Separators and Hypercube Embeddings

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Abstract

Let l and α be constants, l a positive integer, α a real number in the interval (0,1]. Letting the positive integer N vary, we exhibit a sequence of bounded-degree N-node graphs $\{J_N\}$ possessing $O(N^{\alpha})$-separators such that any load-l embedding of J_N into the N-node hypercube has dilation $\Omega(\log N)$. This extends a result of Bhatt et al., *Efficient embeddings of trees in hypercubes*, SIAM J. of Computing (to appear), who considered [redacted] the special case $\alpha = 1$.

Keywords: separator, hypercube, graph embedding, load, dilation, expander graph.

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†This work was done in 1988 while the author was a gradute student at MIT.

1 Introduction

The *hypercube* architecture is one of the most popular architectures for universal parallel computation. The reasons are many. It has a convenient recursive structure that lends itself well to many divide-and-conquer algorithms. It supports efficient message routing schemes which form the backbone of universal parallel computation. It is highly fault-tolerant, able to withstand a very large number of random faults while incurring only a small degradation in performance. And last but not least, many other useful networks (e.g. multi-dimensional grids, complete binary trees, butterflies, pyramids, meshes of trees, etc.) are embedded or nearly embedded in the hypercube as subgraphs. The hypercube can therefore simulate algorithms on these networks with very little or no communication overhead.

Recently, Bhatt et al. [1] showed how to embed any bounded-degree N-node network having $O(1)$-separator (e.g. trees, outerplanar graphs, etc.) into an N-node hypercube with constant load, dilation, and congestion. It follows that the hypercube, using constant communication overhead, can simulate any bounded-degree network possessing $O(1)$-separator. As for interesting lower bounds, they showed that any constant-load embedding of a bounded-degree N-node expander graph (these have $O(N)$-separators) into an N-node hypercube requires dilation $\Omega(\log N)$.

The purpose of this note is to elaborate a little further on the relationship between separator-size of the guest network and its minimum dilation under constant-load embeddings in the hypercube. Specifically, for any fixed α in the real interval (0,1], we exhibit a sequence of bounded-degree graphs $\{J_N\}$ (parameterized by number of vertices) such that J_N has an $O(N^\alpha)$-separator and requires dilation $\Omega(\log N)$ under any constant-load embedding into the N-node hypercube. The J_N's are constructed by taking the disjoint union of a bounded-degree N^α-node expander graph together with $N - N^\alpha$ isolated nodes.

This note has relevance to an open question posed by Bhatt et al., namely can every bounded-degree N-node planar graph graph (these have $O(\sqrt{N})$-separators) be embedded in the N-node hypercube with constant load and dilation? Our result shows that if the answer to this question is yes, then it does not follow merely as a corollary to the fact that bounded-degree N-node planar graphs have $O(\sqrt{N})$-separators.

2 Definitions

An *embedding* $< \phi, \rho >$ of a graph $G = (V_G, E_G)$ into a graph $H = (V_H, E_H)$ is defined by a mapping ϕ from V_G to V_H, together with a mapping ρ that assigns each $(u, v) \in E_G$ onto a path $\rho(\phi(u), \phi(v))$ in H that connects $\phi(u)$ and $\phi(v)$. The *load* on a vertex $v \in H$ is the number of nodes of G that are mapped onto v, and the *load* of the embedding is the maximum load on any vertex in H.

The *dilation* of an edge $(u, v) \in E_G$ under $< \phi, \rho >$ equals the length of the path $\rho(\phi(u), \phi(v))$ in H. The *dilation* of the embedding is the maximim dilation of any edge in G. The *average dilation* of the embedding is the average dilation of all the edges in G.

The *congestion* of a node or edge in H is the number of edges of G that are routed through it. The *congestion* of an embedding is the maximum congestion of any node or edge in H.

A class of graphs closed under the subgraph relation has an $O(f(N))$*-separator* if there exist constants a and b where $0 < a \leq \frac{1}{2}$ and $b > 0$ such that every N-node graph in the class can be partitioned (by the removal of at most $b \cdot f(N)$ edges) into disjoint subgraphs having $a'N$ and $(1 - a')N$ nodes, where $a \leq a' \leq 1 - a$.

A graph $G = (V, E)$ is an *expander graph* if there is a constant α such that every vertex set $X \subset V$ with $|X| < |V|/2$ is adjacent to at least $\alpha|X|$ vertices in $V - X$.

3 Graphs with $O(N^\alpha)$-Separators Requiring Dilation $\Omega(\log N)$

Theorem 3.1. Let $G =< V, E >$ be a bounded-degree expander graph with m nodes. Then any load-$l$embedding of G into the N-node hypercube yields average dilation $\Omega(d)$, where d is the largest integer such that $m/l \geq \sum_{i \leq d} \binom{\log_2 N}{i}$.

Proof. Let $< \phi, \rho >$ be any embedding of G into the N-node hypercube with load l. Let $D(< \phi, \rho >)$ be the sum of all the dilations of all the edges of G. By adding up the number of edges in G routed across each dimension of the hypercube, we will show that $D(<\phi,\rho>) = \Omega(dm)$. This means that the average dilation of the embedding is $\Omega(d)$ since G has $O(m)$ edges.

To begin the argument, enumerate the dimensions of the hypercube $1, 2, \ldots, \log N$ in any manner. Partition the hypercube across some dimension, say dimension i. This partitions $\phi(V)$, and hence V, into two subsets. Let S_i be the smaller subset of V. (In the case of a tie, arbitrarily designate one of the two subsets as smaller.) Since G is a bounded-degree expander graph, the number of edges routed across dimension i is $\Omega(|S_i|)$. Hence

$$D(<\phi,\rho>) = \Omega(\sum_{i \leq \log_2 N} |S_i|).$$

Another way to count the quantity on the right is to add up over all vertices $v \in V$, the number of dimensions that place v in the smaller set. For each $v \in V$, let P_v be a binary $\log N$-vector with a 1 in the ith position iff $v \in S_i$. Let $w(P_v)$ be the number of 1's in P_v. Then

$$\sum_{i \leq \log_2 N} |S_i| = \sum_{v \in V} w(P_v).$$

What is the smallest value that $\sum_{v \in V} w(P_v)$ can assume? We know two things: (1) at most l nodes of V are mapped to the same image, and (2) the relation $\phi(u) \neq \phi(v)$ implies $P_u \neq P_v$. Hence $\sum_{v \in V} w(P_v)$ is minimum when $\phi(V)$ is a ball centered at $0^{\log_2 N}$, and each node of this ball is the image of exactly l nodes of V. Clearly the radius of such a ball is at least as large as the largest integer d such that $m/l \geq \sum_{i \leq d} \binom{\log_2 N}{i}$. Thus we have the lower bound

$$\sum_{v \in V} w(P_v) \geq \sum_{i \leq d} il \binom{\log_2 N}{i}.$$

This latter quantity is easily shown to be $\Omega(dm)$ by considering the separate cases $d > \frac{1}{3}\log_2 N$ and $d \leq \frac{1}{3}\log_2 N$. Hence $D(<\phi,\rho>) = \Omega(dm)$. Since G has $O(m)$ edges, the average dilation of the embedding is $\Omega(d)$. ■

Corollary 3.2 Let l and α be constants, l a positive integer, α a real number in $(0,1]$. There is a sequence $\{J_N\}$ of bounded-degree N-node graphs with $O(N^\alpha)$-separators such that any load-l embedding of J_N into the N-node hypercube requires dilation $\Omega(\log N)$.

Proof. Let J_N be the disjoint union of an N^α-node bounded-degree expander graph, G_{N^α}, and a set of $N - N^\alpha$ isolated nodes. Certainly J_N is a bounded-degree N-node graph with $O(N^\alpha)$-separator. We now show that any load-l embedding of G_{N^α} (and therefore of J_N) into the the N-node hypercube requires dilation $\Omega(\log N)$. By Theorem 3.1, any load-l embedding of G_{N^α} into the N-node hypercube requires dilation $\geq d$ where d is the largest integer satisfying $\sum_{i\leq d}\binom{\log_2 N}{i} \leq N^\alpha/l$. Let $\delta = \min(d, \frac{1}{3}\log_2 N)$. Since $\binom{\log_2 N}{i} \geq 2\binom{\log_2 N}{i-1}$ for all $i \leq \delta$, it follows $\sum_{i\leq\delta}\binom{\log_2 N}{i} < 2\binom{\log_2 N}{\delta}$. Suppose $\delta = \beta\log_2 N$. Then

$$\sum_{i\leq\delta}\binom{\log_2 N}{i} < 2\binom{log_2 N}{\delta} < 2\left(\frac{e\log_2 N}{\delta}\right)^{\delta} = 2\left(\frac{e}{\beta}\right)^{\beta\log_2 N}.$$

Now $2\left(\frac{e}{\beta}\right)^{\beta\log_2 N} \leq N^\alpha/l$ holds iff $\left(\frac{e}{\beta}\right)^{\beta} \leq 2^\alpha(1/2l)^{\frac{1}{\log_2 N}}$. Since l is a constant, there is a fixed value of β in $(0,1)$ that makes the latter inequality hold for all N. Hence $\delta = \Omega(\log N)$, and thus $d = \Omega(\log N)$, since $d \geq \delta$. ■

References

[1] S. Bhatt, F. Chung, T. Leighton, A. Rosenberg, *Efficient embeddings of trees in hypercubes*, SIAM J. of Computing, to appear.