A Note on the Convexity of the Probability of a Full Buffer in the $M/M/1/K$ Queue[1]

Ramesh Nagarajan[2] and Don Towsley[3]

ABSTRACT

The loss probability in queueing systems is a very useful metric in the design and analysis of high-speed communication networks. In this paper, we investigate the convexity properties of this metric for the finite-buffer, single server $M/M/1/K$ queue. We demonstrate that the loss probability in the $M/M/1/K$ queue is convex with respect to the traffic intensity (arrival rate) for values of the traffic intensity below a certain value $\rho^*(K)$ and is concave for values of the traffic intensity larger than $\rho^*(K)$. We establish several useful properties of $\rho^*(K)$. Second, we show that the loss probability is convex with respect to the service rate. Last, we show that the throughput is jointly concave in the arrival and service rates while the loss rate is jointly convex with respect to the same.

KEYWORDS: $M/M/1/K$ QUEUE; CONVEXITY OF PERFORMANCE MEASURES; LOSS SYSTEM.

---

[1]This work is supported in part by the National Science Foundation under grant NCR-9116183
[2]Dept. of Elect. and Comp. Engg., Univ. of Massachusetts, Amherst, MA 01003
[3]Dept. of Computer Science, Univ. of Massachusetts, Amherst, MA 01003

1 Introduction

Convexity properties of performance measures of queueing systems are useful in solving optimization problems in stochastic systems. The loss probability in queueing systems has acquired significance, recently, in the design and analysis of high-speed communication networks [12, 4, 1, 19]. Consequently, we are witnessing an increasing study of optimization problems involving loss performance measures in queueing networks [14, 17, 11]. In this paper, we, hence, examine the convexity properties of the probability of a full buffer or the loss probability in the $M/M/1/K$ queue which will be useful in the design and analysis of high-speed networks.

Several researchers in the past have examined convexity properties of queueing system performance. We consider results for finite buffer queueing systems. Meister and Shantikumar [9] establish the concavity of the throughput of a tandem queueing system with respect to the buffer capacity. The tandem system in [9] is a set of exponential servers with finite buffer storage between servers. The servers block, i.e., service is not initiated, if the downstream buffer is full. Note that the tandem arrangement with two stages can be used to mimic a single finite buffer queue. Harel [5] investigates the convexity of the Erlang loss formula for the $M/G/K/K$ queue with respect to the traffic intensity and service rate. The author shows that the Erlang loss formula is convex with respect to the traffic intensity for values of the traffic intensity below a value which depends on the number of servers and concave for values above. The Erlang loss formula is, however, shown to be convex with respect to the service rate. In this paper, we consider the single-server, finite-buffer $M/M/1/K$ queue and study the second-order monotonicity properties of the loss probability function with respect to the traffic intensity and service rate. We find that the results for the $M/M/1/K$ queue are similar in flavour to those for the $M/G/K/K$ queue discovered in [5] suggesting similar results for other finite buffer queues.

There are a number of convexity results for infinite buffer systems [8, 3, 20, 16, 6, 7] as well. We will not detail these results here but point out that most of the work on infinite buffer systems focuses on the mean and variance of the waiting time in these systems. Finally, note that the $M/M/1/K$ queue can be modeled as a closed queueing network. In this context, Trivedi and Sigmon [15] show that the inverse of the throughput in a closed queueing network which models a linear storage hierarchy is a convex function of the relative utilizations at the nodes or alternately the average service demand (per storage reference) at the nodes.

The *first result* of this paper is that there *exists a buffer-size dependent value of* ρ, $\rho^*(K)$, that *delineates the convex and concave regions of the loss probability function* with respect to the traffic intensity. Further, we outline the numerical computation of $\rho^*(K)$ and present sample results. We also detail a number of useful properties of $\rho^*(K)$ and their implications for the behaviour of the loss probability with respect to the traffic intensity. Second, we show that the loss probability in the $M/M/1/K$ queue is *convex with respect to the service rate.*

Finally, we conclude by establishing *joint convexity and concavity* respectively for related performance measures, viz., *loss rate and throughput*, of the $M/M/1/K$ queue with respect to the arrival and service rate.

2 Convexity of the loss probability

We investigate second-order monotonicity properties of the loss probability, or probability of a full buffer, in the $M/M/1/K$ queue. The probability of loss in the $M/M/1/K$ queue is given as:

$$G(\rho) = \begin{cases} (1-\rho)\rho^K/(1-\rho^{(K+1)}) & \rho \geq 0,\ \rho \neq 1,\ K = 1, 2, 3, \cdots, \\ \frac{1}{K+1} & \rho = 1,\ K = 1, 2, 3, \cdots \end{cases} \tag{1}$$

where, as usual, $\rho = \lambda/\mu$ is the traffic intensity and K is the buffer size.

In the following, we will be first interested in ascertaining if $G(\rho)$ is a convex function of ρ for a fixed value of K. We shall subsequently investigate the convexity of $G(\rho)$ with respect to the service rate, μ, for fixed values of λ and K. Before we proceed to the proofs, we show that $G(\rho)$ is a continuous function of ρ which will prove useful in the subsequent analysis. We will, also, find it useful to adopt the interval notation, $I' = [0, 1]$ and $I = (0, 1)$.

Lemma 1 *$G(\rho)$ is a continuous function of ρ on $[0, \infty)$.*

Proof: Clearly, both $(1-\rho)\rho^K$ and $1-\rho^{K+1}$ are continuous on $(0, \infty)$ [13, Theorem 4.2]. Hence, by Theorem 4.2 [13], $G(\rho)$ is continuous on $(0, \infty)$ except at $\rho = 1.0$ where $1 - \rho^{K+1} = 0$. At $\rho = 1.0$, $G(\rho)$ is also continuous since by application of L'Hospital's rule [13, Theorem 5.11], $\lim_{\rho \to 1} G(\rho) = 1/(K+1)$. Now, consider the left end-point $\rho = 0$. Since $\lim_{\rho \to 0^+} G(\rho) = G(0) = 0$ (easily proved by using the corollary to the fundamental limit theorem [13, Thm 3.4]), we have that $G(\rho)$ is right-continuous at $\rho = 0$ and hence is continuous on $[0, \infty)$ [13, pp.100]. ∎

We, now, restate a theorem from [2, Theorem. 94] which states an important property of convex functions:

Theorem 1 *Suppose that $f(x)$ possesses a second derivative $f''(x)$ in the open interval (x_1, x_2). Then a necessary and sufficient condition that $f(x)$ should be convex in the interval is $f''(x) \geq 0$.*

However, in certain cases we will be interested in proving convexity over a closed interval rather than the open interval. A footnote in [2, pp. 76] states that the above theorem extends to the closed interval, $[x_1, x_2]$, provided the value of the function at the end-points is not less than the limiting value of the function at that end. Note that this latter property is indeed true for functions that are continuous on $[x_1, x_2]$. This implies that all results for the convexity of $G(\rho)$ with respect to ρ need be proved only for open intervals with automatic extension to the corresponding closed intervals. As a final comment on Theorem 1, we note that the reverse inequality is satisfied by concave functions.

2.1 Convexity of the loss probability with respect to traffic intensity

In this section, we first establish that there exists a value $\rho^*(K)$ for the traffic intensity that delineates the convex and concave regions of $G(\rho)$. Next, we outline the numerical computation of $\rho^*(K)$; sample results are deferred to a later discussion. We then proceed to analytically establish various useful properties of $\rho^*(K)$ followed by sample numerical values of $\rho^*(K)$.

Theorem 2 *There exists a value $\rho^*(K) \geq 0$ of ρ such that for $\rho \in [0, \rho^*(K)]$, $G(\rho)$ is convex in ρ and for $\rho \in [\rho^*(K), \infty)$, $G(\rho)$ is concave in ρ.*

Proof:

We will need the second derivative of $G(\rho)$ with respect to ρ in the following proof. Hence, we compute this first. Using Mathematica [18], we have that

$$G''(\rho) = \frac{\rho^{K-2}(2\rho^{2(K+1)} - (2+3K+K^2)\rho^{K+2} + (5K+K^2)\rho^{K+1} - K(K+1)\rho + K(K-1))}{(1-\rho^{K+1})^3} \tag{2}$$

for general K. The above equation can be written more compactly as

$$G''(\rho) = \frac{A_1(\rho)A_2(\rho)}{A_3(\rho)} \tag{3}$$

where

$$\begin{aligned} A_1(\rho) &= \rho^{K-2}, \\ A_2(\rho) &= 2\rho^{2(K+1)} - (2+3K+K^2)\rho^{K+2} + (5K+K^2)\rho^{K+1} - K(K+1)\rho + K(K-1), \\ A_3(\rho) &= (1-\rho^{K+1})^3. \end{aligned} \tag{4}$$

Consider the factor $A_2(\rho)$ in equation (3). Note that $A_2(\rho)$ is a polynomial in ρ of degree $2(K+1)$. We will be interested in determining the real roots of $A_2(\rho)$. First, note that $A_2(1) = 0$. Hence it will be useful to first separate any factors of $\rho - 1$ from $A_2(\rho)$. In the following, we will show that $(\rho-1)^3$ is a factor of $A_2(\rho)$ for general K i.e.,

$$A_2(\rho) = (\rho-1)^3 A_4(\rho) \tag{5}$$

In order to carry out the factorization of $A_2(\rho)$, we adopt *Horner's scheme* [10, pp. 153]:

$$\begin{aligned} A_2(\rho) &= (\rho-1)^3 A_4(\rho) \\ A_4(\rho) &= 2\rho^{2K-1} + 6\rho^{2K-2} + \cdots + K(K+1)\rho^K + \sum_{j=1}^{K-1} jK(j-K)\rho^{K-j-1}. \end{aligned} \tag{6}$$

Also, note that

$$A_3(\rho) = (1-\rho)^3 (\sum_{i=0}^{K} \rho^i)^3. \tag{7}$$

Taking into account the factoring of $A_2(\rho)$ and $A_3(\rho)$ above, we can write the second derivative as:

$$G''(\rho) = \frac{-A_1(\rho)A_4(\rho)}{(\sum_{i=0}^{K} \rho^i)^3} \tag{8}$$

Note that $A_1(\rho) > 0$ and the denominator is also always positive. Hence $G''(\rho) > 0$ when $A_4(\rho) < 0$ and vice-versa. Consider the expression for $A_4(\rho)$ in equation (6). It is useful to consider two separate cases. For $K = 1$, we have $A_4(\rho) = 2\rho$ and since $A_4(\rho) > 0$ for $\rho > 0$, we have $\rho^*(1) = 0$. Now consider the case $K \geq 2$. We note that there is only one change in the sign of the series of coefficients of $A_4(\rho)$. Hence by Descartes' theorem [10, pp. 193], $A_4(\rho)$ has a single positive real-valued root. We claim that this root of $A_4(\rho)$ is precisely the $\rho^*(K)$ that we seek. Note that $A_4(0) = -K(K-1) < 0$ and since $\rho^*(K) > 0$, we have $A_4(\rho) < (>)\ 0.0,\ \rho < (>)\ \rho^*(K)$. This completes the proof. ∎

We now make a brief comment on the computation of $\rho^*(K)$. In order to determine the exact value of ρ which delineates the convex and concave regions of $G(\rho)$, we only need to compute the positive root of $A_4(\rho)$. This can be accomplished by any standard root-finding algorithm since the above proof established that $A_4(\rho)$ has a single positive valued real root. We will present sample values of $\rho^*(K)$ in a later discussion.

Having established the existence of $\rho^*(K)$ above, we will find it useful to consider different properties of $\rho^*(K)$. Specifically, we claim the following bounds on the values of $\rho^*(K)$ for different ranges of values of the buffer size:

1. $\rho^*(1) = 0.0$
2. $0 < \rho^*(2),\ \rho^*(3) < 1.0$
3. $\rho^*(4) = 1.0$
4. $1 < \rho^*(K) \leq h(K) < 1.84,\ K \geq 5$ where $h(K) = (1 + K(K+3)/2)^{1/K}$. Note that this has the interesting implication that $G(\rho)$ is convex for $\rho \in I',\ K \geq 4$ which is typically a regime of practical interest in communication system analysis.

5. $\lim_{K\to\infty} \rho^*(K) = 1.0$.

Proof:

We will consider, as above, different cases.

CASE 1 ($K = 1$):

The proof of this case is embedded in the proof of the theorem for the existence of $\rho^*(K)$ above.

CASE 2 ($K = 2,\ 3$):

For $K = 2$,

$$G''(\rho) = \frac{2(1 - 3\rho^2 - \rho^3)}{(1 + \rho + \rho^2)^3} \tag{9}$$

Now, let $0 < \rho < 0.5$. We then have

$$\begin{aligned} \rho^2(\rho + 3) &< 0.875, \\ &< 1.0. \end{aligned} \tag{10}$$

Hence, we have $G''(\rho) > 0$ for $0 < \rho < 0.5,\ K = 2$. However, it is easily verified numerically that $G''(0.6) = -0.0786$. Hence, it is clear that $0.5 < \rho^*(2) < 0.6$ which implies $0 < \rho^*(2) < 1$. Similarly, for $K = 3$, we can show that $G''(\rho) > 0$ for $0 < \rho < 0.66$ but $G''(0.85) = -0.00759$. Hence, we have $0.66 < \rho^*(3) < 0.85$ which implies the corresponding claim.

CASE 3 ($K = 4$):

Here we have

$$G''(\rho) = \frac{2(1-\rho)\rho^2(6 + 14\rho + 20\rho^2 + 20\rho^3 + 10\rho^4 + 4\rho^5 + \rho^6)}{(1 + \rho + \rho^2 + \rho^3 + \rho^4)^3}. \tag{11}$$

Hence, we have that $G''(0) > 0$ for $\rho \in I$ and $G''(0) < 0$ for $\rho \in (1, \infty)$. Hence, we have $\rho^*(4) = 1.0$.

CASE 4 ($K \geq 5$):

First, we attempt to show that $\rho^*(K) > 1.0$.

It is easily verified numerically that $A_4(0) = -K(K-1) < 0.0$. We will show in the following that $A_4(1) < 0.0$ which implies that $\rho^*(K) > 1.0$.

We have

$$\begin{aligned} A_4(1) &= \sum_{j=0}^{K-1}(j+1)(j+2) + \sum_{j=1}^{K-1} jK(j-K), \\ &= \sum_{j=0}^{K-1} j^2 + 3\sum_{j=0}^{K-1} j + 2K + K\sum_{j=1}^{K-1} j^2 - K^2\sum_{j=1}^{K-1} j, \\ &= (K+1)\sum_{j=1}^{K-1} j^2 + (3-K^2)\sum_{j=1}^{K-1} j + 2K, \\ &= \frac{(K+1)(K-1)K(2K-1)}{6} + \frac{(3-K^2)(K-1)K}{2} + 2K, \\ &= \frac{K}{6}((K-1)(-K^2+K+8)+12). \end{aligned} \tag{12}$$

For $K \geq 5$, we have $(K-1)(K^2-K-8) > 12$ and hence $A_4(1) < 0$. This leaves the second part of the proof, viz., $\rho^*(K) < 1.84$.

Our starting point will be the second derivative for $G(\rho)$ in equation (3). Rearranging terms in $A_2(\rho)$, we have

$$\begin{aligned} A_2(\rho) &= \rho^{K+1}(2\rho^{K+1} + 5K + K^2 - (2+3K+K^2)\rho) + K(K-1) - K(K+1)\rho, \\ &= \rho^{K+1}(2\rho^{K+1} + 3K + K^2 - (2+3K+K^2)\rho) + 2K\rho^{K+1} + K(K-1) - K(K+1)\rho, \\ &= \rho^{K+1}g_1(\rho) + g_2(\rho) \end{aligned} \tag{13}$$

where

$$\begin{aligned} g_1(\rho) &= 2\rho^{K+1} + 3K + K^2 - (2+3K+K^2)\rho, \\ g_2(\rho) &= 2K\rho^{K+1} + K(K-1) - K(K+1)\rho. \end{aligned} \tag{14}$$

We will now demonstrate that $g_1(\rho) \geq 0.0$ and $g_2(\rho) \geq 0.0$ for $\rho \geq h(K) > 1.0$. Rewriting $g_1(\rho)$ we have

$$g_1(\rho) = 2\rho(\rho^K - 1) - (3K+K^2)(\rho-1). \tag{15}$$

Since $\rho > \rho - 1$ and $2(\rho^K - 1) \geq 3K + K^2$ for $\rho \geq h(K)$, we have $g_1(\rho) \geq 0$.

Now rewrite $g_2(\rho)$ as

$$\begin{aligned} g_2(\rho) &= K(2\rho^{K+1} + K - (1+\rho+K\rho)), \\ &= K(K + \rho^{K+1} + \rho^{K+1} - (1+\rho+K\rho)). \end{aligned} \tag{16}$$

Clearly, $K \geq 1$ and $\rho^{K+1} \geq \rho$. Hence it remains to be shown that $\rho^{K+1} \geq K\rho$. Since $\rho \geq h(K)$, we have $\rho^K \geq K(K+3)/2+1 > K$ and $g_2(\rho) \geq 0$. Hence $A_2(\rho) \geq 0.0$ and $G''(\rho) \leq 0.0$.

Clearly, $\rho^*(K) \leq h(K)$. Numerically evaluating $h(K)$ for $K = 5$ yields $h(5) = 1.838$. Further, in Appendix A we establish that $h(K)$ is a decreasing function of K. Hence, we have $\rho^*(K) \leq h(K) < 1.84$ for $K \geq 5$ and this completes the proof.

CASE 5:

Since $h(K) > 1.0$ and is a decreasing function of K (see Appendix A), we have $\lim_{K\to\infty} h(K) = 1.0$ [13, Exercise 2.11]. Further, since $1 < \rho^*(K) \leq h(K)$, we have proved Claim 5 above [13, Theorem 2.2]. ■

Having proved the various bounds on the value of $\rho^*(K)$, we will find it instructive to numerically compute the values of $\rho^*(K)$ for sample buffer sizes. Table 1 shows values for $h(K)$ and $\rho^*(K)$ for different values of K. It can be immediately seen that the sample values conform to the earlier established bounds. Further, the numerical values show that that $h(K)$ is a relatively tight upper bound for $\rho^*(K)$ for large K as expected. We also see, interestingly, that $\rho^*(K)$ appears to first increase and then decrease.

2.2 Convexity of the loss probability with Respect to Service Rate

In this section we examine the convexity of the loss probability in the $M/M/1/K$ queue with respect to the service rate. Our approach will be similar to that in the previous section for the traffic intensity.

Theorem 3 *The loss probability, $G(\rho)$, in the $M/M/1/K$ queue is convex with respect to the service rate for μ, $\lambda \in (0, \infty)$ and fixed λ and K.*

We will use the notation $G'(\rho)$ to denote the derivative with respect to the service rate.

Using Mathematica [18], we find that

$$G''(\rho) = \frac{\lambda^K \mu^{K-1}(K+1)F(\mu)}{(\mu^{K+1} - \lambda^{K+1})^3} \tag{17}$$

where

$$F(\mu) = K\lambda^{K+1}(\mu - \lambda) + K\mu^{K+1}(\mu - \lambda) + 2\lambda\mu(\lambda^K - \mu^K). \tag{18}$$

$F(\mu)$ can be factored according to *Horner's scheme* and it can be shown that

$$\begin{aligned} F(\mu) &= (\mu-\lambda)^3(K\mu^{K-1} + 2\lambda(K-1)\mu^{K-2} + 3\lambda^2(K-2)\mu^{K-3} + \cdots + K\lambda^{K-1}) \\ &= (\mu-\lambda)^3 \sum_{j=1}^{j=K} j(K-(j-1))\lambda^{j-1}\mu^{K-j}. \end{aligned} \tag{19}$$

Hence

$$G''(\rho) = \frac{\lambda^K \mu^{K-1}(K+1)(\mu-\lambda)^3 F_1(\mu)}{(\mu-\lambda)^3(\mu^K + \lambda\mu^{K-1} + \cdots + \lambda^{K-1}\mu + \lambda^K)^3} \tag{20}$$

where

$$F_1(\mu) = \sum_{j=1}^{j=K} j(K-(j-1))\lambda^{j-1}\mu^{K-j} > 0 \tag{21}$$

Hence $G''(\rho) > 0$ and that completes our proof. ∎

Remark: The above result implies that the loss probability is entirely convex with respect to the mean interarrival time with the μ and K fixed.

3 Convexity of Related Performance metrics

In this section, we consider the convexity of two related performance metrics for the $M/M/1/K$ queue, viz., the throughput and the loss rate. We will establish that the loss rate is jointly convex with respect to the arrival and service rates while the throughput is jointly concave.

Theorem 4 *The loss rate, $L = \lambda G(\rho)$, and the throughput, $T = \lambda(1 - G(\rho))$, are jointly convex and concave respectively in (λ, μ).*

Proof:

Proposition 4 in [5] states that if $W(\rho)$ is a convex function of μ with λ fixed, then $\lambda W(\rho)$ is jointly convex in (λ, μ). Since $G(\rho)$ in our case is a convex function of μ, the loss rate, $\lambda G(\rho)$, is jointly convex in (λ, μ). Similarly, $-\lambda G(\rho)$ is jointly concave in (λ, μ) and hence the throughput is jointly concave. ∎

4 Conclusion

In this paper, we considered the convexity of the loss probability in the $M/M/1/K$ queue with respect to the traffic intensity and the service rate of the queue. We also considered the convexity of some related performance measures for the $M/M/1/K$ queue.

With regard to the traffic intensity, we established that there exists a $\rho^*(K)$ which exactly delineates the convex and concave regions of $G(\rho)$. We demonstrated as to how $\rho^*(K)$ can be determined numerically and illustrated the computation for some sample buffer sizes. Further, we investigated properties of $\rho^*(K)$. Of most interest, we have shown that the loss probability in the $M/M/1/K$ queue is convex for $\rho \in [0,1], \quad K \geq 4$, which is a regime of considerable practical interest in communication system analysis. We also showed that for $\rho \geq h(K) > 1.0$, where $h(K)$ is presented in closed-form, the function is entirely concave. The numerical evaluation of $\rho^*(K)$ confirmed that $h(K)$ is a tight upper bound for $\rho^*(K)$ for large K. Finally, the value $\rho = 1.0$ was found to nearly delineate the convex and concave regions of $G(\rho)$ for large K. Second, with respect to the service rate, the loss probability in the $M/M/1/K$ queue was found to be entirely convex. Finally, we established rather trivially the joint convexity of the loss rate and the joint concavity of the throughput in the $M/M/1/K$ queue with respect to the arrival and service rates.

The loss probability in more general queueing systems such as the $M/M/c/K$ queue where closed-form expressions are available could possibly be studied in a similar fashion. This remains a topic for further research.

Acknowledgement:

We would like to thank Bibhu Mohanty for useful discussions and for checking parts of the proof.

References

[1] Andrea Baiocchi et al. Loss performance analysis of an ATM multiplexer loaded with high-speed on-off sources. *IEEE J.Select.Areas Commun.*, 9(3):388–393, April 1991.

[2] J.E. Littlewood G. Hardy and G. Polya. *Inequalities*. Cambridge University Press, 1952.

[3] W. Grassmann. The convexity of the mean queue size of the M/M/c queue with respect to the traffic intensity. *Journal of Applied Probability*, 20:916–919, 1983.

[4] Roch Guerin and Levent Gun. A unified approach to bandwidth allocation and access control in fast packet-switched networks. In *INFOCOM'92*, pages 01–12, 1992.

[5] Arie Harel. Convexity of the erlang loss formula. *Operations Research*, 38:499–505, May-June 1990.

[6] Arie Harel and Paul Zipkin. The convexity of a general performance measure for multiserver queues. *J. Appl. Prob.*, 24:725–736, 1987.

[7] Arie Harel and Paul Zipkin. Strong convexity results for queueing systems. *Operations Research*, 35:405–418, 1987.

[8] Hau Leung Lee and Morris Cohen. A note on the convexity of performance measures of M/M/c queueing systems. *Journal of Applied Probability*, 20:920–923, 1983.

[9] Ludolf Meister and George Shantikumar. Concavity of the throughput of tandem queueing systems with finite buffer storage space. *Advances in Applied Probability*, 22:764–767, 1990.

[10] A. P. Mishina and I. V. Proskuryakov. *Higher Algebra: Linear Algebra, Polynomials, General Algebra.* Pergamon Press, 1965.

[11] Ramesh Nagarajan et al. Local allocation of the end-to-end quality of service in high-speed networks. Manuscript in preparation.

[12] Ramesh Nagarajan et al. Approximation techniques for computing packet loss in finite-buffered voice multiplexers. *IEEE J.Select.Areas Commun.*, 9(3):368–377, April 1991.

[13] William R. Parzinsky and Philip W. Zipse. *Introduction to Mathematical Analysis.* McGraw-Hill Book Company, 1982.

[14] Henning Schulz-Rinne et al. Congestion control by selective packet discarding for real-time traffic in high-speed networks. In *INFOCOM'90*, pages 543–550, June 1990.

[15] Kishor S. Trivedi and Timothy S. Sigmon. Optimal design of linear storage hierarchies. *J. ACM*, 28:270–288, 1981.

[16] Richard Weber. A note on waiting times in single server queues. *Operations Research*, 31:950–951, 1983.

[17] Abel Weinrib and Ramesh Nagarajan. Guaranteeing end-to-end quality of service in connection-oriented packet networks. Technical Report TR 91-51, COINS Dept., UMASS, Amherst, June 1991.

[18] Stephen Wolfram. *Mathematica: A system for doing mathematics by computer.* Addison-Wesley, Inc., 1988.

[19] Gillian M. Woodruff and Rungroj Kositpaiboon. Multimedia traffic management principles for guaranteed ATM network performance. *IEEE J.Select.Areas Commun.*, 8:446, April 1990.

[20] Hung yuan Tu and Hillel Kumin. A convexity result for a class of GI/G/1 queueing systems. *Operations Research*, 31:948–950, 1983.

Appendix A

In Section 2.1, we establish the concavity of $G(\rho)$ with respect to ρ for $\rho \geq h(K)$. In this appendix, we establish that $h(K)$ is a decreasing function of K.

Lemma 2 *$h(K)$ is a decreasing function of K for $K \geq 1$.*

Proof:

Treating K as a real variable, and taking derivatives we have:

$$\frac{dh(K)}{dK} = \left(\frac{K+3/2}{K(1+K(K+3)/2)} - \frac{ln(1+K(K+3)/2)}{K^2}\right)h(K). \quad (22)$$

We now attempt to show that $\frac{dh(K)}{dK} < 0.0$. The proof is by contradiction. Let $\frac{dh(K)}{dK} \geq 0.0$. This implies that

$$\begin{aligned}
\frac{K+3/2}{K(1+K(K+3)/2)} &\geq \frac{ln(1+K(K+3)/2)}{K^2}, \\
K^2 + 3K/2 &\geq (K^2/2 + 3K/2 + 1)ln(1+K(K+3)/2), \\
K^2 + 3K/2 &\geq (K^2/2 + 3K/2)ln(1+K(K+3)/2), \\
K(2 - ln(1+K(K+3)/2)) &\geq 3(ln(1+K(K+3)/2) - 1).
\end{aligned} \quad (23)$$

For $K > 3$, $ln(1+K(K+3)/2) > 2.3$. Hence $2 - ln(1+K(K+3)/2) < 0$ and since $ln(1+K(K+3)/2) - 1 > 0$ we have a contradiction. Hence $\frac{dh(K)}{dK} < 0.0$ for $K \in (3,\infty)$. This implies that $h(K)$ is decreasing in $(3,\infty)$ [13, Exercise 5.14]. Interpreting this result for integer values of K we conclude that $h(j+1) < h(j),\ \ j > 3$. We now simply verify numerically for small values of K that $h(K)$ is indeed decreasing for all K. Numerical evaluation shows that $h(1) = 3.0, h(2) = 2.45, h(3) = 2.15, h(4) = 1.97$ and that completes the proof.
■

Buffer Size (K)	$h(K)$	$\rho^*(K)$
1	3.0	0.0
2	2.45	0.53
3	2.15	0.843
4	1.97	1.0
5	1.84	1.08
6	1.74	1.13
7	1.67	1.15
8	1.61	1.16
9	1.56	1.17
10	1.52	1.174
20	1.31	1.15
50	1.15	1.09
100	1.09	1.05

Table 1: Traffic intensity values that delineate the convex and concave regions of $G(\rho)$