An Optimal Control Formulation and Related Numerical Methods for a Problem in Shape Reconstruction

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Abstract

The main problem considered in this paper is the construction of numerical methods and proofs of their convergence for the problem of "shape from shading." In the first part of the paper it is assumed that the height function that describes the surface to be reconstructed is known at all local minima (or maxima). These points are a subset of the singular points, which are the brightest points in the image. A pair of optimal control problems are defined that provide representations for the height function. Numerical schemes based on these representations are then constructed. While both schemes lead to the same approximation, the pair are developed simultaneously because one leads to more efficient algorithms, while the other is more convenient in the convergence analysis. The proof of convergence is based on a representation of the approximation to the height as a functional of a controlled Markov chain. In the second part of the paper the assumption that the height must be known at all local minima (or maxima) is dropped. An extension of the algorithm is described that is capable of reconstruction without this information. Numerical experiments for both algorithms on synthetic and real data are included.

1 Introduction

Shape from shading has been a central problem in computer vision for the entire history of the field. Initial work dates back to the 1960s with the mapping of the lunar surface from visual images [25]. In the early 1970s, Horn's work on this problem [8] pioneered the new quantitative focus on low level, precognitive vision, which has since dominated vision research. The problem has continued to be of interest over the years, and recently has seen dramatic progress [1, 4, 10, 19, 20, 21, 22, 27, 28, 29, 30].

The problem of shape from shading is to reconstruct the three dimensional shape of a surface from the brightness or intensity variation in a black–and–white photographic image of the surface. If the illumination is mainly unidirectional, and the surface is uniform and untextured, then the brightness at an image point provides information about the surface orientation imaged at that point. However, in general it does not fully specify the surface orientation, even if the illumination and surface reflectance properties are completely known, but only constrains it. Consequently, the task of reconstructing the surface is difficult. It is for this reason that, for much of its history, shape from shading was believed to be an ill–posed problem. However, it has been argued more recently, and is reillustrated here, that the problem is actually well posed under a wide range of conditions [19, 20].

Shape from shading is something that humans do well [17]. It is clearly an essential ingredient in visual interpretation for the many contexts where other cues for shape reconstruction are lacking, e.g. for still photographs, or for distant objects where stereo or motion parallax are not effective. It is an important aid in reconstructing shape corresponding to regions of smoothly varying intensity in images, where the absence of distinguishable features makes stereo or motion reconstruction difficult. Finally, the problem in its idealized form is precise and well–defined mathematically: it amounts to finding the "proper" solution to a particular nonlinear first order partial differential equation.

We consider here the idealized problem. Thus we make the standard assumptions that the surface is matte, rather than mirror–like, and reflects light evenly in all directions ("Lambertian" reflectance); that the illuminating light is from a single known direction; and the surface is distant from the camera ("orthographic" projection). Moreover, we make the standard but stronger assumptions that the surface is uniform and untextured; that there are no cast shadows; and, finally, that all portions of the surface are visible (no "occlusion"). Although there has been some work on the problem relaxing one or another of these assumptions, [2, 7, 10, 16, 18, 23, 28], most work has focused on the problem in this form. Understanding this idealized problem is a prerequisite for attacking the larger problem of how shading, in combination with other information, e.g., from stereo, can be used for image understanding.

Under the assumptions stated above, the intensity $I(\cdot)$ registered on the image plane at coordinates $x \equiv (x_1, x_2)$ is given by

$$I(x) = \langle \gamma, \hat{n}(x) \rangle, \tag{1.1}$$

where $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ is the fixed direction from which the light is coming, and $\hat{n}(\cdot)$ is the surface normal at the corresponding surface point (x_1, x_2, z). Such an equation

1

relating $I(\cdot)$ to the corresponding surface orientation is known as the image irradiance equation [12]. We will always assume here that the surface height $z(x_1, x_2)$ is continuously differentiable (though this is not essential). Then the surface normal is given in terms of $z(\cdot)$ by

$$\hat{n}(x) = \frac{(-z_x(x), 1)}{(1 + \|z_x(x)\|^2)^{1/2}}, \tag{1.2}$$

and the image irradiance equation is a first order nonlinear partial differential equation. In general this equation can have multiple classical sense solutions. The problem of shape from shading is to reconstruct $z(\cdot)$ from the given data $I(\cdot)$.

We briefly summarize previous work on this problem. Most of this work is based on the assumption that the reconstruction problem is not well posed in any reasonable sense. In the early work of Horn [8], the characteristic strip method was used to give a numerical scheme for shape reconstruction. Horn later formulated the problem in the idealized form considered in most subsequent work [9]. So-called variational methods were introduced and studied in, e.g., [2, 10, 11, 13, 31]. Let the image irradiance equation be written in the form $H(x, z_x(x)) = 0$, and $V(\cdot)$ be an approximation for the height function. In the variational approach, a functional of this approximation $\mathcal{F}(V) \equiv \int_{\mathcal{D}} H^2(x, V_x(x))dx$ is defined, where $\mathcal{D}$ is the domain of definition of the intensity function. Since $z(\cdot)$ obeys $H(x, z_x(x)) = 0$ it minimizes this functional. In practice, $V(\cdot)$ is approximated by a function $V^h(\cdot)$ defined on a discrete grid $\mathcal{D}^h$, and the functional by $\mathcal{F}^h(V^h) \equiv \sum_{\mathcal{D}^h} H(x, V_x^h(x))$, where V_x^h is a discrete approximation of the derivative V_x. It is assumed that $z^h(\cdot)$, defined to minimize $\mathcal{F}^h$, is a good approximation to $z(\cdot)$.

The variational approach attempts to locate $z^h(\cdot)$ by letting an initial estimate $V^h(\cdot)$ evolve in the steepest descent direction $-\partial \mathcal{F}^h(V^h)/\partial V^h$. This procedure guarantees convergence to a local minimum of $\mathcal{F}^h$. In practice, however, this often produces a spurious local minimum rather than the desired approximation $z^h(\cdot)$. To ameliorate this problem, a "regularization" term proportional to $\int_{\mathcal{D}} \sum_{ij \in \{1,2\}} V_{x_i, x_j}^2 dx$ is included in $\mathcal{F}(V)$, and discretely approximated in $\mathcal{F}^h$ as before. Because it penalizes those functions $V(\cdot)$ with high curvature, this term destabilizes some of the spurious local minima of $\mathcal{F}^h$, which tend to have multiple regions of high "curvature." Unfortunately, this term also has the effect of perturbing the global minimum of $\mathcal{F}^h$ away from the best approximation to $z(\cdot)$. A potential solution to this problem was discussed in [10].

In a second approach [6, 16, 23], the surface is reconstructed separately in small image patches based on restrictive assumptions about the surface shape there—for instance, that the surface is spherical in each patch. The partial reconstructions are then patched together to give an overall reconstruction.

More recently, Pentland [24] has presented an algorithm that replaces the shape from shading problem with a linearized approximation, which can then be solved by direct integration. However, boundary conditions are required to make the approximating linearized reconstruction unambiguous. In general, proofs of convergence are lacking in all of these approaches.

Finally, in [3, 19, 20, 29, 30], the uniqueness of the solution for $z(\cdot)$ was discussed, under the assumption that $z(\cdot)$ was at least C^2. Reference [19] also considered the constraint on $z(\cdot)$ provided by a visible "occluding boundary," i.e., the set of points on the surface

where the derivative of the map from the surface to the image plane is singular.

In this paper we view the problem of shape-from-shading in an entirely different way. The cornerstone to our approach is a representation for the height function $z(\cdot)$ as the infimal cost for a deterministic optimal control problem [21]. Convergent numerical methods can then be constructed on the basis of this representation. Work by other authors that is closest to ours is [27, 28], which also makes use of the connection to an optimal control problem in order to derive a numerical scheme. The main mathematical tool used in these papers is the theory of viscosity solutions of first order nonlinear PDE. However, the types of assumptions made here on the data available for reconstruction differ from those in [27, 28], as do the particular results, algorithms and methods of analysis.

We conclude this introduction by loosely summarizing the main results of the paper. In the idealized framework mentioned above, the "brightest" points in the image have value $I(x) = 1$. Furthermore, these brightest points include all local maxima, minima, and saddle points of the height $z(\cdot)$ (in the case of oblique light this is with respect to a coordinate system in which the "vertical" direction is the direction from which the light arrives). It turns out that specification of the height function on either the set of local maxima or minima allows, via an optimal control formulation, a representation for $z(\cdot)$ in a region around these points. A more precise statment is as follows. Let an isolated local minimum point x of the surface be given, and define the "domain of attraction" of this point to be a set of points y such that any steepest descent trajectory (with respect to $z(\cdot)$) that starts at y terminates at x. One can then construct an optimal control problem such that the infimal cost for this control problem has $z(\cdot)$ as its value for all points in the domain of attraction. Knowledge of $z(\cdot)$ at a number of local minima allows reconstruction in the union of their domains of attraction. This indicates in a precise way what information is needed to reconstruct a given part of the surface. The optimal control problem itself is easily (formally) derived from the irradiance equation via a dynamic programming equation and convex duality. In fact, different ways of rewriting the irradiance equation give rise to different optimal control representations, and two that are particularly convenient to work with (Control Problem 1 and Control Problem 2) are discussed in Section 2. The proof that the representation is valid for Control Problem 2 is given in Section 3.

In general, the data available for shape reconstruction is not the complete intensity function $I(\cdot)$, but rather a restriction $I^h(\cdot)$ of $I(\cdot)$ to a regular grid of points $\mathcal{D}^h$, where h is the spacing between points. Our schemes for computing approximations to $z(\cdot)$ are based on replacing the original optimal control problems by control problems defined on $\mathcal{D}^h$. It is clearly desirable to preserve as far as possible the characteristics of the original control problems. Thus we leave the cost structure essentially unchanged. However, we must approximate the original deterministic dynamical model by one whose state space is the grid $\mathcal{D}^h$. It turns out that the best model to use is a controlled Markov chain. The reason for this is discussed in Section 2. Thus the approximating control problem takes the form of a stochastic optimal control problem whose state space is the grid $\mathcal{D}^h$. By applying the principle of dynamic programming to this stochastic control problem, we obtain a nonlinear iteration that defines a monotonically nonincreasing sequence of functions defined on $\mathcal{D}^h$. The approximation to $z(\cdot)$ is defined as the limit of these functions, which is a fixed

point of the iteration. This approximation scheme is an example of a widely applicable method of approximation known as the Markov chain approximation method [14]. The basis for the proof of convergence is a representation of the approximation as a functional of a controlled Markov chain.

In Section 2, the numerical schemes associated to both Control Problem 1 and Control Problem 2 are defined and shown to yield equivalent approximations. Because Control Problem 1 has quadratic costs, it gives rise to an efficient algorithm that is simple to construct and implement. However, it is a more difficult problem to work with as far as the representation of the height function and the proof of convergence of the scheme are concerned. This is the reason for the introduction of Control Problem 2. Although the algorithm obtained from Control Problem 2 is not as attractive as that of Control Problem 1, proofs of the representation and convergence theorems are much easier. Thus the equivalence of the approximations is a key observation. The proof of convergence of the approximations to $z(\cdot)$ is given in Section 4. There are a number of nonstandard features associated to these control problems and numerical schemes that make the proof much more difficult that usual. For example, the nonlinear iterations that define the approximations will in general have multiple fixed points, and we must be careful to pick out the "correct" fixed point.

In Subsection 5.1 we present some experimental results and further discussion of the algorithms described in the first part of the paper. For these algorithms it is assumed that the values of $z(\cdot)$ are given on either the set of local maxima or minima in order to obtain an approximation. In Subsection 5.2 we describe a method which eliminates the need for such data. Suppose that the extreme points of $z(\cdot)$ are isolated. As noted above, the optimal control representation tells us that the correct specification of the height at either a local maximum or minimum identifies the height at all points in the domain of attraction, and by continuity in the closure of this domain as well. Suppose that the only data available besides the intensity function is the location of one local minimum point. Under conditions stated in Section 5, there is always at least one local maximum on the boundary of the domain of attraction. An application of the algorithm of Section 2 yields an approximation to $z(\cdot)$, which on the basis of the results of Sections 3 and 4 should be good in the closure of the domain of attraction. Given this approximation, a method is described for identifying from among all the remaining "brightest" points one that is a local maximum on the boundary of the domain of attraction. Since an approximation to the height at this maximum point has already been generated, the algorithm can be applied again (although this time we would use the algorithm that is appropriate when data is given at the local maxima) to obtain an even larger region on which the reconstruction can be expected to be good. The procedure is then repeated until all possible extrema have been identified and their heights approximated. A scheme is also discussed for identifying a starting local minimum or maximum. Thus approximations can be constructed that are based only on the image intensity function. Besides a description of this method, we present experimental results obtained using it.

4

2 Optimal Control Representations and Numerical Procedures

In this section we give two optimal control representations for the unknown height function in the shape from shading problem. Numerical procedures will then be constructed that are based on these control problems.

In the first representation for the height function, the control space is unbounded, the dynamics are very simple, and the running cost is quadratic in the control. In fact, the problem is actually a calculus of variations problem. It turns out in the case of "oblique light" (described below) that there are certain regions in which this control problem must be extended to the form of a differential game. As noted in the introduction, the reason we consider two control problems is that each representation gives rise to a different numerical procedure, and each scheme has some attractive features.

All the schemes we consider will take the following form. For a scalar h and set A, let $hA = \{ha : a \in A\}$. Given $h \in (0, \infty)$ let $\mathcal{D}^h = h\mathbb{Z}^2 \cap \mathcal{D}$, where $\mathcal{D}$ is some open bounded subset of $\mathbb{R}^2$. Assume that we are given as data for the reconstruction problem the restriction $I^h(\cdot)$ of the intensity function $I(\cdot)$ to $\mathcal{D}^h$. Then our goal is to produce an approximation V^h to the restriction of $z(\cdot)$ to $\mathcal{D}^h$, that is based on I^h. The schemes for V^h will always be iterative: given an initial condition V_0^h (to be described below), we define in a recursive fashion

$$V_{n+1}^h = F^h(V_n^h, I^h),$$

and then set $V^h = \lim_{n \to \infty} V_n^h$. In all cases, the iterates V_n^h will be monotonically non-increasing and bounded from below. This implies V^h will always exist and be a fixed point of the iteration $V^h \to F^h(V^h, I^h)$. It will often be the case that the iteration itself has multiple fixed points. However, if the initial condition V_0^h is chosen sufficiently large, then the iteration will be shown to converge to the "correct" fixed point V^h, which gives the desired approximation to $z(\cdot)$. Also, though various forms of F^h will be suggested by the different control representations, it will be shown below that the limit V^h will be the same for any of the schemes we define.

Our reason for introducing the first control problem is that, due to its simple dynamics and cost structure, it results in an algorithm that is easy to construct and iterate. On the other hand, besides computational efficiency, we are also interested in demonstrating the convergence of the approximation $V^h(x) \to z(x)$. This turns out to be simpler using the second control problem, which, as stated above, also yields the same approximation $V^h(x)$. Our proof of convergence will be based on interpreting $V^h(x)$ as the minimal (or maximal, depending on the case) cost for a stochastic control problem whose dynamics "mimic" those of the second control problem. One reason that this is simpler is that the second control problem, unlike the first, does not need to be extended to a differential game in the oblique light case. Furthermore, because the control space is bounded for the second problem a basic compactness property of any sequence of controls indexed by h turns out to be automatic.

To simplify the presentation we consider first the case of vertical light in Subsection 2.1. We then extend to oblique light in Subsection 2.2. In Subsection 2.3 we discuss relationships between the algorithms derived in the previous two subsections.

2.1 THE CASE OF VERTICAL LIGHT.

Control Problem 1. Recall that $\mathcal{D}$ describes the subset of $I\!R^2$ on which the image data is recorded and that $z(\cdot)$ is the height function to be recovered. We begin by describing the control problem and numerical scheme for "vertical light." In this situation, we will assume that the surface $S = \{(x_1, x_2, x_3) : x_3 = z(x_1, x_2)\}$ is illuminated from the positive x_3 direction by a point light source that is infinitely far away in the positive x_3 direction, and that the reflected light is recorded in an imaging plane that is parallel to the plane $\{(x_1, x_2, x_3) : x_3 = 0\}$. Thus the "viewing direction" of the camera and the direction of the incoming light are the same. We take the reflected light as being characterized in terms of the deterministic intensity function $I(x)$, where x identifies the (x_1, x_2) coordinates of a point on the imaging plane. Under a number of additional assumptions, including the assumption that the surface is "Lambertian" [12], the height function $z(\cdot)$ and the intensity function $I(\cdot)$ are related by the equations (1.1) and (1.2). Eliminating $\hat{n}$ from these equations and taking $\gamma = (0, 0, 1)$ yields the equation

$$I(x) = \left(1 + \|z_x(x)\|^2\right)^{-1/2} \tag{2.1}$$

in regions where $z(\cdot)$ is continuously differentiable. Thus, $I(x)$ equals the absolute value of the x_3–component of a unit vector orthogonal to S at $(x, z(x))$. We also have $I(x) \in (0, 1]$. We will refer to the points in $\mathcal{D}$ where $I(x) = 1$, which obviously includes all local maximum and minimum points of $z(\cdot)$ in $\mathcal{D}$, as the *singular points* of $I(\cdot)$ (or $z(\cdot)$). Let $\mathcal{S}$ denote the set of all singular points.

Terminology. In the development below we will want to precisely specify what data is needed to reconstruct a given part of the surface. Because of this we need to introduce some nonstandard terminology. We will usually assume that the set of singular points consists of a finite collection of connected sets, and that the height function is constant on each connected subset (recall $z_x = 0$ at a singular point). We will refer to such a connected subset $\mathcal{S}_C$ as a set of local minima, and to all points in the subset as local minimum points, if there exists $\varepsilon > 0$ such that $d(x, \mathcal{S}_C) < \varepsilon$ and $x \in \mathcal{D}$ imply $z(x) \geq z(y)$ for $y \in \mathcal{S}_C$. An analogous definition is used for local maxima. A connected subset that is neither a set of local maxima or local minima is called a set of saddle points.

We next describe an optimal control representation for $z(\cdot)$ that is based on knowing $I(x)$ for $x \in \mathcal{D}$ and $z(x)$ only on a subset of $\mathcal{S}$. A result analogous to the one stated below is possible when the set of local minima is replaced by the set of local maxima. To give a precise statement, we need the following. We say that a set $A \subset I\!R^2$ is smoothly connected if given any two points x and y in A there is an absolutely continuous path $\phi : [0, 1] \rightarrow A$ such that $\phi(0) = x, \phi(1) = y$.

A2.1 *Assume that $\mathcal{S}$ consists of a finite collection of disjoint, compact, smoothly connected sets, and that $z_x(\cdot)$ is continuous on the closure of $\mathcal{D}$. Let $\mathcal{G} \subset \mathcal{D}$ be compact, and assume $\mathcal{G}$ is of the form $\mathcal{G} = \cap_{j=1}^{J} \mathcal{G}_j, J < \infty$, where each $\mathcal{G}_j$ has a continuously differentiable boundary. Let $\mathcal{M}$ be the set of local minima of $z(\cdot)$ inside $\mathcal{G}$. Then we assume*

that the value of $z(\cdot)$ is known at all points in $\mathcal{M}$. Define $n_j(x)$ to be the inward (with respect to $\mathcal{G}$) normal to $\partial \mathcal{G}_j$ at x. Then we also assume that $\langle z_x(x), n_j(x) \rangle < 0$ for all $x \in \partial \mathcal{G} \cap \partial \mathcal{G}_j, j = 1, ..., J$.

Assume we are given an upper bound B for $\{z(x) : x \in \mathcal{G}\}$. Suppose that $z(\cdot)$ is continuously differentiable on the closure of $\mathcal{D}$, and define

$$g(x) = \begin{cases} z(x) & \text{for } x \in \mathcal{M} \\ B & \text{for } x \notin \mathcal{M} \end{cases} \tag{2.2}$$

and

$$L(x, \beta) = \frac{1}{2} \|\beta\|^2 + \frac{1}{2} \left(\frac{1}{I^2(x)} - 1 \right) = \frac{1}{2} \|\beta\|^2 + \frac{1}{2} \|z_x(x)\|^2. \tag{2.3}$$

Note that $L(x, \beta) \geq 0$, and that $L(x, \beta) = 0$ if and only if $x \in \mathcal{S}$ and $\beta = 0$. We can now define the optimal control problem. As an admissible control we will consider any integrable function $u : [0, \infty) \to \mathbb{R}^2$. Given such a control, the dynamics of the controlled process are simply $\dot{\phi}(s) = u(s)$ (a.s. in s) and $\phi(0) = x$. Let

$$V(x) = \inf \left[\int_0^{\rho \wedge \tau} L(\phi(s), u(s)) ds + g(\phi(\rho \wedge \tau)) \right], \tag{2.4}$$

where $\tau = \inf\{t : \phi(t) \in \partial \mathcal{D} \cup \mathcal{M}\}$ and the infimum is over all $\rho \in [0, \infty)$ and admissible controls. We will follow the convention of defining the integral to be ∞ if the integrand is not integrable.

It turns out that within the set $\mathcal{G}$ the minimizing trajectories for the deterministic control problem given above are essentially the two dimensional projections of the paths of steepest descent on the surface represented by the height function. Thus, the assumptions that are placed on $\mathcal{G}$ in A2.1 guarantee that any minimizing trajectory that starts in $\mathcal{G}$ stays in $\mathcal{G}$. Because of this it can be shown that $V(x) = z(x)$ for all $x \in \mathcal{G}$, where $\mathcal{G}$ is any set that satisfies A2.1. We will never actually use this fact, since the proof of convergence will be based on the analogous theorem for Control Problem 2, which is stated and proved in Section 3. We are mostly interested in Control Problem 1 because it can be used as a basis for the construction of numerical algorithms.

A formal derivation of the equality $V(x) = z(x)$ is as follows. The equation (2.1) can be rewritten as

$$H(x, z_x(x)) = 0 \tag{2.5}$$

where

$$H(x, \alpha) = \frac{1}{2} \left[1 + \|\alpha\|^2 \right] - \frac{1}{2I^2(x)}.$$

Note that $H(x, \alpha)$ is convex in α. The function $L(x, \beta)$ defined in (2.3) is the Legendre transform of $H(x, \alpha)$ in α:

$$L(x, \beta) = \sup_{\alpha \in \mathbb{R}^2} \left[-\langle \alpha, \beta \rangle - H(x, \alpha) \right]. \tag{2.6}$$

(Note that our sign convention results in a slight deviation from the usual definition of the Legendre transform.) Let $u(\cdot)$ be any admissible control, $\phi(t) = x + \int_0^t u(s) ds$, and let

$\rho \in [0, \infty)$. If $g(\phi(\rho \wedge \tau)) = B$ then $V(x) \geq z(x)$ is automatic. If $g(\phi(\rho \wedge \tau)) < B$ then $\tau < \rho$ and $\phi(\tau) \in \mathcal{M}$. Also, by (2.5) and (2.6),

$$L(\phi(t), u(t)) \geq -\langle z_x(\phi(t)), u(t)\rangle - H(\phi(t), z_x(\phi(t))) = -\langle z_x(\phi(t)), u(t)\rangle,$$

and hence

$$\int_0^\tau L(\phi(t), u(t))dt + g(\phi(\tau)) \geq \int_0^\tau -\langle z_x(\phi(t)), \dot\phi(t)\rangle dt + z(\phi(\tau)) = z(x).$$

Thus $V(x) \geq z(x)$.

Next consider a solution $\phi(t)$ to $\dot\phi = -z_x(\phi), \phi(0) = x$. Equation (2.5) and the definition of $L(x, \beta)$ in (2.3) imply the supremum in (2.6) for $L(\phi(t), \dot\phi(t))$ is achieved at $\alpha = z_x(\phi(t))$ for all $t \in [0, \tau]$. Note that A2.1 implies $\phi(t)$ remains in $\mathcal{G}$ for all $t \geq 0$. Since $z(\phi(t))$ is decreasing with time, this solution must tend to the set $\mathcal{S}$ as $t \to \tau$, where τ may be infinite. Assume that $\phi(t)$ actually tends to a point in $\mathcal{M}$. If $\tau < \infty$ then we can take $\rho > \tau$ and $u(t) = -z_x(\phi(t))$ to achieve

$$
\begin{aligned}
V(x) &\leq \int_0^\tau L(\phi(t), u(t))dt + g(\phi(\tau)) \\
&= \int_0^\tau \langle z_x(\phi(t)), z_x(\phi(t))\rangle dt + z(\phi(\tau)) \\
&= \int_0^\tau -\langle z_x(\phi(t)), \dot\phi(t)\rangle dt + z(\phi(\tau)) \\
&= z(x).
\end{aligned}
$$

The first inequality is due to the definition of $V(x)$ as an infimum. The following equality is due to the fact that the supremum is achieved in (2.6) with the given definition of $u(\cdot)$. If $\tau = \infty$, then we can use the same $u(\cdot)$ to get $\phi(\cdot)$ into an arbitrarily small neighborhood of the limit point in finite amount of time. Once we have reached a nearby point we modify the control to move $\phi(\cdot)$ to the limit point in a finite time with small running cost. Although this results in a control that incurs greater running cost than the control that takes infinite time, it can be shown that the additional cost can be made arbitrarily small.

Note that the main facts used in this formal derivation were that $H(x, z_x(x)) = 0$ is equivalent to the basic equation (2.1), and that the running cost $L(x, \beta)$ is related to $H(x, \alpha)$ via the Legendre transform (2.6). Thus we expect that various ways of rewriting (2.1) would give rise to various control representations for $z(x)$, with the different running costs defined via (2.6).

We next consider the construction of numerical schemes. It will be useful to give a description of what we have in mind before actually constructing the scheme. There are two basic ingredients in the control problem just described. The first is the dynamical equation, which is simply $\dot\phi = u$. The second is the cost structure, which includes a running cost and a stopping cost. Since the data in the reconstruction problem are really only given on the discrete set $\mathcal{D}^h$, the best we can hope for is to build an approximation $V^h(x)$ to $V(x)$. To do this, we will pose another control problem that will be defined in terms of the data actually available, which is I^h. Of course there are many ways in

which this could be done. Ideally, for the alternative control problem it should be easy to solve for $V^h(x)$, but we also desire the convergence $V^h(x) \to V(x)$ as $h \to 0$ (and perhaps even a fast rate of convergence). In our approach, we approximate the dynamics $\dot{\phi} = u$ by suitably choosing a controlled Markov chain whose state space is $\mathcal{D}^h$, while keeping essentially the same cost structure as that of $V(x)$. As will be seen below, the Markov property allows one to easily solve for $V^h(x)$, while the "stochasticity" in the dynamics of this new control problem is desirable if we want the approximation to the deterministic dynamics to be of sufficient quality that $V^h(x) \to V(x)$. In fact, for the general case of oblique light this "stochasticity" is necessary for the convergence. Further remarks on this point will be given below. However, we emphasize that $V^h(x)$ will not be random; it will simply have a representation as a functional of a controlled Markov chain.

Dynamics. For each $h > 0$ and $u \in \mathbb{R}^2$, let $p^h(x, y|u)$ be a probability transition function on $h\mathbb{Z}^2$ and let $\Delta t^h(u)$ an interpolation interval, whose use will be made clear momentarily. We assume that $p^h(x, y|u)$ and $\Delta t^h(u)$ satisfy the following "local consistency" condition: for each $x \in \mathcal{D}^h$,

$$\sum_y (y - x) p^h(x, y|u) = u \Delta t^h(u), \tag{2.7}$$

$$\sum_y \left[(y - x) - u \Delta t^h(u) \right] \left[(y - x) - u \Delta t^h(u) \right]' p^h(x, y|u) = o(\|u\| \Delta t^h(u)). \tag{2.8}$$

In the proof of convergence for the numerical procedure we will define a continuous time interpolation of this discrete time Markov chain, and the interpolation intervals used will be $\Delta t^h(u)$. With such a scaling, equation (2.7) states that the average increment obtained when control u is applied is essentially u, while (2.8) implies that the variance about this mean tends to zero as $h \to 0$. Hence the controlled Markov process is a natural replacement for the deterministic controlled dynamics $\dot{\phi} = u$, given that state space of the approximating process should be $h\mathbb{Z}^2$. Note that with deterministic dynamics (i.e., $p^h(x, y(x, u)|u) = 1$, some $y(x, u) \in \mathcal{D}^h$) it is in general impossible to satisfy (2.7) for all $u \in \mathbb{R}^2$ because of the discrete nature of $\mathcal{D}^h$.

Remark. If desired an "error" term of size $o(\|u\| \Delta t^h(u))$ can be allowed in (2.7). However, due to the unbounded nature of the control space care must be taken regarding how the "errors" in the local consistency equations depend on u. The precise meaning of the statement that the error is of size $o(\|u\| \Delta t^h(u))$ in such a case would be

$$\left(\sum_{y \in \mathcal{D}^h} (y - x) p^h(x, y|u) - u \Delta t^h(u) \right) \Big/ \left(\|u\| \Delta t^h(u) \right) \to 0$$

uniformly for $u \in \mathbb{R}^2$. A similar remark applies to the error term in (2.8). We also need to assume that $\Delta t^h(u) \to 0$ for each $u \in \mathbb{R}^2$, and that

$$\lim_{h \to 0} \sup \{ \|y - x\| : p^h(x, y|u) > 0, x \in \mathcal{D}^h, y \in \mathcal{D}^h, u \in \mathbb{R}^2 \} = 0.$$

Example 2.1. Let $t^+ = t \vee 0$ and $t^- = -(t \wedge 0)$. An obvious choice for the transition function is

$$p^h(x, y | u) = \begin{cases} u_i^{\pm} / \sum_j |u_j| & \text{if } y = x \pm he_i \\ 0 & \text{otherwise} \end{cases}$$

and $\Delta t^h(u) = h(\sum_j |u_j|)^{-1}$. This definition actually makes sense only when $u \neq 0$, and we take care of the omitted case by setting

$$p^h(x, y | 0) = \begin{cases} 1 & \text{if } y = x \\ 0 & \text{otherwise.} \end{cases}$$

The sequence $\Delta t^h(0) > 0$ is arbitrary [for the purposes of satisfying (2.7) and (2.8)] and for simplicity we can take $\Delta t^h(0) = h$. ∎

It seems likely that other possibilities, such as a less regular grid, may be important. For example, it may be that the grid spacings in the coordinate directions differ. Such variations are easily accommodated. However, to simplify the presentation, we will use the transition probabilities of Example 2.1 for the rest of the paper.

We next consider algorithms for approximating $V(\cdot)$, and hence $z(\cdot)$. The key tool we will use to produce recursive algorithms is the dynamic programming equation for stochastic optimal control. This equation will be valid if we restrict ourselves to controls under which the controlled process has a Markov property. The appropriate definitions are as follows.

Admissible Control Schemes. Let $(\Omega, \mathcal{F}, P)$ be the underlying probability space on which the controlled Markov chain is defined and suppose that the controlled transition probabilities $p^h(x, y | u)$ are to be used. In general, we could allow this space to depend on h, the control scheme, etc. To simplify the notation this dependence will not be explicitly indicated. A sequence $\{u_i^h, i \in \mathbb{Z}^+\}$ is an admissible control scheme for the controlled chain $\{\xi_i^h, i \in \mathbb{Z}^+\}$ if the following Markov property holds:

$$P\left\{\xi_{i+1}^h = y | (u_j^h, \xi_j^h), j = 0, 1, ..., i\right\} = P\left\{\xi_{i+1}^h = y | (u_i^h, \xi_i^h)\right\} = p^h(\xi_i^h, y | u_i^h).$$

For example, if the control scheme is of the feedback form $u_i^h = f^h(\xi_i^h, i)$, where f^h is a measurable mapping from $h\mathbb{Z}^2 \times \mathbb{Z}^+$ to $\mathbb{R}^2$, then it is admissible.

Admissible Stopping Times. A random time M^h with values in $\mathbb{Z}^+$ is said to be admissible if it does not depend on the future in the sense that for any n, any $m > n$, and any continuous function F,

$$E\left[F(\xi_{n+1}^h, \xi_{n+2}^h, ..., \xi_m^h) | (u_j^h, \xi_j^h), j \leq n, M^h I_{\{M^h \leq n\}}\right] = E\left[F(\xi_{n+1}^h, \xi_{n+2}^h, ..., \xi_m^h) | (u_n^h, \xi_n^h)\right].$$

Thus, if we condition on having stopped the process by time n and all the values of $(u_i^h, \xi_i^h), i \leq n$, then the future evolution of ξ_i^h does not depend on whether or not we have stopped the process by n, and depends on the values $(u_j^h, \xi_j^h), i \leq n$ only through (u_n^h, ξ_n^h). This means that M^h cannot anticipate the future. Note that the definition implies M^h is finite with probability one.

Because the "target set" $\mathcal{M}$ for the deterministic optimal control problem associated to $V(x)$ can possibly contain isolated points that may not be included in $\mathcal{D}^h$, we really need to introduce a "discretized target set" $\mathcal{M}^h \subset \mathcal{D}^h$, and redefine $g(\cdot)$ in the obvious way. We would need that $\mathcal{M}^h \to \mathcal{M}$ in the Hausdorff metric [i.e., $d(x, \mathcal{M}) \leq \varepsilon^h$ for all $x \in \mathcal{M}^h$, $d(x, \mathcal{M}^h) \leq \varepsilon^h$ for all $x \in \mathcal{M}$, and $\varepsilon^h \to 0$]. To simplify the notation we will just assume $\mathcal{M} \subset \mathcal{D}^h$.

The Approximation $V^h(x)$. We now define our approximation $V^h(x)$. For a given admissible control scheme $\{u_i^h, i \in \mathbb{Z}^+\}$ let $\{\xi_i^h, i \in \mathbb{Z}^+\}$ denote the associated controlled process. Define $N^h = \inf\{i : \xi_i^h \notin \mathcal{D} \text{ or } \xi_i^h \in \mathcal{M}\}$. We then set

$$V^h(x) = \inf E_x \left[\sum_{i=0}^{(N^h \wedge M^h)-1} L(\xi_i^h, u_i^h) \Delta t^h(u_i^h) + g(\xi_{(N^h \wedge M^h)}^h) \right], \qquad (2.9)$$

where the infimum is over all admissible control sequences and stopping times and E_x denotes expectation conditioned on $\xi_0^h = x$. It is shown in Theorem 4.1 below that $V^h(x) \to V(x)$ for $x \in \mathcal{G}$, where $\mathcal{G}$ is any set that satisfies A2.1. This is at least intuitively plausible, given the characteristics of the controlled chain $\{\xi_i^h, i \in \mathbb{Z}^+\}$ described after (2.7) and (2.8), and the resemblance of (2.9) to (2.4).

Suppose that instead of minimizing over a potentially unbounded time horizon (as here), we consider the problem

$$V_n^h(x) = \inf E_x \left[\sum_{i=0}^{(N^h \wedge M^h \wedge n)-1} L(\xi_i^h, u_i^h) \Delta t^h(u_i^h) + g(\xi_{(N^h \wedge M^h \wedge n)}^h) \right], \qquad (2.10)$$

where the infimum is over the same controls and stopping times. Then $V_n^h(x)$ is clearly nonincreasing in n and $V_n^h(x) \downarrow V^h(x)$ as $n \to \infty$. By the dynamic programming principle of optimality [14], $V_n^h(x)$ and $V_{n+1}^h(x)$ are related by

$$V_{n+1}^h(x) = \min \left[\inf_{u \in R^2} \left(L(x, u) \Delta t^h(u) + \sum_y p^h(x, y|u) V_n^h(y) \right), g(x) \right] \qquad (2.11)$$

for $x \in \mathcal{D}^h \backslash \mathcal{M}$, while $V_n^h(x) = g(x)$ for all $x \notin \mathcal{D}^h, x \in \mathcal{M}$ and all $n \in \mathbb{Z}^+$.

This recursive equation (as well as a number of variations) will serve as the basis for the computation of $V^h(x)$. Starting with the initial condition $V_0^h(x) = g(x)$, we iterate using (2.11). If $V_0^h(x) \geq g(x)$ then the iteration (2.11) extends to $x \in \mathcal{M}$, i.e., we can use (2.11) to define the algorithm for all $x \in \mathcal{D}^h$. It can be shown in the vertical light case that (2.11) converges in no more than $O(h^{-2})$ steps. Typically, for vertical or oblique light, convergence is effectively achieved after $O(h^{-1})$ steps. Further remarks on this point appear in Section 5.

In numerical analysis parlance, (2.9) corresponds to what would be called a Jacobi iteration, since the calculation of each $V_{n+1}^h(x), x \in \mathcal{D}^h$, is based only on the values $V_n^h(x), x \in \mathcal{D}^h$. An alternative is to always use the most recently updated values $V_{n+1}^h(x)$,

if they are available, rather than $V_n^h(x)$. Such an algorithm will be referred to as a Gauss-Seidel type algorithm. Let $<$ denote an ordering of the states in $\mathcal{D}^h$. Then (2.9) is replaced by

$$V_{n+1}^h(x) = \min \left[\inf_{u \in R^2} \left(L(x,u)\Delta t^h(u) + \sum_{y<x} p^h(x,y|u)V_{n+1}^h(y) + \sum_{y \geq x} p^h(x,y|u)V_n^h(y) \right), g(x) \right]$$
(2.12)

It is usually preferable to use a different ordering for each iteration. One can show that the Gauss-Seidel procedure is never worse than the Jacobi. (In fact this is easily shown by adapting the argument used to prove part 1 of Proposition 2.3 below.) We refer the reader to [14] for further discussion and a description of the Markov chain interpretation of the Gauss-Seidel procedure.

We next present the particular nonlinear iteration obtained when the Jacobi algorithm is used with the transition probabilities of Example 2.1. The Gauss-Seidel version is an obvious modification.

Example 2.1 (Continued). For $x \in \mathcal{D}^h$, let v_1 and v_2 be the smallest values from the sets

$$\{V_n^h(x+h(1,0)), V_n^h(x-h(1,0))\} \text{ and } \{V_n^h(x+h(0,1)), V_n^h(x-h(0,1))\},$$

respectively. Define $m = (1/I^2(x)) - 1$. If $0 \leq h^2 m < (v_1 - v_2)^2$, then we use the recursion

$$V_{n+1}^h(x) = \min \left[g(x), (v_1 \wedge v_2) + hm^{1/2} \right].$$

If $h^2 m \geq (v_1 - v_2)^2$, then we use

$$V_{n+1}^h(x) = \min \left[g(x), \frac{1}{2} \left[\left(2h^2 m - (v_1 - v_2)^2 \right)^{1/2} + (v_1 + v_2) \right] \right].$$

∎

This completes our introduction to Control Problem 1 in the case of vertical light. The algorithms derived above, and in particular the Gauss-Seidel iteration (2.12), are the algorithms the authors have used for computations in the case of vertical light. We next turn to Control Problem 2. On the basis of this control problem, we will derive a second approximation scheme. This scheme will not be used for purposes of calculation, since it is more cumbersome than the scheme obtained from Control Problem 1. However, as we show below, the two schemes actually produce the same approximation $V^h(x)$. Our interest in the scheme obtained from the second control problem is because it allows a simpler proof of $V^h(x) \to V(x)$.

Control Problem 2. Define $g(\cdot)$ as in Control Problem 1, and let

$$L(x,\beta) = \begin{cases} 1 - (I^2(x) - \|\beta\|^2)^{1/2} & \text{if } \|\beta\| \leq I(x) \\ +\infty & \text{otherwise.} \end{cases}$$

12

Note that $L(x, \beta) \geq 0$, and that $L(x, \beta) = 0$ if and only if $x \in \mathcal{S}$ and $\beta = 0$. The function $L(x, \beta)$ is the Legendre transform of $H(x, \alpha) = I(x) \left[1 + \|\alpha\|^2 \right]^{1/2} - 1$. It is easy to show that $H(x, z_x(x)) = 0$ is equivalent to (2.1).

The definition of $V(x)$ here is the same as that for Control Problem 1. The dynamics of the controlled process are again $\dot{\phi}(s) = u(s)$ (a.s. in s) and $\phi(0) = x$, and we set

$$V(x) = \inf \left[\int_0^{\rho \wedge \tau} L(\phi(s), u(s)) ds + g(\phi(\rho \wedge \tau)) \right],$$

where $\tau = \inf \{ t : \phi(t) \in \partial \mathcal{D} \cup \mathcal{M} \}$ and the infimum is over all $\rho \in [0, \infty)$ and admissible controls.

We show in Theorem 3.1 that $V(x) = z(x)$ under the same conditions as those stated above for Control Problem 1. The derivation of an approximation procedure is the same here as for the previous control problem. Given a transition function $p^h(x, y | u)$ and interpolation interval $\Delta t^h(u)$ satisfying the local consistency conditions (2.7) and (2.8), we define $V^h(x)$ by (2.9). Approximations $V_n^h(x)$ can be defined via (2.10), which will satisfy (2.11). The approximation $V^h(x)$ obtained here is the same as that obtained via Control Problem 1. A precise statement of this fact in the general setting of oblique light is in the next subsection.

2.2 THE CASE OF OBLIQUE LIGHT.

We now turn to a generalization of the setup of the last subsection in which the light originates from the direction γ, where $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ is some unit vector with $\gamma_3 > 0$. We will consider only the case where $\gamma_1 = 0$. In practice, the case $\gamma_1 \neq 0$ is handled by a coordinate transformation, i.e., by rotating and reinterpolating the data. It is also possible to develop algorithms and a convergence theory that apply directly to this case. However, it has been the authors' experience that the resulting algorithms are somewhat cumbersome, and that the change of coordinates method is preferable.

For this setup, the image irradiance equation obtained from (1.1) and (1.2) for the intensity becomes

$$I(x) = \left\langle \gamma, \frac{(-z_{x_1}(x), -z_{x_2}(x), 1)}{(1 + \|z_x(x)\|^2)^{1/2}} \right\rangle. \tag{2.13}$$

As for the case of vertical light, we define the set $\mathcal{S}$ of singular points by $\mathcal{S} = \{ x \in \mathcal{D} : I(x) = 1 \}$. Owing to the significance of the singular points, it is best to study this problem in the new coordinate system defined by $x_1' = x_1, x_2' = x_2, x_3' = x_2 \gamma_2 + x_3 \gamma_3$. Thus we define a new height function $f(x_1, x_2)$ that measures height along the light (rather than viewer) direction:

$$f(x_1, x_2) = x_2 \gamma_2 + z(x_1, x_2) \gamma_3.$$

The function $f(\cdot)$ is the natural object of study, in that its extreme points are contained in the singular points of $I(\cdot)$. Of course a reconstruction of $f(\cdot)$ will also give a reconstruction of $z(\cdot)$. Substituting $f_x(\cdot)$ into (2.13) and a little algebra produces $H(x, f_x(x)) = 0$, where

$$H(x, \alpha) = \frac{1}{2} \left[I^2(x) \alpha_1^2 + v(x) \alpha_2^2 + 2(1 - I^2(x)) \gamma_2 \alpha_2 - (1 - I^2(x)) \right], \tag{2.14}$$

and $v(x) = I^2(x) - \gamma_2^2$.

We first consider the subset of the image plane where $v(x) \geq 0$. Note that the interior of this set includes $\mathcal{S}$. In this region, the same heuristic argument as that used for the case of vertical light suggests the following optimal control representation for $f(\cdot)$. Let B be an upper bound for $\{f(x), x \in \mathcal{D}\}$, and define the terminal cost

$$g(x) = \begin{cases} f(x) & \text{for } x \in \mathcal{M} \\ B & \text{for } x \notin \mathcal{M} \end{cases} \tag{2.15}$$

We also define the running cost

$$L(x, \beta) = \frac{1}{2} \frac{|\beta_1|^2}{I^2(x)} + \frac{1}{2} \frac{|\beta_2 + (1 - I^2(x))\gamma_2|^2}{v(x)} + \frac{1}{2}(1 - I^2(x)), \tag{2.16}$$

(where by convention $c/0 = \infty$ if $c > 0$, and $c/0 = 0$ if $c = 0$). Note that when $v(x) \geq 0$ that $L(x, \beta) = 0$ if and only if $x \in \mathcal{S}$ and $\beta = 0$. The optimal control problem that is appropriate for the region where $v(x) \geq 0$ is the same as that for the case of vertical light, save that this new running cost is used. (If desired, the possibility of zeros appearing in denominators in the running cost can be eliminated if one is willing to modify the dynamics to accommodate the change. Such a modification would in fact give the control problem we have previously used for oblique light [21]. However, the algorithms derived will be exactly the same. The control problem chosen here allows a development that more closely parallels that of the previous subsection.)

On the other hand, for regions where $v(x) < 0$ we observe a negative coefficient in front of α_2^2 in (2.14). This means that in order to obtain an analogous representation in such a region we must work with a differential game. Specifically, the control $u = (u_1, u_2)$ splits into two opposing controls, one of which (u_1) seeks to minimize the cost while the other (u_2) seeks to maximize. Since the role of the current optimization problem is only to suggest a scheme, our description of this game will not be precise.

For such regions, the relationship between L and H is now an extension of the Legendre transform in (2.6):

$$\inf_{u_2} \sup_{u_1} \left[-\alpha_1 u_1 - \alpha_2 u_2 - \frac{1}{2} \frac{|u_1|^2}{I^2(x)} - \frac{1}{2} \frac{|u_2 + (1 - I^2(x))\gamma_2|^2}{v(x)} - \frac{1}{2}(1 - I^2(x)) \right]$$

$$= \frac{1}{2} \left[I^2(x)\alpha_1^2 + v(x)\alpha_2^2 + 2(1 - I^2(x))\gamma_2\alpha_2 - (1 - I^2(x)) \right] \equiv H(x, \alpha).$$

Note that for $v(x) < 0$ only one of the zeros of $H(x, \cdot)$ for a given α_1 is actually a valid solution of the image irradiance equation (2.1). The other zero corresponds to a surface normal pointing away from the camera, which is impossible if $f(x)$ is a function. This spurious zero is due to taking a square in the course of rewriting the image irradiance equation in the form (2.14). The valid zero is characterized by its being achieved at a value u_2 such that

$$u_2 \gamma_2 < 0 \tag{2.17}$$

14

Moreover, if $H = 0$, a simple computation shows that the supremum and infimum are achieved at u obeying

$$\frac{u_1^2}{I^2(x)} + \frac{u_2^2}{v(x)} < 0.$$

Together, these facts imply the angular constraint on the control

$$\frac{\gamma_2 u_2}{\|u\|} < -(\gamma_2^2 - I(x)^2)^{1/2}. \tag{2.18}$$

The above suggests that the differential game should be defined with a restriction on the control to pick out the valid zero of $H(x, \cdot)$. The restriction we will impose is (2.17), since this is simple to implement in our numerical scheme, although the stronger constraint on the maximizer $\text{sign}(\gamma_2)u_2 < -|u_1|(-v(x))^{1/2}/I(x)$ could be used instead. These restrictions may be unnecessary to achieve a valid differential game representation, but result in more efficient algorithms, and are needed to prove the equivalence of the fixed points for the two algorithms. For further discussion see the Appendix.

As in the previous subsection the relationship between H and L suggests a numerical scheme. Let $p^h(x, y|u)$ and $\Delta t^h(u)$ be as in Example 2.1. We can then define an analogue of (2.9), in which the control problem becomes a discrete time discrete state stochastic game in the region where $v(x) < 0$. Only the minimizer will be allowed to stop the game and pay the stopping cost. A formal application of the principle of dynamic programming suggests the following algorithm: $V_0^h(x) = g(x)$, and

$$V_{n+1}^h(x) = \min\left[\inf_{u \in R^2}\left(L(x, u)\Delta t^h(u) + \sum_y p^h(x, y|u)V_n^h(y)\right), g(x)\right] \qquad \text{if } v(x) > 0$$

$$V_{n+1}^h(x) = \min\left[\sup_{u_2\gamma_2 < 0}\inf_{u_1 \in R}\left(L(x, u)\Delta t^h(u) + \sum_y p^h(x, y|u)V_n^h(y)\right), g(x)\right] \quad \text{if } v(x) \le 0,$$
$$\tag{2.19}$$

together with boundary conditions as in (2.11).

Example 2.1 (Continued). If we insert $p^h(x, y|u)$ and $\Delta t^h(u)$ from Example 2.1 into the equation above we obtain the following recursive formulas. Fix $x \in \mathcal{D}^h$. Define $m = (1 - I^2(x))/v(x)$, generalizing from the case of vertical light. Also, define

$$v_2 = \begin{cases} h^{-1}\min\{V_n^h(x + h(0, 1)) + m\gamma_2, V_n^h(x - h(0, 1)) - m\gamma_2\} & \text{if } v(x) > 0 \\ h^{-1}v(x - \text{sign}(\gamma_2)h(0, 1)) - m|\gamma_2| & \text{if } v(x) < 0, \end{cases}$$

and

$$v_1 = h^{-1}\min\{V_n^h(x + h(1, 0)), V_n^h(x - h(1, 0))\}, \quad A = v_2 - v_1,$$

$$S = \left(\frac{I(x)|v(x)|^{1/2}}{I^2(x) + v(x)}\right)\left|m\gamma_3^2(I^2(x) + v(x))/v(x) - A^2\right|^{1/2}.$$

For $v(x) > 0$, let

$$\hat{V} = h\begin{cases} v_1 + |m|^{1/2}\gamma_3 & \text{if } A > |m|^{1/2}\gamma_3 \\ S + \frac{I^2(x)v_1 + v(x)v_2}{I^2(x) + v(x)} & \text{if } -|m|^{1/2}\gamma_3(I(x)/v^{1/2}(x)) \le A \le |m|^{1/2}\gamma_3 \\ v_2 + |m|^{1/2}\gamma_3(I(x)/v^{1/2}(x)) & \text{if } A < -|m|^{1/2}\gamma_3(I(x)/v^{1/2}(x)). \end{cases}$$

For $v(x) < 0$, we let

$$\hat{V} = h \begin{cases} v_2 - |m|^{1/2}\gamma_3(I(x)/|v(x)|^{1/2}) & \text{if } A < |m|^{1/2}\gamma_3(I(x)/|v(x)|^{1/2}) \\ S + \frac{I^2(x)v_1 + v(x)v_2}{I^2(x) + v(x)} & \text{otherwise.} \end{cases}$$

The recursions defined above are continuous in $v(x)$, so that the recursions for $v(x) = 0$, $-I(x)^2$ are just given by the appropriate limits. We then set

$$V_{n+1}^h(x) = \min[\hat{V}, g(x)].$$

For $x \notin \mathcal{D}^h$ we set $V_{n+1}^h(x) = g(x) = B$. ∎

If the constraint $u_2\gamma_2 < 0$ were not imposed for $v(x) < 0$, then the definition of v_2 would be altered to:

$$v_2 = h^{-1}\max\{V_n^h(x + h(0,1)) + m\gamma_2, V_n^h(x - h(0,1)) - m\gamma_2\}$$

for $v(x) < 0$, and the recursion would otherwise be unchanged.

As for the vertical light case, we define $V^h(x) = \lim_{n\to\infty} V_n^h(x)$. The existence of such a limit is proved in Proposition 2.3 below, and the convergence of $V^h(x)$ as $h \to 0$ is studied in Section 4. Without the constraint above on the maximizer for $v(x) < 0$, the algorithm would probably converge to the same V^h in most cases. However, computational experience suggests the convergence may be slow over those regions of $\mathcal{D}^h$ in which the maximizer, for at least some number of iterations, can prevent the entry of the controlled path into the domain where there is only a minimizer (i.e., $v(x) \geq 0$). By this delay, the maximizer attempts to win a large positive terminal cost by not allowing the controlled path to terminate in the target set $\mathcal{M}$.

The algorithm above can be implemented in a Gauss–Seidel version just as in the case of vertical light. This is the algorithm actually used in the case of oblique light.

This completes our discussion of Control Problem 1 for the case of oblique light. We next describe Control Problem 2. The equation (2.13) may be rewritten as $H(x, f_x(x)) = 0$, where

$$H(x, \alpha) = I(x)\left(1 + \|\alpha\|^2 - 2\alpha_2\gamma_2\right)^{1/2} + \alpha_2\gamma_2 - 1.$$

This function is convex in α, and its Legendre transform in α is

$$L(x, \beta) = \begin{cases} \gamma_3^2 - \gamma_2\beta_2 - \gamma_3\left(I^2(x) - |\beta_1|^2 - |\beta_2 + \gamma_2|^2\right)^{1/2} & \text{if } |\beta_1|^2 + |\beta_2 + \gamma_2|^2 \leq I^2(x) \\ \infty & \text{if } |\beta_1|^2 + |\beta_2 + \gamma_2|^2 > I^2(x). \end{cases}$$

Once again note that $L(x, \beta) \geq 0$, and that $L(x, \beta) = 0$ if and only if $x \in \mathcal{S}$ and $\beta = 0$. Also, $L(x, \beta) < \infty$ implies the angular constraint on the control

$$\frac{\gamma_2 u_2}{\|u\|} \leq -(\gamma_2^2 - I(x)^2)^{1/2},$$

which is almost the same as (2.18).

16

The derivation of an approximation procedure is the same here as in the previous subsection. Given a transition function $p^h(x, y|u)$ and interpolation interval $\Delta t^h(u)$ satisfying the local consistency conditions (2.7) and (2.8), we define $V^h(x)$ by (2.9). Approximations $V_n^h(x)$ are defined via (2.11). As in the case of vertical light we have the following result, which relates the approximations obtained from the two control problems. The proof is given in the Appendix.

Proposition 2.2 *Consider the transition probability $p^h(x, y|u)$ and interpolation interval $\Delta t^h(u)$ of Example 2.1, as well as the recursive equation (2.19) for Control Problem 1 and (2.11) for Control Problem 2. Then $w(x)$ is a fixed point of (2.19) if and only if it is also a fixed point for (2.11).*

2.3 CHARACTERIZATION AND PROPERTIES OF FIXED POINTS OF THE ALGORITHMS.

In the previous subsections we have shown that the fixed points of Jacobi type algorithms algorithms obtained from various formally derived control representations were all actually the same. It was also noted that the fixed points of the Gauss-Seidel type procedures [e.g. (2.12)] (under any ordering of the states) and those of the Jacobi procedures [e.g. (2.11)] are also identical for either of the control problems. One of the more subtle points in the analysis of these algorithms has to do with the fact that the iterations we have described may have multiple fixed points. The existence of this nonuniqueness is tied to the fact that the running costs $L^{(i)}(x, u)$ are not bounded from below away from zero (which is itself due to the presence of singular points in the region over which the reconstruction is to be done). However, the nonuniqueness also turns out to be quite illuminating, since, as we will see below, its proper resolution leads naturally to algorithms that are quite efficient. The following result will be essential in relating the various algorithms and dealing with the nonuniqueness. Recall that B is an upper bound for $f(\cdot)$ on $\mathcal{G}$, and that $g(\cdot)$ is defined in (2.15).

Proposition 2.3 *Consider any of the recursive algorithms derived in either Subsection 2.1 or Subsection 2.2. Let an initial condition V_0^h be given and define the sequence $\{V_i^h, i \in N\}$ according to either the Jacobi iteration [e.g. (2.11)] or the Gauss-Seidel iteration [e.g. (2.12)], where for the Gauss-Seidel procedure we allow a possibly different ordering to be used for each iteration. Assume that $V_0^h(x) \geq g(x)$ for all $x \in \mathcal{D}^h$. Then the following conclusions hold.*

1. *For each $x \in \mathcal{D}^h$, $V_i^h(x)$ is nonincreasing in i and bounded from below. Define $V^h(x) = \lim_{i \to \infty} V_i^h(x)$. Then the function $V^h(\cdot)$ is a fixed point of (2.11) (or (2.19) if appropriate).*

2. *The function $V^h(\cdot)$ can be uniquely characterized as the largest fixed point of (2.11) (or (2.19) if appropriate) that satisfies $V^h(x) \leq V_0^h(x)$ for all $x \in \mathcal{D}^h$. Thus, whenever the initial condition is sufficiently large, the limits for all the various procedures (Control Problem 1 or Control Problem 2, Jacobi or Gauss-Seidel) are identical.*

17

Remark. In the actual use of the algorithm $g(x)$ is always taken as the initial condition, and thus this assumption in the proposition is automatically satisfied.

Proof. For each fixed $x \in \mathcal{D}^h$ and $i \in \mathbb{N}$, any of the Jacobi and Gauss-Seidel iterations we have defined may be written in one of the following forms:

$$V_{i+1}^h(x) = \min \left[\inf_u \left(c^h(x,u) + \sum_y p^h(x,y|u)w(y) \right), g(x) \right], \qquad (2.20)$$

or

$$V_{i+1}^h(x) = \min \left[\sup_{u_2} \inf_{u_1} \left(c^h(x,u) + \sum_y p^h(x,y|u)w(y) \right), g(x) \right]. \qquad (2.21)$$

Here $c^h(x,u)$ denotes the running cost, and $w(\cdot)$ is a function that depends on the particular type of iteration used as well as (in the Gauss-Seidel case) the ordering of the states. Note that for both (2.20) and (2.21) the right hand sides are monotonically nondecreasing in $w(\cdot)$ if we use the partial ordering of real valued functions on $\mathcal{D}^h$ defined by $w_1(\cdot) \leq w_2(\cdot)$ whenever $w_1(x) \leq w_2(x)$ for all $x \in \mathcal{D}^h$.

First consider the Jacobi iteration. The monotonicity property just described implies that $V_{i+1}^h \leq V_i^h$ whenever $V_i^h \leq V_{i-1}^h$. Since the initial condition satisfies $V_0^h \geq g$, and since $V_1^h \leq g$, we conclude $V_{i+1}^h \leq V_i^h$ for all $i \in \mathbb{N}$ by induction.

Next consider the Gauss-Seidel procedure, with a possibly different ordering for each iteration. Regardless of the ordering used on the first iteration, the fact that $V_0^h \geq g$ implies $V_1^h \leq V_0^h$. We will again complete the proof via an induction argument. Suppose that $V_i^h \leq V_{i-1}^h$. Let $<_i$ denote the ordering that is used on the ith iteration. Let $x_1, x_2, \ldots$ denote the states of $\mathcal{D}^h$, ordered according to $<_{i+1}$. The values $V_i^h(y), y \in \mathcal{D}^h$ are used to define $V_{i+1}^h(x_1)$, while the values $V_i^h(y), y <_i x_1$ and $V_{i-1}^h(y), x_1 <_i y$ were used to define $V_i^h(x_1)$. Since $V_i^h \leq V_{i-1}^h$, $V_{i+1}^h(x_1) \leq V_i^h(x_1)$. We now proceed by induction according to $<_{i+1}$. Fix j, and assume $V_{i+1}^h(x_k) \leq V_i^h(x_k)$ for $k < j$. The values $V_{i+1}^h(y), y <_{i+1} x_j$ and $V_i^h(y), x_j <_{i+1} y$ are used to define $V_{i+1}^h(x_j)$, while the values $V_i^h(y), y <_i x_j$ and $V_{i-1}^h(y), x_j <_i y$ were used to define $V_i^h(x_j)$. In all cases the values used to define $V_{i+1}^h(x_j)$ are no larger than those used to define $V_i^h(x_j)$. Thus $V_{i+1}^h(x_j) \leq V_i^h(x_j)$. By induction on $<_{i+1}$ we conclude $V_{i+1}^h \leq V_i^h$, and by induction on the usual ordering on $\mathbb{N}$ we obtain the monotonicity described in part 1 of the proposition.

Since for Control Problem 2 the running costs are non-negative, the monotonicity proved above establishes the existence of $V^h(x) = \lim_{i \to \infty} V_i^h(x)$ for both the Jacobi and Gauss-Seidel procedures. This is also the case for Control Problem 1 in vertical light, or for $v(x) \geq 0$ for oblique light. When $v(x) < 0$ in Control Problem 1, we first consider the case $w(y) = 0$ in (2.21). A simple calculation shows that

$$\sup_{u_2 \gamma_2 < 0} \inf_{u_1} \left(L^{(1)}(x,u) \right) > 0,$$

and therefore

$$\sup_{u_2 \gamma_2 < 0} \inf_{u_1} \left(L^{(1)}(x,u) \Delta t^h(u) \right) \geq 0.$$

18

Since the probabilities are nonnegative, and $\sum_y p^h(x,y|u) = 1$, this implies

$$\sup_{u_2\gamma_2<0} \inf_{u_1} \left(L^{(1)}(x,u)\Delta t^h + \sum_y p^h(x,y|u)w(y) \right) \geq w_{\min},$$

where $w_{\min} \equiv \min_y w(y)$. Thus for Control Problem 1 and $v(x) < 0$, V_n^h is bounded from below by $\min_x V_0^h(x)$. This gives part 1 of the proposition.

We next turn to part 2. Let $\bar{V}^h$ be any fixed point of (2.11) or (2.19) that satisfies $\bar{V}^h(x) \leq V_0^h(x)$ for all $x \in \mathcal{D}^h$. An argument very similar to the one used to prove part 1 shows that

$$\bar{V}^h(x) \leq V_i^h(x) \Rightarrow \bar{V}^h(x) \leq V_{i+1}^h(x).$$

Therefore by induction $\bar{V}^h(x) \leq V^h(x)$ for all $x \in \mathcal{D}^h$. $\blacksquare$

3 The Representation Theorem

The main purpose of this section is to prove the representation theorem that connects the shape to be reconstructed and a deterministic optimal control problem. In the course of the proof we will construct certain paths that will also be needed in the next section. The properties of these paths are summarized in a lemma that follows the proof of the representation theorem.

We will consider the general case of oblique light. Recall that γ describes the direction from which the incoming light originates, and that the image intensity function $I(\cdot)$ is defined on some open bounded subset $\mathcal{D}$ of $I\!R^2$. Without loss of generality, we assume as before that $\gamma_1 = 0$, $\gamma_3 > 0$, and γ is a unit vector.

In Section 2 we formally related a pair of deterministic optimal control problems to the function $f(\cdot)$ measuring the height along the γ direction:

$$f(x_1,x_2) = x_2\gamma_2 + z(x_1,x_2)\gamma_3,$$

where $z(\cdot)$ is the vertical height. We need consider here only Control Problem 2 of Subsection 2, since it is the basis for the convergence proof in the next section. This problem was formally derived by rewriting (2.13) as $H(x, f_x(x)) = 0$, where

$$H(x,\alpha) = I(x) \left(1 + \|\alpha\|^2 - 2\alpha_2\gamma_2 \right)^{1/2} + \alpha_2\gamma_2 - 1.$$

The running cost was defined to be

$$
\begin{aligned}
L(x,\beta) &= \sup_\alpha \left[-\langle \alpha, \beta \rangle - H(x,\alpha) \right] \\
&= \gamma_3^2 - \gamma_2\beta_2 - \gamma_3 \left(I(x)^2 - |\beta_1|^2 - |\beta_2 + \gamma_2|^2 \right)^{1/2}
\end{aligned}
\tag{3.1}
$$

if $|\beta_1|^2 + |\beta_2 + \gamma_2|^2 \leq I^2(x)$ and ∞ if $|\beta_1|^2 + |\beta_2 + \gamma_2|^2 > I^2(x)$.

Define

$$\mathcal{U}(x) = \left\{ (u_1, u_2) : |u_1|^2 + |u_2 + \gamma_2|^2 \leq I^2(x) \right\}.$$

Thus $\mathcal{U}(x)$ is the domain on which $L(x, \cdot)$ is finite. It is easy to check that $H(x, \cdot)$ and $L(x, \cdot)$ are strictly convex on $\mathbb{R}^2$ and $\mathcal{U}(x)$, respectively. Hence

$$H(x, \alpha) = \sup_{\beta \in \mathcal{U}(x)} \left[-\langle \alpha, \beta \rangle - L(x, \beta) \right], \tag{3.2}$$

and for each $\alpha \in \mathbb{R}^2$ there exists a unique vector $u(x, \alpha)$ such that

$$H(x, \alpha) = -\langle \alpha, u(x, \alpha) \rangle - L(x, u(x, \alpha)).$$

Define $\bar{u}(x)$ for $x \in \mathcal{D}$ by

$$0 = H(x, f_x(x)) = -\langle f_x(x), \bar{u}(x) \rangle - L(x, \bar{u}(x)). \tag{3.3}$$

From (3.1), $\bar{u}(x)$ is given by

$$\bar{u}(x) = -\left. \nabla_\alpha H(x, \alpha) \right|_{f_x(x)}$$

If (as we assume) $f_x(x)$ is continuous, then the continuity of $\nabla_\alpha H(x, \alpha)$ implies $\bar{u}(x)$ is continuous on $\mathcal{D}$. As in the case of vertical light, an explicit calculation shows that $\bar{u}(x)$ is proportional to the projection in the (x_1, x_2) plane of the a steepest descent direction on the surface. Now, however, "steepest descent" is defined with respect to the light direction γ, rather than the vertical direction $(0, 0, 1)$.

We consider subsets $\mathcal{G}$ of $\mathcal{D}$ that satisfy the following extension of A2.1 to the case of oblique light.

A3.1 *Assume that S consists of a finite collection of disjoint, compact, smoothly connected sets, and that $f_x(\cdot)$ is continuous on the closure of $\mathcal{D}$. Let $\mathcal{G} \subset \mathcal{D}$ be a compact set, and assume $\mathcal{G}$ is of the form $\mathcal{G} = \cap_{j=1}^J \mathcal{G}_j, J < \infty$, where each $\mathcal{G}_j$ has a continuously differentiable boundary. Let $\mathcal{M}$ be the set of local minima of $f(\cdot)$ inside $\mathcal{G}$. Then we assume that the value of $f(\cdot)$ is known at all points in $\mathcal{M}$. Let $\bar{u}$ denote the "steepest descent" direction given by (3.3) above. Define $n_j(x)$ to be the inward (with respect to $\mathcal{G}$) normal to $\partial \mathcal{G}_j$ at x. Then we also assume that $\langle \bar{u}(x), n_j(x) \rangle > 0$ for all $x \in \partial \mathcal{G} \cap \partial \mathcal{G}_j, j = 1, ..., J$.*

As noted above the minimizing trajectories for the control problems we consider are essentially the two dimensional projections of the paths of steepest descent on the surface represented by the height function. Thus, the conditions that are placed on $\mathcal{G}$ in this assumption guarantee that any minimizing trajectory that starts in $\mathcal{G}$ stays in $\mathcal{G}$. If we consider an initial point $x \in \mathcal{D}$ such that the minimizing trajectory exits $\mathcal{D}$, then we cannot construct $f(\cdot)$ at x as the infimal cost in the control representation 2.4, since this requires $I(x)$ for values of x outside $\mathcal{D}$. However, as remarked previously, if we assume that the height function is specified at the local maximum points, then $f(\cdot)$ may be constructible at x using an analogous control problem with a maximization. This is possible if there is a maximizing trajectory, corresponding to a path of steepest ascent on the surface, that stays in $\mathcal{D}$. Thus one should consider this part of A3.1 as a means of determining which parts of the surface can be reconstructed from the given data.

Theorem 3.1 *Assume A3.1 and that B is an upper bound for $f(\cdot)$ on $\mathcal{G}$. Define $L(\cdot,\cdot)$ by (3.1), $g(\cdot)$ by (2.15), and $V(x)$ as in Section 2 by (2.4). Then $V(x) = f(x)$ for all $x \in \mathcal{G}$.*

Remarks. Because of (3.3) we have $\bar{u}(x) = 0$ if and only if $f_x(x) = 0$. Let y be an isolated local minimum of $f(\cdot)$ and let O be the domain of attraction of y with respect to $\bar{u}(\cdot)$, i.e., if $\dot{\phi} = \bar{u}(\phi), \phi(0) = x$ and $x \in O$ then $\phi(t) \to y$ as $t \to \infty$. Assume that $O \subset \mathcal{D}$. Further suppose that $f(\cdot)$ is C^2, so that $\bar{u}(\cdot)$ is C^1. In this case O is open and one can find an increasing sequence of sets $\mathcal{G}_i$ such that each $\mathcal{G}_i$ satisfies A3.1 and $\mathcal{G}_i \uparrow O$ as $i \to \infty$. Hence the conclusion of Theorem 3.1 will hold with $\mathcal{G}$ replaced by O. If $V(x)$ is continuous on the closure of O (which is usually easy to verify and holds automatically if the closure of $\mathcal{O}$ is contained in a domain that satisfies A3.1), then the conclusion of Theorem 3.1 will hold with $\mathcal{G}$ replaced by the closure of O. Similar remarks will apply to the conclusion of Theorem 4.1 of the next section. These remarks will find application in the scheme described in Section 5 for eliminating the assumption that the height function must be known at all local minima (or local maxima).

Proof. We first show that $V(x) \geq f(x)$. Let $u(\cdot)$ be any admissible control and define

$$\phi(t) = x + \int_0^t u(s)ds, \ \tau = \inf\{t : \phi(t) \in \partial\mathcal{D} \cap \mathcal{M}\}. \tag{3.4}$$

Since L is the Legendre transform of H and since $H(x, f_x(x)) = 0$ for $x \in \mathcal{D}$,

$$0 \geq -\langle f_x(x), \beta \rangle - L(x, \beta)$$

for all $\beta \in \mathbb{R}^2$, and in particular

$$-\langle f_x(\phi(t)), u(t) \rangle \leq L(\phi(t), u(t))$$

for $t \in [0, \rho \wedge \tau]$ (recall that ρ is the controlled stopping time). This implies that

$$-f(\phi(\rho \wedge \tau)) + f(x) = -\int_0^{\rho \wedge \tau} \langle f_x(\phi(t)), u(t) \rangle dt \leq \int_0^{\rho \wedge \tau} L(\phi(t), u(t))dt,$$

and thus for any admissible control and any $\rho \in [0, \infty)$

$$\int_0^{\rho \wedge \tau} L(\phi(t), u(t))dt + f(\phi(\rho \wedge \tau)) \geq f(x).$$

Since $g(\phi(\rho \wedge \tau)) \geq f(\phi(\rho \wedge \tau))$, we obtain $V(x) \geq f(x)$.

Next we show $V(x) \leq f(x)$. In order to do so we will verify that for each $\varepsilon > 0$ there exists a control $u(\cdot)$ such that for ϕ and τ defined by (3.4) we have $\tau < \infty$, and

$$\int_0^{\tau} L(\phi(t), u(t))dt + g(\phi(\tau)) \leq f(x) + \varepsilon. \tag{3.5}$$

Recall that $\mathcal{U}(x) = \{(u_1, u_2) : |u_1|^2 + |u_2 + \gamma_2|^2 \leq 1\}$ and $L(x, 0) = 0$ for $x \in \mathcal{S}$. Let $\mathcal{S}_C$ be a maximal smoothly connected component of $\mathcal{S}$. We first show that any two

points near $\mathcal{S}_C$ can be connected by a piecewise continuous control with low cost. For any two points x, y in $\mathcal{S}_C$, there exists a control $u(t)$ and a time $t^* < \infty$ such that if $\phi(t) = x + \int_0^t u(s)ds$, then $\phi(t) \in \mathcal{S}_C$ for $t \in [0, t^*]$ and $y = \phi(t)$. Define a new control $u_\lambda(t) \equiv \lambda u(t\lambda)$, where $\lambda > 0$, and let $\phi_\lambda(t) = \phi(t\lambda)$ be the corresponding path. Since

$$\frac{L(x,u)}{\|u\|} \to 0 \text{ as } \|u\| \to 0$$

for x such that $I(x) = 1$, we can choose λ such that

$$\int_0^{\lambda t^*} L(\phi_\lambda(t), u_\lambda(t))dt = \int_0^{t^*} \frac{L(\phi(t), \lambda u(t))}{\lambda}dt \leq \frac{\varepsilon}{3}.$$

Since $L(\cdot, \cdot)$ is continuous on an open neighborhood of $\mathcal{S}_C \times \{0\}$, we can also assume u_λ is continuous. Further, since $|\gamma_2| < 1$, there exists $a > 0$ such that for any component $\mathcal{S}_C$ as above, and x such that $d(x, \mathcal{S}_C) \leq a$, we have the following. Let y be the point in $\mathcal{S}_C$ closest to x. Then there exists a time $t_a \in [0, \infty)$, a constant control $u(t) = (y-x)/t_a$, and corresponding path $\phi(t) = x + \int_0^t u(s)ds$, such that $\phi(t_a) = y$ and $\int_0^{t_a} L(\phi(t), u(t))dt \leq \varepsilon/3$. Finally, this shows that for any $\mathcal{S}_C$, and x, y such that $d(x, \mathcal{S}_C) \leq a$ and $d(y, \mathcal{S}_C) \leq a$, there exists a piecewise continuous control $\tilde{u}_{xy}(t)$ and time $\sigma_{xy} \in [0, \infty)$ such that for the corresponding path $\phi_{xy}(t)$ we have

$$\phi_{xy}(0) = x, \quad \phi_{xy}(\sigma_{xy}) = y,$$

and

$$\int_0^{\sigma_{xy}} L(\phi_{xy}(t), \tilde{u}_{xy}(t))dt \leq \varepsilon.$$

Since f is constant on $\mathcal{S}_C$ we can assume (by choosing $a > 0$ smaller if need be) that $|f(x) - f(y)| \leq \varepsilon$.

We now construct the control that satisfies (3.5). If $x \in \mathcal{S}_C$, and $\mathcal{S}_C \subset \mathcal{M}$, then we simply take $\tau = 0$ and are done. There are then two remaining cases: (1) x is contained in some $\mathcal{S}_C$ with $\mathcal{S}_C \cap \mathcal{M} = \emptyset$, or (2) $x \notin \mathcal{S}$. If case (1) holds then $\mathcal{S}_C$ is either a set of local maxima or saddle points, which implies the existence of a point y such that $f(y) < f(x)$ and $d(y, \mathcal{S}_C) \leq a$. Since A3.1 implies $\mathcal{S} \subset \mathcal{G}^0$, we can assume that $y \in \mathcal{G}$. In this case we will set $u(t) = \tilde{u}_{xy}(t)$ for $t \in [0, \sigma_{xy})$.

Next consider the definition of the control for $t \geq \sigma_{xy}$. For $c > 0$ let $b = \inf\{L(x, u) : x \in \mathcal{G}, d(x, \mathcal{S}) > c, u \in \mathbb{R}^2\}$. The continuity of $I(\cdot)$ and the fact that $I(x) < 1$ for $x \notin \mathcal{S}$ implies $b > 0$. Consider any solution (there may be more than one) to

$$\dot{\phi}(t) = \bar{u}(\phi(t)), \quad \phi(0) = y. \tag{3.6}$$

According to (3.3), for any t such that $\phi(t) \in \mathcal{G}\backslash\mathcal{S}$ and $d(\phi(t), \mathcal{S}) > c$

$$\begin{aligned}
\frac{d}{dt}f(\phi(t)) &= \langle f_x(\phi(t)), \bar{u}(\phi(t))\rangle \\
&= -L(\phi(t), \bar{u}(\phi(t))) \\
&\leq -b.
\end{aligned} \tag{3.7}$$

A3.1 implies $\phi(t)$ cannot exit $\mathcal{G}$. Thus, since $f(x)$ is bounded on $\mathcal{G}$, (3.7) implies that $\phi(t)$ must enter the set $\{x : d(x,\mathcal{S}) \leq c\}$ in finite time, for any $c > 0$. If $\phi(t) \in \mathcal{S}$ for some $t < \infty$ we define $\eta_y = \inf\{t : \phi(t) \in \mathcal{S}\}$ and $w = \phi(\eta_y)$. Otherwise, let t_i be any sequence tending to ∞ as $i \to \infty$. Since $\mathcal{G}$ is compact we can extract a subsequence (again labeled by i) such that $\phi(t_i) \to v$ for some $v \in \mathcal{S}$. Let i^* be large enough that $\|\phi(t_{i^*}) - v\| \leq a$. Since $f(\phi(t_i)) \downarrow f(v)$, we have $f(\phi(t_{i^*})) > f(v)$. For this case we define $\eta_y = t_{i^*}$ and $w = \phi(\eta_y)$.

Integrating (3.7) gives

$$f(y) - f(w) = \int_0^{\eta_y} L(\phi(t), \bar{u}(\phi(t))) dt.$$

We then define the control $u(t)$ to be used for $t \in [\sigma_{xy}, \sigma_{xy} + \eta_y)$ to be $\bar{u}(\phi(t - \sigma_{xy}))$.

We now consider the point w. We first examine the case in which the solution to (3.6) does not enter $\mathcal{S}$ in finite time. Since $\|w - v\| \leq a$, $\tilde{u}_{wv}(t)$ gives a control such that the application of this control moves $\phi(\cdot)$ from w to v with accumulated running cost less than or equal to ε. We define $u(t) = \tilde{u}_{wv}(t - (\eta_y + \sigma_{xy}))$ for $t \in [\sigma_{xy} + \eta_y, \sigma_{xy} + \eta_y + \sigma_{wv})$. If the solution to (3.6) reached $\mathcal{S}$ in finite time we define $w = v$ and $\sigma_{wv} = 0$. Let $\sigma = \sigma_{xy} + \eta_y + \sigma_{wv}$.

Let us summarize the results of this construction. Given any point $x \in \mathcal{S}$ that is not a local minimum we have constructed a piecewise continuous control $u(\cdot)$ and $\sigma < \infty$ such that if $\phi(t) = x + \int_0^\sigma u(s) ds$, then

$$
\begin{aligned}
f(x) - f(\phi(\sigma)) &= f(x) - f(y) + f(y) - f(w) + f(w) - f(v) \\
&\geq \int_{\sigma_{xy}}^{\sigma_{xy} + \eta_y} L(\phi(t), u(t)) dt \\
&\geq -2\varepsilon + \int_0^\sigma L(\phi(t), u(t)) dt.
\end{aligned}
$$

We have also shown that $f(x) > f(v) = f(\phi(\sigma))$, $\phi(\sigma) \in \mathcal{S}$. Thus, either the component $\mathcal{S}_C$ containing $\phi(\sigma)$ satisfies $\mathcal{S}_C \cap \mathcal{M} \neq \emptyset$, and we are done, or we are back into case (1) above and can repeat the procedure. Let K be the number of disjoint compact connected sets that comprise $\mathcal{S}$. Then the strict decrease $f(x) > f(\phi(\sigma))$ and the fact that $f(\cdot)$ is constant on each $\mathcal{S}_C$ imply the procedure can be repeated no more than K times before reaching some $\mathcal{S}_C$ contained in $\mathcal{M}$. If case (2) holds we can use the same procedure, save that the very first step is omitted. Thus, in general, we have exhibited a control $u(\cdot)$ such that

$$\int_0^\tau L(\phi(t), u(t)) dt + g(\phi(\tau)) \leq f(x) + (2K + 1)\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the theorem is proved. ∎

The following lemma will be needed in the next section.

Lemma 3.2 *Assume A3.1, and let $L(x, \beta)$ and $\mathcal{U}(x)$ be defined as above. Assume that $I \in (0, 1]$ on $\mathcal{G}$, let u be the piecewise continuous control constructed in the proof of the Theorem 3.1 that gives the inequality in (3.5), and let ϕ and τ be as defined as in (3.4). Then the following conclusions hold.*

23

1. *The function $L(x, \cdot)$ satisfies the following property: given $B < \infty$ there exists $\delta > 0$ such that $\|\nabla_u L(x, u)\| \leq B$ implies $d(u, \mathcal{U}(x)^c) \geq \delta$, where the superscript c denotes complement.*

2. *The path $\phi(t)$ remains in $\mathcal{G}$ for all $t \in [0, \tau]$, and there exists $\delta > 0$ such that $d(u(t), \mathcal{U}(\phi(t))^c) \geq \delta$ for all $t \in [0, \tau]$.*

The assumption $I(x) > 0$ implies that $\mathcal{U}(x)$ has a nonempty interior, and is clearly needed for part 2 of the lemma. Although we must also assume this condition in the convergence theorem of the next section, by working with a sequence of domains [see the remark after the statement of Theorem 3.1], we can extend the convergence theorem of Section 4 to cover the case $I(x) \in [0, 1]$.

Proof. If part 1 were not true, then there would exist a sequence $u_i \to u \in \partial\mathcal{U}(x)$ for which $\|\nabla_u L(x, u_i)\| \leq B$ (i.e., $L(x, \cdot)$ would not be *essentially smooth* in the terminology of Rockafellar [26]). This can be contradicted either via a direct calculation or the general result given in [26, Theorem 26.3].

Since $f_x(\cdot)$ is bounded on $\mathcal{G}$, equations (3.2), (3.3) and part 1 of the lemma imply that for each $x \in \mathcal{G}$ there exists $\delta > 0$ such that $d(\bar{u}(x), \mathcal{U}(x)^c) \geq \delta$. Since $\bar{u}(x)$ and $\mathcal{U}(x)$ are continuous on $\mathcal{G}$, we can assume that δ is independent of x for $x \in \mathcal{G}$.

It is also straightforward to show that we can take $\delta > 0$ small enough that $d(\tilde{u}_{xy}(t), \mathcal{U}(\phi(t))^c) \geq \delta$ for all $t \in [0, \sigma_{xy}]$ whenever both $d(x, \mathcal{S}_C)$ and $d(y, \mathcal{S}_C)$ are sufficiently small, where $\tilde{u}_{xy}$ and σ_{xy} are as defined in the proof of Theorem 3.1 and ϕ is the associated controlled path. This implies part 2 of the lemma. ∎

4 The Convergence Theorem

In this section we prove the basic convergence theorem for the approximations $V^h(x)$ defined in Section 2. The theorem is presented for the general case of oblique light. According to Theorem 3.1 the minimal cost function $V(\cdot)$ agrees with $f(\cdot)$ on domains that satisfy A3.1.

Theorem 4.1 *Let $\mathcal{G}$ be any domain that satisfies A3.1, and let B be an upper bound for $f(\cdot)$ on $\mathcal{G}$. Define $L(\cdot, \cdot)$ by (3.1), $g(\cdot)$ by (2.15), and define $V(x)$ as in Section 2 by (2.4). Let $V^h(\cdot)$ be any of the approximations to $V(\cdot)$ defined in Subsection 2.2 (which are actually all the same according to Proposition 2.3). Then $V^h(x) \to V(x)$ as $h \to 0$ for all $x \in \mathcal{G}$.*

Remark. By combining the proof given below with an argument by contradiction, one may show the convergence is in fact uniform on $\mathcal{G}$.

Proof. The basis for the proof of the theorem will be the representation for $V^h(x)$ as the minimal cost for a stochastic control problem, as given in Section 2. Thus we have

$$V^h(x) = \inf E_x \left[\sum_{i=0}^{(N^h \wedge M^h) - 1} L(\xi_i^h, u_i^h) \Delta t^h(u_i^h) + g(\xi_{(N^h \wedge M^h)}^h) \right], \tag{4.1}$$

where the infimum is over all admissible control sequences and controlled stopping times M^h, and N^h is the first time of exit from $\mathcal{D}$ or entrance into $\mathcal{M}$. The transition probabilities and interpolation times are those of Example 2.1, and $L(x, u)$ is the running cost for Control Problem 2 as in (3.1). General references for terminology, basic results from weak convergence theory and the properties of martingales that are used are [5, 14]. Let $\{y^n, n \in I\!\!N\}$ and y be random variables that take values in some metric space S. We follow the standard practice of saying that the random variables y^n converge weakly to y (denoted $y^n \Rightarrow y$) if the measures induced by y^n on S converge weakly to the measure induced by y on S.

In order to study $V^h(x)$ we must first put the representation into a form suitable to taking limits. Let ξ_i^h and u_i^h be the state of the controlled chain and the control applied at time i, respectively. Define $t_n^h = \sum_0^{n-1} \Delta t^h(u_i^h)$. Thus t_n^h is the interpolated time up until discrete time n. Define $\tau^h = t_{N^h}^h$ and $\rho^h = t_{M^h}^h$, and also the piecewise constant interpolations

$$\xi^h(t) = \xi_n^h, \quad u^h(t) = u_n^h \text{ for } t \in [t_n^h, t_{n+1}^h). \tag{4.2}$$

We consider the processes $\xi^h(\cdot)$ as taking values in $D([0, \infty) : I\!\!R^2)$, the metric space of $I\!\!R^2$−valued functions that are continuous from the right and have limits from the left. (Our interest in the control actually stops when $i = N^h \wedge M^h$. However, to simplify the notation we may assume the control is defined for all $i \in I\!\!N$. Specifically, its value will be defined as $(0, -\gamma_2)$ when $i \geq N^h \wedge M^h$, so that the running costs per unit time after $N^h \wedge M^h$ are automatically bounded. For points $x \notin \mathcal{D}^h$ we can let $L(x, \beta)$ be $L(y, \beta)$, where y is any point in S.) In order to take limits of the sequences of controls it will be convenient to use an alternative representation for the control processes. For all $t \geq 0$ and $h > 0$ define

$$m_t^h(d\alpha) = \delta_{u^h(t)}(d\alpha) \text{ and } m^h(A \times B) = \int_B m_t^h(A)dt \tag{4.3}$$

where $\delta_u(d\alpha)$ is the probability measure that puts a unit mass at u. The m^h are random and take values in the space of measures on $I\!\!R^2 \times [0, \infty)$. We consider this space as endowed with the following topology: a sequence l_n of measures on $I\!\!R^2 \times [0, \infty)$ converge to l if $\int_{I\!\!R^2 \times [0,\infty)} s(u, t) l_n(du \times dt) \to \int_{I\!\!R^2 \times [0,\infty)} s(u, t) l(du \times dt)$ for every $s \in C(I\!\!R^2 \times [0, \infty))$ with compact support. The random measures $\{m^h, h > 0\}$ will be called tight if the corresponding restrictions of the measures to $I\!\!R^2 \times [0, T]$ are tight in the usual sense for each $T < \infty$. The representation m^h for the control process $u^h(\cdot)$ is known as the *relaxed control representation*. A measure l on $I\!\!R^2 \times [0, \infty)$ will be called a relaxed control process if $l(I\!\!R^2 \times [0, t]) = t$ for all $t \in [0, \infty)$. For such a measure, it follows from the existence of regular conditional probability measures that $l(\cdot)$ has a derivative in the following sense: for each $t \in [0, \infty)$ there exists a probability measure $l_t(\cdot)$ on $I\!\!R^2$ such that $l(A \times B) = \int_B l_t(A)dt$ for all Borel sets $B \subset [0, \infty)$ and $A \subset I\!\!R^2$ (see [5, p. 502]).

With these definitions, we can write

$$V^h(x) = \inf E_x \left[\int_0^{(\rho^h \wedge \tau^h)} L(\xi^h(t), u) m^h(du \times dt) + g(\xi^h(\rho^h \wedge \tau^h)) \right]. \tag{4.4}$$

In the next lemma we derive some properties of the controlled processes under an arbitrary admissible control with bounded running cost. In the lemma's statement, the

expectation operator actually depends on the admissible control strategy that is used. In order to simplify the notation this dependence is not denoted explicitly.

Lemma 4.2 *Consider any sequence of initial conditions $\xi_0^h \in \mathcal{D}$ and admissible controls for which*

$$\limsup_{h \to 0} E_{\xi_0^h} \int_0^T L(\xi^h(t), u) m^h(du \times dt) < \infty \qquad (4.5)$$

for all $T < \infty$. Then the random measures $\{m^h(\cdot), h > 0\}$ are tight. Suppose that a subsequence (again indexed by h) is given such that $m^h(\cdot)$ converges weakly to a limit $m(\cdot)$ and that ξ_0^h converges to x_0. Then $m(\cdot)$ is a relaxed control process (w.p.1), and hence can be written $m(du \times dt) = m_t(du)dt$. Furthermore, the sequence of processes $\{\xi^h(\cdot), h > 0\}$ converge weakly to a process $x(\cdot)$ that satisfies

$$x(t) - x_0 = \int_0^t \int_{\mathbf{R}^2} um(du \times ds) = \int_0^t \int_{\mathbf{R}^2} um_s(du)ds \qquad (4.6)$$

for all $t \in [0, \infty)$, w.p.1.

Proof. If (4.5) holds then the fact that $L(x, u) = \infty$ whenever $\|u\| > 2$ implies for all $h > 0$ that the restrictions of $m^h(\cdot)$ to $\mathbf{R}^2 \times [0, T]$ are supported on the compact set $\{u : \|u\| \leq 2\} \times [0, T]$. Hence the tightness of $\{m^h, h > 0\}$ is automatic. Assume that m^h converges weakly to m. Since $m^h(\mathbf{R}^2 \times [0, t]) = t$ for all ω and $t \in [0, \infty)$, $m(\mathbf{R}^2 \times [0, t]) = t$ for $t \in [0, \infty)$ (w.p.1). Thus $m(\cdot)$ is a relaxed control process (w.p.1). Define the random processes

$$x^h(t) = \int_0^t \int_{\mathbf{R}^2} um^h(du \times ds) + \xi_0^h \quad \text{and} \quad x(t) = \int_0^t \int_{\mathbf{R}^2} um(du \times ds) + x_0.$$

We view these processes as taking values in $C([0, \infty); \mathbf{R}^2)$, the space of continuous functions from $[0, \infty)$ to $\mathbf{R}^2$. This space is equipped with the metric inherited as a subset of $D([0, \infty); \mathbf{R}^2)$, under which convergence is equivalent to uniform convergence on each compact subset of $[0, \infty)$. Then the fact that the second marginal of m is Lebesgue measure together with the uniform (in t, h and ω) boundedness of the supports of the m_t^h implies $x^h \Rightarrow x$.

We next prove the weak convergence $\xi^h \Rightarrow x$. If we use the fact that $x^h \Rightarrow x$, then to prove $\xi^h \Rightarrow x$ it suffices to show for each $T < \infty$ that

$$\sup_{t \in [0,T]} \|\xi^h(t) - x^h(t)\| \to 0 \qquad (4.7)$$

in probability as $h \to 0$. Let $N^h(T) = \inf\{n : t_n^h \geq T\}$. By (2.7)

$$E\left[\xi_{i+1}^h - \xi_i^h - u_i^h \Delta t^h(u_i^h)|(\xi_j^h, u_j^h), j \leq i\right] = 0.$$

This implies that

$$\xi_i^h - \xi_0^h - \int_0^{t_i^h} \int_{\mathbf{R}^2} um^h(du \times ds) \qquad (4.8)$$

26

is a martingale in i. Using (2.8), we compute that

$$E_{\xi_0^h} \left[\xi_{N^h(T)}^h - \xi_0^h - \int_0^{t_{N^h(T)}^h} \int_{R^2} u m^h (du \times ds) \right]^2 \to 0$$

as $h \to 0$. Hence by the submartingale inequality [5, Chapter 2],

$$\sup_{0 \le i \le N^h(T)} \left\| \xi_i^h - \xi_0^h - \int_0^{t_i^h} \int_{R^2} u m^h (du \times ds) \right\| \to 0$$

in probability as $h \to 0$. This clearly implies (4.7), and completes the proof of the lemma. ∎

Proof of the Upper Bound. We first prove $\limsup_{h \to 0} V^h(x) \le V(x)$. We begin by describing a "controllability" property of the chain $\{\xi_i^h, i \in I\!N\}$. An elementary calculation shows $L(x, u) \le 2$ for all $x \in \mathcal{D}$ and $u \in \mathcal{U}(x)$. Let y be any point in $\mathcal{M}$. The running cost at such a point is $\gamma_3^2 - \gamma_2 u_2 - \gamma_3 (1 - |u_1|^2 - |u_2 + \gamma_2|^2)^{1/2}$ for all u such that $|u_1|^2 + |u_2 + \gamma_2|^2 \le 1$, and ∞ otherwise. Since $|\gamma_2| < 1$, there exists $c > 0$ such that $L(y, u) \le 2$ for all u satisfying $\|u\| \le c$. Owing to the continuity of $I(\cdot)$ and the form of $L(x, u)$, we can in fact assume $L(x, u) \le 2$ whenever $\|x - y\| \le c$ and $\|u\| \le c$.

Fix x such that $\|x - y\| \le c$, and define $j_i = |x_i - y_i|/h, i = 1, 2$. Suppose the control $u = ([-\text{sign}(x_1 - y_1)]c, 0)$ is applied for exactly j_1 steps, and that after this the control $(0, [-\text{sign}(x_2 - y_2)c])$ is applied for j_2 steps. It is easy to see that this control admissible. The "deterministic" form of the transition probabilities under these controls guarantees that the controlled chain arrives at y at discrete time $j = j_1 + j_2$. Since for all steps $\Delta t^h(u) = h/c$, the running cost accumulated during this time will be bounded above by

$$2 \left[\frac{1}{h} (|x_1 - y_1| + |x_2 - y_2|) \right] \left[\frac{h}{c} \right] = \frac{2}{c} (|x_1 - y_1| + |x_2 - y_2|). \tag{4.9}$$

We now give the proof of the upper bound. Let $\mathcal{G}$ be any set that satisfies A3.1. According to Lemma 3.2, for each $x \in \mathcal{G}$ and $\varepsilon > 0$ there exists $\delta > 0, \tau < \infty$ and piecewise continuous $u : [0, \tau] \to I\!R^2$ such that if $x(t) = x + \int_0^t u(s) ds$, then

1.

$$\int_0^\tau L(x(t), u(t)) dt + g(x(\tau)) \le V(x) + \varepsilon,$$

2. $x(\tau) \in \mathcal{M}, x(t) \notin \mathcal{M}$ for all $t \in [0, \tau)$,

3. $d(u(t), \mathcal{U}^c(x(t))) \ge \delta$ for all $t \in [0, \tau]$.

Since $\mathcal{U}(x)$ is continuous in x, we can assume $\delta > 0$ has been chosen small enough that

$$d(u(t), \mathcal{U}^c(x)) \ge \delta \text{ for all } x \text{ such that } |x - x(t)| \le \delta \text{ and a.e. } t \in [0, \tau]. \tag{4.10}$$

Since $\mathcal{G} \subset \mathcal{D}$ is compact and $\mathcal{D}$ is open, we can also assume $d(x, \mathcal{D}^c) \ge \delta$ for all $x \in \mathcal{G}$.

Let $y = x(\tau) \in \mathcal{M}$. Fix $c > 0$ so that the "controllability" property and the bound (4.9) on the running cost will hold. We now define an admissible control scheme for the Markov chain in terms of $u(\cdot)$. To apply $u(\cdot)$ to the chain $\{\xi_i^h, i < \infty\}$ we recursively define the control applied at discrete time i by $u_i^h = u(t_i^h)$ and $t_{i+1}^h = t_i^h + \Delta t^h(u_i^h)$. This defines a control until i such that $t_{i+1}^h \geq \tau$. Let $\{\xi_i^h, i < \infty\}$ be the chain that starts at x and uses this control.

Define

$$S^h = \inf\{i : t_i^h \geq \tau \text{ or } \xi_i^h \in \mathcal{M} \text{ or } \xi_i^h \notin \mathcal{D} \text{ or } \|\xi_i^h - x(t_i^h)\| \geq \delta\},$$

and let $\sigma^h = t_{S^h}^h$. By construction the restriction of the measures $m^h(du \times ds)$ to $\mathbb{R}^2 \times [0, \tau]$ converge weakly to the measure $m(du \times ds) = \delta_{u(s)}(du)ds$. Hence by Lemma 4.2, we have $\sup_{0 \leq t \leq \sigma^h} \|\xi^h(t) - x(t)\| \to 0$ in probability, and for each $\theta > 0$ $P_x\{\|\xi_{S^h}^h - x(\tau)\| \geq \theta\} \to 0$ as $h \to 0$.

Assume $\theta \in (0, c)$. If $\|\xi_{S^h}^h - x(\tau)\| \geq \theta$, then we stop the process at discrete time S^h and pay the stopping cost $g(\xi_{S^h}^h) \leq B$. On the set where $\|\xi_{S^h}^h - x(\tau)\| < \theta$, we extend the definition of the control sequence for discrete times larger than S^h according to the discussion at the beginning of the proof. The control applied after S^h will drive ξ_i^h to y in fewer than $2\theta/h$ steps with a running cost that is bounded above by $4\theta/c$.

The total cost for the control scheme and stopping time defined in this way is bounded above by

$$E_x \int_0^{\sigma^h} \int_{\mathbb{R}^2} L(\xi^h(s), u)m^h(du \times ds) + 4\theta/c$$

$$+ P_x\{\|\xi_{S^h}^h - x(\tau)\| \geq \theta\}B + P_x\{\|\xi_{S^h}^h - x(\tau)\| < \theta\} \sup\{g(x) : \|x - y\| \leq \theta, x \in \mathcal{M}\}.$$

According to the Skorokhod representation theorem [5, Theorem 3.1.8] we can assume that the convergence $(\xi^h(\cdot), m^h(\cdot)) \to (x(\cdot), m(\cdot))$ is w.p.1 for the purposes of evaluating the limits of the expectations above, and that x and m satisfy (4.6). Owing to the definition of S^h and (4.10), we can apply the dominated convergence theorem to obtain

$$\limsup_{h \to 0} V^h(x) \leq \int_0^\tau \int_{\mathbb{R}^2} L(x(s), u)m(du \times ds) + \sup\{g(z) : \|z - y\| \leq \theta, z \in \mathcal{M}\} + 4\theta/c$$

$$= \int_0^\tau L(x(s), u(s))ds + \sup\{g(z) : \|z - y\| \leq \theta, z \in \mathcal{M}\} + 4\theta/c.$$

Since $\varepsilon > 0$ and $\theta \in (0, c)$ are arbitrary,

$$\limsup_{h \to 0} V^h(x) \leq V(x).$$

Proof of the Lower Bound. We now prove $\liminf_{h \to 0} V^h(x) \geq V(x)$. Fix $x \in \mathcal{G} - \mathcal{M}$ and $\varepsilon > 0$. Owing to the definition of $V^h(x)$, there is a controlled Markov chain $\{\xi_i^h, i < \infty\}$ with admissible control sequence $\{u_i^h, i < \infty\}$ that satisfies $\xi_0^h = x$, and a stopping time M^h such that

$$V^h(x) \geq E_x \sum_{j=0}^{(N^h \wedge M^h)-1} L(\xi_j^h, u_j^h)\Delta t^h(u_j^h) + E_x g(\xi_{N^h \wedge M^h}^h) - \varepsilon, \tag{4.11}$$

28

where N^h is the time of first exit from $\mathcal{D}$ or entrance into the set $\mathcal{M}$. Let $\xi^h(\cdot)$ and $u^h(\cdot)$ be the continuous parameter interpolations of $\{\xi_i^h, i < \infty\}$ and $\{u_i^h, i < \infty\}$ as defined by (4.2), and define $\rho^h = t_{M^h}^h, \tau^h = t_{N^h}^h$. We can then rewrite (4.11) as

$$V^h(x) \geq E_x \int_0^{\rho_h \wedge \tau_h} \int_{I\!\!R^2} L(\xi^h(s), u) m^h(du \times ds) + E_x g(\xi^h(\rho_h \wedge \tau_h)) - \varepsilon, \qquad (4.12)$$

where $m^h(\cdot)$ is the relaxed control representation of the ordinary control $u^h(\cdot)$.

Let $\mathcal{S}_q, q = 1, ..., Q$ be disjoint compact connected sets such that $\mathcal{S} = \cup_{q=1}^Q \mathcal{S}_q$. The existence of such a decomposition has been assumed in the statement of Theorem 4.1. Now $V(x)$ is constant on each $\mathcal{S}_q$, so there exists $\theta > 0$ such that

$$x \in \mathcal{S}_q, y \in N_\theta(\mathcal{S}_q) \Rightarrow |V(x) - V(y)| \leq \varepsilon \qquad (4.13)$$

and such that the sets $N_\theta(\mathcal{S}_q)$ are separated by a distance greater than θ for distinct q. Because the reflected light intensity $I(\cdot)$ is continuous on the closure of $\mathcal{D}$, there is $c > 0$ such that

$$L(x, u) \geq c \text{ for all } u \in I\!\!R^2, x \in \mathcal{D} - \cup_{q=1}^Q N_{\theta/2}(\mathcal{S}_q). \qquad (4.14)$$

For simplicity, we will consider the proof of the lower bound for the case when the initial condition satisfies $x \in N_{\theta/2}(\mathcal{S}_{q^*})$ for some q^*. The general case follows easily using the same arguments. We define a sequence of stopping times by

$$\begin{aligned} \tau_0^h &= 0, \\ \sigma_j^h &= \inf\{t \geq \tau_j^h : \xi^h(t) \notin \cup_{q=1}^Q N_\theta(\mathcal{S}_q)\}, \\ \tau_j^h &= \inf\{t \geq \sigma_{j-1}^h : \xi^h(t) \in \cup_{q=1}^Q N_{\theta/2}(\mathcal{S}_q) \text{ or } \xi^h(t) \notin \mathcal{D}\}. \end{aligned}$$

Consider the processes

$$\Xi^h(\cdot) = \left(\xi_0^h(\cdot), \xi_1^h(\cdot), ...\right), \quad M^h(\cdot) = \left(m_0^h(\cdot), m_1^h(\cdot), ...\right),$$

where $\xi_j^h(\cdot) = \xi^h(\cdot + \sigma_j^h)$ and where $m_j^h(\cdot)$ is the relaxed control representation of the ordinary control $u^h(\cdot + \sigma_j^h)$. We consider $(\Xi^h(\cdot), M^h(\cdot))$ as taking values in the product space endowed with the usual product space topology, i.e., a sequence converges if and only if each finite subset of components converges.

Consider any subsequence along which $V^h(x)$ converges to a point in $(-\infty, \infty]$. Then it is enough to prove that the limit of this subsequence is no less than $V(x)$. We can eliminate the case where the limit is ∞, since in this case the lower bound is automatic. For the rest of the proof we shall assume that we are working with a subsequence (again labeled by h) along which $V^h(x)$ converges to a bounded limit. Thus the sequence $V^h(x)$ will be uniformly bounded from above.

Lemma 4.2 shows that given any subsequence of $\{(\Xi^h(\cdot), M^h(\cdot)), h > 0\}$, we can extract a further subsequence that converges weakly, and that any limit point

$$(X(\cdot), M(\cdot)) = ((x_0(\cdot), x_1(\cdot), ...), (m_0(\cdot), m_1(\cdot), ...))$$

of such a convergent subsequence satisfies

$$x_j(t) - x_j(0) = \int_0^t \int_{\mathbf{R}^2} u m_j(du \times ds),$$

where each $m_j(\cdot)$ is a relaxed control process. In addition, the definition of the stopping times $\{\sigma_j^h\}$ guarantees that $x_0(0) \in \partial N_\theta(S_{q^*})$, and that for all $j > 0$ either $x_j(0) \in \partial N_\theta(S_q)$ for some q or $x_j(0) \notin \mathcal{D}$.

Let $J^h = \min\{j : \tau_j^h \geq \tau^h\}$, where τ^h has been defined to be the interpolated time at which $\xi^h(\cdot)$ first exited $\mathcal{D}$ or entered $\mathcal{M}$. By construction, if $\xi^h(\tau^h) \in S_q$ for some q (i.e., ξ^h enters $\mathcal{M}$ before it leaves $\mathcal{D}$) then $\xi^h(\tau_{J^h-1}^h) \in N_{\theta/2}(S_q)$ for the same q. It follows from $\lim_{h\to 0} V^h(x) < \infty$ and the uniform bound from below given in (4.14) that

$$\limsup_{h \to 0} E_x \sum_{0 \leq j < J^h} (\tau_{j+1}^h - \sigma_j^h) < \infty. \tag{4.15}$$

Define $s_j^h = \tau_{j+1}^h - \sigma_j^h$ and $S^h = (s_0^h, s_1^h, ...)$. It also follows from (4.14) that there exists $\bar{c} > 0$ such that for all q_1 and q_2,

$$\inf\left\{ \int_0^T L(\phi, \dot{\phi}) ds : \phi(0) \in \partial N_\theta(S_{q_1}), \phi(T) \in N_{\theta/2}(S_{q_2}), \phi(t) \in \overline{\mathcal{D}}, t \in [0,T], T > 0 \right\} \geq \bar{c}, \tag{4.16}$$

and $\tilde{c} > 0$ that is independent of θ for all small $\theta > 0$ such that for all $q_1 \neq q_2$,

$$\inf\left\{ \int_0^T L(\phi, \dot{\phi}) ds : \phi(0) \in \partial N_\theta(S_{q_1}), \phi(T) \in N_{\theta/2}(S_{q_2}), \phi(t) \in \overline{\mathcal{D}}, t \in [0,T], T > 0 \right\} \geq \tilde{c}. \tag{4.17}$$

We now prove the lower bound $\lim_{h\to 0} V^h(x) \geq V(x)$. Extract a subsequence along which

$$\left(\xi^h(\cdot), m^h(\cdot), \rho^h, \tau^h, \Xi^h(\cdot), M^h(\cdot), J^h, S^h \right)$$

converges to a limit

$$(x(\cdot), m(\cdot), \rho, \tau, \Xi(\cdot), M(\cdot), J, S).$$

Note that the bounds (4.15) and (4.16) imply that J and the $s_j, j \in \{0, ..., J - 1\}$ are finite w.p.1. The ρ^h and τ^h are regarded as taking values in the compactified space $[0, \infty]$ in order to guarantee tightness, and hence τ and ρ may take the value ∞. We assume via the Skorokhod representation [5] that the convergence is w.p.1, and consider any ω for which there is convergence. If $\rho < \tau$, i.e., if we choose to stop before entering $\mathcal{M}$ or leaving $\mathcal{D}$, we have

$$\liminf_{h \to 0} g(\xi^h(\rho^h \wedge \tau^h)) = B \geq V(x).$$

Next assume $\tau \leq \rho$. The non-negativity and lower semicontinuity of $L(x, u)$ in (x, u) then imply

$$\liminf_{h \to 0} \int_0^{\rho^h \wedge \tau^h} \int_{\mathbf{R}^2} L(\xi^h(s), u) m^h(du \times ds)$$

$$\geq \liminf_{h \to 0} \sum_{0 \leq j < J^h} \int_{\sigma_j^h}^{\tau_{j+1}^h} \int_{\mathbf{R}^2} L(\xi^h(s), u) m^h(du \times ds) \tag{4.18}$$

$$\geq \sum_{0 \leq j < J} \int_0^{s_j} \int_{\mathbf{R}^2} L(x_j(s), u) m_j(du \times ds).$$

30

Let $j_k, k = 1, ..., K$ index those values of $j \in \{0, ..., J-1\}$ such that $x_j(0) \in N_\theta(S_q)$ and $x_j(s_j) \notin N_{\theta/2}(S_q)$, i.e., the j_k label those trajectories that actually leave the neighborhood of one of the S_q and either enter the neighborhood of some $S_{q'}, q' \neq q$, or else end up at ∂D. Assume that $j_{k_1} < j_{k_2}$ whenever $k_1 < k_2$.

For any $j \in \{0, ..., J-1\}$ and $s \in [0, s_j]$ define $u_j(s) = \int_{R^2} u m_{j,s}(du)$, where $m_{j,s}(\cdot)$ is the derivative of $m_j(\cdot)$. Then $x_j(t) - x_j(0) = \int_0^t u_j(s)ds$ for $t \in [0, s_j]$, and by Jensen's inequality

$$\int_0^{s_j} \int_{R^2} L(x_j(s), u)m_j(du \times ds) \geq \int_0^{s_j} L(x_j(s), u_j(s))ds.$$

From the definition of $V(\cdot)$ and an elementary dynamic programming argument,

$$V(x_j(0)) \leq \int_0^{s_j} L(x_j(s), u_j(s))ds + V(x_j(s_j)).$$

Assembling these inequalities gives that for each $j \in \{0, ..., J-1\}$

$$\int_0^{s_j} \int_{R^2} L(x_j(s), u)m_j(du \times ds) \geq V(x_j(0)) - V(x_j(s_j)). \tag{4.19}$$

According to the definitions of the τ_i^h, σ_i^h, and the indices j_k, if $x_{j_k}(s_{j_k}) \in \partial N_{\theta/2}(S_q)$ then $x_{j_{k+1}}(0) \in N_\theta(S_q)$ for all $k \in \{1, ..., K-1\}$. Thus $|V(x_{j_k}(s_{j_k})) - V(x_{j_{k+1}}(0))| \leq 2\varepsilon$ for all such k. Together with the fact that $x_{j_1}(0) \in N_\theta(S_{q^*})$ (recall that $x \in S_{q^*}$), the last sentence together with (4.18) and (4.19) imply

$$\liminf_{h \to 0} \int_0^{\rho^h \wedge \tau^h} \int_{R^2} L(\xi^h(s), u)m^h(du \times ds)$$

$$\geq \sum_{k \in \{1, ..., K\}} \int_0^{s_{j_k}} \int_{R^2} L(x_{j_k}(s), u)m_{j_k}(du \times ds)$$

$$\geq \sum_{k \in \{1, ..., K\}} [V(x_{j_k}(0)) - V(x_{j_k}(s_{j_k}))] \tag{4.20}$$

$$\geq V(x) - V(x_{j_K}(s_{j_K})) + \sum_{k \in \{2, ..., K\}} \left[V(x_{j_k}(0)) - V(x_{j_{k-1}}(s_{j_{k-1}})) \right] - \varepsilon$$

$$\geq V(x) - V(x_{j_K}(s_{j_K})) - (2K-1)\varepsilon$$

w.p.1.

Next consider $\liminf_{h \to 0} g(\xi^h(\rho^h \wedge \tau^h))$. As noted above, $\liminf_{h \to 0} g(\xi^h(\rho^h \wedge \tau^h)) = B$ if $\rho < \tau$. If $\tau \leq \rho$, there are two possibilities. Recall that ρ^h is the controlled stopping time, and that τ^h is the first time the process enters M or leaves D. If $\rho^h < \tau^h$ or $\rho^h \geq \tau^h$ and $\xi^h(\tau^h) \notin M$, then $g(\xi^h(\rho^h \wedge \tau^h)) = B$. If $\rho^h \geq \tau^h$ and $\xi^h(\tau^h) \in S_q \subset M$, then as observed previously $\xi^h(\tau_{j^h-1}^h) \in N_{\theta/2}(S_q)$ and therefore $V(\xi^h(\tau_{j^h-1}^h)) \leq g(\xi^h(\rho^h \wedge \tau^h)) + \varepsilon$. Now j_K is the last index for which there is a transition between different $N_\theta(S_q)$. Thus, in general,

$$\liminf_{h \to 0} g(\xi^h(\rho^h \wedge \tau^h)) \geq V(x_{j_K}(s_{j_K})) - \varepsilon. \tag{4.21}$$

Combining (4.20) and (4.21) gives

$$\lim_{h \to 0} V^h(x) \geq V(x) - 2\varepsilon E_x K.$$

31

Now by (4.17) we also have

$$\lim_{h \to 0} V^h(x) \geq \tilde{c} E_x K,$$

where $\tilde{c} > 0$ is independent of $\varepsilon > 0$. Thus $E_x K$ has a bound that is independent of $\varepsilon > 0$. Sending $\varepsilon \to 0$ gives the desired lower bound $\lim_{h \to 0} V^h(x) \geq V(x)$. ∎

Remark. P. Dupuis would like to acknowledge an error in the proof of the lower bound as it appears in [14, Chapter 13]. That proof considered convergence in the special case of vertical light only. In the proof the distinction between the paths which move between neighborhoods of different S_q and those that do not was omitted, making the assertion of a uniform upper bound on J incorrect. The correct assertion is the uniform upper bound on K, as noted above.

5 Experiments

In this section we present some experimental results obtained with the algorithms described in Section 2. In all of the discussion so far, we have required that the surface "height" $z(\cdot)$ be known at all points $x \in \mathcal{M}$, which are a subset of the singular points in $\mathcal{G}$. This is equivalent to knowing the value of $f(\cdot)$ at all of its local minima in $\mathcal{G}$. (Recall that we use a more restrictive definition of local minimum than is usual. See the definition in Section 2.) In Subsection 5.2, this restriction is relaxed. An approach is outlined in which the surface can be reconstructed with no boundary data, that is, using only the intensity data $I(x)$ in addition to A3.1. To do this, we must impose some additional conditions, the most important of which is that $f(\cdot)$ is C^2. A fuller discussion of this approach is deferred to future work. Throughout this section we continue to use the transition probabilities of Example 2.1.

In the discussion below we use the term "boundary data" to refer to data that are specified on subsets of the set S of singular points. Of course we never specify data on boundaries of sets such as $\mathcal{D}$ or $\mathcal{G}$. Hopefully, this use of the term "boundary" will not cause confusion.

5.1 BOUNDARY DATA GIVEN ON $\mathcal{M}$

Figure 1 displays a 32 by 32 parabolic surface which is assumed to be imaged from above. The image has one singular point, corresponding to the local minimum of the height $z(\cdot)$. Assuming vertical light, the image intensity was first computed using the forward/backward discretization of the derivative implicit in the algorithm of Example 2.1 (see also the proof of Proposition 2.2). Thus we are defining the data to correspond to the discretized (rather than continuous) surface. With this choice, the original surface is a fixed point of the algorithm and should be reconstructed exactly. Our purpose in doing this is to assess the convergence time of the algorithm. Using Jacobi updates, the algorithm converged to the correct solution to within, on average, one part in 10^7 after 63 iterations. In general, the convergence time is expected to be on the order of the maximum length of an optimal path, i.e., a path of steepest descent. For the controlled transition

probabilities that we use, the Markov chain ξ_t^h jumps one lattice site per discrete time step. Thus, when the number of iterations becomes greater than the maximum optimal path length, all points x should have "learned" their heights from the singular points. For the given surface, the maximal path length is on the order of 32, since paths starting at the image corners must zigzag to the singular point at the center of the image.

Convergence using Gauss–Seidel updating was faster: it was obtained after just 4 iterations. Gauss–Seidel was performed using a row major pass over the image, changing the direction of the pass after each iteration. In this case, the information from the singular point is able to propagate outward to a portion of the boundary in just one iteration, since the ordering of the states corresponds to the steepest ascent direction throughout one quadrant of the image.

If we use for data the values that correspond to the true derivatives of the (continuous) surface at the grid points, the average and maximal errors were .8 and 1.6 (the latter obtained at the image boundary), compared with a range for the surface height of 25.

For comparison, Figures 2 and 3 display the result of applying our implementation of a more traditional algorithm [10] to a similar surface. The intensity was computed using discrete forward derivatives such that the given surface is a fixed point of the algorithm. Even after 3072 iterations, the algorithm has not converged to the correct fixed point solution. We have also implemented other algorithms such as those of [15] and [32], and applied them to this surface with similar results.

Figure 4 shows a more complicated 128 by 128 surface. Under vertical light, the intensity was first computed as for Figure 1 so as make this surface a fixed point of the algorithm. At the local minimum points $\mathcal{M}$, the initial estimate $V_0^h(x)$ and the terminal cost $g(x)$ were set to the known height values $z(x)$, as discussed in Section 2. The algorithm converged to a perfect reconstruction of the original surface in 100 iterations. As expected, the convergence time is on the order of the longest optimal path. Using Gauss–Seidel, convergence was achieved in 10 iterations. When the intensity was derived analytically, the algorithm again converged in 10 iterations using Gauss–Seidel, with an average error of 1.7 compared to a surface range of 51 (Figure 5). Because the surface does not obey the condition in A3.1 that it be decreasing in from the boundary, the reconstruction is incorrect (as it should be) in some regions near the boundary of $\mathcal{D}$, though it is good in the interior. This is clear in Figure 6, which displays the difference between the reconstruction and the original surface. This surface was also reconstructed assuming oblique light at an angle of 17.5° to the vertical. For an intensity derived so that the surface is a fixed point, convergence to within one part in 10^{-7} was obtained using the Jacobi scheme in 120 iterations. Using Gauss–Seidel, convergence was obtained in 11 iterations. Reconstruction for the analytically–derived intensity function was obtained in 14 iterations, with an average error of 2.2. As previously, the reconstruction was good in the interior but incorrect along one boundary (Figure 7). Note that the domains which satisfy A3.1 will in general vary with the direction of the incoming light.

Finally, our algorithm has been applied to the real 200 × 200 image shown in Figure 8. The data for this problem was kindly provided by the authors of [15]. The light is from above at (0, .488, .873). This image is a photograph of a mannequin. The particular object was chosen since its reflectance properties seem to be reasonably in line with the

33

assumed Lambertian model. For this example, one of course specifies the height at the local maxima, rather than minima. For the reconstruction, boundary conditions were specified only at the point that corresponds to the nose of the mannequin, though $\mathcal{M}$ actually contains several other points which correspond to shallow local maxima. Figure 9 shows the reconstruction obtained using Gauss–Seidel after 6 iterations, illuminated from the same direction as the original. Convergence has essentially been achieved over the face. Since a fixed point of the algorithm must correspond to the original intensity data (modulo the effects of any neglected local maxima), this image essentially reproduces Figure 8. However, errors due to incorrect modeling of the surface as Lambertian and the neglect of the shallow local maxima show up as spurious lines in Figure 9. Their occurrence suggests that the truly Lambertian surface corresponding to the intensity data in Figure 8, and with the prescribed boundary conditions, is not everywhere C^1, but has at least line discontinuities in f_x. Figure 10 shows the reconstruction illuminated from below. Since this image is not guaranteed to correspond to the original intensity, it better displays the errors due to incorrect modeling, neglect of local maxima, and discretization effects (i.e., noninfinitesimal h). Finally, Figure 11 shows the surface reconstruction. This reconstruction took about 9 seconds of CPU time on a DEC 5000 workstation. Standard variational algorithms typically require thousands of iterations [10].

For comparison, Figures 12 and 13 display the reconstruction obtained by the authors of [15] using a more standard variational method, developed for the purpose of including stereo information. Stereo information was used as an initial condition for this reconstruction.

5.2 RECONSTRUCTION WITHOUT BOUNDARY DATA

We present a preliminary description of our approach, and some experimental results employing it. For simplicity, our description will be for the case of vertical light, although our method is also valid in the more general case of oblique light. Our experimental results are presented for oblique light.

Reconstruction without boundary data is possible only if some additional assumptions are placed on the surface, the most important of which is that $z(\cdot)$ is C^2. In addition, we assume that the extremal points of $z(\cdot)$ are isolated, and that the surface has nonzero curvature at these points. (These assumptions are not necessarily respected in our experiments.) The above two conditions are implied by the following assumption on the intensity data $I(\cdot)$:

A5.1 *The set S consists of isolated singular points. At these points, the matrix of second derivatives of $I(\cdot)$ has nonzero, unequal eigenvalues.*

In addition, we assume as before that $I(\cdot)$ is defined over a region $\mathcal{D}$ and that we are considering the reconstruction in a subset $\mathcal{G}$ that satisfies A3.1. Finally, we will assume for simplicity that the set $\mathcal{G}$ is simply connected. The uniqueness of $z(\cdot)$ given $I(\cdot)$ under these assumptions was investigated in [19, 20].

Suppose that a point x_0 is known to be an isolated, local minimum of the height $z(\cdot)$ (the discussion is similar if x_0 is a local maximum). We define the domain of attraction

of x_0 to be the set A_{x_0} that consists of all points $y \in \mathcal{G}$ such that the projected steepest descent curve passing through y converges to x_0. These curves are uniquely determined since $z(\cdot)$ is C^2. Suppose that we set $g(x_0) = z(x_0)$ and $g(x) = B$ for all $x \neq x_0$, where B is an upper bound for $z(\cdot)$ on $\mathcal{G}$. The algorithm with g as its initial data will accurately reconstruct the surface in the set A_{x_0} (although the reconstruction will generally be poor elsewhere). This follows by considering an increasing sequence of domains $\mathcal{G}_i \subset A_{x_0}$ with $\mathcal{G}_i \uparrow A_{x_0}$ as $i \to \infty$ and each $\mathcal{G}_i$ satisfying A3.1, and by applying the convergence and representation theorems on each of these domains.

Typically, there will be other singular points in ∂A_{x_0}. By continuity, the heights of these points will be correctly approximated by the algorithm. From A5.1, these points are either local maxima or saddle points of $z(\cdot)$. Suppose it is possible to identify one of them (say x_1) as a local maximum. The algorithm can then be applied again with the approximate height provided as initial datum at the new singular point x_1 (as well as any other points $x_2, ..., x_n$ on ∂A_x that are identified as local maxima), which extends the computed approximation $V^h(\cdot)$ over the region $\cup_{i=1}^n B_{x_i} \cup A_{x_0}$ where B_{x_i} is the analogous domain of attraction for the local maximum point x_i. Again by continuity, the heights of any local minima on the boundary of this region will be correctly approximated, and the whole process can be repeated. It is argued in [19, 20] that every singular point in $\mathcal{G}$ is connected to every other singular point by a collection of steepest descent curves. Thus, iterating the above process should eventually identify all local minima and maxima, and lead to a computed approximation to $z(\cdot)$ over the entire domain $\mathcal{G}$.

For the above strategy to work, all that we need is a method for identifying as such the local maxima or minima on ∂A_{x_0}. This is nontrivial, since for the computed $V^h(\cdot)$ that is based on just one singular point, all other singular points will be reconstructed as though they were inflection points.

We now sketch such a strategy. There are a number of possible variations, and it is not clear that the approach we now outline is the best in any sense. Let $x \in \mathcal{M}$, and assume $A_{x_0} \subset \mathcal{G}$ and $\partial \mathcal{G} \cap \partial A_{x_0} = \emptyset$. From A5.1, $\mathcal{S} \cup \partial A_{x_0}$ consists of a finite set of points. It can be shown that it contains at least two, and at least one local maximum. Since $z(\cdot)$ is C^2, the steepest ascent paths are uniquely determined, and fill out A_{x_0}. It is easy to show that all but a finite number converge to local maximum points in $\mathcal{S} \cup \partial A_{x_0}$. The finite number of exceptions converge to saddle points in $\mathcal{S} \cup \partial A_{x_0}$, exactly one to each saddle point.

The following discussion is purely formal, since our purpose here is simply to suggest a numerical scheme. According to the discussion in the previous paragraph, if we start a particle off from x_0 in a random direction selected according to the uniform distribution and let it proceed along a steepest ascent path, then it has probability one of ending up at a local maximum point. Suppose we could compute the probability distribution $\mu(dy)$ on $\partial \mathcal{G}$ that describes the distribution of the particle in the limit $t \to \infty$. If $x_i, i = 1, 2, ..., n$ denote the local maxima in ∂A_{x_0}, then $\mu(\{x_1, ..., x_n\}) = 1$, and obviously $\mu(\partial A_{x_0} \backslash \{x_1, ..., x_n\}) = 0$. In particular, for at least one $i \in \{1, ..., n\}$ we must have $\mu(\{x_i\}) > 0$. Thus, the distribution μ allows the identification of at least one local maximum point in ∂A_{x_0}.

A scheme for computing an approximation to μ on $\mathcal{G}^h$ can be developed that is similar in spirit to our reconstruction algorithms. However, it is much simpler. Let $V^h(\cdot)$ be

the reconstruction obtained by the algorithm using only the value $z(x_0)$ as the initial boundary datum. Let ξ_i^h be a Markov chain defined on $\mathcal{G}^h$ such that $\xi_0^h = x_0$. The transition probabilities are defined so that the discrete time evolution of ξ_i^h approximates via $V^h(\cdot)$ a steepest ascent curve on $z(\cdot)$. A simple choice for these probabilities is the following. At x_0, let $p^h(x_0, x_0 \pm h(1,0)) = p^h(x_0, x_0 \pm h(0,1)) = 1/4$, giving an equal probability of emerging from x_0 in any of the compass directions. We can also allow diagonal transitions to give a better approximation to the uniform distribution. Now consider $y \neq x_0$, and define

$$s_1 = \text{sign}\left[V^h(y + h(1,0)) - V^h(y - h(1,0))\right],$$

$$s_2 = \text{sign}\left[V^h(y + h(0,1)) - V^h(y - h(0,1))\right].$$

Also, let v_1 and v_2 be the largest values from the sets

$$\{V^h(y + h(1,0)), V^h(y - h(1,0))\} \text{ and } \{V^h(y + h(0,1)), V^h(y - h(0,1))\},$$

respectively, and

$$d_{x,i} \equiv \max\left(0, \; h^{-1}s_i(v_i - V^h(y))\right),$$

for $i = 1, 2$. The $d_{x,i}$ are essentially forward or backward derivatives, depending on which direction gives the steepest ascent. For $d_{x,1} = d_{x,2} = 0$ define $p^h(y,y) = 1$. Otherwise, define

$$p^h(y, y + s_1 h(1,0)) = \frac{d_{x,1}}{d_{x,1} + d_{x,2}},$$

$$p^h(y, y + s_2 h(0,1)) = \frac{d_{x,2}}{d_{x,1} + d_{x,2}},$$

with all other probabilities zero.

Let $\{\xi_i^h, i \in \mathbb{N}\}$ be the Markov chain that is defined by the construction outlined above. For $y \in \mathcal{D}^h$ set

$$P_n^h(y) \equiv P\{\xi_n^h = y\},$$

The Markov property gives the recursion

$$P_{n+1}^h(y) = \sum_x p^h(x,y)P_n^h(x) \tag{5.1}$$

We would like to use $\lim_{n \to \infty} P_n^h(y)$ as the approximation to μ. However, because the reconstruction $V^h(\cdot)$ is computed with $g(y) < B$ only at $y = x_0$, the infimal expected cost $V^h(y)$ for $y \in \mathcal{G} \setminus A_{x_0}$ will approximately satisfy $V^h(y) \geq z(y)$ and also $V^h(y) \geq \inf_{y' \in \partial A_{x_0}} V^h(y')$. Since the evolution of the chain approximates steepest ascent on $V^h(\cdot)$, and because of the discretization, it is likely that ξ_n^h will evolve outside a neighborhood of the set A_{x_0}, and that the support of $P_n^h(y)$ as defined above will eventually extend beyond A_{x_0}. A direct use of this distribution therefore gives a poor approximation to μ as $n \to \infty$, and may result in spurious concentrations of probability at singular points in $\mathcal{G} \setminus \bar{A}_{x_0}$, leading to the false identification of these points as local maxima.

36

We avoid this problem by modifying the transition probabilities. Since the steepest ascent paths converge on the local maxima points in ∂A_{x_0}, ξ_i^h should pass near a point in the set $\{x_1, ..., x_n\}$ with probability that approaches 1 as $h \to 0$. Let $m > 0$ be such that $m < (1/2)\min\{\|y - y'\| : y, y' \in \mathcal{S}\}$. We modify the probabilities so that the chain is stopped as soon as it enters the set $\{y' : \|y' - y\| < m, y \in \mathcal{S}\backslash x_0\}$, i.e., all points $x \in \mathcal{D}^h \cap \{y' : \|y' - y\| < m, y \in \mathcal{S}\backslash x_0\}$ are taken to be absorbing, so that $p^h(x, x) = 1$. With this modification, the strength of $P_n^h(y)$ at singular points outside ∂A_{x_0} should be sharply reduced. Let $P^h(y) = \lim_{n \to \infty} P_n^h(y)$. This limit should be well defined at all points, and is obviously well defined because of monotonicity for $x \in \mathcal{D}^h \cap \{y' : \|y' - y\| < m, y \in \mathcal{S}\backslash x_0\}$. Since the structure of the recursion (5.1) is similar to that of the reconstruction algorithm itself, the number of iterations required for convergence to P^h is expected to be similar to the number of iterations used in the convergence to V^h. Once an approximation to $P^h(y)$ has been computed, those singular points near which this approximation has a large value can be identified as local maxima.

In our actual experiments, a variation of this procedure was used. Rather than an approximation to the probability distribution μ, we consider the sum

$$Q_n^h(y) \equiv \sum_{i=0}^{n} P_i^h(y),$$

ξ_i^h is the same Markov chain as before, except that at y such that $d_{x,1} = d_{x,2} = 0$, we define the transition probability to be $p^h(y, \mathcal{T}) = 1$, where $\mathcal{T}$ is a "fictitious" absorbing point adjoined to $\mathcal{D}^h$. We also define $p^h(y, \mathcal{T}) = 1$ for $y \notin \mathcal{D}^h$. At points $y \in \mathcal{D}^h\backslash\{x_0\}$, $Q_n^h(y)$ satisfies the same recursion (5.1) as before, while $Q_n^h(x_0) = 1$ for all n. The advantage of working with the summed distribution is that it is monotonically nondecreasing in n for all y. Define $Q^h(y) = \lim_{n \to \infty} Q_n^h(y)$. As before, this limit should exist at all points in $\mathcal{D}^h$ because of the altered definition of the chain and since, apart from the case $d_{x,1} = d_{x,2} = 0$, transitions are uphill with respect to $V^h(\cdot)$. Due to the monotonicity, $Q^h(y)$ can be calculated using a Gauss–Seidel iteration similar to that of the reconstruction algorithm, where the ordering of the states is changed after each iteration. As before, the number of iterations necessary for convergence is expected to be the same as for the original reconstruction of $V^h(\cdot)$. Also, it is again necessary to modify the transition probabilities so as to reduce the probability of the process exiting a neighborhood of the set A_{x_0}. In this approach, local maxima in $z(\cdot)$ are identified by looking for a singular point near a local maximum of $\bar{P}^h(y)$. More precisely, parameters $\theta_1 > 0$ and $\theta_2 > 0$ are chosen and we identify a singular point as a local maximum if there is a local maximum of $Q^h(y)$ with value greater than θ_1 within θ_2 of the singular point.

The algorithm as outlined above still requires that at least one singular point be identified as a local maximum or minimum point. Since the height $z(\cdot)$ is ambiguous up to the addition of an overall constant, the height of this point is not needed to start the procedure. However, for any singular point x_0, it is often possible to determine a priori whether it is a local maximum, minimum, or saddle point, as we now discuss.

Let $I'(\cdot)$ be an arbitrary function in some region around x_0 such that $I'(x_0) = 1$ and $I'(y) \in (0, 1)$, $y \neq x_0$. Suppose $g(x_0) = z(x_0)$ and $g(x) = B$ for all $x \neq x_0$, where B is an upper bound for $z(\cdot)$ on $\mathcal{G}$. The corresponding height function $V(x)$, given by the control

representation of (2.4), will in general not be C^1. In the typical case, $V_x(\cdot)$ will have "line" discontinuities, and the $V^h(\cdot)$ reconstructed by our algorithm will approximately reproduce these discontinuities. For $I(\cdot)$, our experiments show in general that if $V^h(\cdot)$ is computed using a singular point x_0 under the incorrect assumption that it is a local minimum (or maximum), then the discrete "derivatives", e.g., $\Delta V_{x_1}^h(y) \equiv V^h(y+h(0,1)) - V^h(y)$, will display abrupt changes that can be interpreted as "discontinuities". These can be easily detected. We have used the occurrence of these "discontinuities" both to determine an initial local minimum point, and to check the assignments of singular points as local minima, maxima, or saddle points, as determined by our iterative procedure above.

We have applied this procedure to the surface of Figure 4, under the oblique lighting used previously. In order to reduce the effects of discretization, the surface was first scaled by a factor .5, so that the height range was approximately 25 units compared to a range for x_1 and x_2 of 128 units. Using no boundary data other than $I(\cdot)$, the surface shown in Figure 14 was reconstructed by this procedure. The reconstruction took four cycles: that is, a surface was first reconstructed using a single local minima, then again using the recovered local maxima, again using local minima, and the result of reconstructing a final time using the final recovered local maxima is displayed in the figure. The reconstruction took less than 30 seconds of CPU time on a DEC 5000 workstation.

Figure 15 illustrates the reconstruction error, the magnitude of the difference between the original surface height and that of the reconstruction. The reconstruction is good except near the edges of the image. This is due in part to the fact that, as in the previous subsection, the surface does not obey the condition A3.1 for the entire set $\mathcal{D}$ (or its analog for reconstruction based on local maxima). The average reconstruction error in the interior of the image with $20 \leq x_2 \leq 105$ is 0.5 units, or about 2% of the height range. Figures 16 and 17 show the surface and its reconstruction, respectively, over this region, illustrating the accuracy of the reconstruction there.

A second surface (illuminated as before) and its reconstruction are displayed in Figure 18 and 19, respectively. The reconstruction error is shown in Figure 20. Only three cycles, again starting from a local minimum singular point, were enough to give this reconstruction. The surface was also reconstructed starting from a different local minimum singular point, with comparably good results. The average height error in the interior of the image is 1 unit, in comparison to the overall height range for this surface of 44 units. As before, the accuracy of reconstruction is on the order of 2%.

Figure 21 depicts the summed probability distribution Q^h generated from Figure 18 after one cycle, together with the singular points in the image of this surface (shown as dark isolated clusters). Larger magnitudes for Q^h are indicated by increasing darkness. Q^h achieves its maximum at the central local minimum singular point used to generate this distribution. The Figure clearly shows the dominant evolution of the chain towards four other singular points which in fact correspond to local maxima of the height function. There are local maxima of Q^h near each of the four singular points, as indicated by the shaded patches near these points. A fifth point has a shaded patch nearby, but is not identified as a maximum since it does not satisfy the threshold rule.

We have also studied the effect of adding noise to the image of Figure 18. The noise

added had a uniform distribution on the interval $[-.1, .1]$, and was independent for different lattice points. Since the maximum image intensity is only $I = 1$, this is a large noise of $\pm 10\%$. The reconstruction based on the image with added noise is shown in Figure 22. The surface shown was generated using the computed local minima, and required three cycles. Although there are large errors in some parts of the image, the reconstruction is still good over much of the image. The error in the height is displayed in Figure 23, where saturated white represents a height error of 3. The error is less than 3 units over most of the image. In the region of the image with $127 > x_{1,2} > 40$, the mean height error is just 1.6. This represents a surprising immunity to the large image noise.

In Figure 24 the error is shown for a different reconstruction from the same noisy image, with the same scale as before. In this case, the surface was generated from the local maxima after just 2 cycles. As expected, near the boundary of the image, the region of accurate reconstruction for the maxima–based method is complementary to that of the minima–based one. Since the image boundary does not respect A3.1 (for either method), the maximum–based method does better at those points near the boundary where the steepest descent direction is outward, while the minima–based one does better where this direction is inward. Together, the two methods give reconstruction with error less than three units over most of the image.

6 Appendix

Proof of Proposition 2.2. It is sufficient to consider only the case $\gamma_2 \leq 0$. For convenience we recall the various functions of interest. We have

$$L^{(1)}(x, \beta) = \frac{1}{2} \frac{|\beta_1|^2}{I^2(x)} + \frac{1}{2} \frac{|\beta_2 + (1 - I^2(x))\gamma_2|^2}{v(x)} + \frac{1}{2}(1 - I^2(x)), \tag{6.1}$$

$$L^{(2)}(x, \beta) = \begin{cases} \gamma_3^2 - \gamma_2 \beta_2 - \gamma_3 \left(I^2(x) - |\beta_1|^2 - |\beta_2 + \gamma_2|^2\right)^{1/2} & \text{if } |\beta_1|^2 + |\beta_2 + \gamma_2|^2 \leq I^2(x) \\ \infty & \text{if } |\beta_1|^2 + |\beta_2 + \gamma_2|^2 > I^2(x), \end{cases} \tag{6.2}$$

$$\begin{aligned} H^{(1)}(x, \alpha) &= \sup_{\beta \in \mathbb{R}^2} \left[-\langle \alpha, \beta \rangle - L^{(1)}(x, \beta) \right] \\ &= \frac{1}{2} \left[I^2(x)\alpha_1{}^2 + v(x)\alpha_2{}^2 + 2(1 - I^2(x))\gamma_2 \alpha_2 - (1 - I^2(x)) \right]. \end{aligned} \tag{6.3}$$

if $v(x) = I^2(x) - \gamma_2^2 \geq 0$,

$$\begin{aligned} H^{(1)}(x, \alpha) &= \inf_{\beta_2} \sup_{\beta_1} \left[-\langle \alpha, \beta \rangle - L^{(1)}(x, \beta) \right] \\ &= \frac{1}{2} \left[I^2(x)\alpha_1{}^2 + v(x)\alpha_2{}^2 + 2(1 - I^2(x))\gamma_2 \alpha_2 - (1 - I^2(x)) \right]. \end{aligned} \tag{6.4}$$

if $v(x) < 0$, and finally

$$\begin{aligned} H^{(2)}(x, \alpha) &= \sup_{\beta \in \mathbb{R}^2} \left[-\langle \alpha, \beta \rangle - L^{(2)}(x, \beta) \right] \\ &= I(x) \left(1 + \|\alpha\|^2 - 2\alpha_2 \gamma_2\right)^{1/2} + \alpha_2 \gamma_2 - 1. \end{aligned} \tag{6.5}$$

We also recall that the functions $H^{(1)}$ and $H^{(2)}$ arise as two ways of rewriting the image irradiance equation (2.13). The key to establishing the equivalence of the fixed points will be to relate the discrete equations that characterize a fixed point back to these functions. For points x where $v(x) > 0$, it is easy to see that the convexity of both $H^{(1)}$ and $H^{(2)}$ in α imply

$$H^{(1)}(x, \alpha) \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0 \iff H^{(2)}(x, \alpha) \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0. \tag{6.6}$$

For the points where $v(x) \le 0$, the set $\{\alpha : H^{(1)}(x, \alpha) = 0\}$ takes the form of a hyperbola with two branches: one that opens in the positive α_2 direction and one that opens in the negative α_2 direction. For the case $\gamma_2 \le 0$ considered here, the image irradiance equation must be satisfied with $f_x(x)$ taking a value in the branch of the hyperbola that opens in the positive α_2 direction. This is because a value in the branch that opens in the negative α_2 direction corresponds to a surface normal that points away from the camera direction, which is impossible if $f(\cdot)$ is a function. Even more to the point is that the positive branch coincides precisely with the zeros of $H^{(2)}(x, \cdot)$.

A consequence of the convex duality formula used in the demonstration of (6.15) below is that

$$\begin{aligned} \tilde{H}^{(1)}(x, \alpha) &\equiv \inf_{\beta_2 \ge 0} \sup_{\beta_1} \left[-\langle \alpha, \beta \rangle - L^{(1)}(x, \beta) \right] \\ &= \sup_{\alpha_2^* \ge 0} H^{(1)}(x, \alpha + (0, \alpha_2^*)). \end{aligned} \tag{6.7}$$

The function $\tilde{H}^{(1)}(x, \alpha)$ is automatically nonincreasing in α_2 for each fixed α_1. Note that the zeros of the function $\tilde{H}^{(1)}(x, \alpha)$ correspond exactly to the branch of the hyperbola that opens in the positive α_2 direction. It follows that

$$\tilde{H}^{(1)}(x, \alpha) \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0 \iff H^{(2)}(x, \alpha) \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0. \tag{6.8}$$

(It would indeed be possible to construct the algorithm of Control Problem 1 directly from the equation $\tilde{H}^{(1)}(x, f_x(x)) = 0$, which by (6.8) is equivalent to the image irradiance equation.)

We now consider the equation for a fixed point of (2.19) with running cost $L^{(1)}(x, \beta)$, and that for a fixed point of (2.11) with running cost $L^{(2)}(x, \beta)$. By the statement "(2.19) [or (2.11)] holds with $w(\cdot)$ at x" we mean that (2.19) [or (2.11)] holds at x with V_n^h and V_{n+1}^h replaced by w. It will be convenient to introduce some new notation. Define $Q_1 = \{x : x_1 \ge 0, x_2 \ge 0\}$, $Q_2 = \{x : x_1 \le 0, x_2 \ge 0\}$, $Q_3 = \{x : x_1 \le 0, x_2 \le 0\}$ and $Q_4 = \{x : x_1 \ge 0, x_2 \le 0\}$. We set $w_1(x) = (w(x + h(1, 0)) - w(x), w(x + h(0, 1)) - w(x))$, $w_2(x) = (w(x + h(-1, 0)) - w(x), w(x + h(0, 1)) - w(x))$, and define $w_3(x)$ and $w_4(x)$ analogously.

First assume $w(x) < g(x)$. If we insert the transition probabilities and interpolation interval of Example 2.1, (2.11) implies

$$\left(\min_{i=1,2,3,4} \left[\inf_{u \in Q_i \setminus \{0\}} \frac{L^{(2)}(x, u) + \langle u, w_i(x) \rangle}{|u_1| + |u_2|} \right] \right) \wedge \left(L^{(2)}(x, 0) \right) = 0. \tag{6.9}$$

Recall that $L^{(2)}(x,\beta) \geq 0$, and that $L^{(2)}(x,\beta) = 0$ if and only if $(x,u) \in \mathcal{S} \times \{0\}$. Thus for $x \in \mathcal{S}$ we obtain

$$\min_{i=1,2,3,4} \left[\inf_{u \in Q_i \backslash \{0\}} \frac{L^{(2)}(x,u) + \langle u, w_i(x) \rangle}{|u_1| + |u_2|} \right] \geq 0, \tag{6.10}$$

while for $x \notin \mathcal{S}$ we have

$$\min_{i=1,2,3,4} \left[\inf_{u \in Q_i \backslash \{0\}} \frac{L^{(2)}(x,u) + \langle u, w_i(x) \rangle}{|u_1| + |u_2|} \right] = 0, \tag{6.11}$$

Next consider the case $w(x) = g(x)$. For this case we deduce that (6.10) must hold.

Consider any $x \in \mathcal{S}$. Then for each $i = 1,2,3,4$

$$\inf_{u \in Q_i \backslash \{0\}} \frac{L^{(2)}(x,u) + \langle u, w_i(x) \rangle}{|u_1| + |u_2|} \geq 0 \iff \inf_{u \in Q_i \backslash \{0\}} \left[L^{(2)}(x,u) + \langle u, w_i(x) \rangle \right] \geq 0. \tag{6.12}$$

Now consider any point $x \notin \mathcal{S}$. Owing to the strictly positive lower bound on $L^{(2)}(x,\cdot)$ for such points, for each $i = 1,2,3,4$

$$\inf_{u \in Q_i \backslash \{0\}} \frac{L^{(2)}(x,u) + \langle u, w_i(x) \rangle}{|u_1| + |u_2|} \left\{ \begin{matrix} > \\ = \end{matrix} \right\} 0 \iff \inf_{u \in Q_i \backslash \{0\}} \left[L^{(2)}(x,u) + \langle u, w_i(x) \rangle \right] \left\{ \begin{matrix} > \\ = \end{matrix} \right\} 0. \tag{6.13}$$

It will be demonstrated below that (6.6), (6.8), and a convex duality formula imply

$$\inf_{u \in Q_i \backslash \{0\}} \left[L^{(2)}(x,u) + \langle u, w_i(x) \rangle \right] \left\{ \begin{matrix} > \\ = \end{matrix} \right\} 0 \iff \inf_{u \in Q_i \backslash \{0\}} \left[L^{(1)}(x,u) + \langle u, w_i(x) \rangle \right] \left\{ \begin{matrix} > \\ = \end{matrix} \right\} 0 \tag{6.14}$$

for $i = 1,2,3,4$ when $v(x) > 0$ and

$$\inf_{u \in Q_i \backslash \{0\}} \left[L^{(2)}(x,u) + \langle u, w_i(x) \rangle \right] \left\{ \begin{matrix} > \\ = \end{matrix} \right\} 0 \iff \sup_{u_2 > 0} \inf_{(-1)^{i+1} u_1 \geq 0} \left[L^{(1)}(x,u) + \langle u, w_i(x) \rangle \right] \left\{ \begin{matrix} > \\ = \end{matrix} \right\} 0 \tag{6.15}$$

for $i = 1,2$ when $v(x) \leq 0$. Postponing temporarily the proofs of (6.14) and (6.15), we now complete the proof of the equivalence of the fixed points. It will be convenient to separate the cases $\mathcal{A}_1 = \{x : x \in \mathcal{S}\}$, $\mathcal{A}_2 = \{x : x \notin \mathcal{S}, w(x) < g(x), v(x) > 0\}$, $\mathcal{A}_3 = \{x : x \notin \mathcal{S}, w(x) < g(x), v(x) \leq 0\}$, $\mathcal{A}_4 = \{x : x \notin \mathcal{S}, w(x) = g(x), v(x) > 0\}$ and $\mathcal{A}_5 = \{x : x \notin \mathcal{S}, w(x) = g(x), v(x) \leq 0\}$.

Assume (2.19) holds with $w(\cdot)$ at x. Note that (6.12) holds for $L^{(1)}$ as well as $L^{(2)}$. For $x \in \mathcal{A}_1$ it is always true that (6.10) holds, and by (6.12) with $L^{(2)}$ replaced by $L^{(1)}$ and (6.14) equation (6.10) holds with $L^{(2)}$ replaced by $L^{(1)}$. Since $L^{(2)}(x,0) = L^{(1)}(x,0) = 0$, equation (2.19) is satisfied for $x \in \mathcal{A}_1$. Next consider $x \in \mathcal{A}_2$. In this case (6.11) holds. Since $L^{(1)}(x,\cdot)$ has a strictly positive lower bound for $x \in \mathcal{A}_2$, equation (6.13) holds with $L^{(2)}$ replaced by $L^{(1)}$. If we now use (6.13) with $L^{(2)}$ replaced by $L^{(1)}$ and (6.14) we find that (6.11) holds with $L^{(2)}$ replaced by $L^{(1)}$. Since $L^{(1)}(x,0) > 0$ for $x \in \mathcal{A}_2$, (2.19) holds for $x \in \mathcal{A}_2$.

41

Next consider $x \in \mathcal{A}_3$. The analogue of (6.12) that is appropriate for $L^{(1)}$ is

$$\sup_{u_2 > 0} \inf_{(-1)^{i+1}u_1 \geq 0} \frac{L^{(1)}(x,u) + \langle u, w_i(x) \rangle}{|u_1| + |u_2|} \left\{ \begin{array}{c} > \\ = \end{array} \right\} 0 \iff \sup_{u_2 > 0} \inf_{(-1)^{i+1}u_1 \geq 0} \left[L^{(1)}(x,u) + \langle u, w_i(x) \rangle \right] \left\{ \begin{array}{c} > \\ = \end{array} \right\} 0$$

(6.16)

for $i = 1, 2$. This equation is most easily verified by showing that strict positivity (respectively strict negativity) of the left hand side implies strict positivity (respectively strict negativity) of the right hand side, and conversely. For $x \in \mathcal{A}_3$ we have (6.11) for $i = 1, 2$, which by (6.15) and (6.16) implies

$$\sup_{u_2 > 0} \inf_{(-1)^{i+1}u_1 \geq 0} \frac{L^{(1)}(x,u) + \langle u, w_i(x) \rangle}{|u_1| + |u_2|} = 0$$

(6.17)

for $i = 1, 2$. Thus (2.19) holds for $x \in \mathcal{A}_3$.

The proofs for $x \in \mathcal{A}_4$ and $x \in \mathcal{A}_5$ are very similar to those above and omitted. Thus we have proved that any fixed point of (2.11) is also a fixed point of (2.19). The proof of the reverse implication is essentially the same, and the sake of brevity we give details only for the case $x \in \mathcal{A}_3$. For such points (2.19) implies (6.17) for $i = 1, 2$. Tracing back through (6.15) and (6.16) we see that (6.11) holds for $i = 1, 2$. Since for $x \in \mathcal{A}_3$ $L^{(2)}(x,u) = \infty$ for $u \in Q_3 \cup Q_4$, this implies (2.11) at the point x.

Thus all that remains are the proofs of (6.14) and (6.15). Consider first (6.14) and the case $i = 1$. All other cases of i may be treated similarly. Owing to continuity of the bracketed quantity in the equation below,

$$\inf_{u \in Q_1 \setminus \{0\}} \left[L^{(j)}(x,u) + \langle u, w_i(x) \rangle \right] = \inf_{u \in Q_1} \left[L^{(j)}(x,u) + \langle u, w_i(x) \rangle \right]$$

(6.18)

for $j = 1, 2$. Thus we can consider the infima over Q_1. Define

$$I_{Q_1}(\beta) = \left\{ \begin{array}{ll} 0 & \beta \in Q_1 \\ +\infty & x \notin Q_1. \end{array} \right.$$

We can then write

$$\inf_{u \in Q_1} \left[L^{(j)}(x,u) + \langle u, w_i(x) \rangle \right] = \inf_{u \in \mathbb{R}^2} \left[\left(L^{(j)}(x,u) + I_{Q_1}(u) \right) + \langle u, w_i(x) \rangle \right]$$

for $j = 1, 2$. Define the Legendre transforms

$$L^{(j),*}(x,\alpha) = \inf_{u \in \mathbb{R}^2} \left[L^{(j)}(x,u) + \langle u, \alpha \rangle \right],$$

$$I_{Q_1}^*(\alpha) = \inf_{u \in \mathbb{R}^2} \left[I_{Q_1}(u) + \langle u, \alpha \rangle \right].$$

Note that the Legendre transform considered here differs by a sign convention from the one used previously. The relationship between the two transforms is simply

$$H^{(j)}(x,\alpha) = -L^{(j),*}(x,\alpha)$$

(6.19)

for $j = 1, 2$. One can easily calculate

$$I_{Q_1}^*(\alpha) = \begin{cases} -\infty & \alpha_1 < 0 \text{ or } \alpha_2 < 0 \\ 0 & \text{else.} \end{cases}$$

For x such that $v(x) > 0$, the domains of finiteness of $L^{(j)}(x, \cdot)$ and $I_{Q_1}(\cdot)$ have nonempty intersection for $j = 1, 2$. We can therefore apply the convex duality formula for the Legendre transform of a sum [26, Theorem 16.4] to obtain

$$\inf_{u \in Q_1} \left[L^{(j)}(x, u) + \langle u, \alpha \rangle \right] = \sup_{\alpha^*} \left[L^{(j),*}(x, \alpha - \alpha^*) + I_{Q_1}^*(\alpha^*) \right]$$

$$= \sup_{\alpha_1^* \geq 0, \alpha_2^* \geq 0} L^{(j),*}(x, \alpha - \alpha^*).$$

The last equation, (6.6), (6.18) and (6.19) then imply (6.14).

The proof of (6.15) is very similar. We consider only $i = 1$, since $i = 2$ is treated in the same way. An application of the same convex duality formula gives

$$\inf_{u \in Q_1 \setminus \{0\}} \left[L^{(2)}(x, u) + \langle u, \alpha \rangle \right] = \sup_{\alpha_1^* \geq 0, \alpha_2^* \geq 0} -H^{(2)}(x, \alpha - \alpha^*)$$

$$= -\inf_{\alpha_1^* \geq 0} H^{(2)}(x, (\alpha_1 - \alpha_1^*, \alpha_2)),$$

where the last equality follows from the fact that $H^{(2)}(x, \alpha)$ is nonincreasing in α_2 for each fixed α_1. A second application gives

$$\sup_{u_2 > 0} \inf_{(-1)^{i+1} u_1 \geq 0} \left[L^{(1)}(x, u) + \langle u, \alpha \rangle \right] = -\sup_{\alpha_2^* \geq 0} \inf_{\alpha_1^* \geq 0} H^{(1)}(x, (\alpha_1 - \alpha_1^*, \alpha_2 + \alpha_2^*))$$

$$= -\inf_{\alpha_1^* \geq 0} \tilde{H}^{(1)}(x, (\alpha_1 - \alpha_1^*, \alpha_2)).$$

Thus (6.15) follows from (6.8), which completes the proof. ∎

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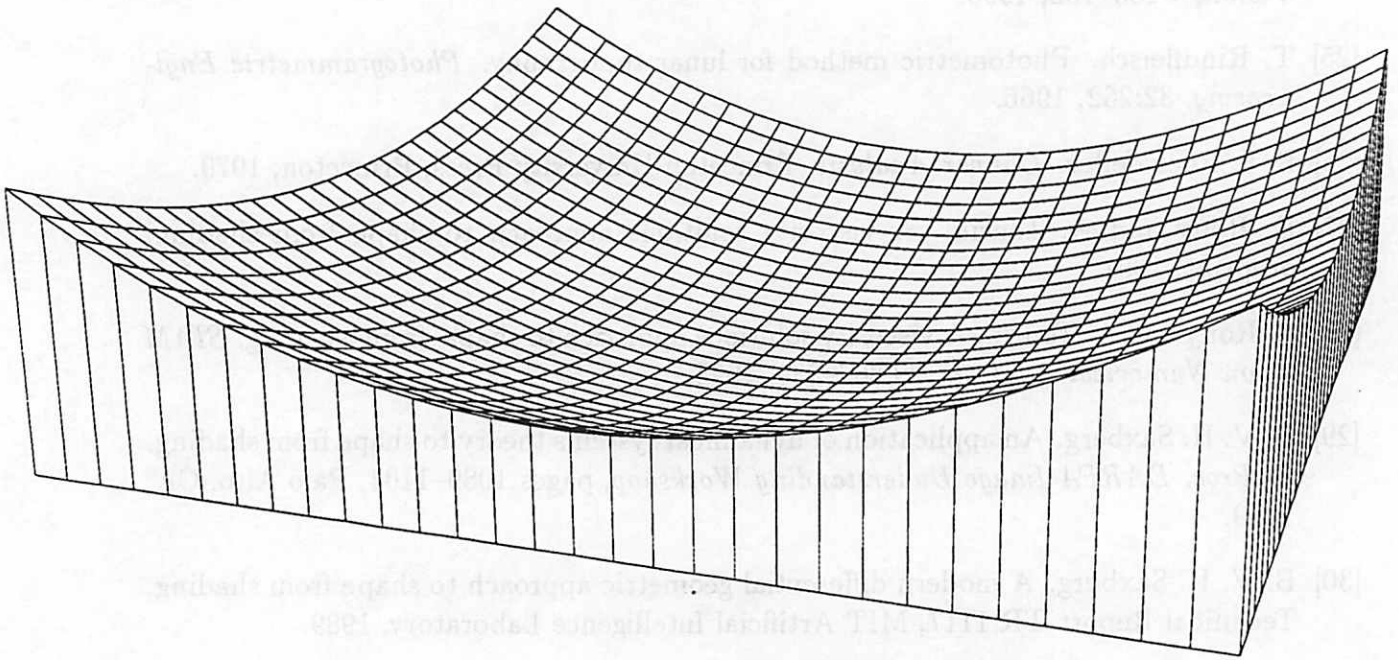
Fig. 1 Parabolic surface.
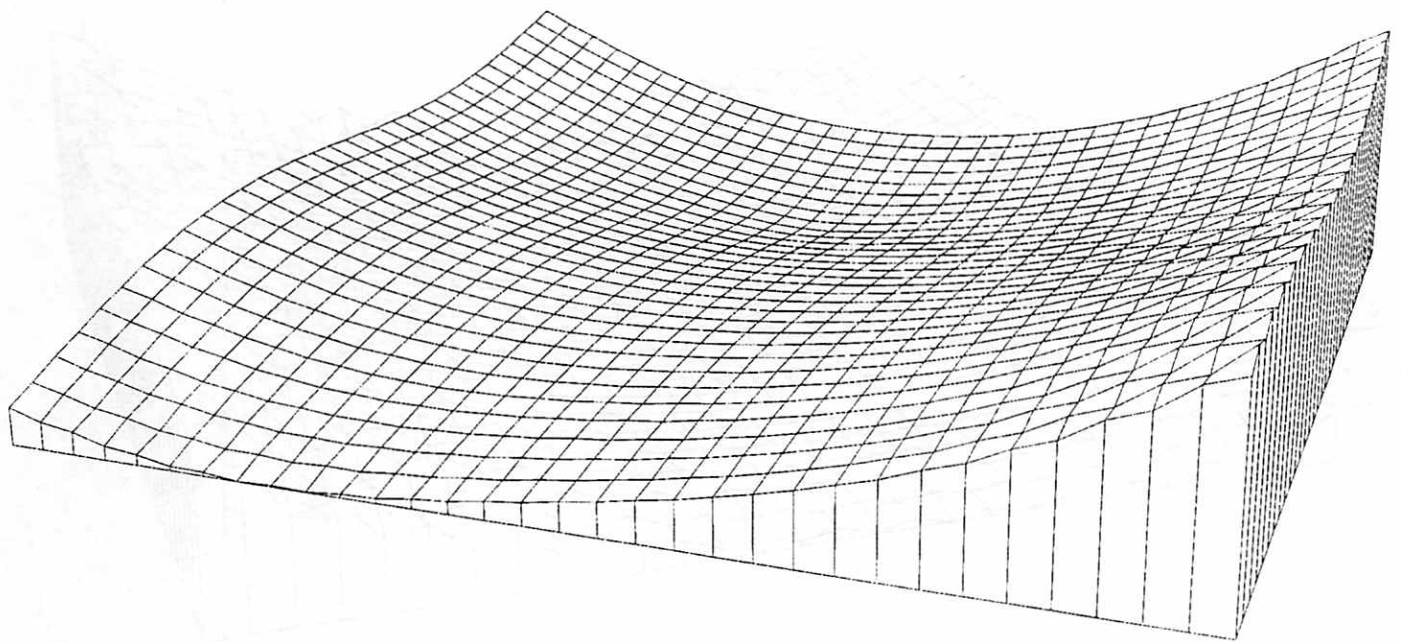
Fig. 2 Horn's algorithm: 128 iterations.
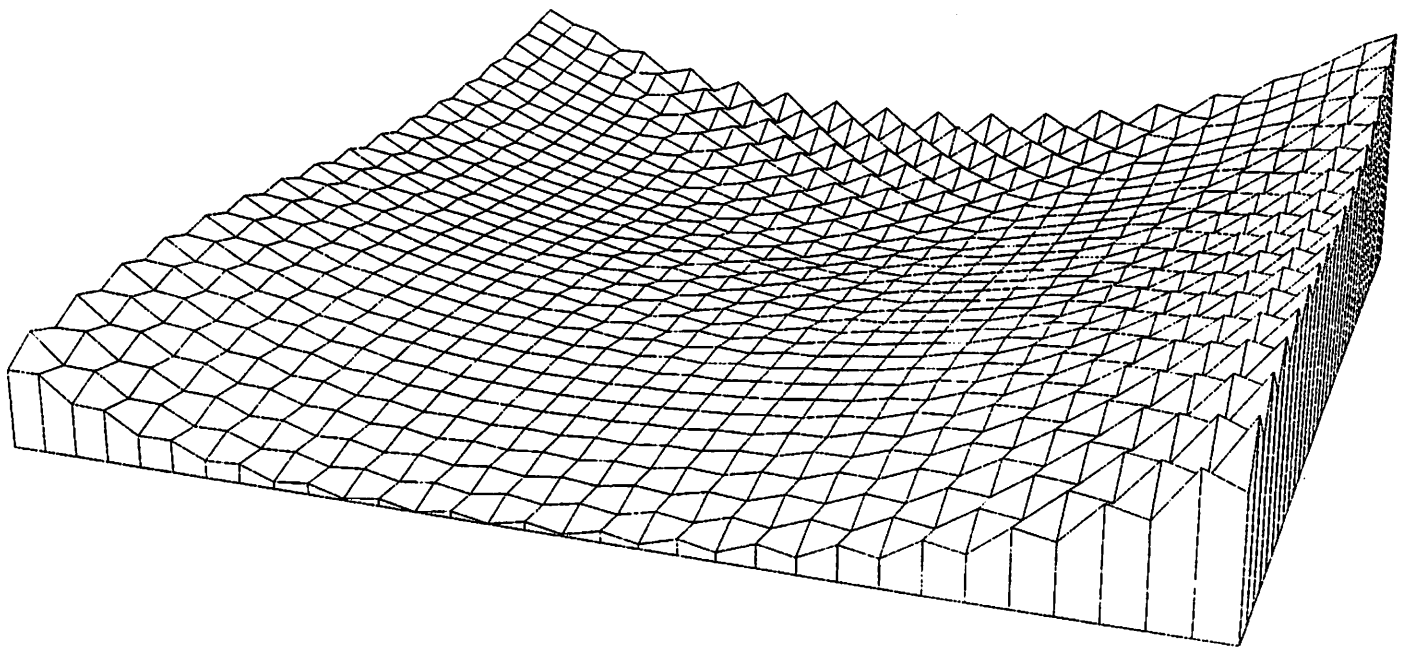
Fig. 3 Horn's algorithm: 3072 iterations.
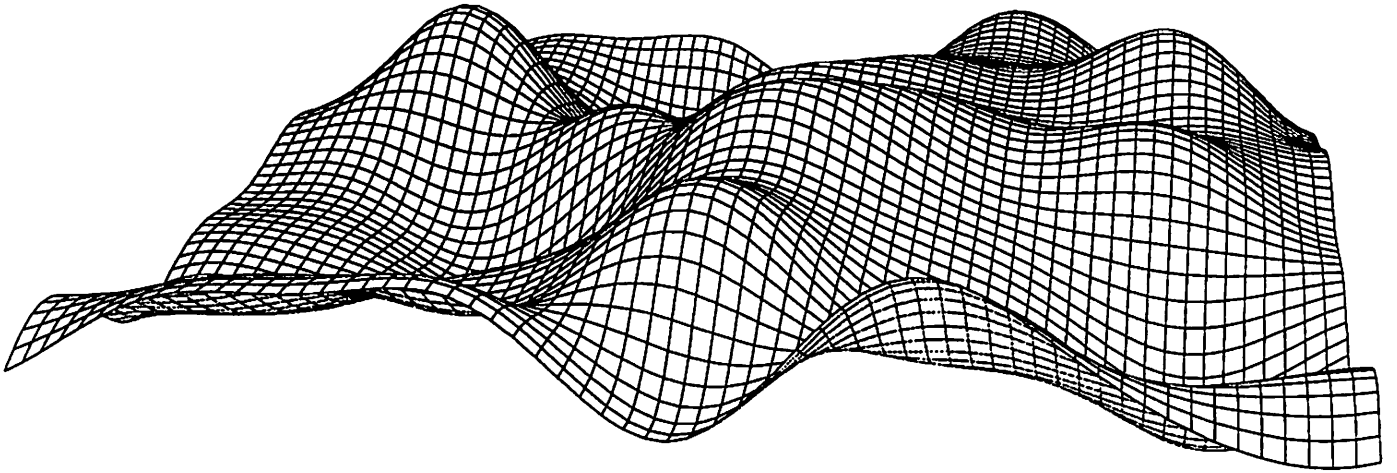
Fig. 4   Complex surface.
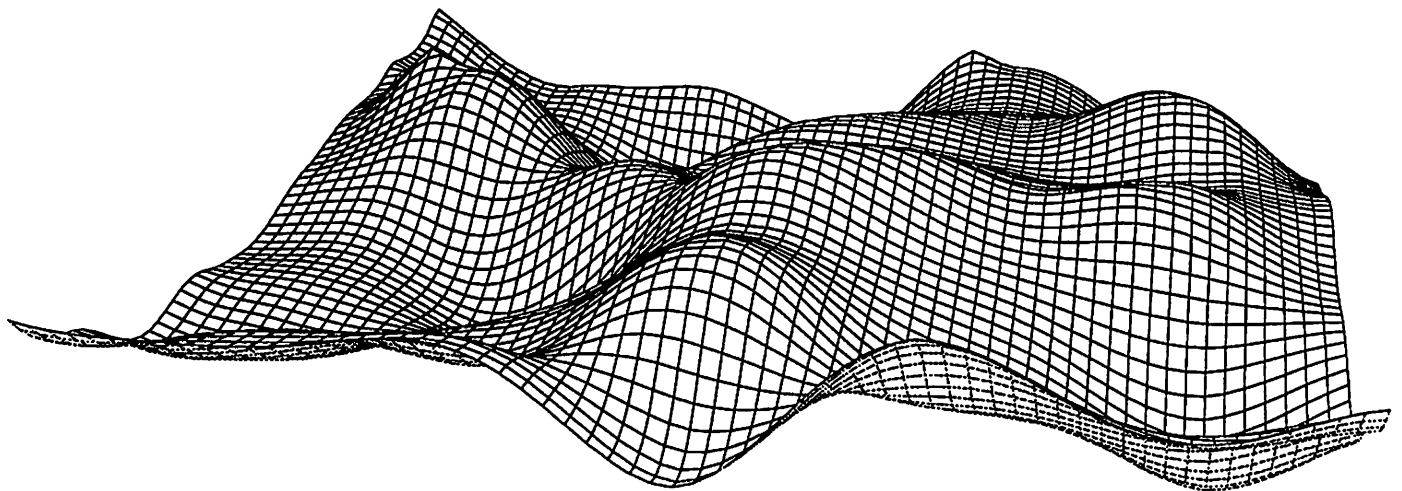
Fig. 5 Reconstruction.
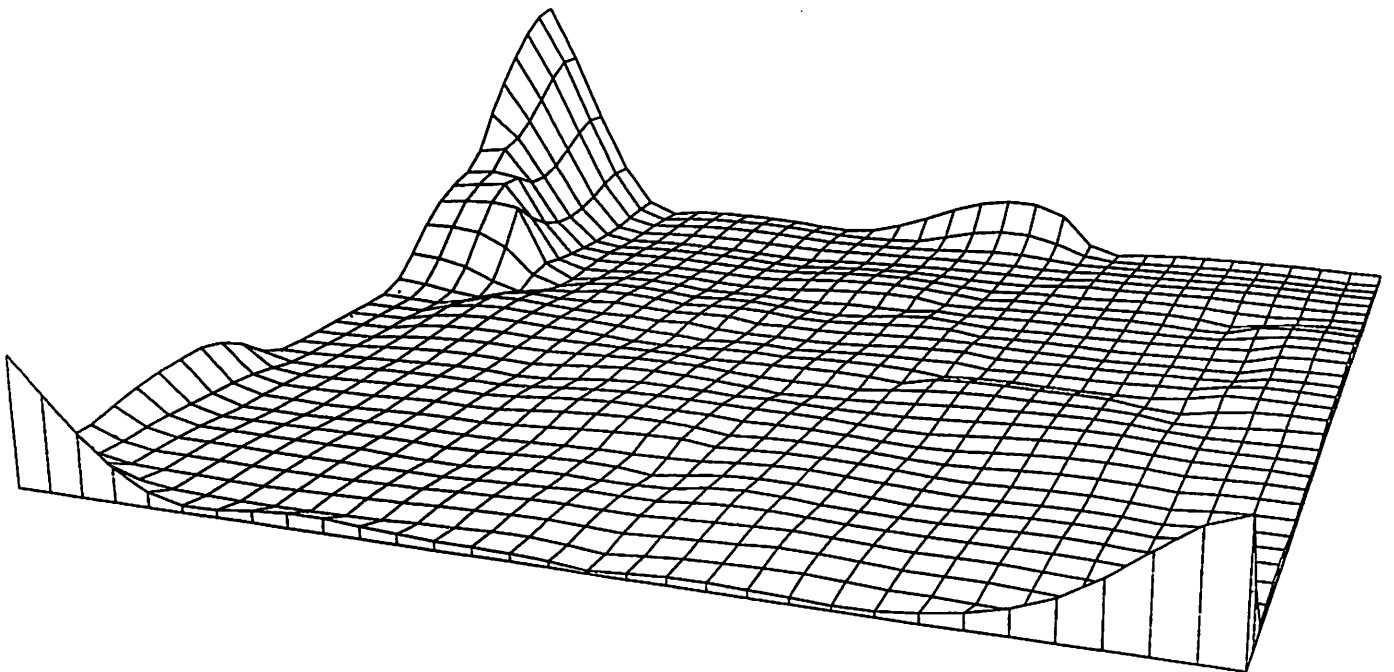
Fig. 6 Difference Surface.
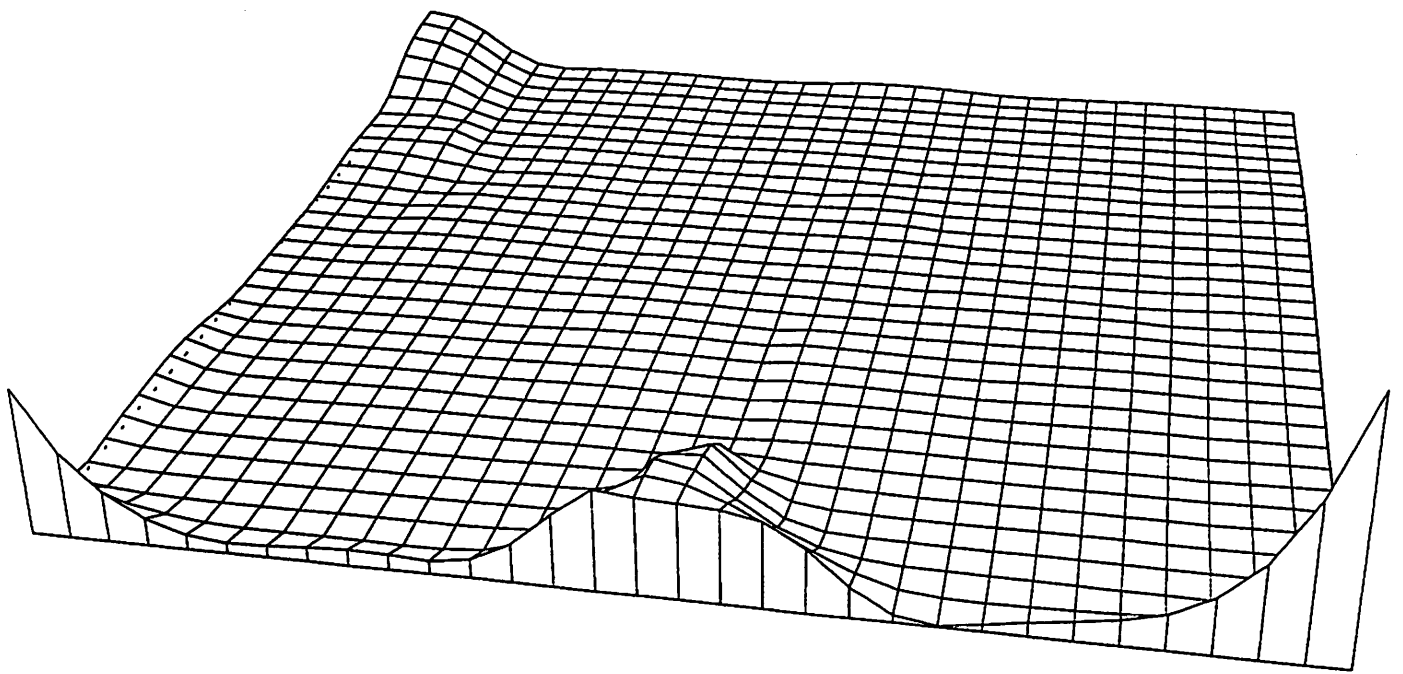
Fig. 7   Difference Surface.
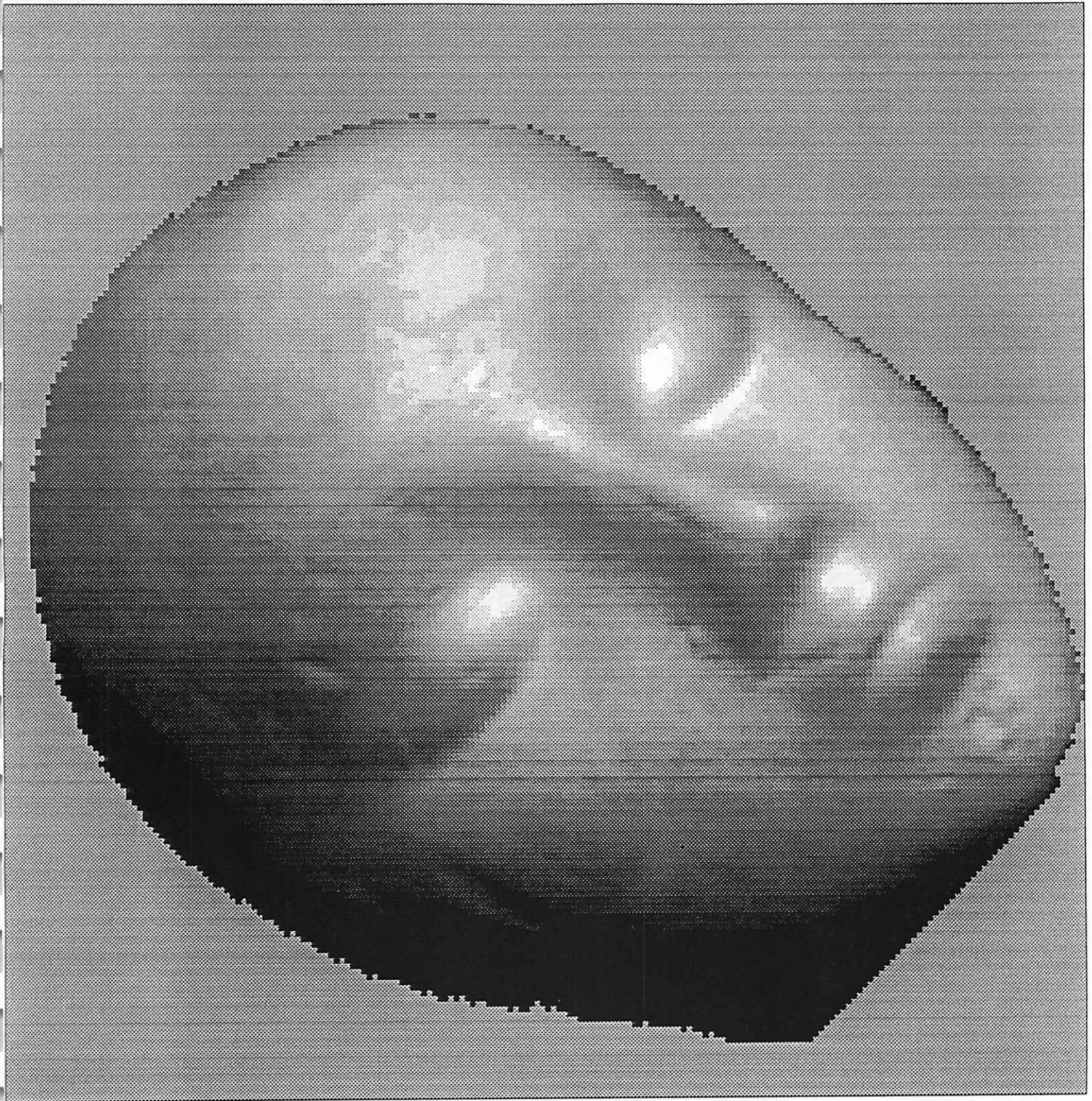
Fig. 8 Mannequin image.
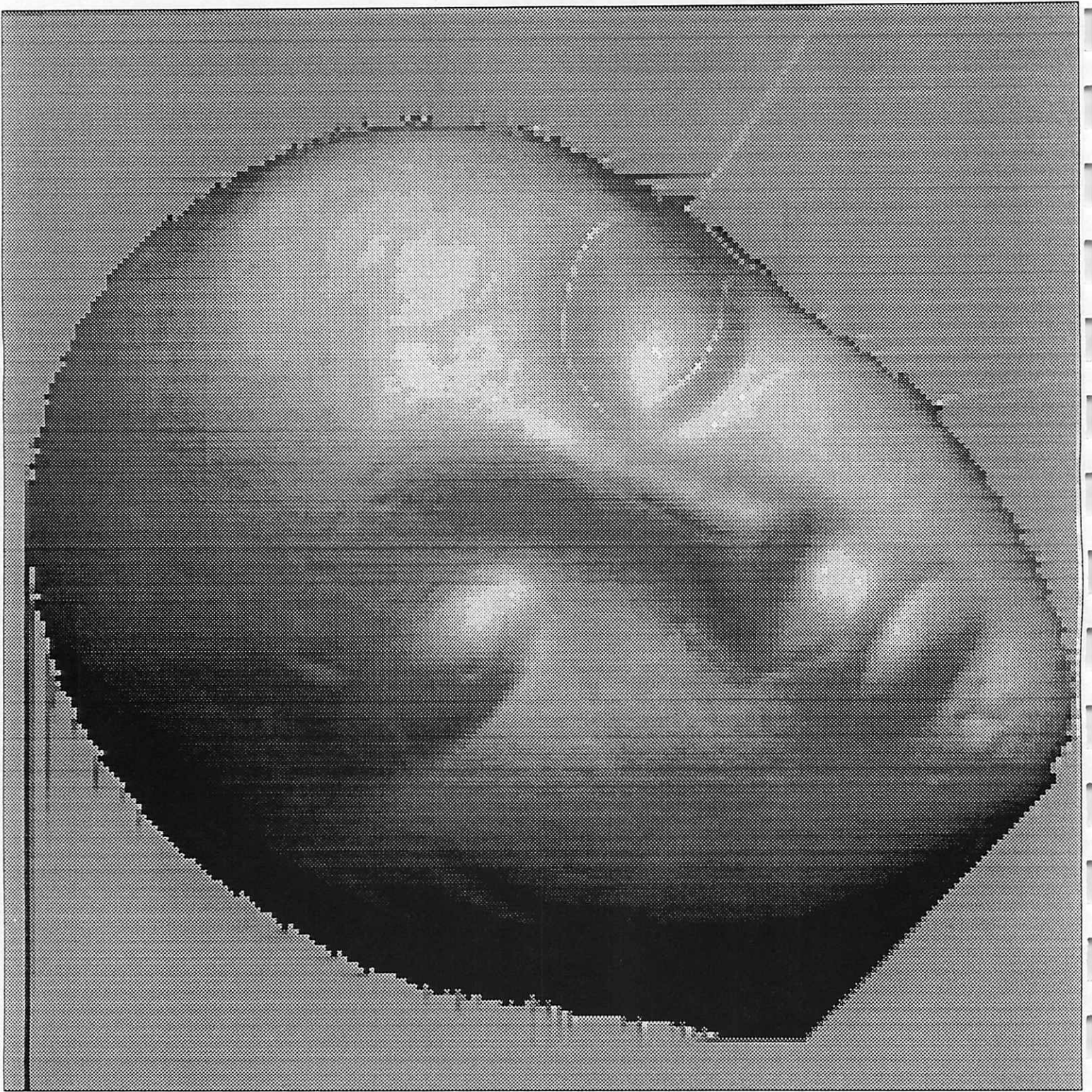
Fig. 9 Reconstruction lighted from above.

Fig. 10 Reconstruction lighted from below.
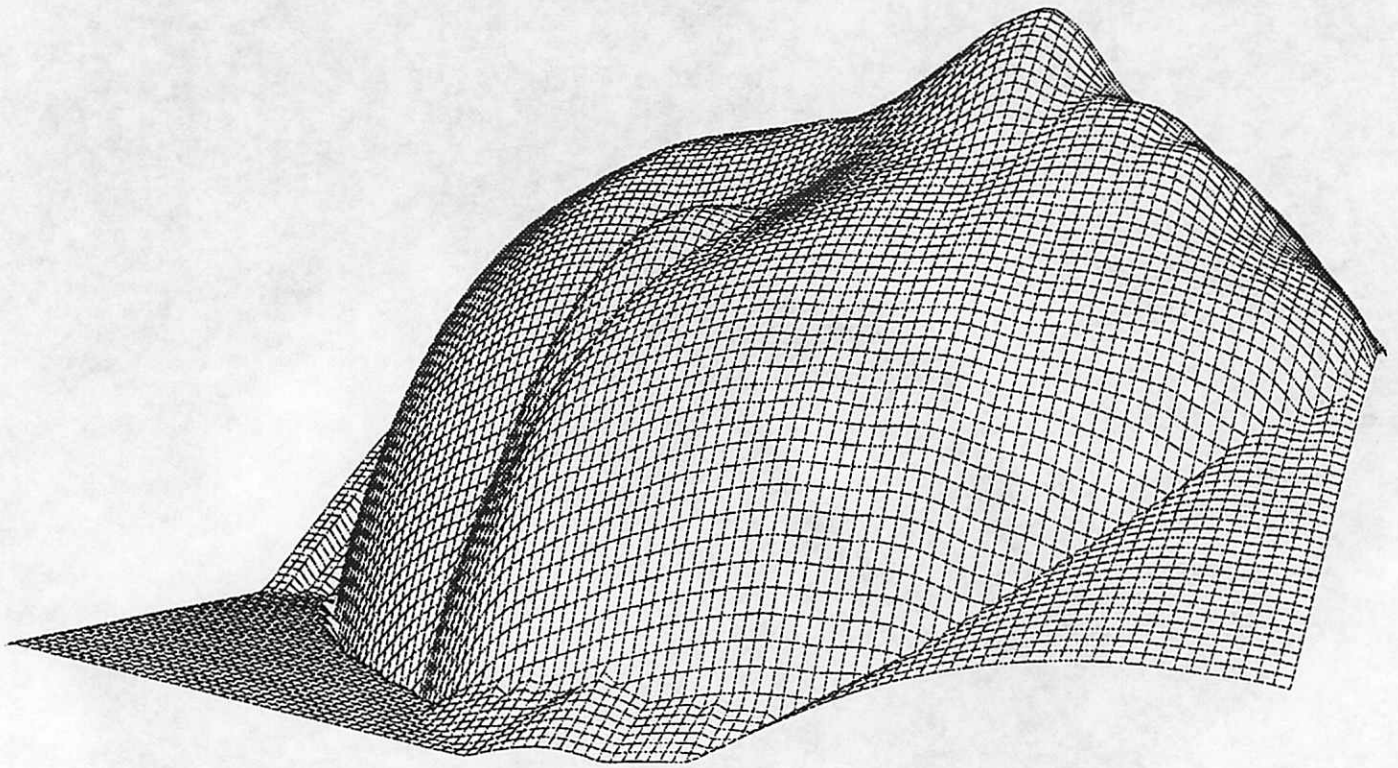
Fig. 11 Surface reconstruction.

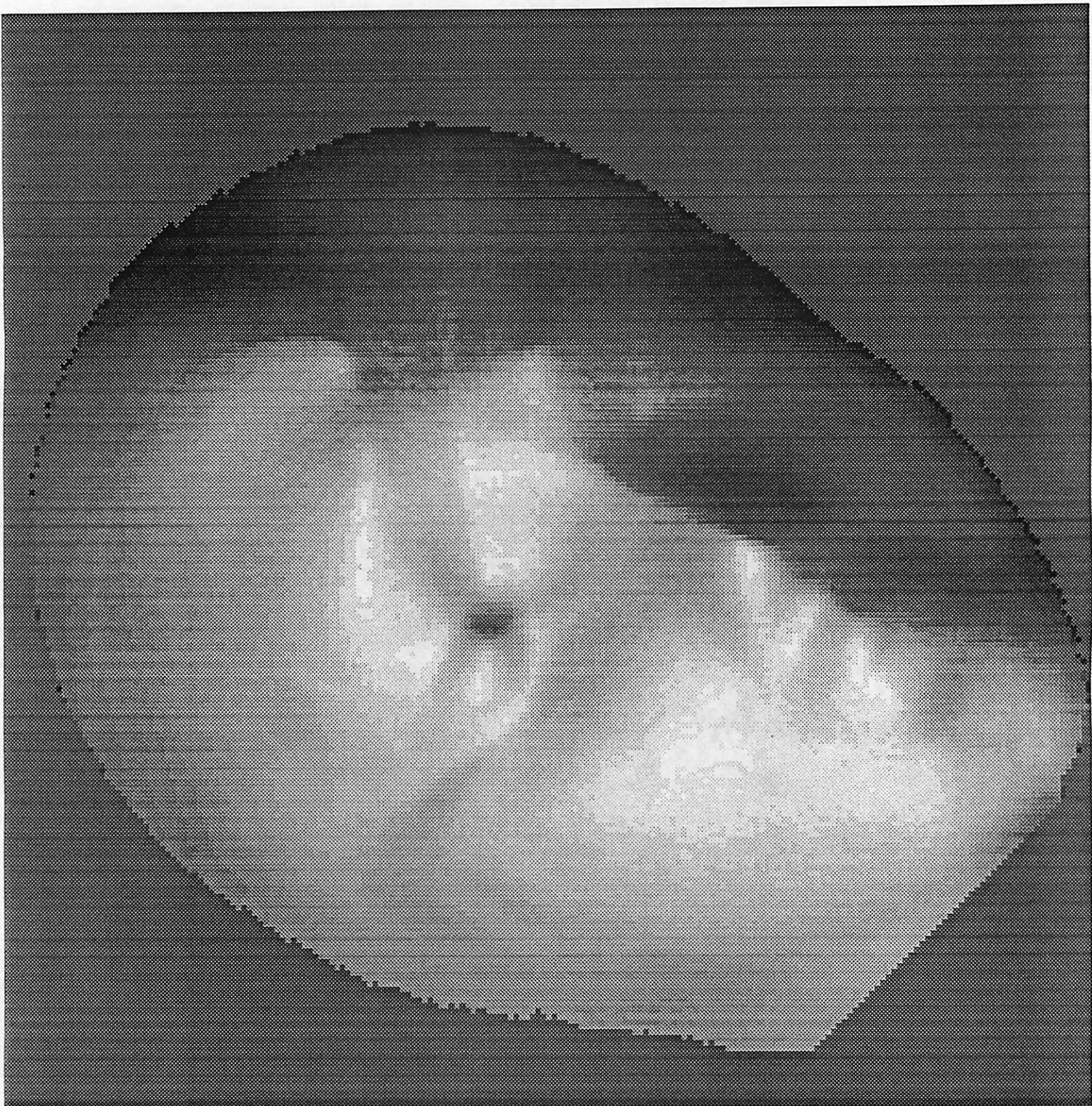Fig. 12 Reconstruction lighted from above.
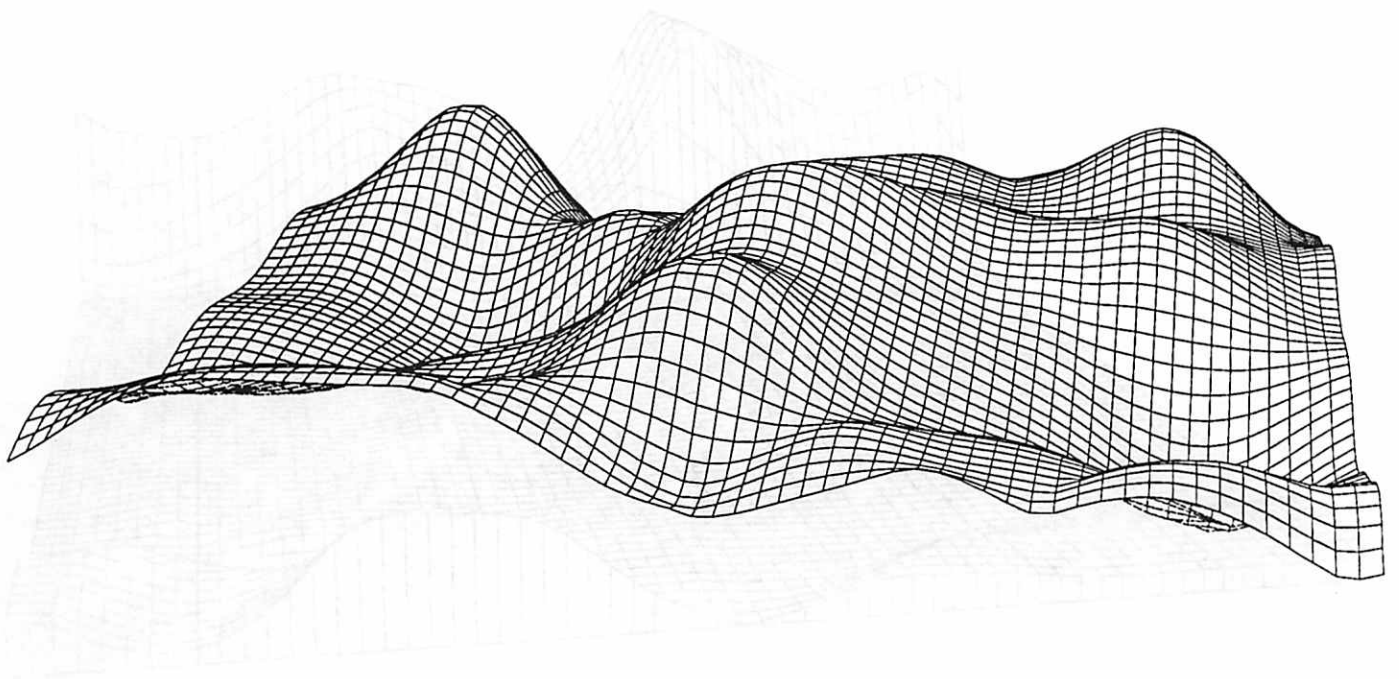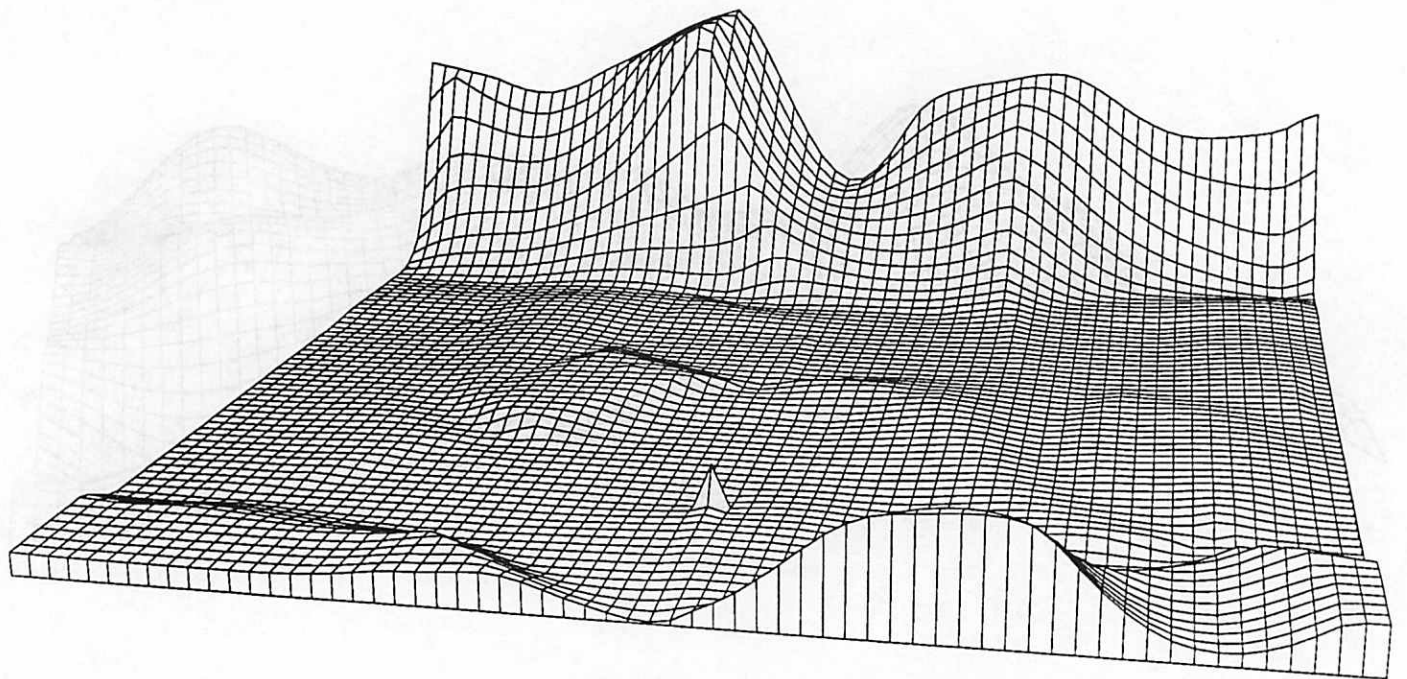
Fig. 13 Reconstruction lighted from below.

Figure 14 Reconstruction.

Figure 15 Difference surface.
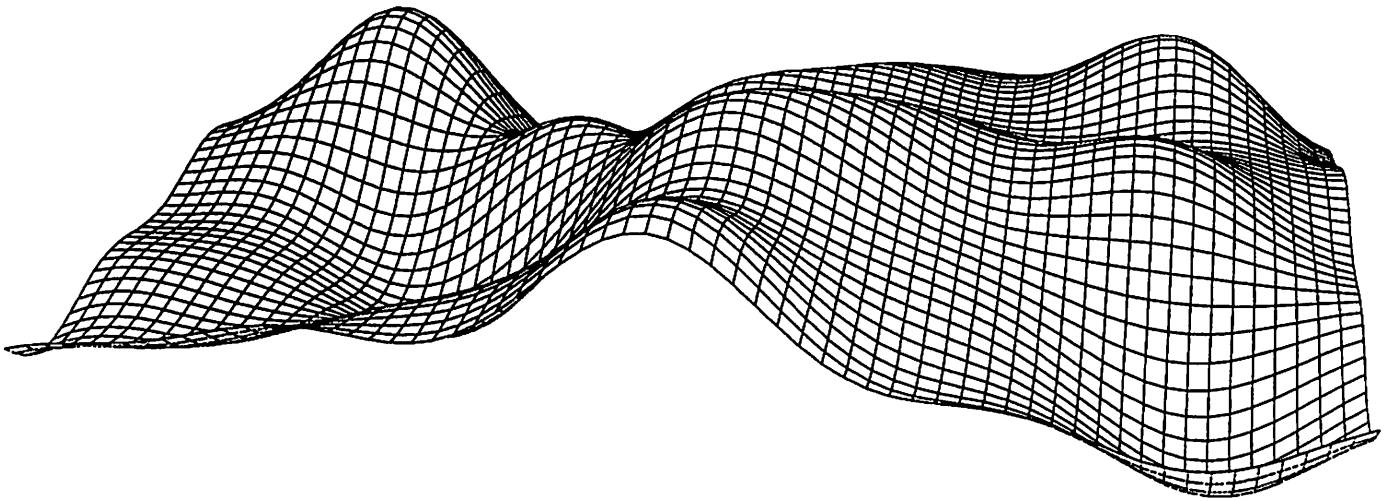
Figure 16 Surface section.
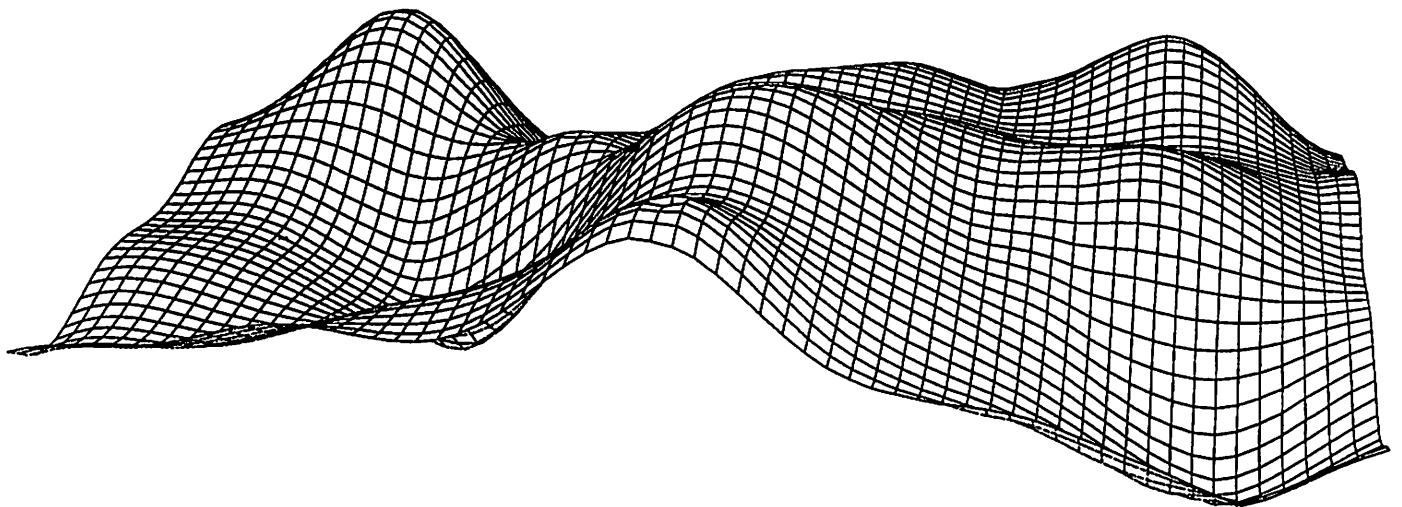
Figure 17 Reconstructed surface section.
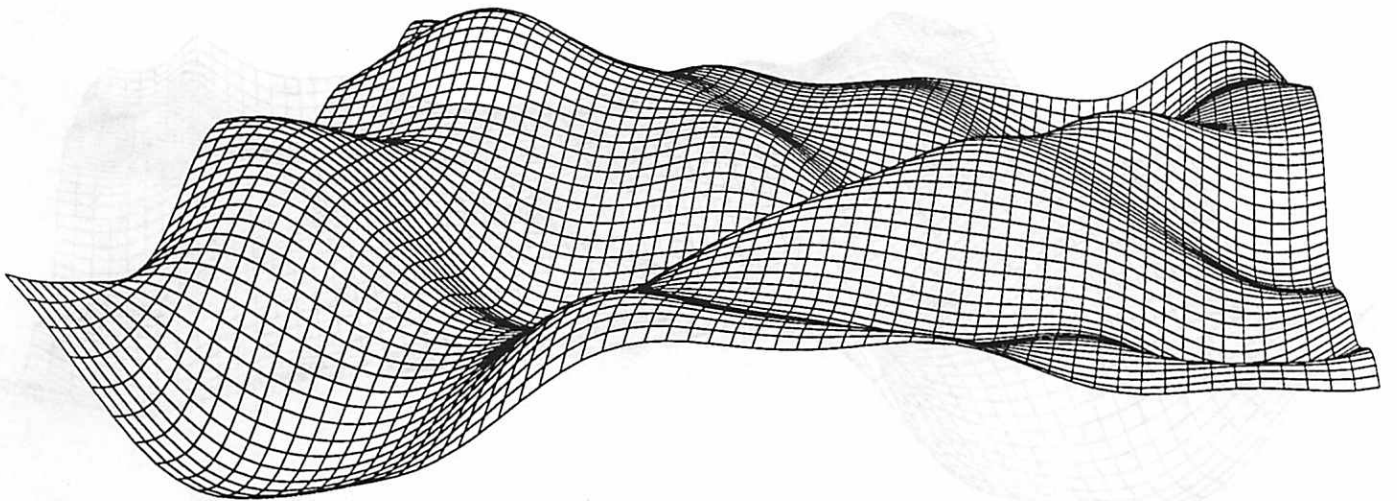
Figure 18 Second surface.
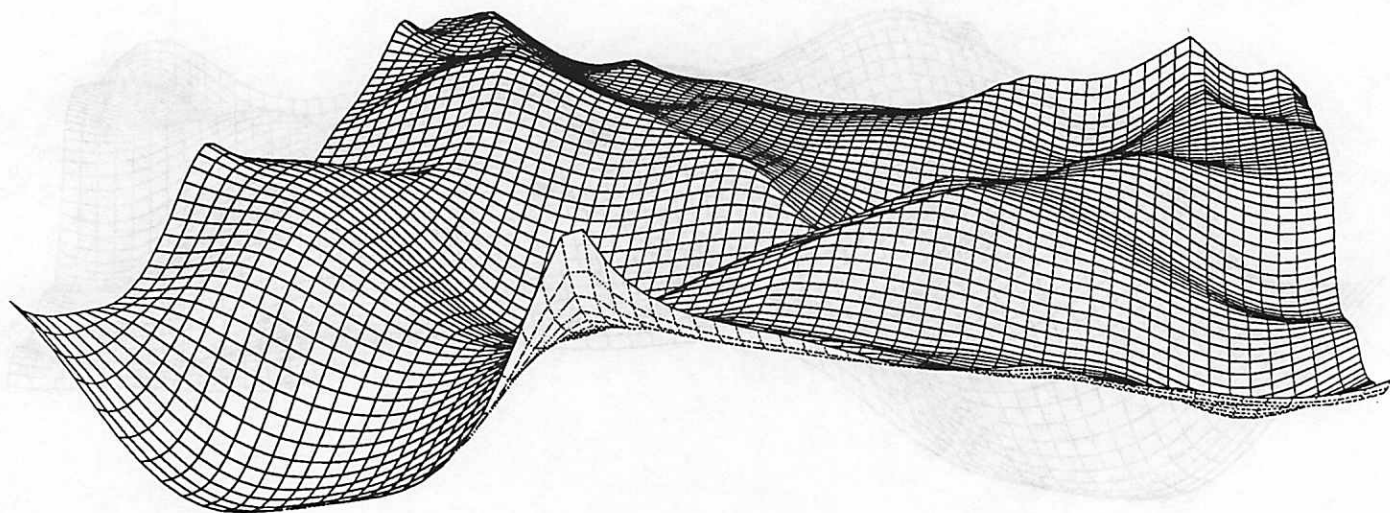
Figure 19 Reconstructed second surface.
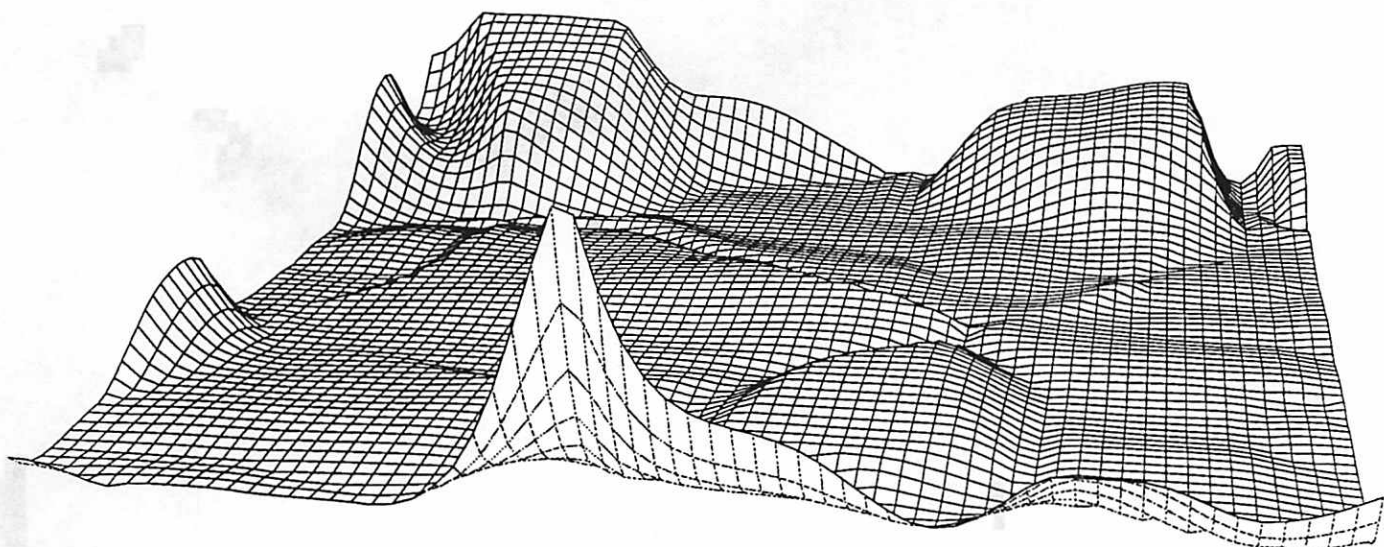
Figure 20 Error surface.
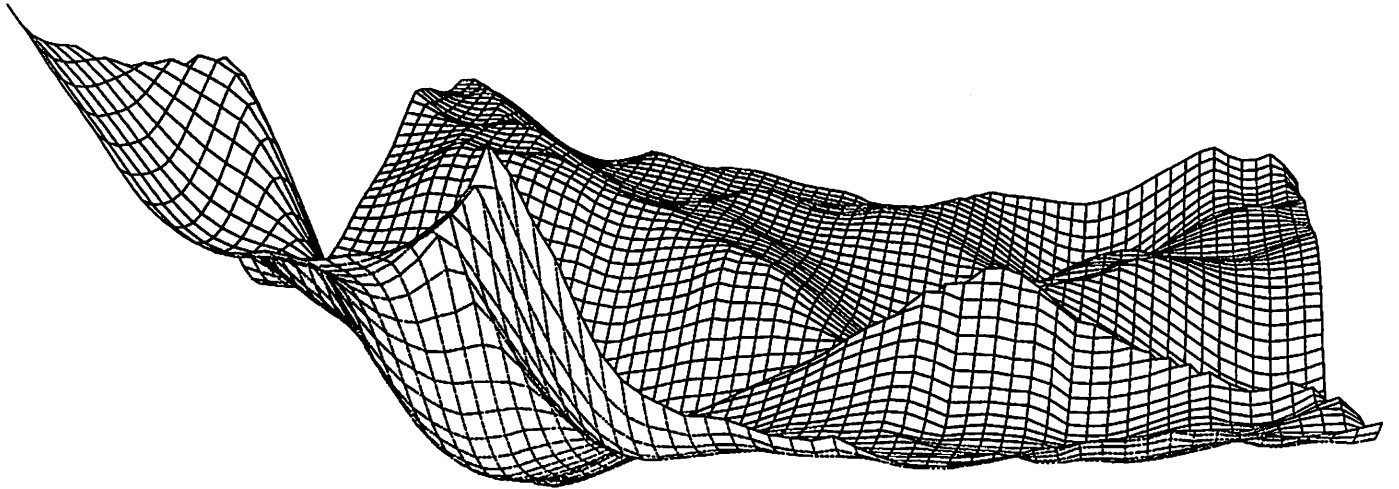
Figure 21. Summed probability distribution.
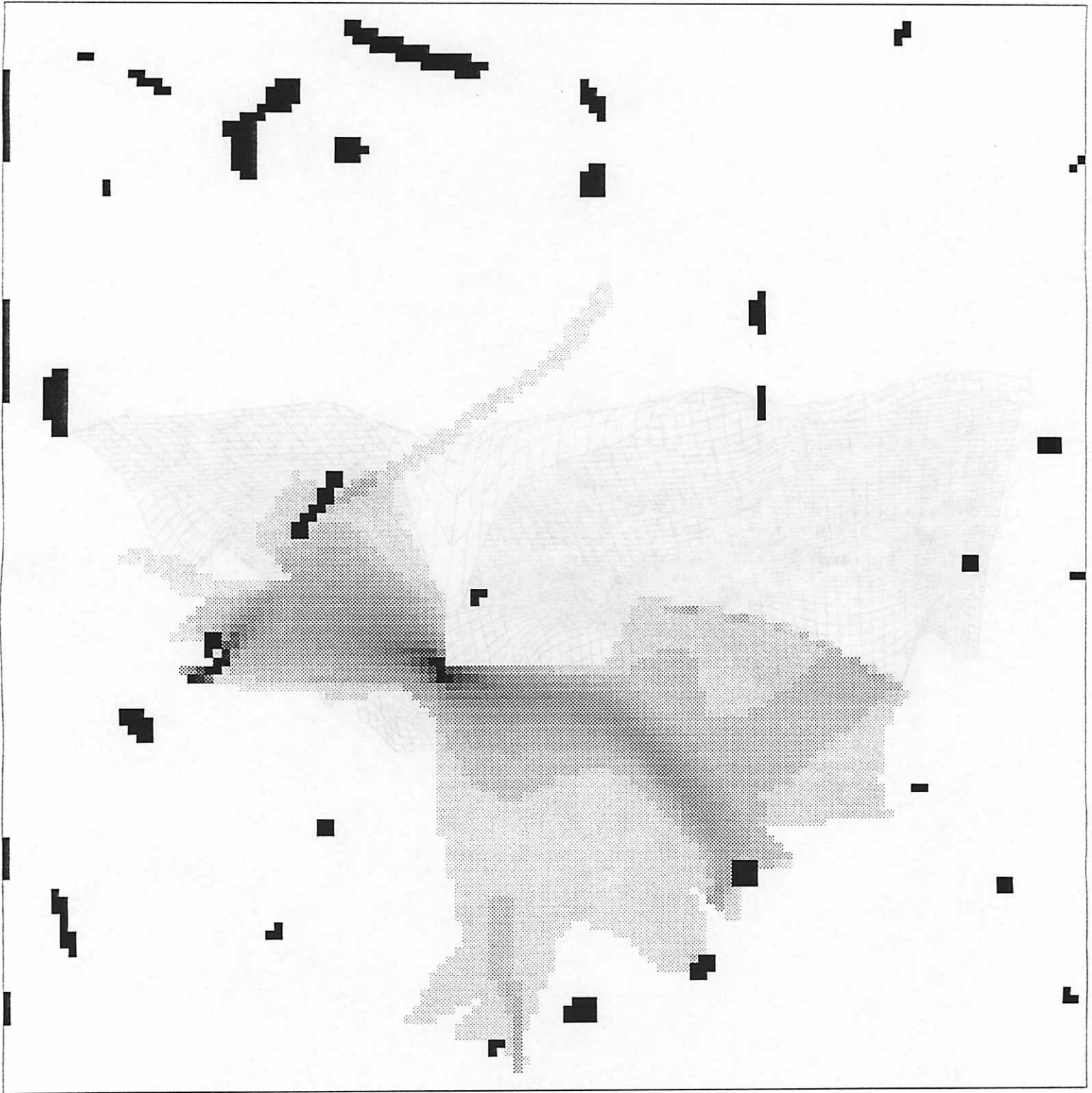
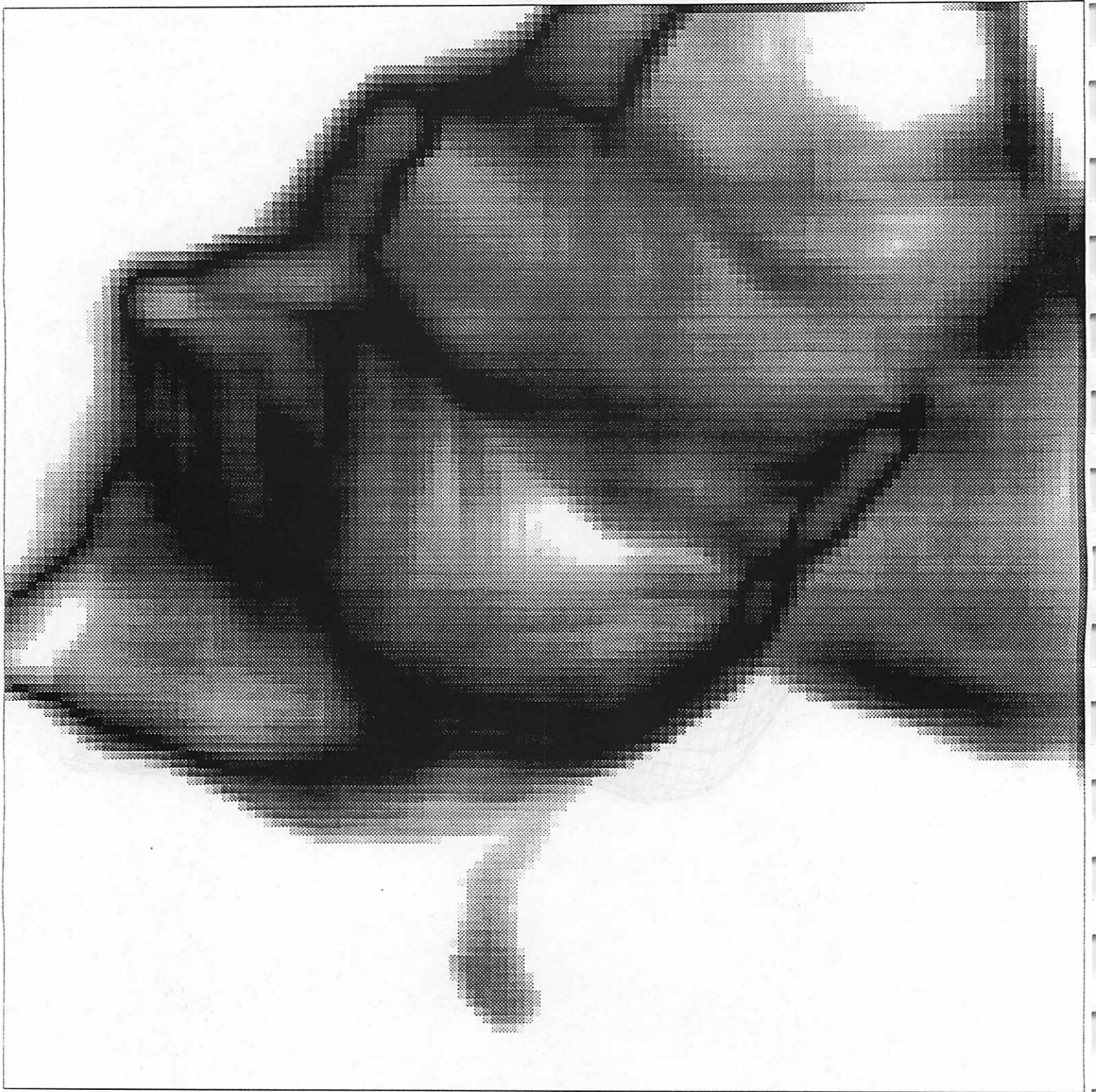Figure 22. Reconstruction with noise.

Figure 23. Reconstruction error with noise.

Figure 24. Reconstruction error with noise.