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**ABSTRACT**

The surface contacts between solids are always associated with a set of symmetries of the contacting surfaces. These symmetries form a group, the *symmetry group* of the surface.

In this paper we develop a group theoretic formalization for describing surface contact between solids. In particular we define:

- Primitive and compound features of a solid;
- A topological characterization of these features;
- The symmetry groups of primitive features and compound features;

The symmetry group of a feature is a descriptor of the feature that is at once both abstract and quantitative. We show how to use group theory to capture the symmetries relevant to surface contacts between solids as well as the symmetries of solids. The central result of this paper is to prove:

- when primitive features of a solid are mutually distinct, 1-congruent or 2-congruent, the symmetry group of a compound feature can be expressed in terms of the intersection of the symmetry groups of its primitive features;
- when two solids have surface contact, their relative positions can be expressed as a coset of their common symmetry group, which in turn can be expressed as an intersection of the symmetry groups of the primitive features involved in this contact.

These results show that using group theory to formalize surface contacts between solids is a general approach for specifying spatial relationships, and forms a sound basis for the automation of robotic task planning. In a companion paper, a geometric representation for symmetry groups and an efficient group intersection algorithm using characteristic invariants is presented, which makes this approach computationally tractable as well.

# 1 Introduction

Human beings readily appreciate symmetries of objects. A *symmetry* of a subset $S$ in Euclidean space is a rotation, reflection and/or translation, including the *trivial symmetry* (identity mapping), which brings $S$ into coincidence with itself. Here $S$ can be a discrete point set, a curve, a surface or a solid. The term *proper symmetry* is often used to exclude mirror symmetries (reflections), which, although useful for describing the appearance of objects, cannot in general be realized by any physical movement and are thus of limited relevance for describing the effect of manipulations. In the rest of this paper the term 'symmetry' will be used to mean proper symmetry, and the term 'solid' to mean a three-dimensional, connected and closed subset of Euclidean space.

In the early nineteenth century, Galois introduced a powerful mathematical tool for describing symmetry, namely *group theory* [12, 14]. Typically, elements in a group are invertible mappings over some domain. The composition and inverse of the mappings determine the algebraic structure of the group. In particular, all the distance and handedness preserving mappings in Euclidean space, i.e. the *proper isometries*, form a group. This group is called the *Proper Euclidean group* $\mathcal{E}^+$. It can easily be proven that the set of all the symmetries of a set $S$ in $\Re^3$ has a group structure and is thus a subgroup of $\mathcal{E}^+$. This group is called the *symmetry group* of $S$. Symmetry groups can be either finite or infinite, discrete or continuous. Examples of finite groups are the symmetry groups of the platonic solids (See Figure 1). Examples of continuous groups are illustrated in Figure 2 (from [13]) where the symmetry groups for all six mechanical *lower pair* joints, joints that have *surface* (areal) contacts, are shown.

An *assembly* is a set of solids physically related to each other via *point*, *edge* and/or *surface* contacts. Each pair of contacting solids in an assembly can be seen as part of a kinematic chain (Figure 2). Among the three kinds of contacts between solids, surface contacts are the most common and stable[1] [2]. In order to describe an assembly, one needs a basic vocabulary to address such contacts. A *primitive feature* of a solid is defined as the algebraic surface which locally coincides with a bounded face of the solid. The primitive features of a cube, for example, are the six infinite planes which bound the solid volume. Each primitive feature has a symmetry group which keeps the feature, although not necessarily the corresponding bounded face of the solid, setwise invariant. It has to be made clear that the coincidence of a pair of primitive features, each belonging to a different solid, does not necessarily imply a physical contact of the solids (See Figure 3). This is not a weakness of the primitive feature definition, on the contrary this is *exactly* what one means by specifying a surface contact between only one pair of planar primitive features. In

[1] In terms of control, surface contacts are desirable because: (1) there are less degrees of freedom left than for point or line contacts, so less motion specifications are needed. (2) if you change the contact force and/or moment a little bit, the relative position of the two contacting bodies remains unchanged, IF there is a compliance in the system which can "absorb" the force/moment differences. If you do the same thing with a line or point contact, the relative position of both bodies will change easily.

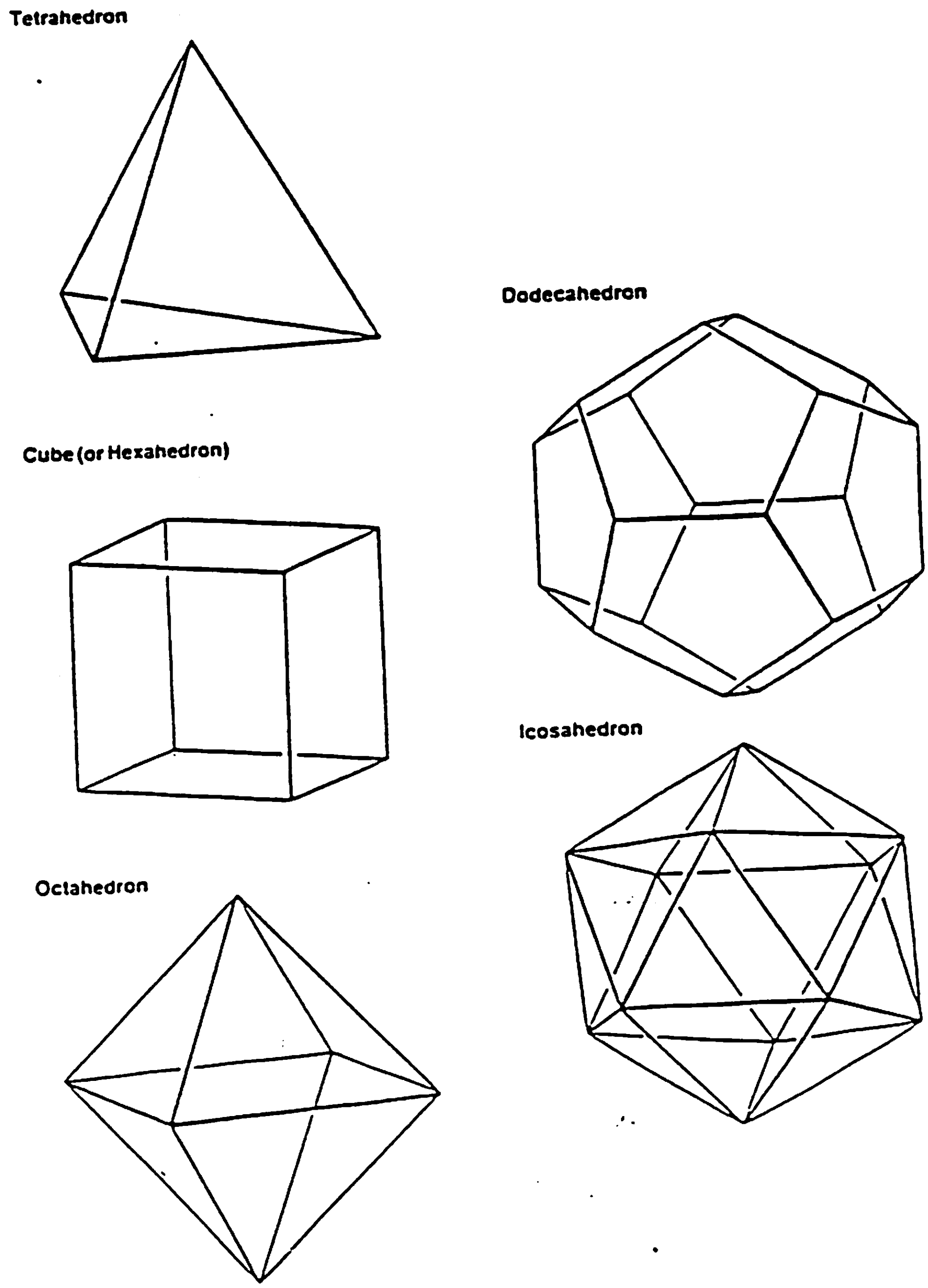


Figure 1: Platonic Solids

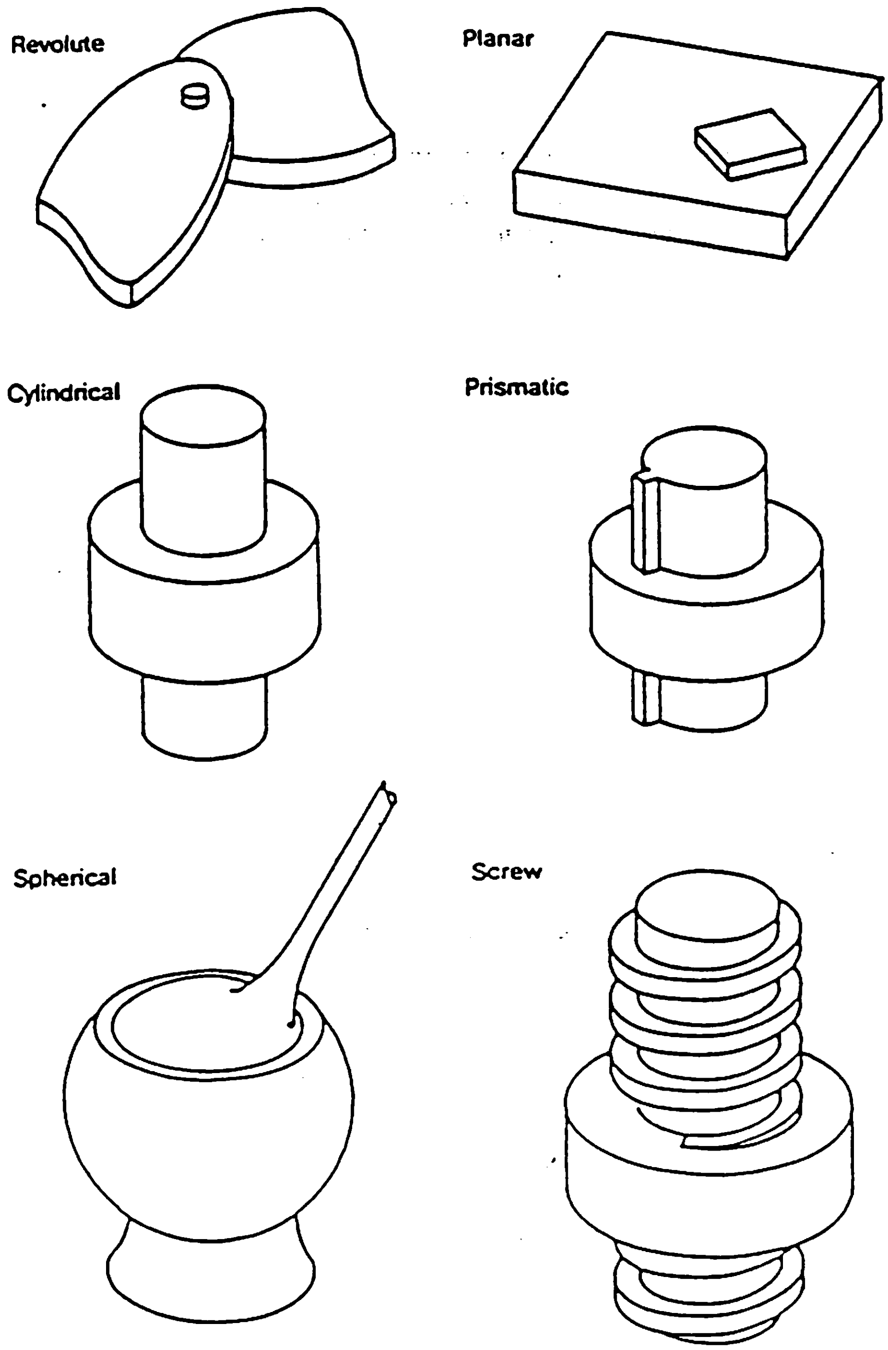


Figure 2: Lower Pairs

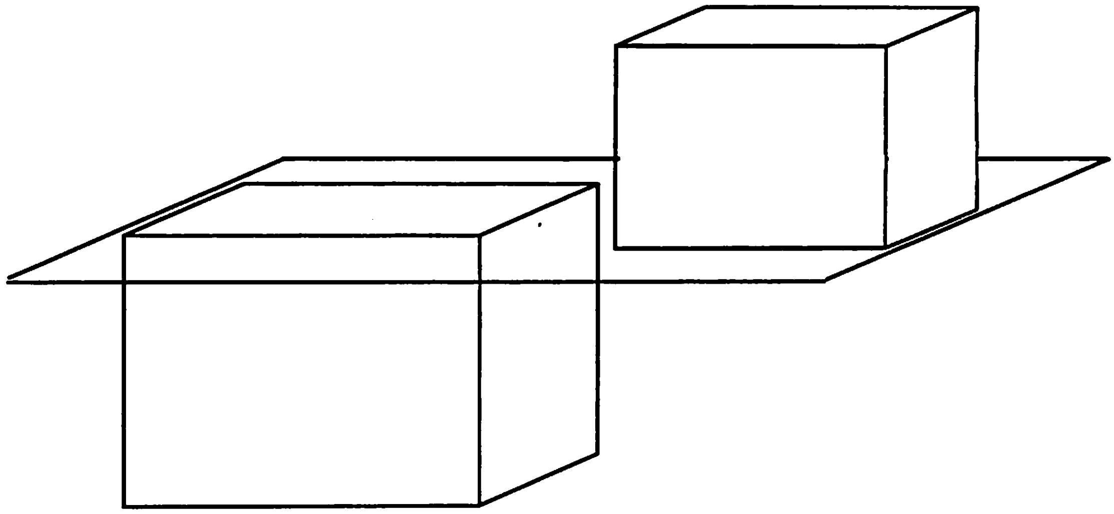

Figure 3: Two solids share a planar primitive feature

order to guarantee a physical contact additional constraints have to be present. For example, using the primitive feature concept just defined, we can uniquely and completely describe three different surface contacts shown in Figure 4, where block $A$ has one primitive feature in contact with the environment, block $B$ has two and block $C$ has three. The union of several such primitive features is called a *compound feature*.

Describing the interactions among objects is a central issue for roboticists. One obvious approach towards a user-friendly human-robot interface is to give commands not in terms of how and how much a robot arm should move, but rather, in terms of how objects are to be moved. These latter kind of instructions are called *task specifications*.

Feature symmetry groups are useful in the support of systems for task-level planning. Robotic *task-level planning* [5, 10] refers to the study of translating such a set of task specifications, automatically, into a plan which can be executed by a robot(s).

Current existing robot task-level assembly planners require, as part of their input, the final assembly configuration to be described [8, 11, 21]. Such task specifications include the relative positions and orientations of the solids to be assembled, and the possible nominal interactions between solids during an assembly process. Even at the abstract level, planning the sequence of assembly requires the relationships among assembly components to be specified [4, 7, 20]. However, from a mechanical design, it is not always trivial to derive an assembly configuration specification that is complete and unambiguous.

The difficulty of generating a complete and unambiguous task specification often arises from the *symmetries* of assembly components:

- The *discrete symmetries* cause the unnecessary consideration of redundant

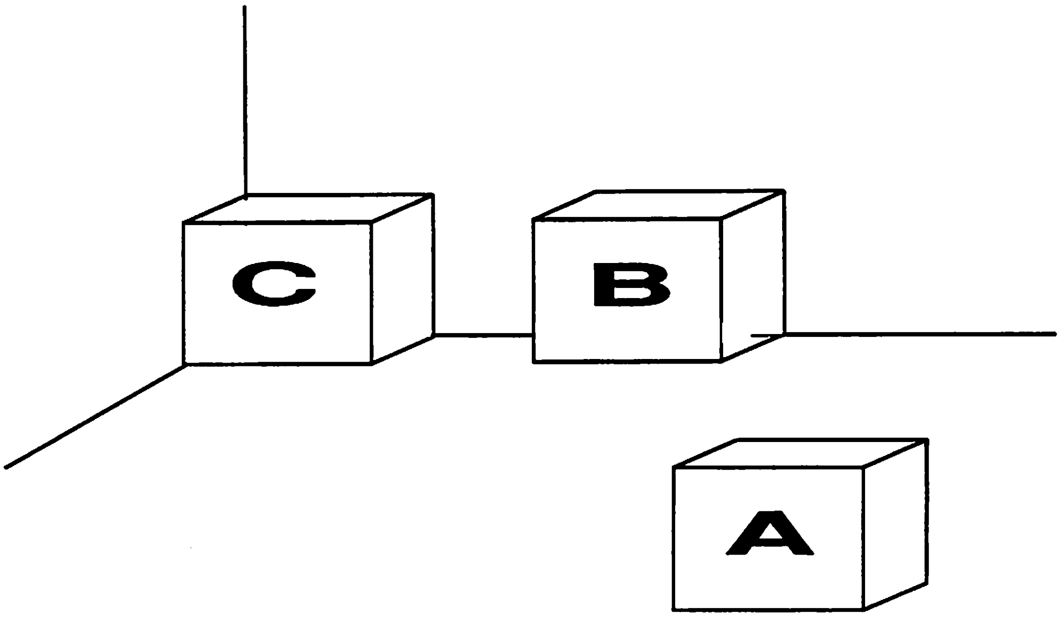


Figure 4: Block $A$ has one contacting primitive feature pair, block $B$ has two and block $C$ has three

configurations. For instance, when using a socket wrench to turn a hexagonal bolt-head, there are six different but spatially equivalent positions for the socket wrench to be in. The explicit enumeration of such possibilities can lead to a computational complexity in assembly planning exponential in number of operations.

- On the other hand, *continuous symmetries* may cause underconstrained task specifications. For example, a shaft has to be cylindrical for revolute motions. In this case, the set of relative locations of the shaft with respect to its bearing is infinite. This cannot be represented satisfactorily in current CAD systems. Typically, one chooses to ignore the distinction of one position over the other.

There are two kinds of relevant symmetries in describing interactions among solids:

- **Body symmetries**: For example, each solid in Figure 1 has a non trivial body symmetry group.
- **Feature symmetries**: These may or may not map the whole body (solid) into itself (Figure 2).

For example, in Figure 4, Block $C$'s location is unambiguously determined up to a finite number of equivalent states since block $C$ as a whole has a non-trivial symmetry group. This simple example shows that the relative position(s) of two solids is determined firstly by the symmetries of the contacting feature pairs, and secondly by the symmetries of each solid as a whole. Even when a solid as a whole has a trivial symmetry (its symmetry group is the identity group), each primitive feature which is in surface contact with another solid often has a non-trivial symmetry group (Figure 2).

The aforementioned problems and facts call for an explicit treatment of the symmetries of a solid and its features. It is the objective of this paper to explore how the algebraic concept of groups and the geometric entities, solids and features, are intertwined, especially when surface contact occurs between solids.

To outline this paper:

1) Section 2 reviews work related to using group theory in spatial reasoning.

2) Section 3 gives a formal definition of set primitive features, compound features and their symmetry groups. Several propositions are proved to show how the symmetry group of a compound feature is related to the symmetry groups of its primitive features.

3) Section 4 gives a formal definition of oriented features. This provides a more precise description of surface contacts.

4) Section 5 shows how the relative positions of two solids can be expressed in terms of the intersection of contacting feature symmetry groups, and their cosets.

5) Section 6 concludes this paper by discussing several relevant issues.

# 2 Related Work

While kinematic and geometric problems are common in robotics, very few papers can be found in the literature that use group theory as an analytical tool.

## 2.1 Group Theory and Mechanisms

In [6]Hervé introduced a rational classification of mechanisms by applying the theory of continuous groups[6]. Since each lower-pair allows a set of relative motions of the two coupled bodies, these motions can be regarded as subgroups of $\mathcal{E}^+$. The independent variables required to define the relative position of two coupled links are referred to as their *degrees of freedom.* The concept of degree of freedom of a kinematic pair can be extended to a subgroup of $\mathcal{E}^+$ in 3-space, the corresponding concept being that of *dimension.*

Hervé represented the intersection and composition of constraints in terms of groups. If there are two relations $R_1, R_2$ between bodies $B_1$ and $B_2$ then the conjoined relation of $B_1, B_2$ is $R_1 \cap R_2$. When relations are composed one has the following relationship between the dimensions:

$$dim(R(i,k)) = dim(R(i,j)) + dim(R(j,k)) - dim(R(i,j) \cap R(j,k)) \qquad (1)$$

where $i, j, k$ refer to three distinct bodies. Equation (1) shows the usefulness of subgroup intersections.

A recent paper by Tchoń [18] also investigated the relationship of subgroups of Lie groups and the inverse kinematic problem for redundant robot manipulators.

## 2.2 Group Theory and Robotics

A paper by one of us [16] relates robotics and group theory by pointing out:

- The symmetry group of a **feature** is a useful descriptor, more important for manipulation and assembly than the symmetry group of a body.
- Not only continuous groups (as previously used in describing kinematic joints) but also finite and discrete groups should be handled.
- Spatial relations can be described in terms of cosets of symmetry groups, conjoining relations requires the intersection of cosets which is either a coset or null.
- One advantage of this formulation is to avoid combinatorics arising from multiple relationships.

Our current work extends many of these ideas. In particular, the representation of a well-defined family of subgroups of Euclidean group $\mathcal{E}^+$, namely the **TR** groups, has been realized which makes efficient group intersection possible[9].

Thomas and Torras [19] dealt with the problem of finding configurations of a set of rigid bodies from a given set of d.o.f. constraints on the bodies. A **constraint** between two bodies is the chain product of a symbolic $4 \times 4$ matrix which is pre- or post-multiplied by constant displacements, i.e. a two sided coset. Therefore the problem of finding values for the variables (d.o.f.) associated with constraints can be reduced to the problem of obtaining the cycles appearing in a directed graph whose nodes are bodies and whose arcs are constraints, and solving their corresponding matrix equations. Thomas and Torras tabulate the outcomes of intersection and multiplication of certain continuous subgroups of $\mathcal{E}^+$ (based on tables from Hervé [6]). In their algorithm, the tables are used to simplify certain algebraic equations. This approach is simpler than using a large number of rewriting rules [1] and permits a uniform treatment of some special cases.

The work of Thomas and Torras is a useful contribution in combining the group theoretic formulation of continuous groups developed by Hervé with the RAPT [1, 17] formation of constraints and equation solving approach. As in RAPT, they still need to use symbolic manipulations, which can be very slow. They do not treat discrete or finite symmetry groups.

# 3 Set Features and Their Symmetry Groups

In this section we provide the basic vocabulary for describing surface contacts among solids. We formally define features of a solid, the symmetries of a feature, and then show that they form a group, called the symmetry group of the feature. This provides the basic justification for the use of group theory in describing surface contact.

## 3.1 Primitive Set Features and Their Symmetry Groups

Here primitive features are treated as subsets in $\Re^3$. We shall define oriented features in Section 4. This section provides several propositions stating the relationship between the symmetry group of a single feature and that of a set of features. We write the composition of the group elements $g_1, g_2$ as $g_1 g_2$. With a matrix representation of isometries, this is just a matrix product.

**Definition 3.1** *A* **primitive feature** *$F$ is a connected, irreducible and non-trivial algebraic surface of $\Re^3$.*

By definition, a primitive feature is a closed algebraic surface. A primitive feature may contain one or more bounded faces of a solid.

For a set of points $S$ in Euclidean space we define a symmetry of $S$ as:

**Definition 3.2** *An isometry $g \in \mathcal{E}^+$ is a* **proper symmetry of a set** *$S$ if and only if $g(S) = S$.*

**Proposition 3.3** *The proper symmetries of a set $S \subseteq \Re^3$ form a subgroup of $\mathcal{E}^+$.*

*Proof*:

Let $G$ denote the set of the symmetries of $S \subset \Re^3$. Obviously, $1(S) = S$, so $1 \in G$. If $g \in G$ then $g(S) = S$, multiplying by $g^{-1}$ we have $g^{-1}g(S) = g^{-1}(S)$ therefore $g^{-1}(S) = S$ and so $g^{-1} \in G$. Finally, if $g_1, g_2 \in G$ then $(g_1 g_2)(S) = g_1(g_2(S)) = g_1(S) = S$ therefore $g_1 g_2 \in G$. By the definition of a subgroup $G$ is a subgroup of $\mathcal{E}^+$. □

**Definition 3.4** *When $S$ is a feature, the above group $G$ associated with $S$ is called the* **symmetry group of the feature** *$S$.*

**Definition 3.5** *Two primitive features $F_1, F_2$ are said to be*

- **Distinct:** if for all the open subsets[2] $S'_1 \subset F_1, S'_2 \subset F_2$, no $g \in \mathcal{E}^+$ exists such that $g(S'_1) \subset F_2$ or $g(S'_2) \subset F_1$. See Figure 5 for an example of two distinct features.

- **1-congruent:** **(weakly-congruent)** if there exists at least one $g \in \mathcal{E}^+$ such that $g(F_1) = F_2$, but for all such $g, g(F_2) \neq F_1$ simultaneously. For an example see Figure 6.

[2]They are open with respect to the induced topology from $\Re^3$

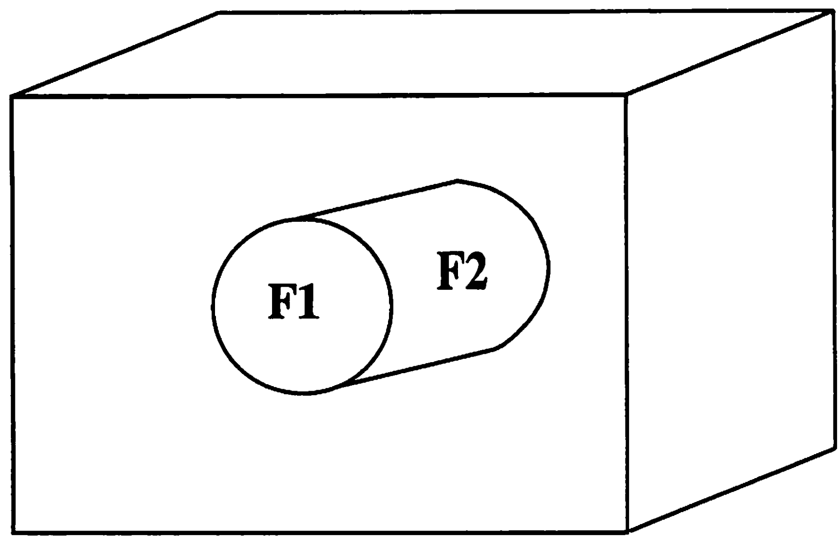


Figure 5: A pair of **distinct** features $F_1, F_2$

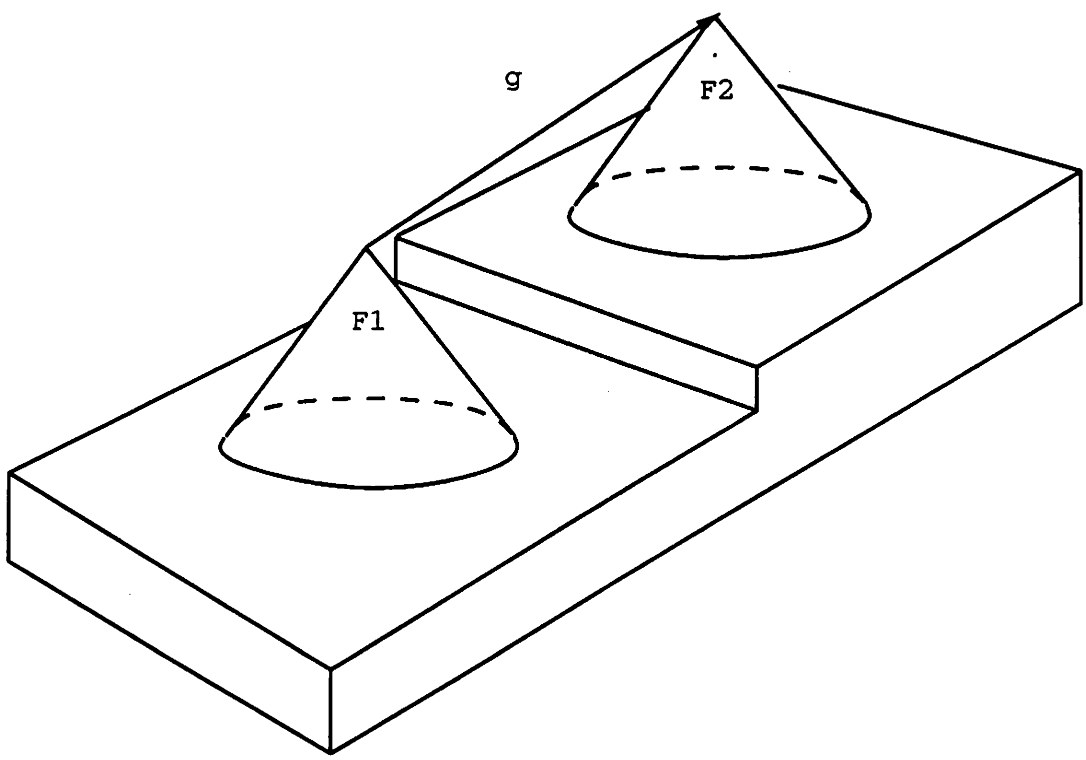


Figure 6: Two conic features $F_1, F_2$ which are **weakly congruent** to each other

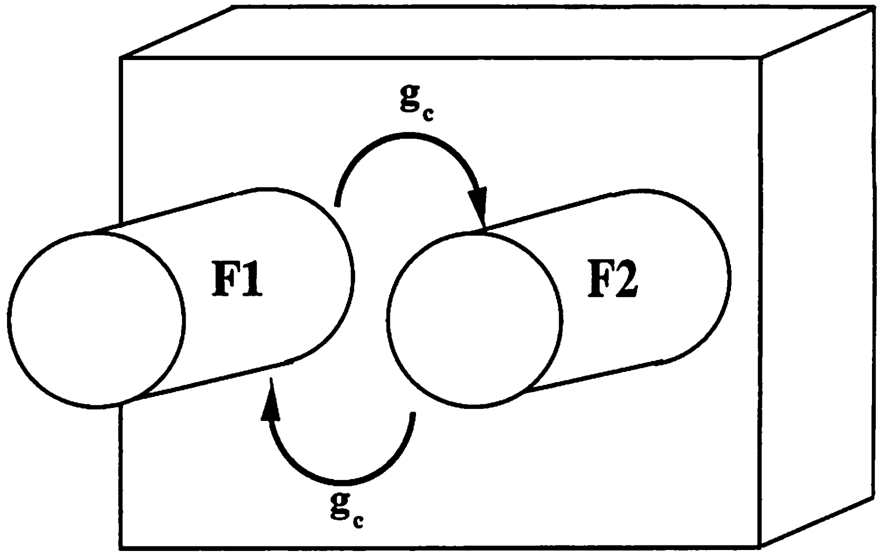


Figure 7: Two cylindrical features $F_1, F_2$ which are **strongly congruent** to each other

- **2-congruent (strongly-congruent)** via $g_c$ if there exists $g_c \in \mathcal{E}^+$ such that $g_c(F_1) = F_2$ and $g_c(F_2) = F_1$. For an example see Figure 7.

The following proposition shows how features and their symmetry groups are related.

**Proposition 3.6** *If $G$ is the symmetry group of $S$ then for any rigid transformation $a$ in $\mathcal{E}^+, aGa^{-1}$ is the symmetry group of $a(S)$.*

*Proof*:
Let $H = aGa^{-1}$ and let $H_a$ be the symmetry group of $a(S) \subset \Re^3$. If $h \in H$ then there exists a $g \in G$ such that $h = aga^{-1}$, and moreover $g(S) = S$. Then $h(a(S)) = aga^{-1}(a(S)) = ag(S) = a(S)$. Thus $h$ is a symmetry of $a(S)$, and so $h \in H_a$. Thus we can conclude $H \subseteq H_a$.

Conversely, if $h_a \in H_a$ then $h_a(a(S)) = a(S)$ and so $a^{-1}h_a a(S) = S$, i.e. it is a symmetry of $S$. Thus $a^{-1}h_a a = g \in G$ and $h_a = aga^{-1} \in H$ therefore $H_a \subseteq H$.

Thus we conclude $H = H_a$. □

By Proposition 3.6, when a feature is relocated by a transformation $a$, the symmetry group of the relocated feature will be the conjugation by $a$ of the symmetry group of the original feature. This suggests that a conjugation class of a subgroup can be represented in a compact way, that is to represent any feature symmetry group by making it a conjugate of **a canonical symmetry group**. There is one canonical symmetry group in each conjugation class of a symmetry group in $\mathcal{E}^+$. These canonical groups are chosen in a systematic way — if they have a single axis of rotation it is chosen to be the $Z$-axis, if they leave a single point of 3-space fixed,

Table 1: Some important subgroups of $\mathcal{E}^+$

| Canonical Groups | Definition of Canonical Group Members |
|---|---|
| $\mathcal{G}_{id}$ | $\{1\}$ |
| $\mathcal{T}^1$ | $\{\mathbf{trans}(0,0,z) \mid z \in \Re\}$ |
| $\mathcal{T}^2$ | $\{\mathbf{trans}(x,y,0) \mid x,y \in \Re\}$ |
| $\mathcal{T}^3$ | $\{\mathbf{trans}(x,y,z) \mid x,y,z \in \Re\}$ |
| $SO(3)$ | $\{\mathbf{rot}(\mathbf{i},\theta)\mathbf{rot}(\mathbf{j},\sigma)\mathbf{rot}(\mathbf{k},\phi) \mid \theta,\sigma,\phi \in \Re\}$ |
| $SO(2)$ | $\{\mathbf{rot}(\mathbf{k},\theta) \mid \theta \in \Re\}$ |
| $O(2)$ | $\{\mathbf{rot}(\mathbf{k},\theta)\mathbf{rot}(\mathbf{i},n\pi) \mid \theta \in \Re, n \in \mathcal{N}\}$ |
| $\mathcal{G}_{cyl}$ | $\{\mathbf{trans}(0,0,z)\mathbf{rot}(\mathbf{k},\theta)\mathbf{rot}(\mathbf{i},n\pi) \mid n \in \mathcal{N},\theta,z \in \Re\}$ |
| $\mathcal{G}_{dir\_cyl}$ | $\{\mathbf{trans}(0,0,z)\mathbf{rot}(\mathbf{k},\theta) \mid z,\theta \in \Re\}$ |
| $\mathcal{G}_{plane}$ | $\{\mathbf{trans}(x,y,0)\mathbf{rot}(\mathbf{k},\theta) \mid x,y,\theta \in \Re\}$ |
| $\mathcal{G}_{screw}(p)$ | $\{\mathbf{trans}(0,0,z)\mathbf{rot}(\mathbf{k},2z\pi/p) \mid z \in \Re\}$ |
| $\mathcal{G}_{T_1C_2}$ | $\{\mathbf{trans}(0,0,z)\mathbf{rot}(\mathbf{i},n\pi) \mid n \in \mathcal{N}, z \in \Re\}$ |
| $D_{2n}$ | $\{\mathbf{rot}(\mathbf{k},2\pi/n)\mathbf{rot}(\mathbf{i},m\pi) \mid m,n \in \mathcal{N}\}$ |
| $C_n$ | $\{\mathbf{rot}(\mathbf{k},2\pi/n) \mid n \in \mathcal{N}\}$ |
| $\mathcal{E}^+$ | $\{\mathbf{trans}(x,y,z)\mathbf{rot}(\mathbf{i},\theta)\mathbf{rot}(\mathbf{j},\sigma)\mathbf{rot}(\mathbf{k},\phi) \mid x,y,z,\theta,\sigma,\phi \in \Re\}$ |

that point is chosen to be the origin. If they leave a plane set-wise invariant the plane is chosen as the X-Y plane. A list of some important canonical subgroups of $\mathcal{E}^+$ with their definitions is given in Table 1 where $\mathbf{i},\mathbf{j},\mathbf{k}$ are unit vectors along axes X, Y, Z.. Table 2 gives some of the correspondences between subsets of $\Re^3$ and their canonical symmetry groups.

## 3.2 Compound Set Features and Their Symmetry Groups

Now let us formally define a compound feature.

Table 2: Correspondence between shapes and their symmetry groups

| Subset $S \subset \Re^3$ | surface name | Symmetry group |
|---|---|---|
| $\mathbf{H}$ | half plane | $\mathcal{G}_{plane}$ |
| $\mathbf{Cyl(r)}$ | cylinder | $\mathcal{G}_{cyl}$ |
| $\mathbf{Sph(r)}$ | sphere | $SO(3)$ |
| $\mathbf{Screw(p,r)}$ | screw | $\mathcal{G}_{screw}(p)$ |
| $\mathbf{Gear(p_r,p_a,n)}$ | gear | $D_{2n}$ |
| $\mathbf{Cone}(\theta)$ | cone | $SO(2)$ |

**Definition 3.1** *A* **compound feature** *$F$ is a set union of $n$ primitive features $F_1, ..., F_n$, i.e. $F = F_1 \cup ... \cup F_n, n > 1$.*

In the next few propositions we shall explore how the symmetry group of a compound feature is related to the symmetry groups of its component primitive features.

**Proposition 3.2** *Let $F_1, ..., F_n$ be pairwise distinct primitive features with symmetry groups $G_1, ... G_n$ respectively and $F = F_1 \cup ... \cup F_n$ be a compound feature with symmetry group $G$. Then $G = G_1 \cap ... \cap G_n$.*

*Proof*:

Let $g \in G$, then $g(F) = F$. Thus $g(F_1 \cup ... \cup F_n) = g(F_1) \cup ... \cup g(F_n) = F_1 \cup ... \cup F_n$. Then $g(F_i) \subseteq F_1 \cup ... \cup F_n$. Suppose $g(F_i) \not\subseteq F_i$. Then there exists a point $y \in g(F_i)$ and for some $j \neq i, y \in F_j - F_i$.

Now let $\epsilon$ be the distance between $y$ and $F_i$. Since $F_i$ is closed, $\epsilon > 0$. Since $y \in g(F_i)$, there must exist $x \in F_i$ such that $y = g(x)$ (Figure 8).

Let $O_x \subset F_i$ be a neighborhood of $x$ with radius equal to $\epsilon$. There must exist a point $x_1$ such that $x_1 \in O_x$ and $x_1 \notin F_q$ for some $q \neq i$, since if no such point $x_1$ exists then $F_i$ and $F_q$ share the open set $O_x$ and thus are not distinct. Now take a neighborhood $O_{x_1}$ of $x_1$ and repeat the same argument. Since there are a finite number of features, $F_1, ..., F_n$, we shall eventually find a point $x_0$ and its neighborhood $O$ contained only in $F_i$, i.e. $O \subset O_x \subset F_i$ and $O \not\subset F_j$ for $j \neq i$ (Figure 8).

Since $g$ is an isometry (a distance preserving map), every point of $O$ must be mapped by $g$ to another point within distance $\epsilon$ of the point $g(x) = y \in F_j$; but the point is not necessarily mapped to $F_j$. Clearly, $g(O) \cap F_i = \emptyset$. Suppose a point $p \in O$ has a neighborhood in $O$ with radius $\epsilon_0$. Then $p$ is mapped by $g$ to some feature $F_k, k \neq i$, i.e. $g(p) \in F_k$. Then $g(p)$ is contained in a neighborhood $O_k$ of $F_k$, let the radius of $O_k$ be less than $\epsilon_0/2$. Since $g$ is an isometry, every point of $O_k$ must be mapped by $g^{-1}$ to a point within distance $\epsilon_0/2$ of point $p$ thus it has to be contained in $O \subset F_i$. Hence $g^{-1}(O_k) \subset F_i$. So $F_i$ and $F_k$ are not distinct, a contradiction.

Therefore $g(F_i) \subseteq F_i$. Since $g$ is a bijection, $g(F_i) = F_i \Rightarrow g \in G_i$ for $i = 1, ..., n$. Thus $g \in G_1 \cap ... \cap G_n \Rightarrow G \subseteq G_1 \cap ... \cap G_n$.

For all $g \in G_1 \cap ... \cap G_n, g(F) = g(F_1 \cup ... \cup F_n) = g(F_1) \cup ... \cup g(F_n) = F_1 \cup ... \cup F_n = F \Rightarrow g \in G \Rightarrow G_1 \cap ... \cap G_n \subseteq G$.

Therefore $G = G_1 \cap ... \cap G_n$. □

**Lemma 3.3** *Let a compound feature $F = F_1 \cup F_2$ have symmetry group $G$, where $F_1, F_2$ are primitive features with symmetry groups $G_1, G_2$ respectively. Let $F_1, F_2$ be separated and 1-congruent. Then for all $g \in G, g(F_1) = F_1$ and $g(F_2) = F_2$.*

*Proof*:

For all $g \in G, g(F) = F$, i.e. $g(F_1 \cup F_2) = g(F_1) \cup g(F_2) = F_1 \cup F_2$. $g(F_1)$ is a connected subset of $F_1 \cup F_2$ (Theorem 7.2). By Theorem 7.3 (see Appendix 7),

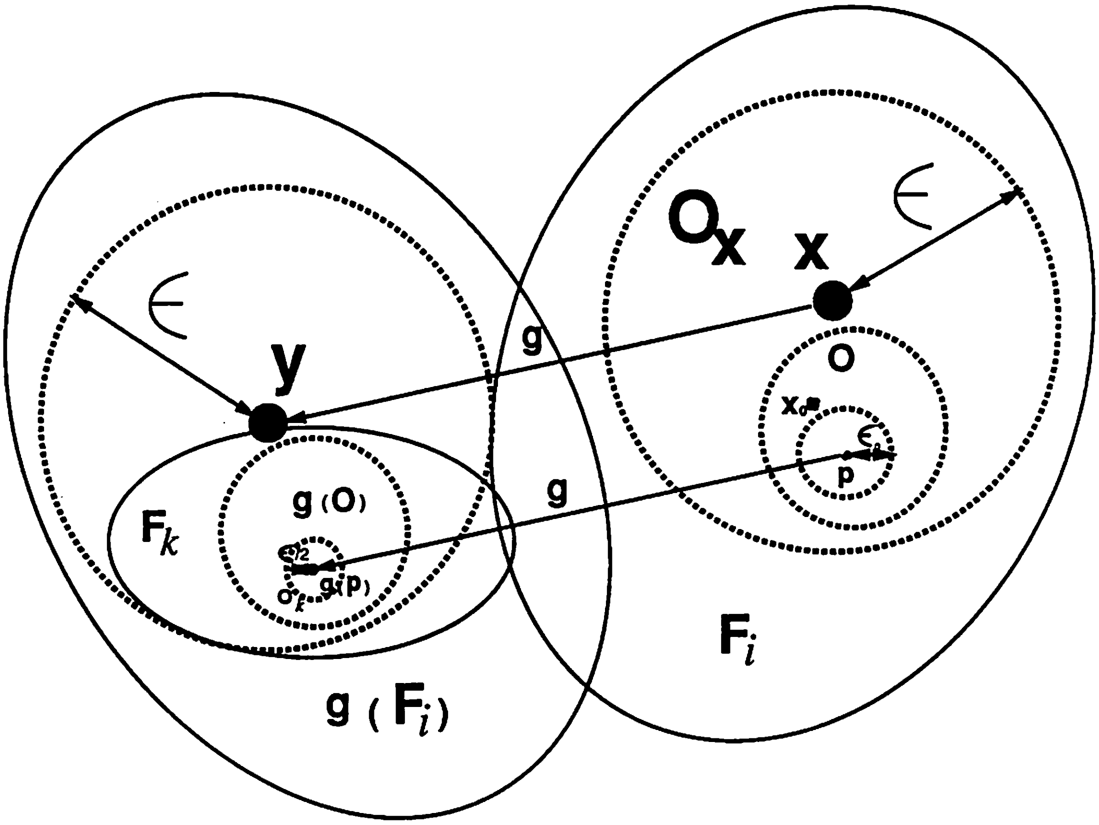


Figure 8: Distinct features and relations among their open sets.

$g(F_1) \subseteq F_1$ or $g(F_1) \subseteq F_2$. Because $F_2$ is connected $g(F_1) \subseteq F_1 \Rightarrow g(F_1) = F_1$ and $g(F_1) \subseteq F_2 \Rightarrow g(F_1) = F_2$. However if $g(F_1) = F_2$ then $g(F_2) = F_1$. That is to say that $F_1, F_2$ are 2-congruent by definition, a contradiction. Therefore $g(F_1) = F_1$ and $g(F_2) = F_2$. □

**Proposition 3.4** *Let a compound feature $F = F_1 \cup F_2$ have symmetry group $G$, where $F_1, F_2$ are primitive features with symmetry groups $G_1, G_2$ respectively, and $F_1, F_2$ are separated and 1-congruent. Then $G = G_1 \cap G_2$.*

*Proof*:

By Lemma 3.3, for all $g \in G, g(F_1) = F_1, g(F_2) = F_2$. Then $g \in G_1 \cap G_2$. So we have $G \subseteq G_1 \cap G_2$. For all $g \in G_1 \cap G_2$, $g(F) = g(F_1 \cup F_2) = g(F_1) \cup g(F_2) = F_1 \cup F_2 = F$. Therefore $g \in G \Rightarrow G_1 \cap G_2 \subseteq G$. We conclude $G = G_1 \cap G_2$. □

**Lemma 3.5** *Let a compound feature $F = F_1 \cup F_2$ have symmetry group $G$, where $F_1, F_2$ are primitive features with symmetry groups $G_1, G_2$ respectively, and $F_1, F_2$ are separated and* 2-congruent *by $g \in \mathcal{E}^+$. Then for all $g \in G$, either $g(F_1) = F_1$ and $g(F_2) = F_2$ or $g(F_1) = F_2$ and $g(F_2) = F_1$.*

*Proof*:

For all $g \in G, g(F) = F$, i.e. $g(F_1 \cup F_2) = g(F_1) \cup g(F_2) = F_1 \cup F_2$. By Theorem 7.3 (Appendix 7), $g(F_1) \subseteq F_1$ or $g(F_1) \subseteq F_2$. Because of the connectivity of $F_1$ and $F_2$, if $g(F_1) \subseteq F_1$ then $g(F_1) = F_1$ and $g(F_2) = F_2$; if $g(F_1) \subseteq F_2$ then $g(F_1) = F_2$ and $g(F_2) = F_1$. □

**Proposition 3.6** *Let a compound feature $F = F_1 \cup F_2$ have symmetry group $G$, where $F_1, F_2$ are primitive features with symmetry groups $G_1, G_2$ respectively, and $F_1, F_2$ are separated and 2-congruent by $g_c \in \mathcal{E}^+$ i.e. $g_c(F_1) = F_2$ and $g_c(F_2) = F_1$. Then $G = <g_c> (G_1 \cap G_2)$ where $<g_c>$ denotes the subgroup of $\mathcal{E}^+$ generated by $g_c$.*

*Proof*:

If $g \in G$ then by Lemma 3.5 either $g(F_1) = F_1$ and $g(F_2) = F_2$ then $g \in G_1$ and $g \in G_2 \Rightarrow g \in G_1 \cap G_2$; or $g(F_1) = F_2$ and $g(F_2) = F_1$ then $g$ can be written as $g = g_c g_c^{-1} g$. Let $g_0 = g_c^{-1} g$ then $g(F_1) = g_c g_0(F_1) = F_2 \Rightarrow g_0(F_1) = g_c^{-1}(F_2) = F_1$ therefore $g_0 \in G_1$ and similarly, $g_0 \in G_2$. Thus $g_0 \in G_1 \cap G_2$. Then $g = g_c g_0 \in <g_c> (G_1 \cap G_2)$.

If $g \in <g_c> (G_1 \cap G_2)$ then $g = g' g_{12}$ where $g' \in <g_c>$ and $g_{12} \in G_1 \cap G_2$. Then $g(F) = g(F_1 \cup F_2) = g(F_1) \cup g(F_2) = g' g_{12}(F_1) \cup g' g_{12}(F_2) = g'(F_1) \cup g'(F_2)$. Since $<g_c>$ is generated from $g_c$, for all the members $g'$ in $<g_c>$ either $g'(F_1) \cup g'(F_2) = F_1 \cup F_2 = F$ or $g'(F_1) \cup g'(F_2) = F_2 \cup F_1 = F$. In either case $g \in G \Rightarrow <g_c> (G_1 \cap G_2) \subseteq G$. Thus we conclude $G = <g_c> (G_1 \cap G_2)$. □

# 4 Oriented Features and their Symmetry Groups

In our treatment of features in this paper so far we have ignored one aspect of the faces of real world bodies, namely that faces are boundaries between solid matter and air: the surfaces which we have treated mathematically as subsets of $\Re^3$ have no intrinsic *inside and outside*. In this section we introduce the concept of *oriented features* which remedy this by defining a set of outward-pointing normal vectors for each surface point. The polynomial which we use to express an algebraic surface does define implicitly such normal vectors.

In general, whether a feature has orientations or not, or which orientation it has, does not make a difference in regards to the symmetries of the feature. The only exception is the plane surface: when it is treated as a set there are flipping symmetries which do not exist for oriented planes. A spherical surface, on the other hand, treated as a set, or with orientation vectors pointing inward, has the same symmetries as the spherical surface with orientation vectors pointing outward.This is why the treatment of primitive features as sets (given planer surface being the only special case) is sufficient as far as their symmetries are concerned. This is also why *same symmetry group* is a necessary condition for two solids at the the surface contact.

Let $S_2$ be the unit sphere at the origin, each point of which corresponds to a unit vector in $\Re^3$. Thus Definition 3.1 for primitive features can be extended as follows:

**Definition 4.1** *An* **oriented primitive feature** $F = (S, \rho)$ *is an oriented surface where*

*1)* $S \subset \Re^3$ *is a connected, irreducible and non-trivial algebraic surface of* $\Re^3$.

*2)* $\rho \subset S \times S_2$ *is a relation such that for all* $s \in S$, $(s, v) \in \rho$ *where* $v \in S_2$ *is one of the normal vectors of surface* $S$ *at point* $s$.

Intuitively speaking, a feature is composed of both "skin", $S$ , and "hair", the set of normal vectors which correspond to the points on $S_2$. Note, there may be more than one 'normal vector' at one point of a surface. For example, at the extreme point of a conic shaped surface.

We now define how an isometry acts on the relation $\rho$:

**Definition 4.2** *Any isometry* $g = tr$ *of* $\mathcal{E}^+, t \in \mathbf{T}^3, r \in SO(3)$ *acts on* $\rho$ *in such a way that* $g * \rho \ni (gs, rv) \Leftrightarrow (s, v) \in \rho$ *where* $s \in S, v \in S_2$.

Next we prove the associativity of isometries when they act on the relation $\rho$.

**Lemma 4.3** *For all* $g_1, g_2 \in \mathcal{E}^+, (g_1 g_2) * \rho = g_1 * (g_2 * \rho)$.

*Proof:*

Let $g_1 = t_1 r_2, g_2 = t_2 r_2$ where $t_1, t_2 \in \mathbf{T}^3, r_1, r_2 \in SO(3)$. Since $g_1 g_2 = t_1 r_1 t_2 r_2 = t_1 t' r_1 r_2$ ($\mathbf{T}^3$ is a normal subgroup of $\mathcal{E}^+$), for all $(s, v) \in \rho, (g_1 g_2 s, r_1 r_2 v) \in (g_1 g_2) * \rho$. On the other hand, for all $(s, v) \in \rho, (g_2 s, r_2 v) \in g_2 * \rho$ and $(g_1 g_2 s, r_1 r_2 v) \in g_1 * (g_2 * \rho)$. Therefore, $(g_1 g_2) * \rho = g_1 * (g_2 * \rho)$. □

For a feature defined in Definition 4.1, its symmetries are different from the symmetries of a set:

**Definition 4.4** *An isometry $g \in \mathcal{E}^+$ is a* **proper symmetry of a feature** *$F = (S, \rho)$ if and only if $g(S) = S$* **and** *$g * \rho = \rho$.*

There is, therefore, an extra demand on the symmetries for an oriented feature, namely, these isometries not only keep the feature setwise invariant but also preserve its orientations. Since orientations are points on $S_2$, such symmetries keep **two** sets of points setwise invariant *simultaneously*.

**Proposition 4.5** *The symmetries of a feature $F = (S, \rho)$ form a group, called the* **symmetry group of feature** *$F$.*

*Proof*:

Let $G$ denote the set of the symmetries of $F$. Since it has shown in Proposition 3.3 that the proposition is true for set $S$, here we only state about $\rho$.

Obviously, $1 * \rho = \rho$, so $1 \in G$. If $g \in G$ then $(g * \rho) = \rho$. Multiplying by $g^{-1}$ we have $g^{-1}(g * \rho) = g^{-1} * \rho$. Using Lemma 4.3 we have $g^{-1} * \rho = \rho$ and so $g^{-1} \in G$. Finally, if $g_1, g_2 \in G$ then $(g_1 g_2) * \rho = g_1 * (g_2 * \rho) = g_1 * \rho = \rho$ therefore $g_1 g_2 \in G$. Hence $G$ is a group. $\mathcal{E}^+$. □

Now we need to redefine our definition for distinct features, taking orientations into consideration.

**Definition 4.6** *Two oriented primitive features $F_1 = (S_1, \rho_1), F_2 = (S_2, \rho_2)$ are said to be*

- **Complement:** if there exists $a \in \mathcal{E}^+$ such that $a(S_1) = S_2$ and $\forall (s, v) \in a * \rho_1, \exists (s, -v) \in \rho_2$, and $\forall (s, v) \in \rho_2, \exists (s, -v) \in a * \rho_1$.

- **Distinct:** if for all the open subsets $S_1' \subset S_1, S_2' \subset S_2$, no $g = tr \in \mathcal{E}^+$ exists such that $g(S_1') \subset S_2$ or $g(S_2') \subset S_1$ (same as for the set features).

- **1-congruent (weakly-congruent)**: if there exists at least one $g \in \mathcal{E}^+$ such that $g(S_1) = S_2$ and $g * \rho_1 = \rho_2$, but for all such $g, g(S_2) \neq S_1$ or $g * \rho_2 \neq g * \rho_1$ simultaneously. Once again the conic shape features shown in Figure 6 can serve as an example, or two parallel planar surfaces with normal vectors pointing to the same direction.

- **2-congruent (strongly-congruent) via $g_c$**: if there exists $g_c \in \mathcal{E}^+$ such that $g_c(S_1) = S_2, g_c(S_2) = S_1, g_c * \rho_1 = \rho_2$ and $g_c * \rho_2 = \rho_1$. For example, two parallel cylindrical surfaces with the same radius, or two parallel planar surfaces with normal vectors pointing to the opposite directions.

The definition for oriented features allows us to distinguish a feature from its complement which we cannot do for set features (Definition 3.1). In general, complement features are not 2-congruent features except for a pair of planar surfaces. Nevertheless, complement features have the same symmetry group. When two solids have a surface contact it implies that a pair of complement features is formed.

The following proposition proves that all the complement features have the same symmetry group which justifies the necessary condition in surface contact of solids.

**Proposition 4.7** *If features $F_1 = (S_1, \rho_1), F_2 = (S_2, \rho_2)$ are complements of each other, where $a(S_1) = S_2$, and $G_1, G_2$ are the symmetry groups of $F_1, F_2$ respectatively, then the two symmetry groups are conjugate via a i.e. $aG_1a^{-1} = G_2$. In particular, if $S_1 = S_2$ then $G_1 = G_2$.*

*Proof :*

For all $g \in G_1, aga^{-1}(S_2) = ag(S_1) = a(S_1) = S_2$. Then for all $g \in aG_1a^{-1}, g(S_2) = S_2$.

For all $(s, v) \in \rho_2$ by definition of complement features $(s, -v) \in a * \rho_1$. For all $g \in G_1, a * \rho_1 = (aga^{-1}a) * \rho_1 = g'(a * \rho_1)$, where $g' = aga^{-1} = t'r'$. Then there must be $(s', -v') \in a * \rho_1$ such that $(g's', -r'v') \in a * \rho_1$. By definition of complement features $(s', v') \in \rho_2$. Then $(s, v) = (g's', r'v') \in g' * \rho_2 = aga^{-1} * \rho_2$. Then $\rho_2 \subseteq aga^{-1} * \rho_2$.

For all $g' = aga^{-1} = t'r' \in aG_1a^{-1}, (s, v) \in \rho_2, (g's, r'v) \in g' * \rho_2$ and $(g's, -r'v) \in g'a * \rho_1 = aga^{-1}a * \rho_1 = a * \rho_1$. By the definition of complement features again, $(g's, r'v) \in \rho_2$. Then $aga^{-1} * \rho_2 \subseteq \rho_2$.

Therefore forall $g \in aG_1a^{-1}, g * \rho_2 = \rho_2$. Hence $aG_1a^{-1} \subseteq G_2$.

Now we need to prove: For all $G_2 \subseteq aG_1a^{-1}$.

If $g = tr \in G_2$ then first consider how it acts on the set $g(S_2) = S_2 = g(a(S_1)) = a(S_1) \Rightarrow a^{-1}ga(S_1) = S_1$. Now let us consider how $g$ acts on the orientations. For all $(s, v) \in \rho_2, (s, -v) \in a * \rho_1$, then $(gs, rv) \in g * \rho_2 = \rho_2 \Rightarrow (gs, -rv) \in a * \rho_1$, but also $(gs, -rv) \in g(a * \rho_1)$ so $a * \rho_1 \subseteq g(a * \rho_1)$. On the other hand, $\forall (s, v) \in a * \rho_1, \exists (s, -v) \in \rho_2$. $(gs, rv) \in g(a * |rho_1)$ and $(gs, -rv) \in g * \rho_2 = \rho_2 \Rightarrow (gs, rv) \in a * \rho_1 \Rightarrow g(a * \rho_1) \subseteq a * \rho_1$. Then one can conclude $g(a * \rho_1) = a * \rho_1$, or $a^{-1}ga * \rho_1 = \rho_1$. Therefore $a^{-1}ga \in G_1 \Rightarrow g \in aG_1a^{-1} \Rightarrow G_2 \subseteq aG_1a^{-1}$.

Hence $G_2 = aG_1a^{-1}$. □

**Definition 4.8** *A* **compound feature** $F = (S, \rho)$ *of primitive features $F_1, ..., F_n$, is defined to be*

- $S = S_1 \cup ... \cup S_n$
- $\rho = \rho_1 \cup ... \cup \rho_n$

The reason why a relation $\rho$ instead of a function is chosen to denote the orientation of a feature becomes more clear here. When two primitive features are combined, there may be two distinct normal directions at one point of the feature, such as the edge where two planes meet (even for primitive features the normal vector at one point may not be unique, e.g. at the pointed end of a conic shape).

**Proposition 4.9** *Given a compound feature $F = (S, \rho)$ of primitive features $F_1, ..., F_n$, is defined to be*

- $S = S_1 \cup ... \cup S_n$
- $\rho = \rho_1 \cup ... \cup \rho_n$

*Let $F_1, ..., F_n$ be pairwise distinct primitive features with symmetry groups $G_1, ...G_n$ respectively. Then the symmetry group $G$ of $F$ is $G = G_1 \cap ... \cap G_n$.*

*Proof*:
Let $g \in G$, then $g(S) = S$ and $g*\rho = \rho$. Thus $g(S_1 \cup ... \cup S_n) = g(S_1) \cup ... \cup g(S_n) = S_1 \cup ... \cup S_n$. Then $g(S_i) \subseteq S_1 \cup ... \cup S_n$. Suppose $g(S_i) \not\subseteq S_i$. Then there exists a point $y \in g(S_i)$ and for some $j \neq i, y \in S_j - S_i$. Now the proof for the set features (Proposition 3.2) can be applied to get a contradiction.

Since $g * \rho = g * (\rho_1 \cup ... \cup \rho_n) = g * \rho_1 \cup ... \cup g * \rho_n = \rho_1 \cup ... \cup \rho_n$, assume $g * \rho_i \not\subseteq \rho_i$ then there must a pair $(gs, rv) = p \in g * \rho_i$ such that $p \notin \rho_i$ and $p \in \rho_j$ where $i \neq j$. Then there are two possibilities, $g(s) \notin S_i$ or $g(s) \in S_i$:

- For the first case the same proof as in (Proposition 3.2) can be given to derive a contradiction.
- For the second case, $g(s) \in S_i$ but $(gs, rv) \notin \rho_i$. Since $g(s) \in S_i$, by Definition 4.1 there must be $v' \in S_2$ such that $(gs, v') \in \rho_i$. Since the orientation at a point of an algebraic surface is determined by the neighbourhood of the point, and isometry $g$ transforms the neighbourhood $O_s$ of $s$ to be a neihgbourhood around $g(s)$. Now if $rv \neq v'$ which is the orientation at $g(s)$ determined by the neighbourhood of $g(s)$ of $S_i$ then there must exist a point $y = g(x) \in g(O_s), x \in O_s \subset S_i$ but $y \notin S_i$. Then the same proof as in (Proposition 3.2) can be given to derive a contradiction.

Therefore $g(S_i) \subseteq S_i$ and $g * \rho_i \subseteq \rho_i$. Since $g$ is a bijection, $g(S_i) = S_i$ and $g * \rho_i = \rho_i \Rightarrow g \in G_i$ for $i = 1, ..., n$. Thus $g \in G_1 \cap ... \cap G_n \Rightarrow G \subseteq G_1 \cap ... \cap G_n$.

For all $g \in G_1 \cap ... \cap G_n, g(F) = g(F_1 \cup ... \cup F_n) = g(F_1) \cup ... \cup g(F_n) = F_1 \cup ... \cup F_n = F$ and $g * \rho = g * (\rho_1 \cup ... \cup \rho_n) = g * (\rho_1) \cup ... \cup g * (\rho_n) = \rho_1 \cup ... \rho_n = \rho \Rightarrow g \in G \Rightarrow G_1 \cap ... \cap G_n \subseteq G$.

Therefore $G = G_1 \cap ... \cap G_n$. □

# 5 Spatial Relationships among Solids

An assembly is *a collection of bodies which are connected by their features.* When two bodies are connected, they do not make contact over their whole surface, rather certain *features* of each body are in contact. Therefore, although the symmetries of a body affect the final assembly configurations, the symmetries of the features in contact play a much more crucial role.

Let $B_1$ and $B_2$ be two bodies, with primitive features $F_1$ and $F_2$ which are in contact and have symmetry groups $sym(F_1), sym(F_2)$ respectively. Suppose $l_1, l_2$ specify the locations of bodies $B_1, B_2$ in the world coordinate system and $f_1$ and $f_2$ specify the locations of features $F_1, F_2$ in their respective body coordinates. Consider what we can infer about the *relative location* of two bodies that have two features in contact. There are three possible kinds of contacts between $B_1$ and $B_2$, they are *point* contact, *line* contact and *surface* or *areal* contact. Regardless of what kind of contacts occur between $F_1$ and $F_2$, by the definition of symmetry groups, it is clear that if we move $B_1$ or $B_2$ by a member of $sym(F_1)$ or $sym(F_2)$ respectively, the relationship between the features is preserved. A spatial relation between two bodies in contact is a binary relation $\tau \subset \mathcal{E}^+ \times \mathcal{E}^+$, where each pair $(l_1, l_2) \in \tau$ specifies one pair of possible positions for $B_1$ and $B_2$. In particular, when the two bodies have an areal contact via $F_1, F_2$, the contacting features coincide and thus their symmetry groups are identical. The spatial relationship then can be expressed as:

$$\tau = \{(l_1, l_2) | l_1^{-1} l_2 \in f_1 G f_2\} \tag{2}$$

where $G = sym(F_1) = sym(F_2)$.

The relative location of body $B_1$ with respect to body $B_2$ can be expressed as:

$$l_1^{-1} l_2 \in f_1 G f_2^{-1}. \tag{3}$$

We can summarize this by saying that *if a primitive feature of one body fits a primitive feature of another body then the relative location of the two bodies belong to a two-sided coset of the common symmetry group of the features.* This coset is an infinite set when the symmetry group is of infinite order.

Two bodies in an assembly are typically related to each other through multiple primitive features. If the above two bodies are related by surface contact of two pairs of features, i.e. $F_{11}$ fits $F_{21}$ and $F_{12}$ fits $F_{22}$ with feature locations in their body coordinate systems $f_{11}, f_{21}, f_{12}, f_{22}$ correspondingly, then the relative location of body $B_1$ to body $B_2$ should be constrained by both relations expressed in the form (3) *simultaneously*. That is:

$$l_1^{-1} l_2 \in f_{11} G_1 f_{21}^{-1} \bigcap f_{12} G_2 f_{22}^{-1} \tag{4}$$

i.e. a member of the intersection of two two-sided cosets, where $G_1 = sym(F_{11}), G_2 = sym(F_{12})$.

Since each two-sided coset can be rewritten as a one-sided coset as follows

$$g_1 H g_2 = g_1 H g_1^{-1} (g_1 g_2) \tag{5}$$

where $H \subset G, g_1, g_2 \in G$, we can modify (4) into the format of the intersection of two one-sided cosets.

$$l_1^{-1} l_2 \in (f_{11} G_1 f_{11}^{-1}) f_{11} f_{21}^{-1} \bigcap (f_{12} G_2 f_{12}^{-1}) f_{12} f_{22}^{-1} = (G_1' \bigcap G_2' f') f'' \tag{6}$$

where $G_1' = f_{11} G_1 f_{11}^{-1}, G_2' = f_{12} G_2 f_{12}^{-1}, f' = f_{12} f_{22}^{-1} f_{21} f_{11}^{-1}, f'' = f_{11} f_{21}^{-1}$. Further from proposition 2 of [16] :

Table 3: Continuous Group and Degree of Freedom

| Dimension (d.o.f.) | Symmetry Group (constraint) | Associated Lower pair |
|---|---|---|
| 1 | $T^1$ | *Prismatic* |
| 1 | $SO2$ | *Revolute* |
| 1 | $\mathcal{G}_{screw}$ | *Screw* |
| 2 | $\mathcal{G}_{cylinder}$ | *Cylindrical* |
| 3 | $\mathcal{G}_{plane}$ | *Planar* |
| 3 | $SO3$ | *Spherical* |

*If $H_1$ and $H_2$ are subgroups of $G$ and $g \in G$, then $H_1 \cap H_2 g$ is either null or is a coset of $H_1 \cap H_2$.*

we have

$$l_1^{-1} l_2 \in (G_1' \bigcap G_2')g \tag{7}$$

where $g \in (G_1' \bigcap G_2' f')f''$.

When the intersection is null, it implies that the specified spatial relationship is impossible, i.e. no values for positions $l_1, l_2$ can realize the required contacts. When the intersection is not null, the relative position can be obtained by calculating a group intersection and choosing a particular element $g$.

When the spatial relationship is realizable, the two primitive features of $B_1$ can be viewed as one compound feature $F_1 = F_{11} \cup F_{12}$ fitting with another compound feature $F_2 = F_{21} \cup F_{22}$ of $B_2$. The common symmetry group of these two compound features can be obtained from $sym(F_{11}), sym(F_{12})$ or $sym(F_{21}), sym(F_{22})$. Following (3) we have:

$$l_1^{-1} l_2 \in f_1 sym(F_1) f_2 \tag{8}$$

where $f_1, f_2$ are the locations of the compound features $F_1, F_2$ with respect to their body coordinate systems. When the primitive features are distinct from each other, by Proposition 3.2,

$$l_1^{-1} l_2 \in f_1(sym(F_{11}) \cap sym(F_{12}))f_2. \tag{9}$$

In the case where $sym(F_{11}) \cap sym(F_{12})$ is the identity group, $\{1\}$, we have

$$l_1^{-1} l_2 = f_1 f_2^{-1}, \tag{10}$$

and the relative position of $B_1$ to $B_2$ is uniquely determined. Interestingly, the most asymmetrical case appears in the simplest form under this formulation.

Surface contact relationships are quite common in assembly and are of primary concern in this paper. Table 3 exhibits all the kinematic joints that are formed by surface contacts (lower pairs as shown in Figure 2) and the associated symmetry groups of the contacting features.

# 6 Conclusion

We have formalized the surface contacts between solids in terms of the contacting feature symmetry group. The results show that this formalization is general, in that:

- all the surface contacts between primitive features can be represented by a symmetry group;
- all the surface contacts between compound features can be expressed in terms of the intersection of the symmetry groups of their primitive features;
- the relative positions of any pair of solids which have a surface contact can be expressed as the coset of their common symmetry group, or as the coset of the symmetry group intersection of their primitive features which are in contact.

These results also imply that that group theoretic formalization of surface contact presented here will only be computationally attractive if symmetry group intersections can be carried out efficiently. Such a method for efficient group intersections is the topic of a companion paper entitled "A Geometric Approach for Representing and Intersecting TR Subgroups of the Euclidean Group".

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# 7 Appendix

These are some topology definitions and theorems taken from [3, 15]. They are listed here as a quick reference for the readers.

**Definition 7.1** *A* **topology** *for a set $X$ is a family $T$ of subsets of $X$ satisfying the following three properties:*

- The set $X$ and the empty set $\emptyset$ are in $T$.
- The union of any family of members of $T$ is in $T$.
- The intersection of any finite family of members of $T$ is in $T$.

**Definition 7.2** *The members of $T$ are called* **open sets.**

**Definition 7.3** *A* **neighborhood** *of a point $x \in X$ is an open set containing $x$.*

**Definition 7.4** *A point $x$ is a* **limit point** *of a subset $A$ of $X$ means that every neighborhood of $x$ contains a point of $A$ distinct from $x$. A* **closure** *of a set $A$ is the set $\bar{A}$, the union of $A$ with its set of limit points. The* **boundary** *of $A$ is the intersect of $\bar{A}$ with $X \setminus A$.*

**Definition 7.5** *A function $f : X \to Y$ from a space $X$ to a space $Y$ is* **continuous** *provided that for each open set $U$ in $Y$ the inverse image*

$$f^{-1}(U) = \{x \in X | f(x) \in U\}$$

*is open in $X$.*

**Definition 7.6** *A one-to-one correspondence $f : X \to Y$ for which both $f$ and the inverse function $f^{-1}$ are continuous is called a* **homeomorphism**; *in this case $X$ and $Y$ are said to be* **homeomorphic.**

**Definition 7.7** *A function $g : X \to Y$ is* **open** *provided that $g(O)$ is open in $Y$ for each open subset $O$ of $X$.* **Closed function** *is defined analogously.*

**Definition 7.8** *A* **path,** *is a space $[X, \mathcal{O}]$ is a mapping $p : [a, b] \to X$, where $[a, b]$ is a closed interval in $\Re$. If $p(a) = P$ and $p(b) = Q$, then $p$ is a* **path from** $P$ **to** $Q$.

**Definition 7.9** *An* $n$**-manifold** *is a separable metric space $M^n$ in which every point has a neighborhood homeomorphic to $\Re^3$.*

**Definition 7.10** *Two sets $H, K$ are* **separated** *if*

$$\bar{H} \cap K = H \cap \bar{K} = \emptyset.$$

**Theorem 7.1** *A set $M \subset X$ is* **connected** *if and only if $M$ is not the union of two nonempty separated sets.*

**Theorem 7.2** *For sets, connectivity is preserved by surjective mappings.*

**Theorem 7.3** *If $H$ and $K$ are separated, then every connected subset $M$ of $H \cup K$ lies either in $H$ or in $K$.*

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