Bounds on Chain Lengths for Collections of Non-negative Lattice Points in n-Space

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Abstract

Consider the poset of non-negative lattice points in the plane, where $A = (a_x, a_y) \leq B = (b_x, b_y) \iff a_x \leq b_x$ and $a_y \leq b_y$. An S-chain is a set of S points which are pairwise comparable. Consider the family of lines $x + y = k$, where $k \geq 0$. Select a point from each of these lines starting with the 0th line (the origin). Given $S > 0$, we obtain optimal upper bounds on the number of points that need to be selected in order to guarantee the existence of an S-chain among the selected points. In n-dimensions, the family of lines above is replaced by a family of planes: a point belongs to the kth plane if the sum of its coordinates equals k, $k \geq 0$. Now, select a point from each of these planes starting with the 0th plane. Given $S > 0$, we obtain upper bounds on the number of points that need to be selected in order to guarantee the existence of an S-chain among the selected points. For $n > 2$, however, the bounds obtained are not optimal.

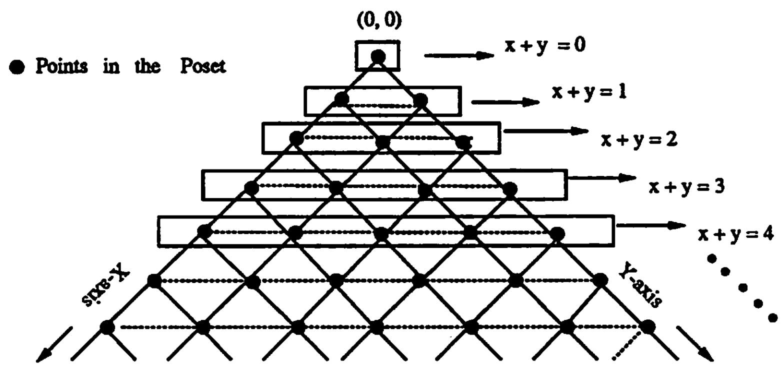

Figure 1: The 2-D Poset and its Diagonals

1 Introduction

Consider points in the first quadrant in 2-dimensions with non-negative integer coordinates. These points form a poset. Point $A = (a_x, a_y)$ is $\leq B = (b_x, b_y)$ if and only if $a_x \leq b_x$ and $a_y \leq b_y$. Points A and B are comparable if and only if either $A \leq B$ or $B \leq A$; otherwise they are incomparable. An S-chain in this poset is a set of S points any two of which are comparable. We define the kth ($k \geq 0$) diagonal in this poset as the line with $(k+1)$ points that satisfies the equation

$$x + y = k$$

For $k \geq 0$, the points on these diagonals partition the poset.

Figure 1 shows the 2-d poset.

Now consider the following problem, which we formulate as a game G, and which we state for the above 2-d poset. (Its generalization to n-dimensions is considered in subsequent sections). G consists of several steps. At the ith step,

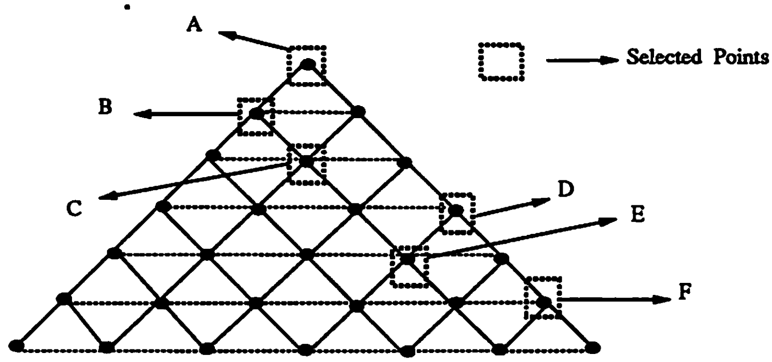

Figure 2: A Game of 6 moves on a 2-d Poset

a point is selected from the $(i-1)$th diagonal. G stops when there exists an S-chain among the points selected.

Figure 2 shows a game up to its sixth step. (Note: If the value of S was 4, the game would have stopped after 5 steps — points A, B, C and E form a chain of length 4).

A "clever" choice of points will prolong G (increase the number of steps of G) while a bad choice will force G to terminate quickly. We seek upper bounds (and if possible, optimal upper bounds) on the number of steps in G.

This paper is organized as follows. In section 2, we describe out notation. In section 3, we present by means of a recurrence relation, an upper bound for the general version of the above problem. In section 4, we show that the optimal upper bound on the number of steps in G for the above 2-d poset is 2^{S-1}.

2 Notation

Consider points with non-negative integer coordinates in the first orthant in n dimensions. These points form a poset $\mathcal{P}$. A point $A = (a_1, a_2, \ldots, a_n)$ is $\leq$ $B = (b_1, b_2, \ldots, b_n)$ if and only if $a_i \leq b_i$ $(1 \leq i \leq n)$. Two points A and B are "comparable" if and only if either $A \leq B$ or $B \leq A$; otherwise they are "incomparable".

An S-**chain** is a set of S points any two of which are comparable.

We identify the n axes by variables $x_1, x_2 \ldots x_n$.

For $k, k' \geq 0$ and i, $(1 \leq i \leq n)$, a **wall** is an infinite set of points which satisfy the equation

$$x_i = k, \quad x_1 + x_2 + \ldots + x_{i-1} + x_{i+1} + \ldots + x_n \geq k' \tag{1}$$

Bounding walls are given by equations

$$x_i = 0, \quad x_1 + x_2 + \ldots + x_{i-1} + x_{i+1} + \ldots + x_n \geq 0, \quad (1 \leq i \leq n)$$

Note that there are n bounding walls, and all points in $\mathcal{P}$ lie inside the region formed by these walls.

For any $k \geq 0$, the k**th base** (H_k) is the finite set of points $A = (a_1, a_2, \ldots, a_n)$ given by

$$H_k = \{A = (a_1, a_2, \ldots, a_n) \,|a_1 + a_2 + \ldots + a_n = k\} \tag{2}$$

The **kth plateau** (P_k) is the infinite set of points given by

$$P_k = \bigcup_{j \geq k} H_j \tag{3}$$

Clearly,

$$P_k = \{A = (a_1, a_2, \ldots, a_n) \,|\, a_1 + a_2 + \ldots + a_n \geq k\} \tag{4}$$

We say plateau P_k has height k. Let $\mathcal{H}$ be the function that computes height: $\mathcal{H}(P_k) = k$.

The **ideal** of C (I_C) is the infinite set of points A satisfying $A \geq C$:

$$I_C = \{A \,|\, A \geq C\} \tag{5}$$

(Note that I_C is isomorphic to the original poset $\mathcal{P}$, and that if $C \in H_k$, then $I_C \subseteq P_k$.)

The **co-ideal** of $C \in H_k$, (D_C), is the infinite set of points given by the equation

$$D_C = P_k - I_C \tag{6}$$

Thus, I_C contains only those points in P_k which are $\geq C$, while D_C contains those points in P_k which are incomparable to C.

Note that in all the terms defined above, n, the dimension of $\mathcal{P}$ has been suppressed. We write H_k^n, P_k^n, D_C^n, etc. only when needed.

We state the following fact without proof.

Fact 1 *Let W be any wall in n dimensions whose equation is given by (1). Then the poset structure of W is isomorphic to the poset structure of $P_{k'}^{n-1}$.*

We now state a lemma which describes D_C for an n-dimensional poset in terms of plateaus of $(n-1)$ dimensions.

Lemma 1 *Let $C = (c_1, c_2, c_3, \ldots, c_n)$ be a point in H_k^n. Then D_C^n is a union of k $(n-1)$-dimensional plateaus. Further, let P denote any of these k plateaus. Then $\mathcal{H}(P) \leq \mathcal{H}(P_k^{n-1}) = k$.*

Proof: Since $C \in H_k^n$, $\sum_{j=1}^{n} c_j = k$. Let $\mathcal{W}_i,\ 1 \leq i \leq n$ be the following set of walls.

$$\mathcal{W}_i = \{x_i = j,\ x_1 + x_2 + \ldots + x_{i-1} + x_{i+1} + \ldots + x_n \geq (k-j),\ \ 0 \leq j \leq (c_i - 1)\} \tag{7}$$

These are walls "parallel" to the bounding wall $x_i = 0$ and bounded above by the wall $x_i = c_i - 1$. There are a total of c_i walls in $\mathcal{W}_i$. By Fact 1, the first wall in $\mathcal{W}_i$ (substitute $j = 0$ in the definition of $\mathcal{W}_i$) is isomorphic to P_k^{n-1}, the second wall ($j = 1$) is isomorphic to P_{k-1}^{n-1} and so on up to the c_ith wall, which is isomorphic to $P_{k-c_i+1}^{n-1}$.

Let $\mathcal{W} = \bigcup_{i=1}^{n} \mathcal{W}_i$. The cardinality of $\mathcal{W}$ is

$$\begin{aligned} |\mathcal{W}| &= |\mathcal{W}_1| + |\mathcal{W}_2| + \ldots + |\mathcal{W}_n| \\ &= c_1 + c_2 + \ldots + c_n \\ &= k \end{aligned}$$

We will show that $D_C^n = \mathcal{W}$. Let $A = (a_1, a_2, \ldots, a_n) \in D_C^n$. Since A is incomparable to C and since $A \in P_k^n$, there exists $p,\ 1 \leq p \leq n,$ such that $a_p < c_p$ and $a_1 + a_2 + \ldots + a_{p-1} + a_{p+1} + \ldots + a_n \geq k - a_p$. Thus, A satisfies the equation of the wall $x_p = a_p$, $x_1 + x_2 \ldots x_{p-1} + x_{p+1} + \ldots + x_n \geq k - a_p$. But this is a wall in $\mathcal{W}_p$. The converse can be proved similarly. Now, since every wall in $\mathcal{W}$ isomorphic to a plateau in $(n-1)$-dimensions and since $|\mathcal{W}| = k$, the first part of the lemma is proved. Also, as noted earlier, the first wall in any $\mathcal{W}_i$ is isomorphic to P_k^{n-1}, the second wall is isomorphic to P_{k-1}^{n-1} and so on. Thus, the height of any plateau in $\mathcal{W}$ cannot be greater than k. □

3 Problem Description

We now describe the general version of the problem presented in section 1. It consists of playing a game G such that at step i, $i \geq 1$, a point is selected from base H_{i-1}. G stops when there exists an S-chain among the selected points. The problem then is to find an upper bound on the number of steps of G.

We first derive an upper bound for a slightly different game G'. G' is essentially G with the exception that at the first step, a point from base H_k ($k \geq 0$) is selected (and not necessarily from base H_0 as in G). At the ith step, we pick a point from base H_{k+i-1}. G' is thus played on P_k. Again, G' stops when an S-chain exists among the points selected. We denote this bound by $L(n, k, 1, S)$ where n is the dimension number, k is the first base from which a point is selected and S is the chain length which stops G'. (The significance of "1" in $L(n, k, 1, S)$ will become clear in the next paragraph).

In order to compute upper bounds for higher dimensions from the previous bounds for lower dimensions, we will need to play G'' — which is essentially G' played with some c ($c \geq 1$) copies of P_k^n (i.e. c plateaus in n dimensions, each with height k). At the first step of G'', a point is picked from base H_k from any of the c plateaus. At any subsequent ith step ($i > 1$), we first select a copy (from among the c copies) and then choose a point from base H_{k+i-1} in that copy. G'' stops when an S-chain exists among the points from some copy. We denote this bound by $L(n, k, c, S)$ where n, k and S are as above, and c is the number of copies of the plateau on which G'' is played.

(Note: The upper bound for G is just a special case of the upper bound for G'' and is given by $L(n, 0, 1, S)$).

Lemma 2 *Suppose we play G'' on c copies of n-dimensional plateaus with possibly different heights. Let the upper bound for G'' on these copies be U_1. Let k be the maximum height of all these copies. Let U_2 be the upper bound for G'' on c copies of n-dimensional plateaus all having the same height k. Then, $U_1 \leq U_2$.*

Proof: The proof follows from the simple observation that an n-dimensional plateau with height $< k$ is contained in an n-dimensional plateau with height k. Thus, every game G'' played on the earlier set of c copies can be duplicated on the latter set. □

3.1 The Base Case — $L(1, k, c, S)$

Lemma 3 *For all* $S \geq 1,\ k \geq 0$

$$L(1, k, 1, S) = S \tag{8}$$

Proof: Any two points in a 1-d poset are comparable. Hence, starting with any base, the first S points that are selected form a chain. □

Lemma 4 *For all* $S \geq 1,\ k \geq 0$ *and* $c \geq 1$

$$L(1, k, c, S) = (S-1)c + 1 \tag{9}$$

Proof: Consider playing G'' for $((S-1)c+1)$ steps on c copies of 1-d plateaus starting with any base k. One of the copies must have been selected for S or more steps, yielding an S-chain. □

3.2 Breaking up an $(N+1)$ dimensional region

We have so far obtained bounds for any S-chain in 1-dimension. We now show how to obtain bounds for higher dimensions.

Suppose we know $L(n, k, c, S)$ for all $n \leq n_0,\ k \geq 0,\ c \geq 1$ and $S \geq 1$.

We want to obtain $L(n_0+1, k, 1, S)$. Let us define the following recurrence

$$w_i = \begin{cases} 1 & \text{if } i = 0 \\ L(n_0, k, k, S) & \text{if } i = 1 \\ L\left(n_0, \left(k + \sum_{j=1}^{i-1} w_j\right), \left(k + \sum_{j=1}^{i-1} w_j\right), S\right) & \text{otherwise } (i \geq 2) \end{cases} \tag{10}$$

Lemma 5

$$L(n_0+1, k, 1, S) \leq w_0 + w_1 + w_2 + \ldots + w_{S-1} \tag{11}$$

where the w_i's are given by (10).

Proof: Consider playing G' on $P_k^{n_0+1}$ for $w_0 + w_1 + w_2 + \ldots + w_{S-1}$ steps. Let $t_i = \sum_{j=0}^{i} w_j,\ 0 \leq i \leq (S-1)$. Let $\mathcal{I}$ be the following invariant.

$\mathcal{C}_1$: Either G' has stopped before t_i steps because there exists an S-chain among the points already selected in $P_k^{n_0+1}$; or

$\mathcal{C}_2$: There exists a chain in $P_k^{n_0+1}$ whose size is $(i+1)$ after t_i steps of G'.

$\mathcal{I}$ is true provided either of $\mathcal{C}_1$ or $\mathcal{C}_2$ is true. We will show that $\mathcal{I}$ holds for G' after every block of w_i steps.

The base step, $i = 0$, is immediate, since the point selected at the first step of G' ensures a chain of length one. So suppose that $\mathcal{I}$ holds up to $t_r = \sum_{j=0}^{r} w_j$ steps. If G' has stopped before t_r steps (i.e. $\mathcal{C}_1$ is true), then $\mathcal{I}$ is trivially true for any future steps. So suppose $\mathcal{C}_2$ is true. Then there exists a chain of length $(r+1)$ among the points selected so far. Let A be the last point on this chain. Note that A was selected either at the t_rth step or before. In either case, I_A, the ideal of A, contains at least one point (A') such that $A' \in H_{k+t_r-1}$. (Note that A and A' may be the same point if A was chosen at the t_rth step.) Also, at the next step (step $t_r + 1$), a point will be selected from H_{k+t_r}.

Consider $D_{A'}$, the co-ideal of A'. By Lemma 1, $D_{A'}$ is a union of $(k+t_r-1)$ copies of n_0-dimensional plateaus, each of height at most $(k+t_r-1)$.

Now play G' for w_{r+1} more steps. By (10)

$$\begin{aligned} w_{r+1} &= L\left(n_0, \left(k + \sum_{j=1}^{r} w_j\right), \left(k + \sum_{j=1}^{r} w_j\right), S\right) \\ &= L\left(n_0, \left(k + (\sum_{j=0}^{r} w_j) - 1\right), \left(k + (\sum_{j=0}^{r} w_j) - 1\right), S\right) \\ &= L\left(n_0, (k+t_r-1), (k+t_r-1), S\right) \end{aligned}$$

The following cases arise.

Case 1: At least one point has been selected from the ideal of A' in the next w_{r+1} steps. In this case, the chain of length $(r+1)$ ending at A at the previous inductive step has now increased by one (since $I_{A'} \subseteq I_A$). Thus, $\mathcal{C}_2$ (and hence $\mathcal{I}$) holds.

Case 2: All next w_{r+1} points have been selected from $D_{A'}$. By the previous bounds for n_0 dimensions, there exists a chain of length S inside one of the n_0-dimensional plateaus constituting $D_{A'}$. In this case, G' stops, and $\mathcal{C}_1$ (and hence $\mathcal{I}$) holds.

To complete the proof, note that if $\mathcal{I}$ holds after the last block of w_{S-1} steps, then either $\mathcal{C}_1$ or $\mathcal{C}_2$ holds. In either case, there exists a chain of length S among the selected points. □

3.3 Increasing the number of copies

Now we show how to obtain $L(n_0, k, c_0+1, S)$ once we have obtained

1. $L(n, k, c, S)$ for all $n < n_0, \quad k \geq 0, \quad c \geq 1, \quad S \geq 1$; and

2. $L(n_0, k, c, S)$ for all $k \geq 0, \quad c \leq c_0, \quad S \geq 1$.

Let us define the following recurrence

$$v_i = \begin{cases} L(n_0, k, c_0, S) & \text{if } i = 1 \\ L\left(n_0 - 1, \left(k + \sum_{j=1}^{i-1} v_j\right), (c_0+1)\left(k + \sum_{j=1}^{i-1} v_j\right), S\right) & i > 1 \end{cases} \tag{12}$$

Lemma 6 *Let* $u = (c_0+1) * (S-1) + 1$. *Then*

$$L(n_0, k, c_0+1, S) \leq v_1 + v_2 + \ldots + v_u \tag{13}$$

where the v_i's are given by (12).

Proof: Consider playing G'' on (c_0+1) copies of $P_k^{n_0}$ for $v_1+v_1+\ldots+v_u$ steps.

Let $t_i = \sum_{j=1}^{i} v_j$, $1 \leq i \leq u$. We will show that the following invariant $\mathcal{I}$ holds for G'' after t_i steps.

$\mathcal{C}_1$: Either G'' has stopped before t_i steps because there exists an S-chain in some copy among previously selected points; or

$\mathcal{C}_2$: For $1 \leq j \leq (c_0+1)$, there exists a chain of length $S_j \geq 1$ in the jth copy such that

$$\sum_{j=1}^{c_0+1} S_j \geq c_0 + i \tag{14}$$

Thus, Condition $\mathcal{C}_2$ claims that after every block of v_i steps, there exists a chain in copy 1, another in copy 2, etc. such that the sum of the lengths of all these chains is at least c_0+i. The S_j's above depend on i.

$\mathcal{I}$ is true if either $\mathcal{C}_1$ or $\mathcal{C}_2$ is true.

Consider the base case when $t_1 = v_1$. For these first v_1 steps, either all the (c_0+1) copies have been considered for selection at least once, or some copy has been ignored. In the former case $\mathcal{C}_2$ is true ($S_j \geq 1$ for all j). In the latter case, by previous bounds, there exists an S-chain in one of the copies. In this case, $\mathcal{C}_1$ is true. Thus, $\mathcal{I}$ holds for the base case.

Suppose $\mathcal{I}$ is true after $t_r = \sum_{j=1}^{r} v_j$ steps, $1 \leq r < u$. Now, if $\mathcal{C}_1$ is true, then G'' has stopped before t_r steps. In this case, $\mathcal{I}$ holds trivially for all future steps.

So suppose $\mathcal{C}_2$ holds. Then there exists a chain of length $S_j \geq 1$ in the jth copy such that the S_j's satisfy (14). Let A_j, $1 \leq j \leq (c_0 + 1)$ be the last point in the S_j-chain for copy j. (A_j was selected by G'' at some prior step). So far, the last base from which a point was selected is H_{t_r+k-1}. Also, there exists at least one point $A_j{}'$ in base H_{t_r+k} for each copy such that $A_j{}' \in I_{A_j}$. Applying Lemma 1, $\bigcup_{j=1}^{c_0+1} D_{A_j{}'}$ is a union of $(c_0+1)*(t_r+k)$ copies of (n_0-1)-dimensional plateaus each of height at most $(t_r + k)$.

Now consider playing G'' for v_{r+1} more steps. By (12),

$$\begin{aligned} v_{r+1} &= L\left(n_0 - 1, \left(k + \sum_{j=1}^{r} v_j\right), (c_0 + 1)\left(k + \sum_{j=1}^{r} v_j\right), S\right) \\ &= L\left(n_0 - 1, (k + t_r), (c_0 + 1) * (k + t_r), S\right) \end{aligned}$$

Either all of these v_{r+1} selected points belong to $\bigcup_{j=1}^{c_0+1} D_{A_j{}'}$, or there exists a point which belongs to the ideal of some $A_j{}'$. In the former case, $\mathcal{C}_1$ is true by previous bounds for copies of $(n_0 - 1)$-dimensional plateaus. In the latter case, at least one of the S_j's increases by one. Therefore $\mathcal{C}_2$ holds.

Thus, after $t_u = \sum_{j=1}^{u} v_j$ steps, $\mathcal{I}$ holds. If $\mathcal{C}_1$ is true, then there exists an S-chain among the selected points. If $\mathcal{C}_2$ is true, then there exists a S_j-chain in the jth copy such that

$$\sum_{j=1}^{c_0+1} S_j \geq c_0 + u$$

Therefore, there exists at least one S_j such that

$$\begin{aligned} S_j &\geq \left\lceil \frac{c_0 + u}{c_0 + 1} \right\rceil \\ S_j &\geq \left\lceil \frac{c_0 + (c_0 + 1)(S - 1) + 1}{c_0 + 1} \right\rceil \\ &\geq S \end{aligned}$$

Hence the result. □.

We can now state the following theorem whose proof follows immediately from the earlier lemmas.

Theorem 1 *The number of steps of G described in section 3 is bounded above by $L(n, 0, 1, S)$. This bound can be computed from equations (8), (9), (11), and (13).* □.

4 Optimal Upper Bound for the 2-D Poset

We now return to the 2-d poset presented in section 1. For this poset, its diagonal lines are bases, and successive diagonal lines starting with some kth diagonal form plateau P_k (refer to Figure 1). Note that the kth diagonal has $(k + 1)$ points. Also, any point in this poset can be uniquely identified by its (x, y) coordinates.

We will show that the optimal upper bound for an S-chain on the number of steps of G for the 2-d poset is 2^{S-1}.

The proof is divided in two parts. First, we prove that if the game continues for 2^{S-1} or more steps, then there exists an S-chain among the points selected.

To establish optimality, we then show that there is a way of selecting $(2^{S-1}-1)$ points (a game with $2^{S-1}-1$ steps) such that the maximal chain length among these points is at most $(S-1)$.

In order to prove the above bounds, we need to consider the following game (denoted by $\hat{G}$) which is a slightly different version of G'. $\hat{G}$ starts by picking a point from some pth ($p \geq 0$) diagonal. Suppose at the ith ($i \geq 1$) step we have selected a point from diagonal p'. Then at the $(i+1)$th step we select a point from any diagonal p'' such that $p'' > p'$. Thus in $\hat{G}$ we relax the restriction of selecting at successive steps a point from successive diagonals. $\hat{G}$ continues provided at the next step a point can be selected which is incomparable to all the previously selected points.

We prove the following lemma for $\hat{G}$.

Lemma 7 *$\hat{G}$ has at most $(p+1)$ steps. Further, this bound is optimal.*

Proof: For any point C in the pth diagonal there are two points greater than C in the $(p+1)$th diagonal, three points greater than C in the $(p+2)$th diagonal and so on (In general there are g points greater than C in the $(p+g-1)$th diagonal.) Thus, once a point A is selected in the pth diagonal at the first step, for any subsequent diagonal p', $p' > p$, there are at most $(p'+1-(p'-(p+1))) = p$ points incomparable to A. Select any point B from among these p points on some $p'(> p)$th diagonal at the second step. Now, for any subsequent diagonal $p'' > p'$, there are at most $(p-1)$ points which are incomparable to both A and B. It is easy to see that the number of points in subsequent diagonals which are incomparable to the points selected at previous steps decrease by one for every selection. (It could reduce by more than one provided one does a "bad" selection at some step). This gives us the required upper bound of $(p+1)$.

To show that this bound is optimal, select the following $(p+1)$ points (given by their (x, y) coordinates): $(p, 0)$, $(p-1, 2)$, $(p-2, 4)$, ...and $(0, 2*p)$. All of these points are incomparable to each other and lie in successive diagonals starting with the pth diagonal. □

4.1 A Directed Acyclic Graph representation of a Game

We now introduce a Directed Acyclic Graph (DAG) $\mathcal{G}$ corresponding to m steps of a game. For each of the m points selected there is a corresponding node in $\mathcal{G}$. There is a directed edge from a node N_1 to node N_2 in $\mathcal{G}$ if and only if

1. the points C_1 and C_2 corresponding to N_1 and N_2 are such that $C_2 > C_1$ and
2. there does not exist a selected point C' such that $(C' \neq C_1)$ and $(C' \neq C_2)$ and $C_2 > C' > C_1$.

Figure 3 shows the DAG for the game in Figure 2.

In any game G, the origin $(0, 0)$ is the first point selected. It becomes the root of $\mathcal{G}$. Nodes in $\mathcal{G}$ can be partitioned into "levels" depending upon their distance from the root node. The root node forms level 1, nodes at distance one from the root form level 2 and so on. The **MLN**, the maximum level number of the DAG, is the number of its maximum non-empty level.

In Figure 3, the MLN of the DAG is 4.

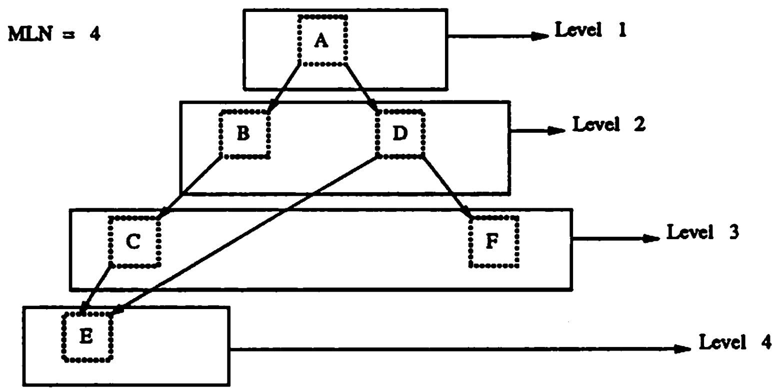

Figure 3: DAG for the Game in Figure 2

In what follows, we use the terms "points" and "nodes" interchangeably to refer both to points selected along diagonals and their corresponding nodes in the DAG.

The following two facts follow directly from the definitions.

Fact 2 *For any game G, the length of the maximal chain among the points selected equals the MLN of the DAG of G.*

Fact 3 *For any DAG corresponding to a game G, all nodes in a level are incomparable to each other.*

4.2 An upper bound for 2-d games

Consider the incremental construction of a DAG for any game of m steps. After the first step the DAG has just the root node. After every step, the number of nodes in the DAG increase by one. A new node added in the DAG (corresponding to a newly selected point) either falls in a new level (in which case the MLN of the DAG increases by one) or it is assigned a level formed at a previous step.

For any game of m steps, let $a_i, (1 \leq a_i \leq m)$ be the first step which increases the MLN of the DAG to i. Let $a_0 = 0$. Clearly, $a_1 = 1$, and $a_2 = 2$. Also we must have $i \leq a_i$ since every step can increase the MLN by at most one.

Lemma 8 *If the MLN of a DAG is h, then the following relationship holds for all a_i, $1 \leq i \leq h$.*

$$a_i \leq \left(\sum_{j=0}^{i-1} a_j \right) + 1 \tag{15}$$

Proof: Consider any game of m steps and its corresponding DAG. At step a_i, let B be the point chosen from diagonal $a_i - 1$. By definition of a_i, the MLN of the DAG increases to i after B is selected.

We first show that the number of nodes in level i is at most a_i. Suppose not: i.e. assume that there are $r > a_i$ nodes in level i. By Fact 3, the r points corresponding to these nodes are mutually incomparable. These r points constitute a game $\hat{G}$ of more than a_i steps starting with diagonal $a_i - 1$. This contradicts Lemma 7.

Thus the maximum number of selected points that can fall in level i is a_i. We can "delay" the increase in MLN of the DAG to $(i+1)$ by at most the total number of nodes that are possible in all the previous i levels. Therefore, a_{i+1} can be no bigger than the sum of all the previous a_j's, $1 \leq j \leq i$ plus one. □.

We now state a corollary (the proof of which follows from Lemma 2).

Corollary 1 *For all $i \geq 1$,*

$$a_i \leq 2^{(i-1)} \tag{16}$$

Proof by Induction: For the base case $a_1 = 1$. Therefore, suppose the result holds for all $i \leq t$. Then by Lemma 8, we have

$$\begin{aligned} a_{t+1} &\leq \left(\sum_{j=0}^{j=t} a_j\right) + 1 \\ &\leq 0 + 1 + 2 + 2^2 \ldots 2^{t-1} + 1 \\ &\leq 2^t - 1 + 1 \\ &\leq 2^t \end{aligned}$$

□

The above corollary and Fact 2 gives us the the required lower bound which we now state in the following lemma.

Lemma 9 *For any game with m steps played on a 2-d poset, there exists an S-chain among the m selected points provided $m \geq 2^{(S-1)}$.*

4.3 Optimality

To establish optimality, we need to prove the following lemma.

Lemma 10 *For any given positive integer S, there exists a game with $(2^S - 1)$ steps such that the MLN of its DAG is at most S.*

Proof: For every i, $1 \leq i \leq S$, let T_i be the set of points as given below.

$$T_i = \{(2^{(i-1)} - 1, 0), (2^{(i-1)} - 2, 2), (2^{(i-1)} - 3, 4), \ldots (0, 2 * (2^{(i-1)} - 1))\}$$

Clearly, $|T_i| = 2^{i-1}$.

Figure 3 shows T_1, T_2 and T_3.

(Note: All the points in any T_i are incomparable to each other.)

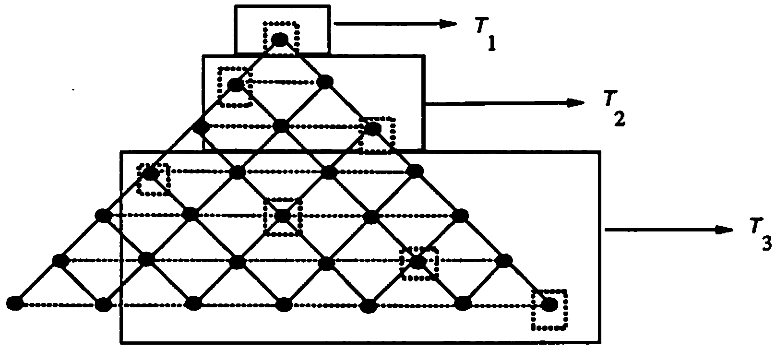

Figure 4: T_1, T_2 and T_3

We now prove that if all points in T_1 followed by all points in T_2 and so on up to all points in T_S are selected (a game with $(2^S - 1)$ steps), the maximum chain length is at most S. We prove this by induction. T_1 contains just the origin. This constitutes a chain of length one. Suppose the induction hypothesis is true up to r. Thus, up to $2^r - 1$ steps, all points from $T_1, T_2 \ldots T_r$ have been selected, and the maximum chain length is at most r. Now, since all points in T_{r+1} are incomparable to each other and since the maximum chain length so far is r, selecting all the points in T_{r+1} can increment the maximum chain length at most by one. Hence after $(2^{r+1} - 1)$ steps the maximum chain length is at most $(r + 1)$. □

We can now state the following theorem which follows from Lemmas 9 and 10.

Theorem 2 *To ensure a chain of length S in the 2-d poset, one has to play G for at least 2^{S-1} steps. Further, this bound is optimal.*

5 Conclusion

We have shown that the number of steps of G played on $\mathcal{P}$ is bounded. We have also shown a dynamic programming method to obtain the bound. These bounds grow tremendously with n. Further, we have shown that the optimal upper bound for G on the 2-d poset is 2^{S-1}.

6 Acknowledgement

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