But by the definition of β and γ, we have $\gamma \geq 3+2\sqrt{3}$ and $1+\frac{1}{\gamma} \geq \beta$ and so the induction goes through.

Hence the radii of D_{d-4} and D_{d-3} are at most $\frac{1}{\beta^{d-5}}$ and $\frac{1}{\beta^{d-4}}$, respectively. Therefore the angle at C in $\Gamma_P(G_d)$ induced by the discs C, D_{d-4}, and D_{d-3} is at most μ^d for a suitably chosen constant $\mu \in (0,1)$. ■

4 The Angular Resolution of Dual Graphs

Let G be a triangulated planar graph of maximum degree d and $G^\star$ its dual graph obtained by placing a vertex in each *interior* face of G and connecting two such vertices whenever the corresponding faces of G are adjacent. Clearly, $G^\star$ has maximum degree 3, and every interior face has at most d vertices. In the present section, we show that $G^\star$ admits a planar straight-line drawing in which every angle is $\Omega(1/d^2)$ radians, and all interior faces are convex. To obtain this drawing of $G^\star$, we first use Theorem 2.1 to get a disc-packing P for G, and then draw $G^\star$ perpendicular to $\Gamma_P(G)$ as shown in Figure 6. In this drawing, all interior faces of $G^\star$ are convex since all the edges in a given face are tangent to the same disc of P. What now remains for us to show is that there is a lower bound of $\Omega(1/d^2)$ on all the angles of $G^\star$.

Consider three mutually adjacent discs A, B, C in P with radii $r_A \leq r_B \leq r_C$, respectively. Let us assume that C is an interior disc of P and therefore the hub of a wheel of length at most d. (The case where C is not an interior disc of P is argued similarly.) Since C is adjacent to A and B, two discs of smaller radius, it must be that $d \geq 4$. Consider, once again, the drawing of $G^\star$ shown in Figure 6. Nestled in the cranny of A, B, C, there lies a vertex v of $G^\star$. Let T denote the three edges of $G^\star$ incident to v. Call T a *tri-spoke*. Notice, the edges of T are perpendicular to the edges of the triangle Δ whose vertices are at the centers of the discs A, B, C. Let α_Δ be the angle in Δ incident to disc A. Let α_T be the angle in T that faces α_Δ. Obviously, α_Δ is the largest angle in Δ since $r_A \leq r_B \leq r_C$. Hence α_T is the smallest angle in T as $\alpha_T = \pi - \alpha_\Delta$. The following two lemmas will used to bound α_Δ from above, and thus bound α_T from below.

Lemma 4.1 *Given $d \geq 4$, fix $t \in \{2, 3, \ldots, d-2\}$. Let the ordered disc packing $W = (C, A, D_1, \ldots, D_t, B)$ be a wheel in P of length $t + 2$ with hub C. Let r_A, r_B, r_C denote the radii of discs A, B, and C respectively, and assume $r_A \leq r_B \leq r_C$. Then $r_A \geq \frac{r_B}{4t(t-1)}$.*

Proof. Consider an infinitely long strip of discs like that shown in Figure 7(a) where the discs are numbered $\ldots, -2, -1, 0, 1, 2, \ldots$ in sequence. Here the discs are all the same radius and adjacent to the lines $y = 1/r_C$ and $y = -1/r_B$. Let the origin of the complex plane be located symetrically between disc 0 and disc 1. Now apply the complex inversive map $1/z$, to yield Figure 7(b). In this figure, the discs labled B and C have radii r_B and r_C respectively.

We claim that r_A is no smaller than the radius of disc $t - 1$ in Figure 7(b). For if it were smaller, then the discs $A, D_1, \ldots, D_t, B$ would not be able to grow in radius quickly enough to make a cycle all the way around disc C.

Now the radius of disc $t-1$ in Figure 7(b) is greater than $\frac{1}{2}\left[\frac{1}{(t-1)(1/r_B+1/r_C)} - \frac{1}{t(1/r_B+1/r_C)}\right]$ which simplifies to $\left[\frac{r_B r_C}{r_B+r_C}\right]\frac{1}{2t(t-1)}$. Since $r_C \geq r_B$, this latter quantity is at least as large as (substituting r_B for r_C) $\frac{r_B}{4t(t-1)}$. Thus $r_A \geq \frac{r_B}{4t(t-1)}$. ■

Before proceeding with the next lemma, we note that $\frac{r_B}{4t(t-1)}$ is smallest when $t = d-2$. Hence, for fixed $r_B \leq r_C$, the angle α_Δ is largest when we take $r_A = \frac{r_B}{4(d-2)(d-3)}$.

Lemma 4.2 *Let A, B, and C be three mutually adjacent discs with respective radii $r_A \leq r_B \leq r_C$. Let $r_A = \frac{r_B}{4(d-2)(d-3)}$. Let Δ be the triangle whose vertices are at the centers of the three discs A, B, C. Let α_Δ denote the angle of Δ incident to disc A. Then $\alpha_\Delta \leq \pi - \frac{\pi}{5(d-2)(d-3)}$ radians.*

Proof. Clearly, for fixed r_A and r_B, the angle α_Δ is largest when $r_C = \infty$. See Figure 8. Let $\beta = \alpha_\Delta - \frac{\pi}{2}$. Then $\frac{2\beta}{\pi} \leq \sin\beta = \frac{r_B - r_A}{r_B + r_A}$. Thus $\beta \leq \frac{\pi}{2}\left[1 - \frac{2r_A}{r_B+r_A}\right] \leq \frac{\pi}{2} - \frac{\pi}{5(d-2)(d-3)}$. Hence $\alpha_\Delta \leq \pi - \frac{\pi}{5(d-2)(d-3)}$. ■

It follows from Lemma 4.2 that the tri-spoke angle α_T which faces α_Δ is at least $\frac{\pi}{5(d-2)(d-3)}$ radians. But every vertex of G^* is the center of a tri-spoke, and we have just argued that the minimum angle in any tri-spoke is at least $\frac{\pi}{5(d-2)(d-3)}$ radians. We have therefore proven

Theorem 4.1 *Let G^* be the dual graph of a triangulated planar graph G of maximum degree d. Then G^* has a planar straight-line drawing such that every angle is $\Omega(1/d^2)$ and all interior faces are convex.*

5 A Linear Program and its Evaluation

In this section we define a very natural linear program associated with a triangulated planar graph G of maximum degree d whose optimal objective value is an upper bound on the angular resolution of G. By casting this LP as a max-flow problem and applying the Max-Flow Min-Cut Theorem we prove that its optimal value is at least $\frac{\pi}{3(d-1)}$ for all G as above.

Let $G = \langle V, E \rangle$ be a triangulated planar graph with maximum degree $d \geq 2$ and specified exterior face f_{ext}. Introduce a collection of variables called "angles" representing angle-values in a drawing of G. The linear program that we are interested in says maximize the minimum "angle" subject to the following constraints: (i) the "angles" at every vertex sum to 2π; (ii) the "angles" inside any interior face sum to π; (iii) all "angles" are greater than or equal to zero.

Theorem 5.1 *The LP above has optimal value at least* $\frac{\pi}{3(d-1)}$.

Proof. Let F be the collection of all faces of G including f_{ext}. Let $\deg(v)$ denote the degree of vertex v in G. Let $V_{\text{ext}} \subseteq V$ be the subset of vertices that lie on f_{ext}. Let H denote the directed bipartite graph $\langle V, F, I \rangle$ where $(v, f) \in I$ iff vertex $v \in V$ and face $f \in F$ are incident in G.

We now describe a max-flow problem corresponding to the LP above. Start with the digraph $H = \langle V, F, I \rangle$. Let all arcs of H have infinite capacity. Add to H two new vertices, s and t. Fix a real number $\alpha \in [0, \frac{\pi}{d}]$. Create an arc from s to each vertex $v \in V$ with capacity $2\pi - \alpha\deg(v)$. Create an arc from each vertex $f \in F - f_{\text{ext}}$ to vertex t with capacity $\pi - 3\alpha$. Create an arc from f_{ext} to t with capacity $5\pi - 3\alpha$. Call this capacitated network K. Clearly, K has an $s - t$ flow saturating all s-arcs iff α is a lower bound for the LP.

Now by the Max-Flow Min-Cut Theorem, K has a flow saturating all s-arcs iff every $s-t$ cut has capacity $\geq \sum_{v \in V} 2\pi - \alpha\deg(v)$. Obviously, the cut separating s from everybody else has this property, and the cut separating t from everybody else has this property. Let us consider any other finite-capacity cut. Such a cut is of the form $(S, \overline{S})$ where $S = \{s\} \cup V' \cup F'$ where $V' \subset V$, and $F' \subseteq F$ denotes a set of F-vertices containing all those incident to V'. Clearly it suffices to consider only those cuts in which F' actually *equals* the set of F-vertices incident to V', since such cuts have lower capacity than those which do not satisfy this condition. Henceforth, we shall denote F' by $F(V')$. If S contains f_{ext}, then by the previous remark, we may assume S also contains at least one member of V_{ext}. However, if S does not contain all members of V_{ext} in this circumstance, a cut with lower capacity can be obtained by removing f_{ext} and all members of V_{ext} from S. Hence, in the cut $(S, \overline{S})$ with least capacity, it must be that S either contains no vertices among $V_{\text{ext}} \cup \{f_{\text{ext}}\}$, or contains all the vertices of this set.

To start, let us assume the former holds. In this case, the capacity of the cut $(S, \overline{S})$ is

$$\sum_{v \notin V'} 2\pi - \alpha\deg(v) + \sum_{f \in F(V')} \pi - 3\alpha.$$

For what values of α is this capacity always greater than or equal to $\sum_{v \in V} 2\pi - \alpha \deg(v)$ (the saturation flow value for K) irregardless of which $V' \subseteq V - V_{\text{ext}}$ is chosen? First, let us simplify. The desired inequality is equivalent to

$$\sum_{f \in F(V')} \pi - 3\alpha \geq \sum_{v \in V'} 2\pi - \alpha \deg(v).$$

Simplifying further, we can restate the question as follows: for what values of α is

$$(\pi - 3\alpha)|F(V')| - 2\pi|V'| + \alpha \deg(V') \tag{2}$$

greater than or equal to zero irregardless of which $V' \subseteq V - V_{\text{ext}}$ is chosen? (Here $\deg(V')$ denotes the sum of the degrees of all the vertices in V'.) Let $G_{F(V')}$ be the subgraph of G induced by $F(V')$ (viewed now as faces in G rather than vertices in K). Let $T_{F(V')}$ be a graph obtained from $G_{F(V')}$ by identifying various pairs of edges (preserving planarity) on the boundary of $G_{F(V')}$, and then at the end contracting various edges, so that the result is a fully triangulated planar graph whose interior vertices are exactly V' and whose interior faces number at most $|F(V')|$. Observe, the quantity $\deg(V')$ does not increase in going from $G_{F(V')}$ to $T_{F(V')}$. Hence, for fixed $V' \subseteq V_{\text{ext}}$, quantity (2) is minimized when $G_{F(V')}$ is assumed to be a fully triangulated planar graph whose interior vertices are exactly V'.

We now compute quantity (2) under this assumption. Let V'' denote the vertices of G incident to the faces $F(V')$. Let $B = V'' - V'$. Let $\deg(B)$ be the sum of the degrees (in the graph $G_{F(V')}$) of the vertices of B. Let us recall a few simple facts: (a) in any graph with, the sum of the vertex degrees equals twice the number of edges; (b) any triangulated planar graph on p vertices has exactly $3p - 6$ edges; (c) in any biconnected planar graph, three times the number of faces (remember to include the exterior face) equals two times the number of edges. Assuming the conditions stated at the end of the last paragraph, it follows from (a) and (b) that

$$\begin{aligned} \deg(V') + \deg(B) &= 2[3(|V'| + 3) - 6] \\ &= 6|V'| + 6. \end{aligned}$$

Additionally, it follows from (c) that

$$3(|F(V')| + 1) = 2(3|V'| + 3)$$

or by rearrangement

$$|F(V')| = 2|V'| + 1.$$

The quantity (2) now evaluates to

$$\begin{aligned} 2\pi|V'| + \pi - 6\alpha|V'| - 3\alpha &- 2\pi|V'| + 6\alpha|V'| + 6\alpha - \alpha\deg(B) \\ &= \pi + 3\alpha - \alpha\deg(B) \\ &\geq \pi + 3\alpha - 3\alpha d. \end{aligned}$$

The latter quantity is greater than or equal to zero iff $\alpha \leq \frac{\pi}{3(d-1)}$. This finishes the case where S contains no members of $V_{\text{ext}} \cup \{f_{\text{ext}}\}$.

Let us now assume that S contains all the members of $V_{\text{ext}} \cup \{f_{\text{ext}}\}$. In this case, $f_{\text{ext}} \in F(V')$ and the capacity of the cut $(S, \overline{S})$ is

$$4\pi + \sum_{v \notin V'} 2\pi - \alpha\deg(v) + \sum_{f \in F(V')} \pi - 3\alpha.$$

We are interested in the values of α for which this capacity is always greater than or equal to $\sum_{v \in V} 2\pi - \alpha\deg(v)$ irregardless of which $V' \subseteq V$ containing V_{ext} is chosen. That is, we are asking for what values of α is

$$4\pi + (\pi - 3\alpha)|F(V')| - 2\pi|V'| + \alpha\deg(V') \tag{3}$$

greater than or equal to zero irregardless of which $V' \subseteq V$ containing V_{ext} is chosen? Arguing in a manner similar to the case considered previously, the quantity (3) is minimized when we assume that $G_{F(V')}$ is a fully triangulated planar graph with vertex set V'. Under this assumption, we find that (3) is greater than or equal to zero (and actually equals zero) for all α. This finishes the case where S contains all the members of $V_{\text{ext}} \cup \{f_{\text{ext}}\}$.

We have now shown that all cuts $(S, \overline{S})$ have capacity $\geq \sum_{v \in V} 2\pi - \alpha\deg(v)$ as long as $\alpha \leq \frac{\pi}{3(d-1)}$. Under this assumption, K has a flow that saturates all s-arcs. Thus $\alpha \leq \frac{\pi}{3(d-1)}$ is a lower bound for the optimal value of the LP. ■

6 The Angular Resolution of Outerplanar Graphs

In this section, we prove that every triangulated outerplanar graph with maximum degree d has a planar straight-line drawing in which all angles are $\Omega(\frac{1}{d})$, and all interior faces are similar isosceles triangles.

First, we introduce some notation. Let AB denote a line segment in the plane with endpoints labeled A, B. Let $\triangle ABC$ denote a triangle with vertices labeled A, B, C. Let $\angle ABC$ denote the *magnitude* of the angle between the line segments BA and BC. We begin with a technical lemma.

Lemma 6.1 *Fix any integer $t \geq 2$. Let $\alpha = \frac{\pi}{2(t+2)}$. Draw $\triangle ABC$ as an isosceles triangle with edge AB of length 1 lying on the x-axis, vertex C above the x-axis, and $\angle CAB = \angle CBA = \alpha$. Let s be the height of vertex C above the x-axis, and let r be the length of the edge AC. Let R_A be a ray emanating from A passing through a point Q_A directly above C at a height $l = s\sum_{i=0}^{\infty} r^i$ above the x-axis. Then the angle β between R_A and the edge AC satisfies $(t-1)\alpha + \beta < \frac{\pi}{2}$. (See Figure 9).*

Proof. The result follows from a short sequence of equalities and inequalities.

$$
\begin{aligned}
\beta + \alpha &= \arctan 2l \\
&= \arctan \frac{2s}{1-r} \\
&< \frac{2s}{1-r} \quad \text{(because } \tan x > x \text{ for } x > 0) \\
&= \frac{2\sin\alpha}{2\cos\alpha - 1} \quad \text{(because } r\cos\alpha = \tfrac{1}{2} \text{ and } r\sin\alpha = s) \\
&< \frac{2\alpha}{2(1-\frac{\alpha^2}{2}) - 1} \\
&= \frac{2\alpha}{1-\alpha^2} \\
&\leq 4\alpha \quad \text{(as long as } \alpha \leq \tfrac{1}{\sqrt{2}}).
\end{aligned}
$$

Since $\alpha = \frac{\pi}{2(t+2)} < 1/\sqrt{2}$, we have from above, $\beta < 3\alpha$. Therefore, $(t-1)\alpha + \beta < (t+2)\alpha \leq \frac{\pi}{2}$. ■

To state and prove the theorem of this section, we need two more definitions.

Given an isosceles triangle $\triangle ABC$ with axis of symmetry passing through C, define the *base angle* to be the value $\angle CAB$ (or equivalently $\angle CBA$.)

Given a triangulated outerplanar graph G, the *dual graph* for G is obtained by placing a vertex in each interior face of G and connecting two vertices by an edge when the corresponding faces of G share an edge. Clearly, the dual graph of an outerplanar graph is always a tree.

Theorem 6.1 *Let G be a triangulated outerplanar graph with exterior face F, and maximum degree d. Let $\triangle ABC$ be any face of G where edge AB lies on F. Let δ_A denote the degree of vertex A in G, and δ_B denote the degree of vertex B in G. Then G admits a planar straight-line drawing $D(G)$ in which all interior faces are similar isosceles triangles with base angle $\alpha = \frac{\pi}{2(d+2)}$. Assume that in this drawing, the edge AB of triangle $\triangle ABC$ is* *of length 1 and lies on the x-axis. Let β be the angle defined in Lemma 5.1. Then $D(G)$ lies inside a triangle $\triangle ABP$ where $\angle PAB \leq (\delta_A - 1)\alpha + \beta < \frac{\pi}{2}$ and $\angle PBA \leq (\delta_B - 1)\alpha + \beta < \frac{\pi}{2}$.*

Proof Let T be the dual tree of G, and suppose T is rooted at $\triangle ABC$. We prove the theorem by induction on the height of T.

If the height of T is 0 (so G consists only of the face $\triangle ABC$), the theorem clearly holds.

So suppose the theorem holds when the height of T is $\leq h$. We now show the theorem holds when the height of T equals $h + 1$.

Consider the triangle $\triangle ABC$. By removing the edge AB from G and duplicating the vertex C (let B_1 and B_2 denote the copies of C) (see Figure 10), the graph G is split into two triangulated outerplanar subgraphs G_1 and G_2. Let $A_1 = A$ and $A_2 = B$. Let T_1 be the dual tree for G_1 rooted at the face $\triangle A_1B_1C_1$ and T_2 be the dual tree for G_2 rooted at the face $\triangle A_2B_2C_2$. The height of both T_1 and T_2 is at most h. So by the induction hypothesis, there are planar straight-line drawings $D(G_1)$ and $D(G_2)$ satisfying the conclusions of the theorem for each of G_1 and G_2 respectively.

Now we construct the drawing $D(G)$ for G and show that it satisfies the conclusions of the theorem. Take the naked triangle $\triangle ABC$ and draw it isosceles with base angle α. Put the edge AB on the x-axis and give it length 1. Next, scale and rotate the drawings $D(G_1)$ and $D(G_2)$ so they abutt the triangle $\triangle ABC$ as shown in Figure 11. Let $D(G)$ denote the resulting drawing of G. Clearly $D(G)$ is planar because by the induction hypothesis, each of $D(G_1)$ and $D(G_2)$ is planar, and each lies inside a triangle (meaning they will not conflict with each other in the drawing $D(G)$.)

It remains to show that $D(G)$ lies inside a triangle $\triangle ABP$ with the desired angles. Let $r < 1$ be the length of the edge AC in the triangle $\triangle ABC$. Let s be the height of C above the x-axis. It is not difficult to see that the highest vertex in the drawing $D(G)$ has height less than $s\sum_{i=0}^{h-1} r^i < s\sum_{i=0}^{\infty} r^i$. Let l denote this value $s\sum_{i=0}^{\infty} r^i$. Consider the ray R_A emanating from A defined as follows. If $G_1 = \phi$, then R_A passes through a point Q_A directly above C at a height l above the x-axis. (The angle between R_A and AC is precisely β from Lemma 5.1.) If $G_1 \neq \phi$, then R_A coincides with the edge A_1P_1 of the triangle $\triangle A_1B_1P_1$ that contains $D(G_1)$. Similarly define the ray R_B. By the induction hypothesis, the angle between the ray R_A and the segment AB is at most $(\delta_A - 2)\alpha + \beta + \alpha = (\delta_A - 1)\alpha + \beta < \frac{\pi}{2}$. Similarly, the angle between the ray R_B and the segment BA is at most $(\delta_B - 2)\alpha + \beta + \alpha = (\delta_B - 1)\alpha + \beta < \frac{\pi}{2}$. Hence the two rays R_A and R_B cross at some point P, and $D(G)$ lies in the triangle $\triangle ABP$ with the desired angles. ■

Figure 12 gives a picture of the planar straight-line drawing generated by Theorem 5.1 on a specific example. We would have drawn more triangles had we a thinner line and a good microscope. The purpose would be to demonstrate an interesting aspect of our drawing—for certain outerplanar graphs with a large number of vertices, the triangles in the drawing turn through arbitrarily large angles. In other words, the boundary of the drawing may not be nearly so dry as bland as we have drawn, but rather be contoured into tapering, branching, spirals remeniscent of fractals.

7 Comments on Three Drawing Algorithms

Let G be any 3-connected planar graph. Tutte [15] shows that any such graph has a planar straight-line drawing in which every face, including the exterior face, is a convex polygon. Call such a drawing a *convex* drawing. In this section, we consider three methods from the literature for generating convex drawings of 3-connected planar graphs, and examine them with respect to angular resolution. The methods we consider are those of Tutte [15], Becker and Hotz [2], and Becker and Osthof [4]. We will exhibit an infinite sequence of 3-regular 3-connected planar graphs for which all three methods fail to insure angular resolution bounded away from zero.

Let $G_1, G_2, G_3, \ldots$ be the infinite sequence of 3-regular 3-connected planar graphs depicted in Figure 13. Take the exterior face of each G_i to be it's bounding triangle. Also, let e_i and θ_i be the indicated edge and angle for G_i.

Given a 3-connected planar graph G with exterior face f_{ext}, Tutte's method starts with a prescribed drawing of f_{ext} as a convex polygon, and then places the interior vertices of G inside f_{ext} so that each is at the center of mass of its neighbors. Tutte shows that the locations of the interior vertices are uniquely determined and distinct, and that all faces of this drawing are convex. Say we draw the exterior face of each G_i as an equilateral triangle with unit-length edges, and consider the drawing of G_i obtained by Tutte's method. Since each interior vertex is at the center of mass of its neighbors, and each face is convex, it is rather clear that the length of the edge e_i must be going to zero as $i \to \infty$. Hence the angle θ_i must be going to zero as $i \to \infty$.

Fix $1 \leq p < \infty$. Given a 3-connected planar graph G with exterior face f_{ext}, the method of Becker and Hotz starts with a prescribed drawing of f_{ext} as a convex polygon, and then positions the interior vertices of G inside f_{ext} so as to minimize $\sum_e |e|^p$, where the sum is taken over all edges $e \in G$ and $|e|$ denotes the Euclidean length of e. Becker and Hotz show that if $1 < p < \infty$, then the locations of the interior vertices are uniquely determined and distinct, and that all faces of the drawing are convex. It is not difficult to show that for $p = 2$, the Becker and Hotz drawing is identical to that of Tutte. So for the case $p = 2$, the Becker and Hotz drawing fails to insure angular resolution bounded away from zero on the sequence G_i (assuming the exterior face of each G_i is an equilateral triangle with unit edge-lengths.)

Let G be a 3-connected planar graph with exterior face f_{ext} which has been given a prescribed drawing as a convex polygon. For $p = 2, 3, 4, \ldots$, let $D_p(G)$ denote the drawing of G obtained by the Becker and Hotz method. In Becker and Osthof [4], it is shown that $D_p(G)$ approaches a limiting drawing $D_\infty(G)$ as $p \to \infty$ where $D_\infty(G)$ minimizes the maximum Euclidean edge length, and maintains convexity of all the faces. In general, the locations of the interior vertices of G are not necessarily distinct in $D_\infty(G)$, but in the case of the G_i in Figure 10, the interior vertices occupy distinct locations in $D_\infty(G_i)$. So it is rather clear that the length of the edge e_i in $D_\infty(G_i)$ must be going to zero as $i \to \infty$. Hence the angle θ_i must be going to zero as $i \to \infty$.

We suspect that θ_i exhibits the same behavior for any fixed p with $1 < p < \infty$. (Note that for $p = 1$, the drawing $D_1(G_i)$ collapses all the interior vertices of G_i to a single point.)

8 Open Questions

We list the following open questions:

- Let G be an arbitrary 3-connected planar graph with exterior face f_{ext}. By a theorem of Tutte [15], any such graph has a planar straight-line drawing in which every face, including the exterior face f_{ext}, is a convex polygon. Call such a drawing of G a *convex* drawing. Let Γ_0 and Γ_1 be two convex drawings of G.

  Question: Is there a continuous deformation $C(t)$, $t \in [0,1]$ taking $\Gamma_0 = C(0)$ into $\Gamma_1 = C(1)$ such that: (1) for all t, $C(t)$ is a convex drawing of G; (2) for each angle ν in G, $\nu(t)$ has no local minimum on the *open* interval (0,1)?

  Notice that condition (2) is equivalent to saying that any local maximum of $\nu(t)$ on the closed interval [0,1] is a global maximum. Let C be the space of all convex drawings of G—a space easily described algebraically by considering all 3-tuples of consecutive vertices along faces of G. For a point $z \in C$, let $\beta(z)$ be the value of the minimum angle in the convex drawing z for G. If the answer to the above question is yes, then it is easily shown that any local maximum of β on C is a global maximum. This would mean that a convex drawing with maximum angular resolution (under the restriction of convex faces) could be obtained for G by a gradient ascent (hill-climbing) algorithm.

- Is it the case that for any 3-connected planar graph G with exterior face f_{ext}, G has a convex drawing with maximum angular resolution (under the restriction of convex faces) where the vertices of this drawing are at distinct locations?

- Let G be an arbitrary planar graph with maximum degree d and exterior face f_{ext}. Let Γ be a convex drawing of G with maximum angular resolution. Does Γ have any special combinatorial properties that could be exploited to prove a better lower bound on angular resolution than σ^d, σ a constant in (0,1)?

- Can Theorem 2.1 and Lemma 2.2 be generalized to 3 dimensions and higher?

- **Miller, Teng, Vavasis [13]:** Is there an efficient algorithm to obtain the disc-packing of Theorem 2.1?

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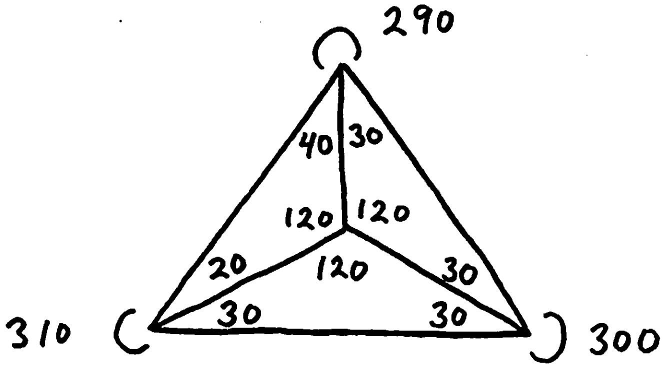

290
40
30
120
120
20
120
30
30
30
310
300

FIG. 1

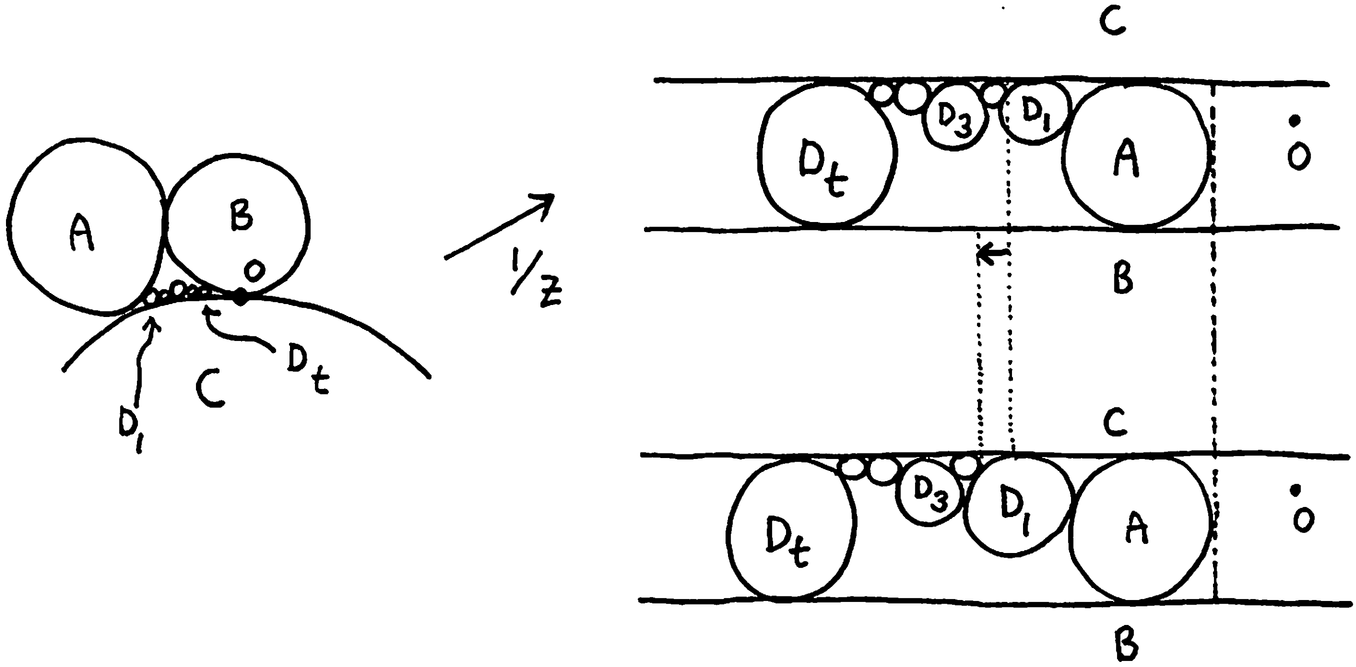
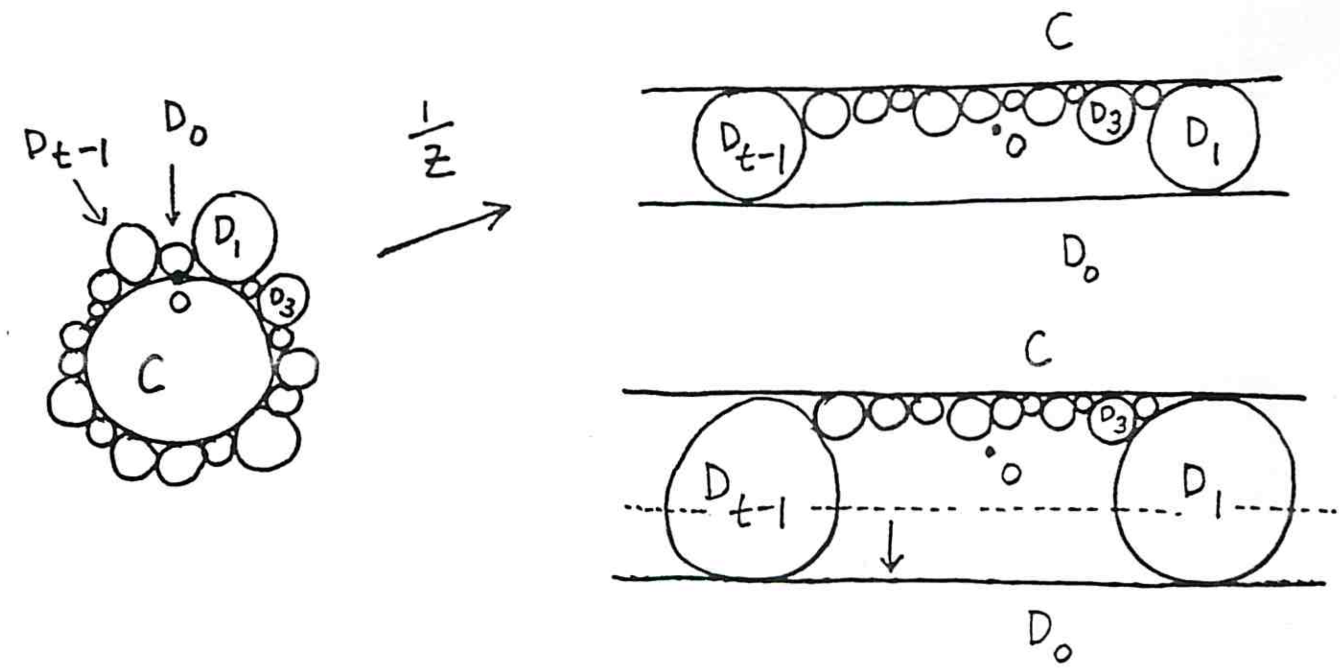
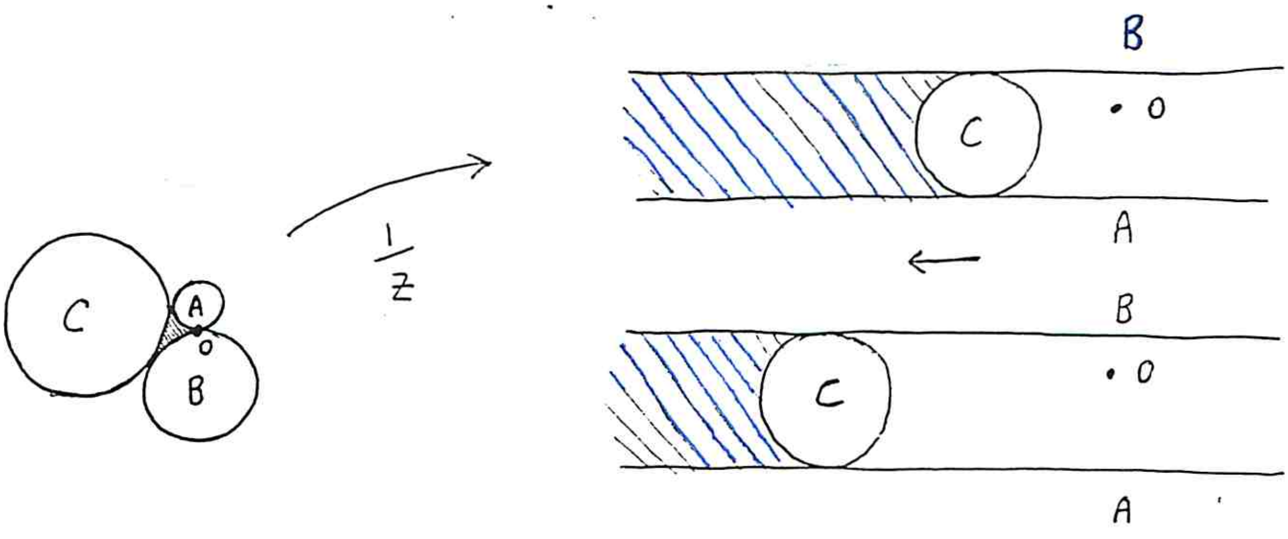

A
B
o
C
D_t
D_1
1/2
C
D_3
D_1
D_t
A
o
B
C
D_3
D_1
D_t
A
o
B

FIG. 2



D_{t-1}
D_0
D_1
0
D_3
C
$\frac{1}{z}$
C
D_{t-1}
0
D_3
D_1
D_0
C
D_3
0
D_{t-1}
D_1
D_0

FIG. 3



C
A
0
B
$\frac{1}{z}$
B
C
0
A
B
0
C
A

FIG. 4

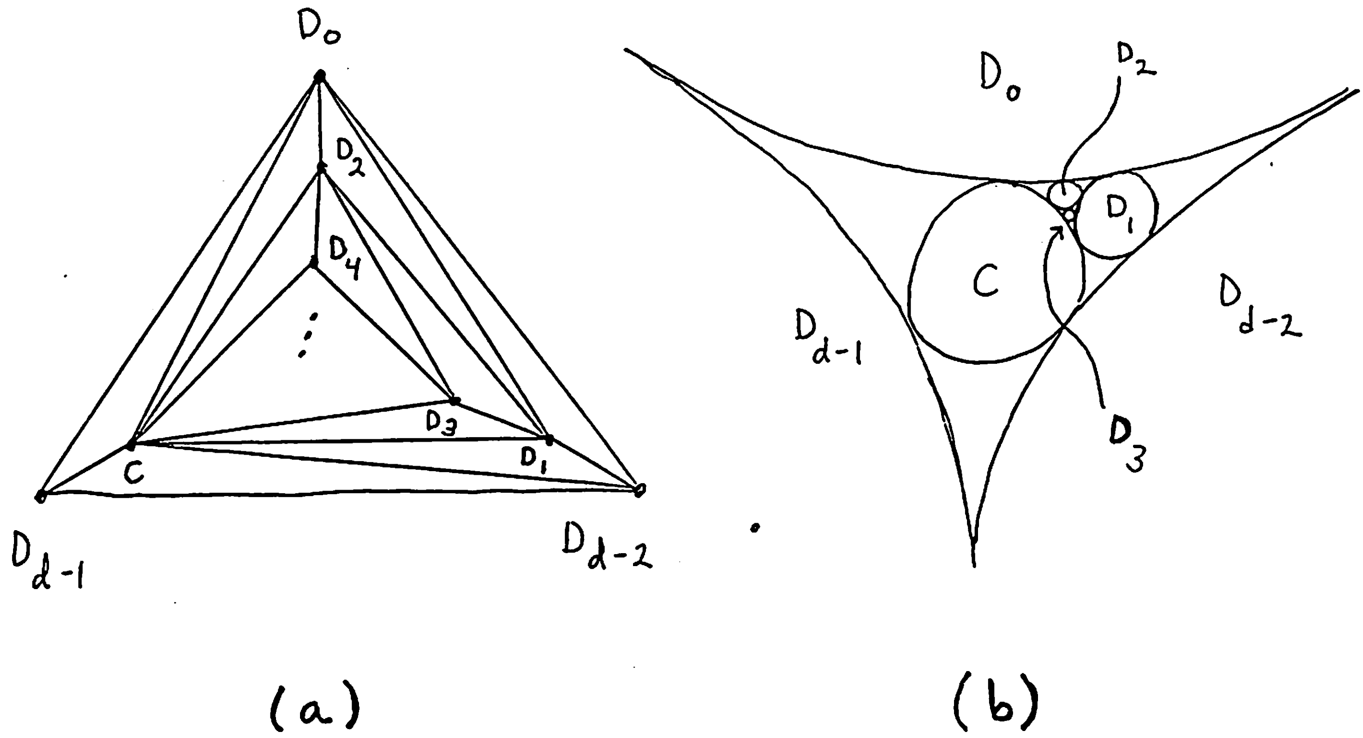


FIG. 5

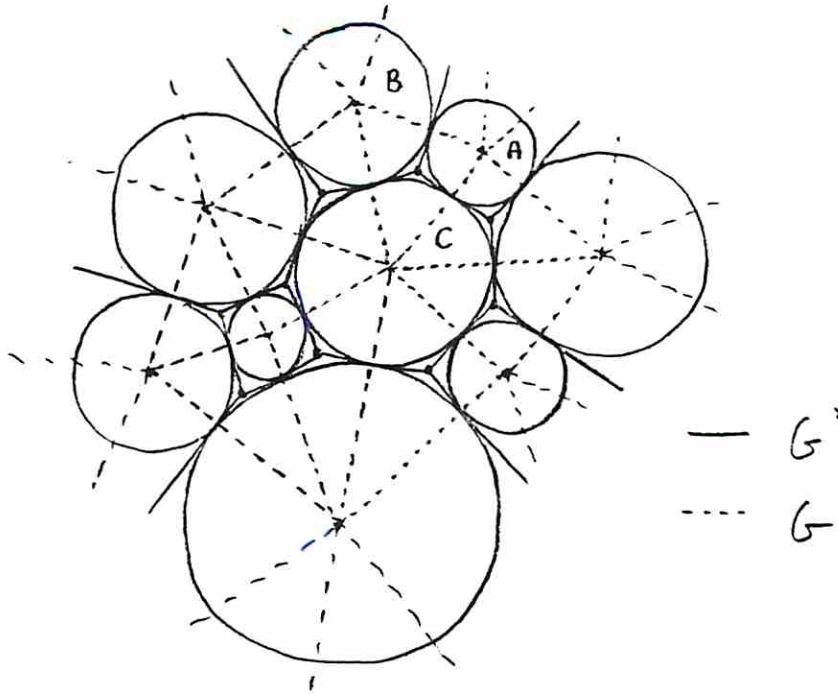


FIG. 6

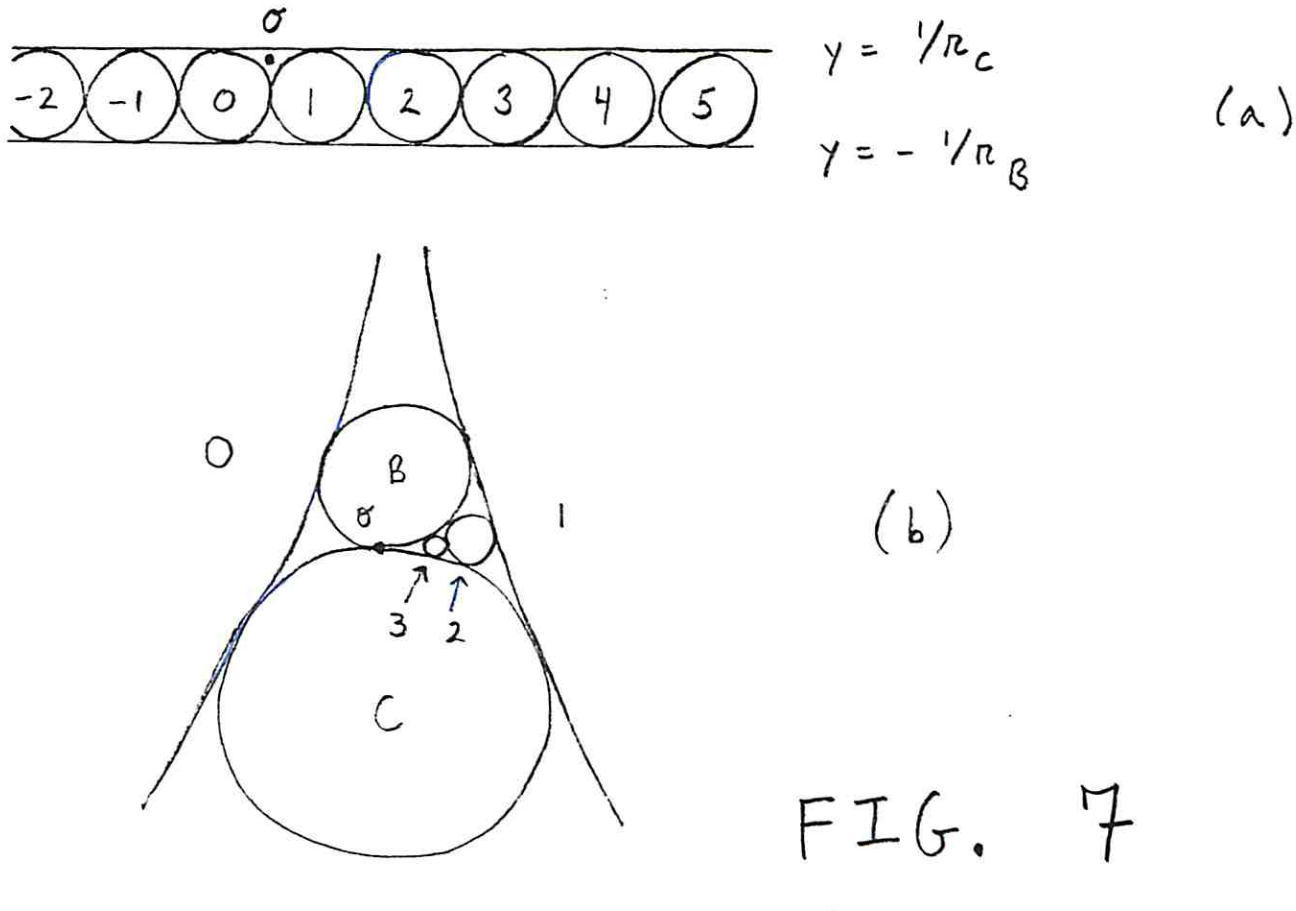


FIG. 7

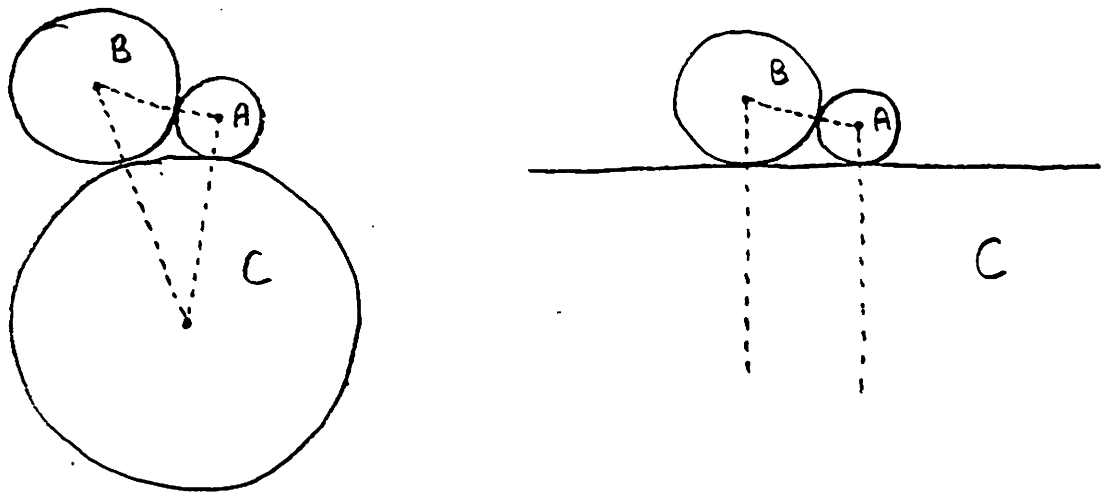


FIG. 8

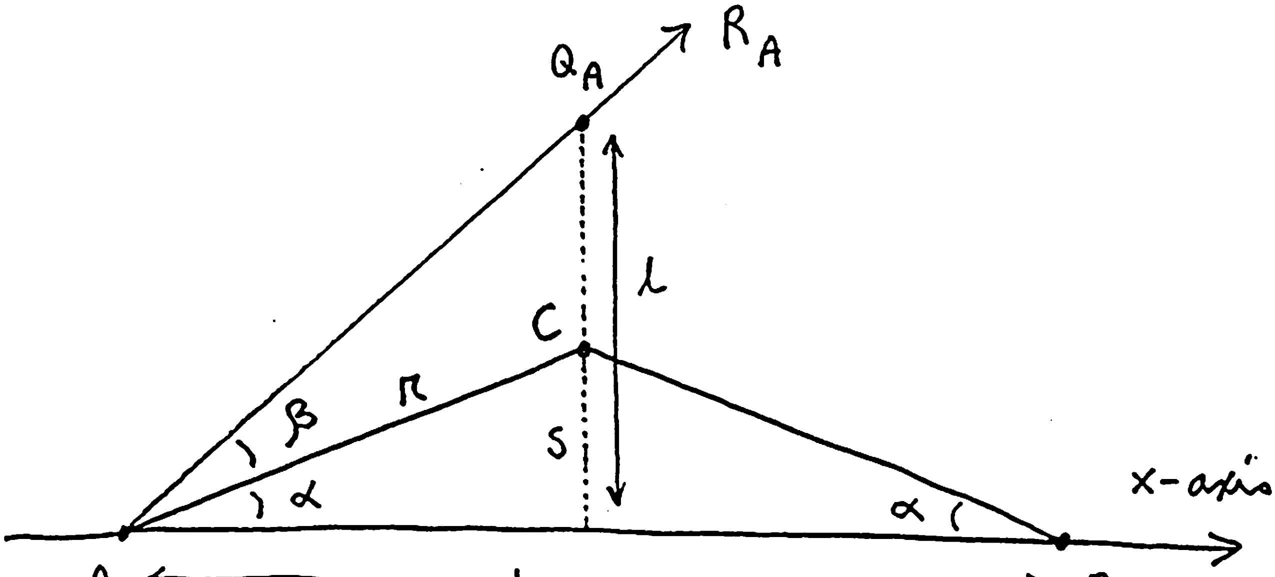


FIG. 9

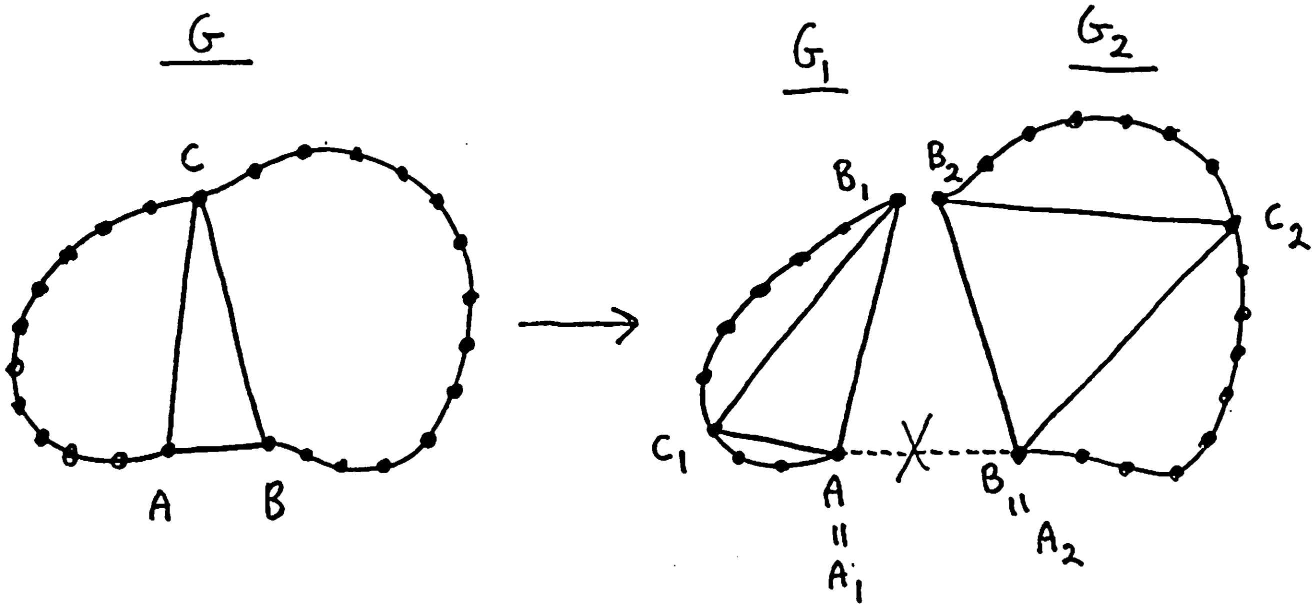


FIG. 10

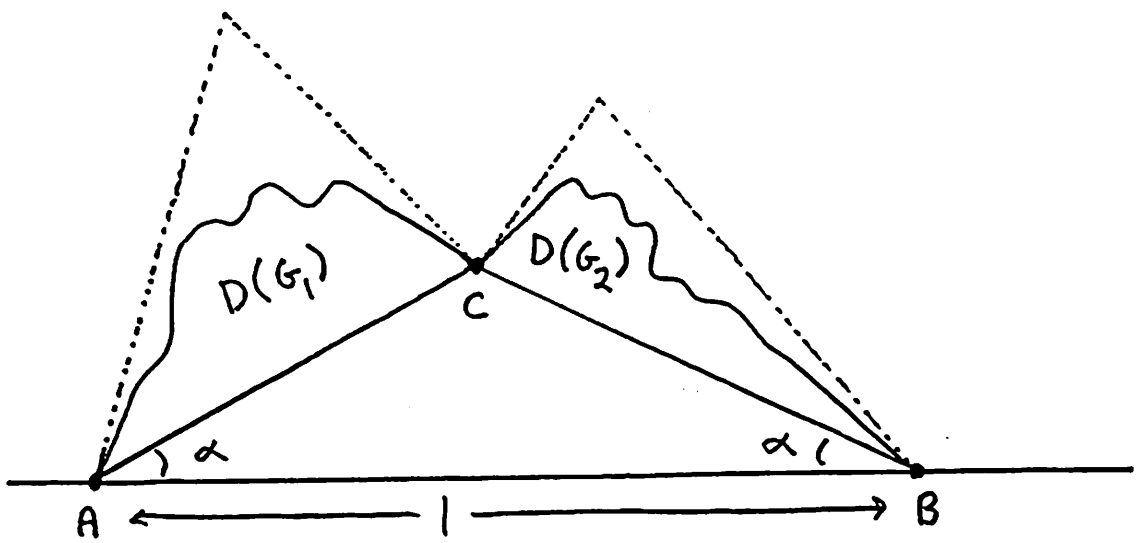


FIG. 11

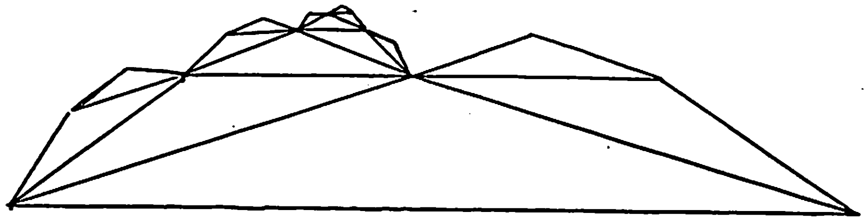


FIG. 12

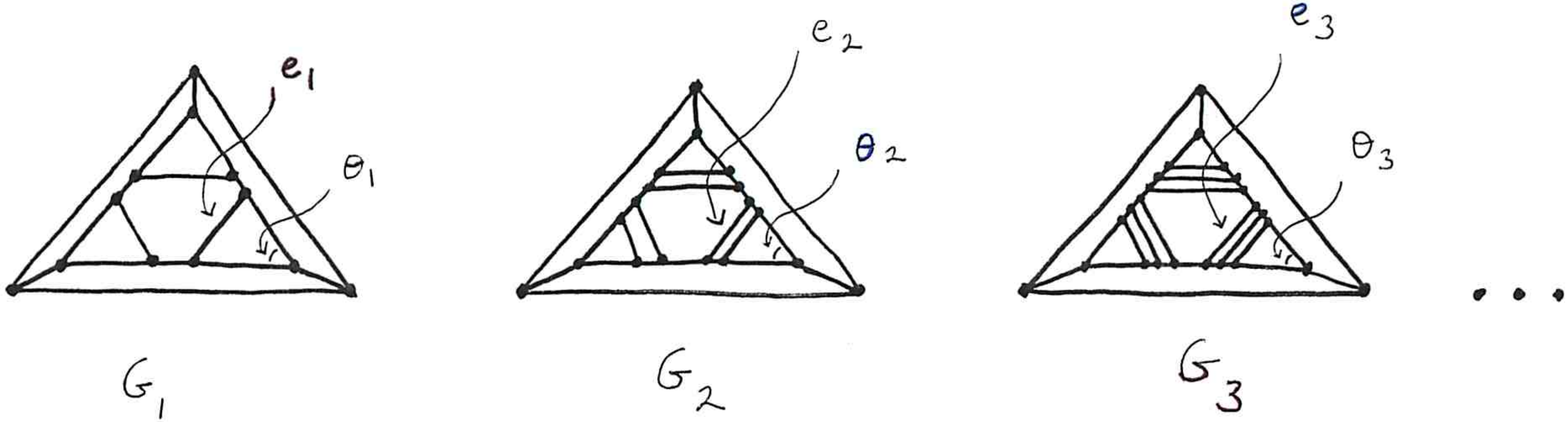


FIG. 13