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ABSTRACT

The first step in establishing a group theoretical framework in robotics research is to formulate the precise basic vocabulary for describing surfaces of solids and their relationships. In a previous paper surface features of a solid are treated as sets in Euclidean space with no orientations taken into consideration. In this paper we show the problems caused by treating surfaces as such sets, then we formally characterize the oriented surfaces of a solid. The surface contacts between solids are always associated with a set of symmetries of the contacting surfaces. These symmetries form a group, called the *symmetry group* of the surface. When the orientation of a surface is taken into account the symmetries of a surface need to be redefined. In this paper we formalize the following important concepts:

- Primitive and compound oriented features of a solid;
- A complete topological characterization of these features, in particular, introduction of the concept of *complementary* relationship between a pair of features;
- The symmetry groups of oriented primitive features and compound features.

The central result of this paper is to prove:

- the given characterization of oriented features is complete and mutually exclusive;
- the symmetry groups of complementary features are conjugations of each other;
- when a compound feature of a solid is composed of a set of distinct, 1-congruent or 2-congruent primitive oriented features, its symmetry group can be expressed in terms of the intersection of the symmetry groups of its primitive features.

These results lay out a realistic and precise group theoretic framework for characterizing surfaces of solids and capture the very nature of surface contact — the state of being complementary. Under this formalization surface contact can be treated conceptually effectively and computationally efficiently.

1 Introduction

In robotics and mechanical design, contacts among solids are of particular interest. These contacts add to the otherwise free-standing (free-flying) individual part a set of constraints which are often referred to as *kinematic constraints.* Since contacts among solids happen via the contacts of the surfaces of the solids, the representation and characterization of each surface constitutes the foundation of any formalization for solid contacts. In [6] a group theoretic framework was proposed with surfaces of a solid treated as sets in Euclidean space. A set-feature of a solid is defined in [6] as:

Definition 1.0.1 *A* **primitive feature** *F of a solid M is a connected, irreducible and non-trivial algebraic surface that partially or completely coincides with one or more finite bounded faces of M.*

We maintain that symmetries of a surface play a crucial role in characterizing the solid which the surface bounds and determining the relative motions/positions of the solid with respect to other solids in contact. A *symmetry* is an isometry (a distance preserving mapping) in Euclidean space. When one considers with only those transformations which are physically realizable (excluding reflections), i.e. the *proper symmetries*, a symmetry is defined as [6]:

Definition 1.0.2 *A proper isometry g is a* **proper symmetry of a set** *S if and only if $g(S) = S$.*

All the symmetries of a set-feature form a group mathematically, the concept of *the symmetry group of a feature* is thus defined in [6] as:

Definition 1.0.3 *When S is a feature, its symmetry group G is called the* **symmetry group of the feature** *S.*

It is argued that "*the treatment of primitive features as sets (planar surfaces being the only special case) is sufficient as far as their symmetries are concerned* [6]. It is true that in general whether a primitive feature has orientations or not, or which orientation it has, does not make a difference in regards to the symmetries of the feature[1]. A spherical surface, treated as a set or with orientation vectors pointing inward, has the same symmetries as the spherical surface with orientation vectors pointing outward. However, in practice it is rare that a primitive feature is considered in isolation. In a mechanical assembly, it is often the case that several primitive features of one solid are in contact with several primitive features of other solids. This is a situation where features treated as simple sets could run into problems. For example, Figure 1 shows two adjacent planar surfaces S_1, S_2 of a

[1] The only exception is the planar surface: when it is treated as a set there are flipping symmetries which do not exist for oriented planes. In applications this can be easily treated as the only special case.

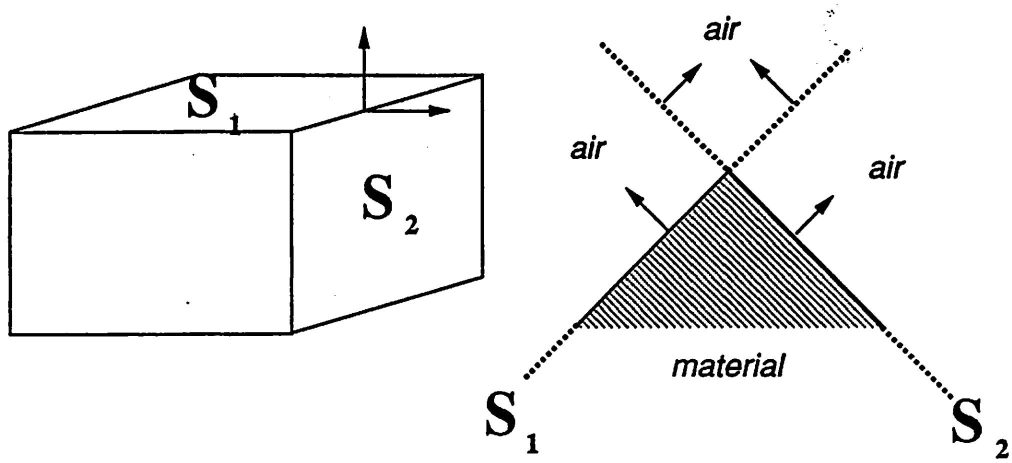

Figure 1: Two adjacent planes on a cube

block. If the two features are treated as sets the symmetries of the two planes (a compound feature) include a 90^0 rotation about the line of the intersection of the two planes which is not a symmetry in reality, if one takes into consideration the fact that one side of the plane is the material of the solid and the other is the air. Another example of such non-existing symmetries is illustrated in Figure 2. If the two cylindrical surfaces S_1, S_2 are treated as sets then one cannot distinguish the two cases (a) and (b). In case (b) the cylindrical hole S_1 and the cylinder S_2, though they have the same radius, are not interchangeable if one takes their orientations into consideration.

The aforementioned problems call for a more precise characterization of surface features of a solid, i.e. taking the orientations of a feature into consideration. This addition to a set-feature will require that the symmetries of the feature keep both the points on the surface and the orientations of the surface, respectively, setwise invariant.

It is the objective of this paper to improve the group theoretic framework for describing surface contact by addressing the orientations of a surface explicitly. Meanwhile we justify that the results on the symmetry groups of compound features proved in [6] hold for oriented compound features as well, with fewer restrictions.

2 Oriented Features and Symmetry Groups

In this section we provide the basic vocabulary for describing surface contacts among solids. We formally define oriented (primitive and compound) features of a solid and

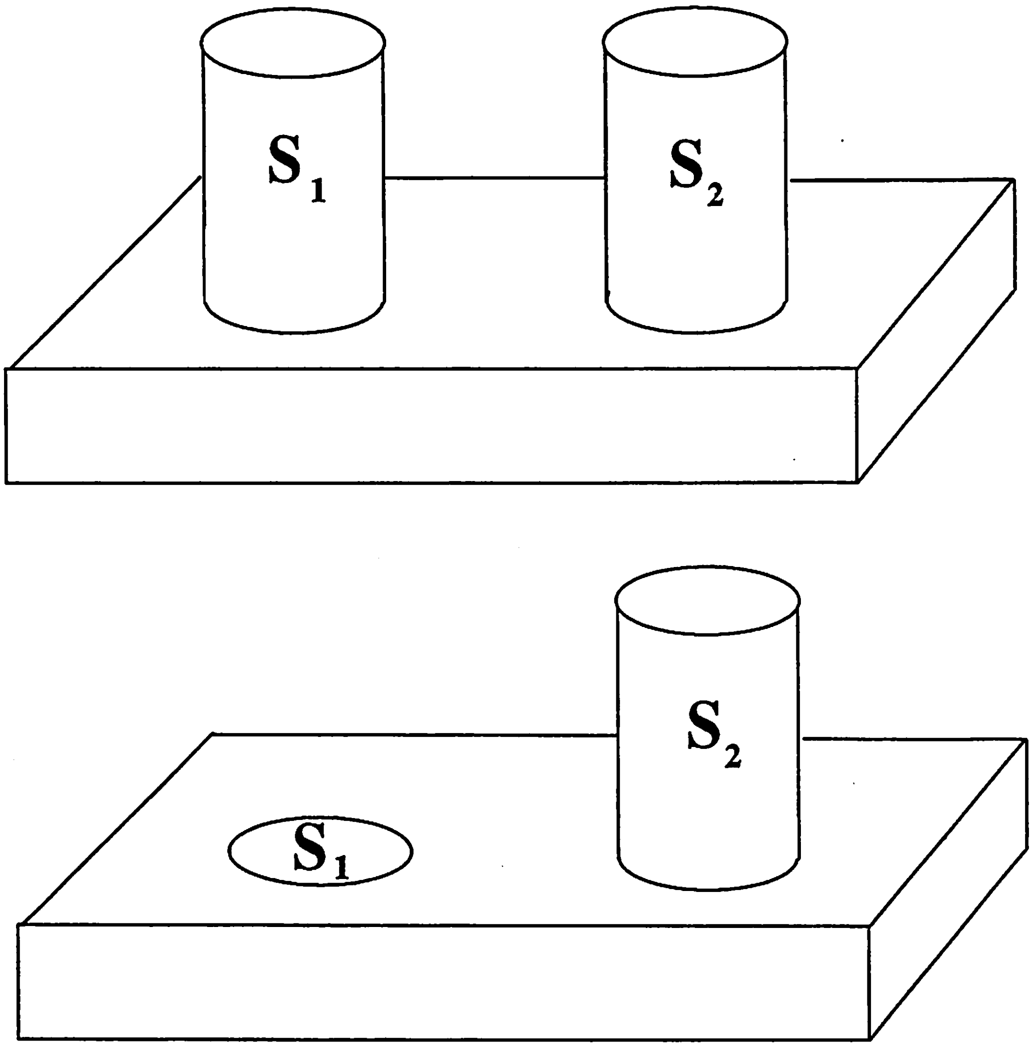

Figure 2: S_1 and S_2 are 2-congruent if they are treated as sets.

the symmetries of a feature. We show that these symmetries form a group, called the symmetry group of the feature. A complete and mutually exclusive characterization of relationships among primitive features is given. Furthermore we prove that when several primitive features on a solid are being considered collectively (often the case in an assembly), the symmetry group of the compound feature can be expressed in terms of the intersection of the symmetry group of each primitive feature involved. The format of such an expression may differ depending on the relationship of the relevant primitive features.

2.1 Primitive Features and Their Symmetry Groups

In our discussion of features [6] we have ignored one aspect of the faces of real world solids, namely that faces are boundaries between solid matter and air. The surfaces which we have treated mathematically as subsets of $\Re^3$ have no intrinsic *inside and outside.* To remedy this we introduce the concept of *oriented features* by defining a set of outward-pointing normal vectors for each surface point of a solid. Let S^2 be the unit sphere at the origin, each point of which corresponds to a unit vector in $\Re^3$.

Definition 2.1.1 *A solid M is a connected, rigid, three dimensional subset of Euclidean space $\Re^3$.*

Definition 2.1.2 *An* **oriented primitive feature** $F = (S, \rho)$ *of a solid M is an oriented surface where*

1) $S \subset \Re^3$ is a connected, irreducible[2] and continuous algebraic surface which partially or completely coincides with one or more finite oriented faces of M;

2) $\rho \subset S \times S^2$ is a continuous relation. For each $s \in S$ if s is a non-singular point of surface S (p.78 [3]) then $v \in S^2$ is one of the normals of the tangent plane at point s such that $(s,v) \in \rho$; if s is a singular point of S (e.g. at the apex of a cone) then, for all v where $v \in S^2$ is the limit of the orientations of its neighborhood, $(s,v) \in \rho$.

3) For all $s \in M, (s,v) \in \rho, v$ points away from M.

Intuitively speaking, a feature is composed of both "skin", S, and "hair", the set of normal vectors which correspond to the points on S^2. Each element of relation ρ is a correspondence between a point on S and a vector on S^2. Note, there may be more than one 'normal vector' at one point of a surface, e.g. at the apex of a conic shaped surface.

See Table 1 for definitions of some important subgroups of the Euclidean group defined with respect to an arbitrary Cartesian coordinate system in Euclidean space, where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are orthogonal unit vectors along axes X, Y and Z, $\mathbf{trans}(x, y, z)$ is a

[2] Here *irreducible* implies that a primitive feature cannot be composed of any other *more basic* surfaces.

Table 1: Some Canonical Subgroups of $\mathcal{E}^+$

Canonical Groups	Definition of Group Members
$\mathcal{G}_{id}$	$\{1\}$
$\mathcal{T}^3$	$\{\mathbf{trans}(x,y,z) \mid x,y,z \in \Re\}$
$SO(3)$	$\{\mathbf{rot}(\mathbf{i},\theta)\mathbf{rot}(\mathbf{j},\sigma)\mathbf{rot}(\mathbf{k},\phi) \mid \theta,\sigma,\phi \in \Re\}$
$\mathcal{E}^+$	$\{\mathbf{trans}(x,y,z)\mathbf{rot}(\mathbf{i},\theta)\mathbf{rot}(\mathbf{j},\sigma)\mathbf{rot}(\mathbf{k},\phi) \mid x,y,z,\theta,\sigma,\phi \in \Re\}$

translation and $\mathbf{rot}(a,b)$ is a rotation about axis a for angle b. Let $\mathcal{E}^+$ be the proper Euclidean group which contains all the rotations and translations in $\Re^3$, and T^3 be the maximum translation subgroup of $\mathcal{E}^+$. We now define how an isometry acts on the relation ρ:

Definition 2.1.3 *Any isometry $g = tr$ of $\mathcal{E}^+, t \in \mathrm{T}^3, r \in SO(3)$ acts on ρ in such a way that $(s,v) \in \rho \Leftrightarrow (gs, rv) \in g * \rho$.*

Next we prove the associativity of isometries when they act on the relation ρ.

Lemma 2.1.4 *For all $g_1, g_2 \in \mathcal{E}^+, (g_1g_2) * \rho = g_1 * (g_2 * \rho)$.*

Proof:

Let $g_1 = t_1r_2, g_2 = t_2r_2$ where $t_1, t_2 \in \mathrm{T}^3, r_1, r_2 \in SO(3)$. Since $g_1g_2 = t_1r_1t_2r_2 = t_1t'r_1r_2$ (T^3 is a normal subgroup of $\mathcal{E}^+$), for all $(s,v) \in \rho, (g_1g_2s, r_1r_2v) \in (g_1g_2) * \rho$. On the other hand, for all $(s,v) \in \rho, (g_2s, r_2v) \in g_2 * \rho$ and $(g_1g_2s, r_1r_2v) \in g_1 * (g_2 * \rho)$. Therefore, $(g_1g_2) * \rho = g_1 * (g_2 * \rho)$. □

For a feature defined in Definition 2.1.2, its symmetries are different from the symmetries of a set (Definition 1.0.2):

Definition 2.1.5 *An isometry $g \in \mathcal{E}^+$ is a* **proper symmetry of a feature** *$F = (S, \rho)$ if and only if $g(S) = S$* **and** *$g * \rho = \rho$.*

There is, therefore, an extra demand on a symmetry for an oriented feature, namely, it has to preserve the orientations of the feature as well. Since orientations are points on S^2, symmetries of an oriented feature have to keep two sets of points, or one set of 5-tuple points, setwise invariant. Let us first prove that the symmetries for a set form a group:

Proposition 2.1.6 *The proper symmetries of a set $S \subseteq \Re^3$ form a subgroup of $\mathcal{E}^+$.*

Proof:

Let G denote the set of the symmetries of $S \subset \Re^3$. Obviously, $1(S) = S$, so $1 \in G$. If $g \in G$ then $g(S) = S$, multiplying by g^{-1} we have $g^{-1}g(S) = g^{-1}(S)$ therefore $g^{-1}(S) = S$ and so $g^{-1} \in G$. Finally, if $g_1, g_2 \in G$ then $(g_1g_2)(S) =$

$g_1(g_2(S)) = g_1(S) = S$ therefore $g_1 g_2 \in G$. By the definition of a subgroup G is a subgroup of $\mathcal{E}^+$. □

Now let us prove that the symmetries for an oriented surface form a group as well:

Proposition 2.1.7 *The symmetries of an oriented feature $F = (S, \rho)$ form a subgroup of $\mathcal{E}^+$, called the* **symmetry group of feature** F.

Proof:

Let G denote the set of the symmetries of F. Since it has been shown in Proposition 2.1.6 that it is true for set S, here we only state about ρ.

Obviously, $1 * \rho = \rho$, so $1 \in G$. If $g \in G$ then $(g * \rho) = \rho$ (By the definition of symmetries). Multiplying by g^{-1} we have $g^{-1}(g * \rho) = g^{-1} * \rho$. Using Lemma 2.1.4 we have $g^{-1} * \rho = \rho$ and so $g^{-1} \in G$. Finally, if $g_1, g_2 \in G$ then $(g_1 g_2) * \rho = g_1 * (g_2 * \rho) = g_1 * \rho = \rho$ therefore $g_1 g_2 \in G$. Hence G is a subgroup of $\mathcal{E}^+$. □

2.2 Compound Features and their Symmetry Groups

It is often the case in an assembly that several features of a solid are in contact with one or more other solids. The possible motions of this solid under these contacts are determined by the symmetry group of the contacting features considered together. How do we obtain the symmetry group of several primitive features considered collectively? First we need a denotation for such a collection of primitive features.

Definition 2.2.1 *A* **compound feature** $F = (S, \rho)$ *of primitive features* $F_1 = (S_1, \rho_1), ..., F_n = (S_n, \rho_n)$, *is defined to be*

- $S = S_1 \cup ... \cup S_n$
- $\rho = \rho_1 \cup ... \cup \rho_n$

The advantage of using a relation ρ to denote the orientations of a feature (Definition 2.1.2) becomes more obvious for compound features. When two primitive features are combined, there often are multiple normal directions at the points where the surfaces meet. For example Figure 3 shows a combination of a cylindrical feature and a planar feature. There are two normals for each point at the intersection of the two primitive features.

In order to determine the symmetry group of a compound feature systematically, we start with the simplest case — a compound feature composed of only one pair of primitive features. See Figure 3, 4 and Figure 5 for examples of these simple compound features (Note that only a finite face on the primitive feature is drawn). Given a pair of primitive features, what kind of relationship holds between the two features and what is the effect of such a relationship in terms of their collective symmetries? The following definition gives such a characterization of the relationships between a pair of primitive features:

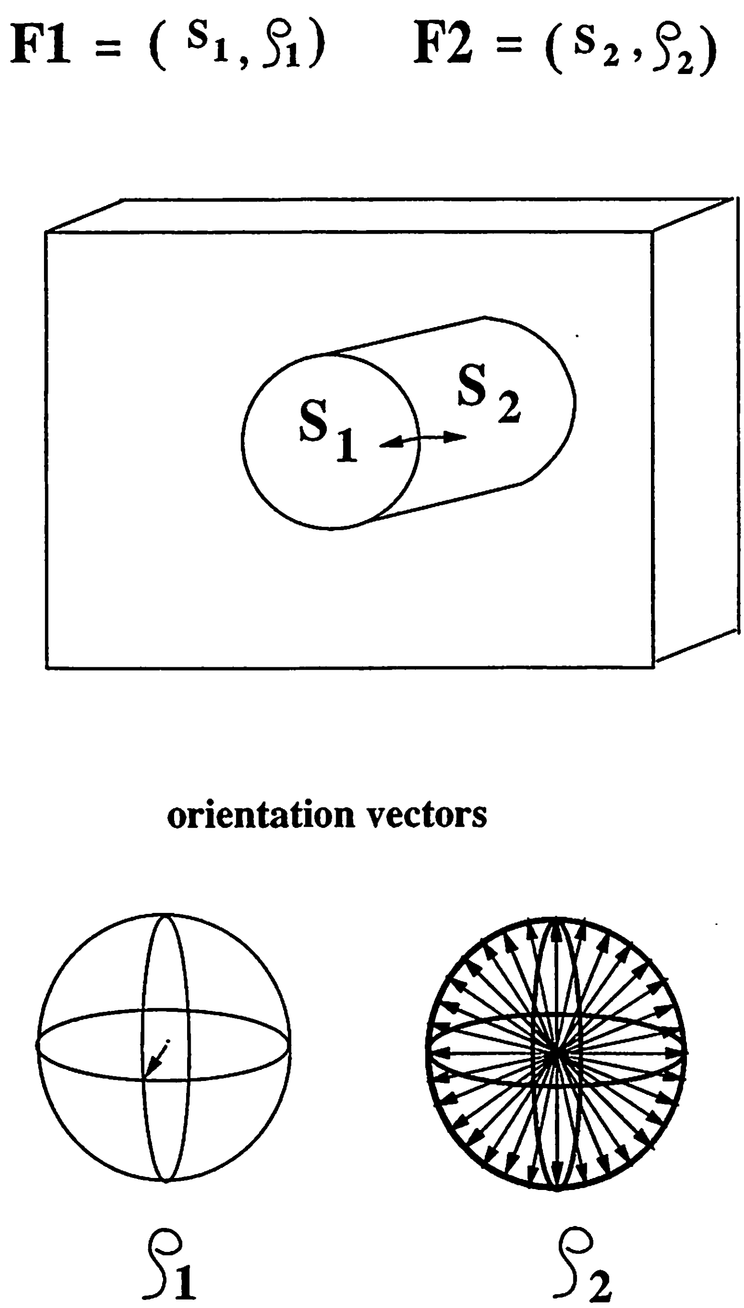

Figure 3: A pair of distinct features F_1, F_2

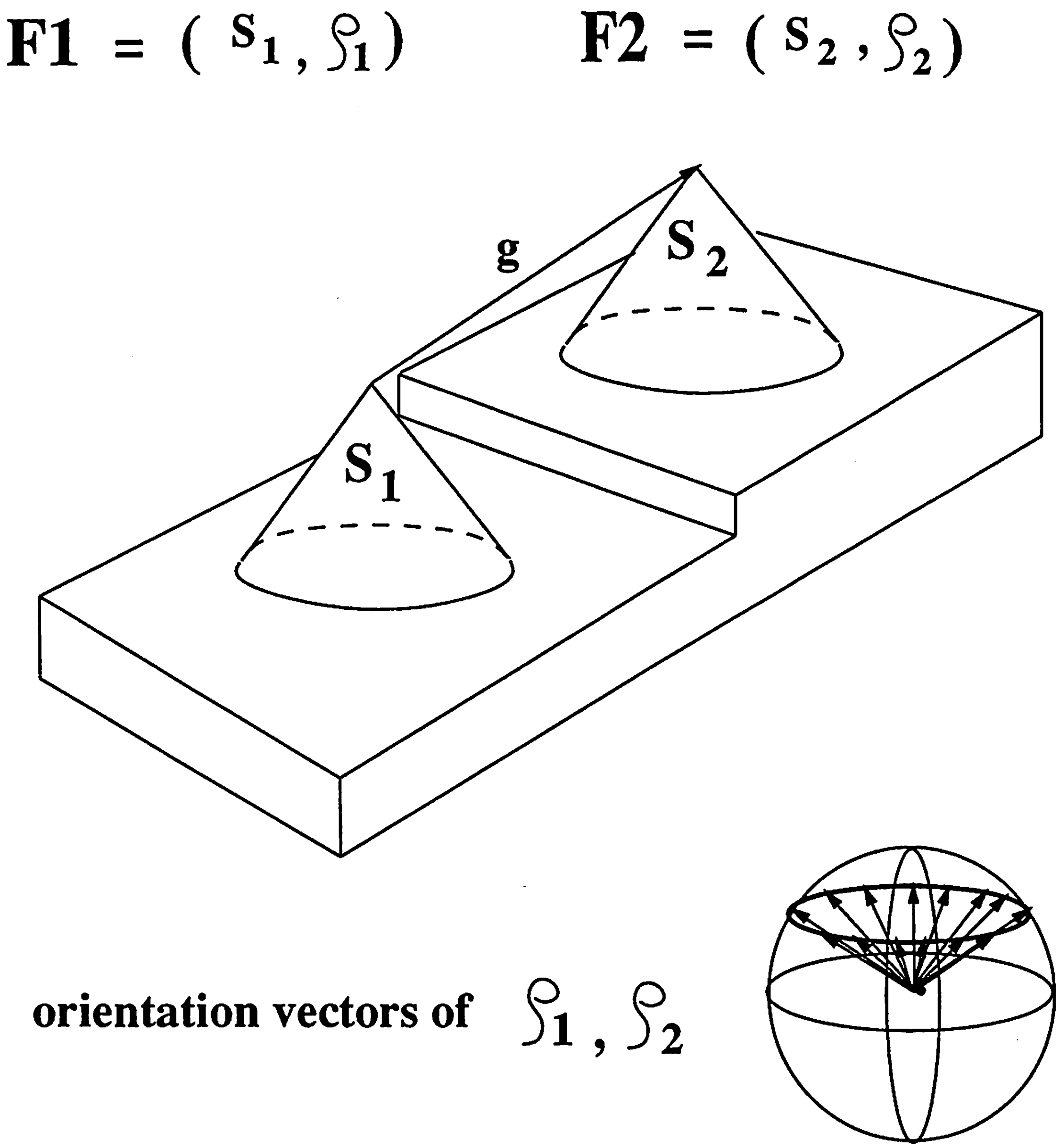

Figure 4: Two conic features F_1, F_2 which are **1-congruent** to each other

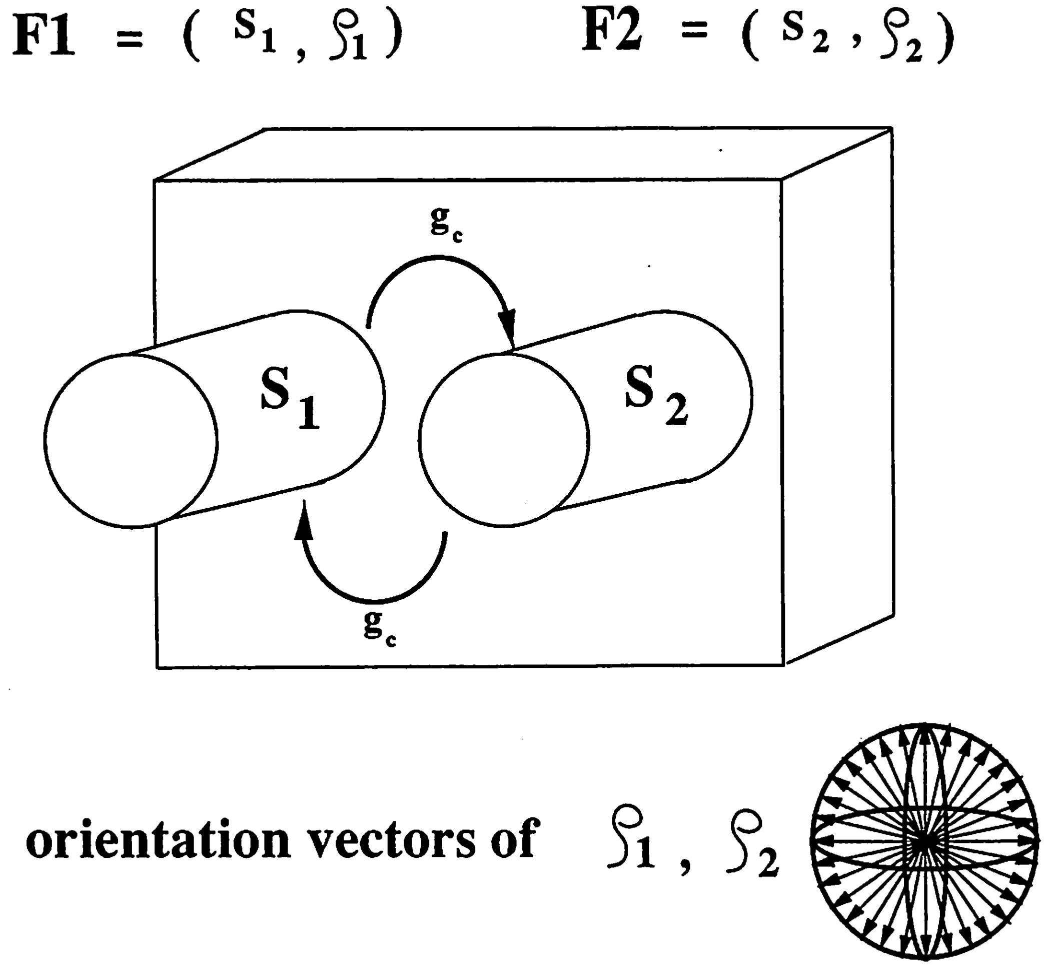

Figure 5: Two cylindrical features F_1, F_2 which are 2-congruent to each other

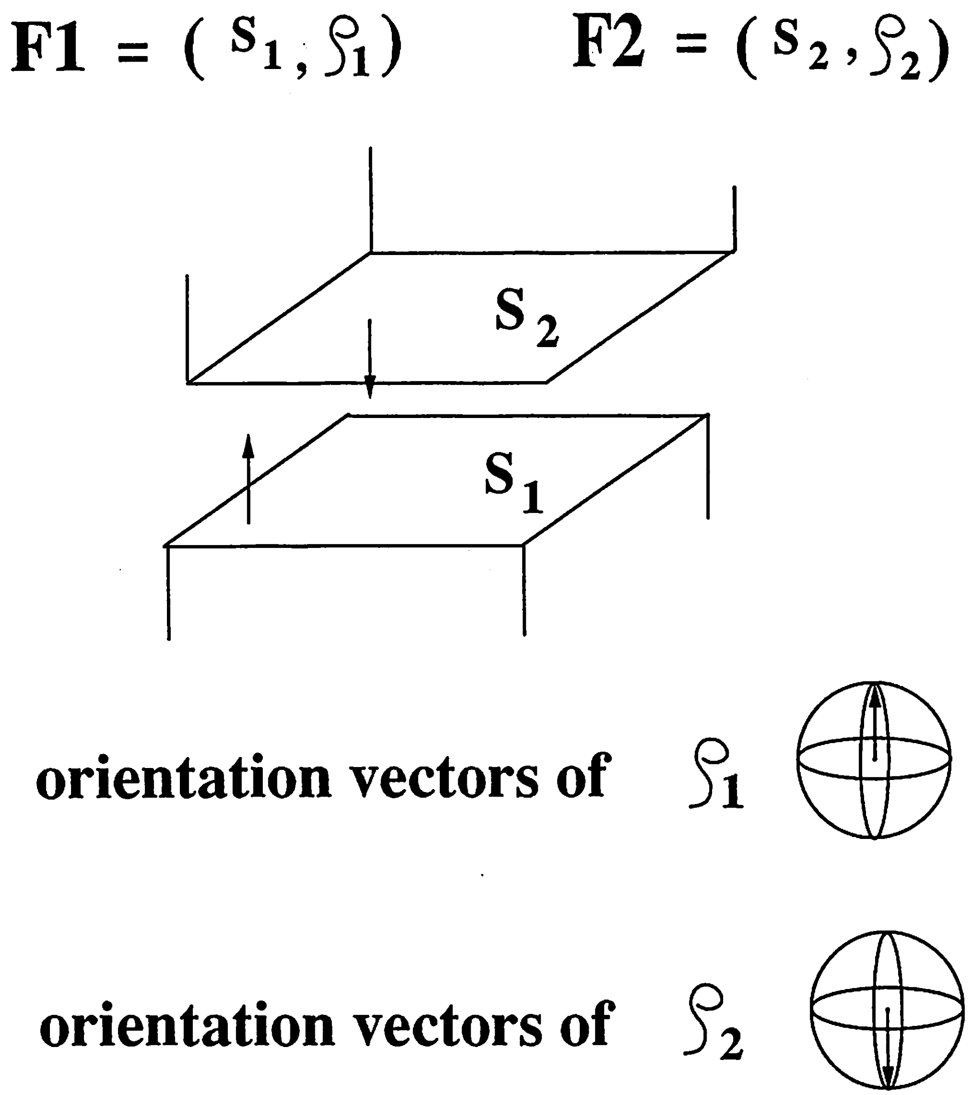

Figure 6: Two complementary features F_1, F_2

Definition 2.2.2 *Two oriented primitive features $F_1 = (S_1, \rho_1), F_2 = (S_2, \rho_2)$ are said to be*

- **Distinct:** if for any open subsets $S_1' \subset S_1, S_2' \subset S_2$, no $g = tr \in \mathcal{E}^+$ exists such that $g(S_1') \subset S_2$ or $g(S_2') \subset S_1$. See Figure 3 for an example of a pair of distinct features F_1, F_2.
- **1-congruent**: if there exists at least one $g \in \mathcal{E}^+$ such that $g(S_1) = S_2$ and $g * \rho_1 = \rho_2$, but for all such $g, g(S_2) \neq S_1$. For an example see Figure 4. Another example is two parallel planar surfaces with normal vectors pointing in the same direction.
- **2-congruent**: if there exists $g_c \in \mathcal{E}^+$ such that $g_c(S_1) = S_2, g_c(S_2) = S_1, g_c * \rho_1 = \rho_2$ and $g_c * \rho_2 = \rho_1$. For an example, consider two parallel cylindrical surfaces having the same radius and normal vectors pointing away from their center lines, as in Figure 5. Also, two parallel planar surfaces with normal vectors pointing to the opposite directions serve as examples of a pair of 2-congruent features.
- **Complementary:** if there exists $g \in \mathcal{E}^+$ such that $g(S_1) = S_2$ and $g*\rho_1 = -\rho_2$ where $-\rho_2 = \{(s, -v) | (s, v) \in \rho_2\}$; in other words, $\forall(s, v) \in g * \rho_1, \exists(s, -v) \in \rho_2$, and $\forall(s, v) \in \rho_2, \exists(s, -v) \in g * \rho_1$. See Figure 6 for an example.

It is easy to verify that these relationships are symmetrical relations. Immediately we can prove that this characterization has exhaustively enumerated all the possible cases between a pair of oriented primitive features. First, let us prove an important lemma:

Lemma 2.2.3 *Given two primitive features $F_1 = (S_1, \rho_1), F_2 = (S_2, \rho_2)$. If there exists an open set O such that $O \subset S_1 \cap S_2$ then $S_1 = S_2$. In another words, if S_1, S_2 are locally identical then they are identical globally.*

Proof :

If one knows an open set in a surface, then one knows all of its derivatives at a point in the open set. Therefore one can write down the equation(s) for the surface. Therefore S_1 and S_2 are locally identical.

Furthermore, analytic functions have the property that if they are locally identical then they are globally identical [1]. In the definition of primitive features (Definition 2.1.2), S_1, S_2 are defined by irreducible algebraic functions, which form a subset of the analytic functions, and thus they inherit the property. Therefore if S_1 and S_2 share an open set then $S_1 = S_2$. □

Lemma 2.2.4 *Given two primitive features $F_1 = (S_1, \rho_1)$, and $F_2 = (S_2, \rho_2)$. If there exists $g \in \mathcal{E}^+$ such that $g(S_1) = S_2$ then either $g * \rho_1 = \rho_2$ or $g * \rho_1 = -\rho_2$.*

Proof :

By Definition 2.1.2, any point s on a primitive feature has either

- a unique tangent plane:

  there are two possible antipodal normals for each plane, say $v, -v$. By the definition of a primitive feature either $(s, v) = (s_2, v_2) \in \rho_2$ or $(s, -v) = (s_2, v_2) \in \rho_2$. Since ρ_1, ρ_2 are continuous mappings and isometry g does not change their continuity, for $(s_1, v_1) \in \rho_1$,

  - if $rv_1 = v_2$ then $g * \rho_1 = \rho_2$,
  - if $rv_1 = -v_2$ then $g * \rho_1 = -\rho_2$; or

- an infinite number of "tangent planes" (the set is isomorphic to the set of reals):

  there is an infinite set of normals which are determined by the neighborhoods of the singular point s. Each of such neighborhoods is composed of non-singular points, Thus the above argument also applies.

□

Proposition 2.2.5 Distinct, 1-congruent, 2-congruent *and* complementary *are the only possible relationships between a pair of primitive features.*

Proof :
Given two primitive features $F_1 = (S_1, \rho_1), F_2 = (S_2, \rho_2)$. Lemma 2.2.3 suggests that either there exists a $g \in \mathcal{E}^+$ such that $g(S_1) = S_2$ or no such g exists. Now let us check each case.

Note that any two planar surfaces are complementary of each other, and are either 2-congruent (when the planes intersect or are parallel with their normals pointing in the opposite directions), or 1-congruent (when the planes are parallel with their normals pointing in the same direction). In the following discussion we exclude the case of a pair of planar surfaces.

- If there exists at least one $g \in \mathcal{E}^+$ such that $g(S_1) = S_2$:

  - If $g(S_2) = S_1$ also, then $g(S_1) = g(g(S_2)) = S_2 \Rightarrow g^2 = 1$. Now there are two cases in terms of their orientations (Lemma 2.2.4):

    (1) If $g * \rho_1 = \rho_2$ then $g * \rho_2 = g * g * \rho_1 = \rho_1$. This is the definition of **2-congruent**.

    (2) If $g * \rho_1 = -\rho_2$ then , this falls into the definition of **complementary**.
  - If $g(S_2) \neq S_1$ then

    (3) If $g * \rho_1 = \rho_2$ this is the definition of **1-congruent**.

    (4) If $g * \rho_1 = -\rho_2$, this is the definition of **complementary**.

- If for any $g \in \mathcal{E}^+, g(S_1) \neq S_2$ (Lemma 2.2.3):

  This is the definition of **distinct**.

□

Corollary 2.2.6 *Except for a pair of planar surface primitive features,* distinct, 1-congruent, 2-congruent *and* complementary *relationships are mutually exclusive relations between a pair of primitive features.*

Proof : As has been shown in the proof of proposition 2.2.5. □

The definition for oriented features allows us to distinguish a feature from its complement which we cannot do for features treated only as sets. In general the relationship between two primitive features can be either distinct, 1-congruent, 2-congruent or complementary, except for a pair of planar surfaces of solids which are always complementary of each other and at the same time can be either 1-congruent or 2-congruent.

When two solids have a surface contact, it is the case that two features which are complementary of each other are brought into coincidence. The following proposition states how the symmetry groups of a pair of complementary features are related to each other.

Proposition 2.2.7 *If features $F_1 = (S_1, \rho_1), F_2 = (S_2, \rho_2)$ are complementary of each other, where $a(S_1) = S_2, a \in \mathcal{E}^+$, and G_1, G_2 are the symmetry groups of F_1, F_2 respectively, then the two symmetry groups are conjugate via a i.e. $aG_1a^{-1} = G_2$. In particular, if $S_1 = S_2$ then $G_1 = G_2$* **(the necessary condition for surface contact).**

Proof :

For all $g = ag_1a^{-1} \in aG_1a^{-1}, g(S_2) = ag_1a^{-1}(S_2) = ag_1(S_1) = a(S_1) = S_2$.

For all $(s, v) \in \rho_2$ by definition of complementary features $(s, -v) \in a * \rho_1 = (ag_1a^{-1}a) * \rho_1 = g(a * \rho_1)$, where $g = aga^{-1} = tr \in aG_1a^{-1}$. Thus $(g^{-1}s, r^{-1}v) \in a * \rho_1$. By the definition of complementary features $(g^{-1}s, r^{-1}v) \in \rho_2$. Then $(s, v) \in g * \rho_2$. Therefore $\rho_2 \subseteq g * \rho_2$.

On the other hand, $\forall (gs, rv) \in g * \rho_2, (s, v) \in \rho_2$. By the definition of complementary features $(s, -v) \in a * \rho_1$. Then $(gs, -rv) \in g(a * \rho_1) = a * \rho_1$. By the definition of complementary features again, $(gs, rv) \in \rho_2$. So $g * \rho_2 \subseteq \rho_2$.

Therefore forall $g \in aG_1a^{-1}, g * \rho_2 = \rho_2$. That is $aG - 1a^{-1}$ is a symmetry group for F_2. Hence $aG_1a^{-1} \subseteq G_2$.

Now we need to prove: $G_2 \subseteq aG_1a^{-1}$, i.e. G_2 is a symmetry group of $a(S_1)$.

If $g = tr \in G_2$ then first consider how it acts on the set $g(a(S_1)) = g(S_2) = S_2 = a(S_1)$. Now let us consider how g acts on the orientations. For all $(s, v) \in a * \rho_1, \exists (s, -v) \in \rho_2 = g * \rho_2$, then $(g^{-1}s, r^{-1}v) \in \rho_2 \Rightarrow (g^{-1}s, -r^{-1}v) \in a * \rho_1 \Rightarrow (s, v) \in g(a * \rho_1)$. So $a * \rho_1 \subseteq g(a * \rho_1)$. On the other hand, $\forall (gs, rv) \in g(a * \rho_1), \exists (s, v) \in a * \rho_1 \Rightarrow (s, -v) \in \rho_2 \Rightarrow (gs, -rv) \in g * \rho_2 = \rho_2 \Rightarrow (gs, rv) \in a * \rho_1$. So $g(a * \rho_1) \subseteq a * \rho_1$. One can conclude $g(a * \rho_1) = a * \rho_1$. Therefore $G_2 \subseteq aG_1a^{-1}$.

Hence $G_2 = aG_1a^{-1}$. In case $a = 1, G_1 = G_2$. □

The following lemma shows that for any non-planar primitive features, symmetries for the set-features are the symmetries for the oriented features.

Lemma 2.2.8 *For any non-planar primitive feature $F = (S, \rho)$, if there exists an isometry $g = tr$ such that $g(S) = S$ then $g * \rho = \rho$.*

Proof:

By lemma 2.2.4, $g * \rho = \rho$ or $g * \rho = -\rho$. To prove by contradiction let us assume that $g * \rho = -\rho$. By definition 2.2.2 F is complementary with itself.

Since in Euclidean space a rotation cannot inverse more than two independent vectors simultaneously[3] [7, 4], an oriented surface F has to have less than or equal to two normals in order for all of its normals to be inversed by a rotation. The only such surface, or even a surface which has finite number of normals, is a planar surface.

Thus F is a planar surface, a contradiction. □

This lemma could be seen as a justification for treating oriented surfaces as subsets of Euclidean space but unfortunately this result does not hold for compound features. Consider the compound feature which is composed of two cylindrical surfaces in case (b) of Figure 2, any transformations which interchange the two surfaces (symmetries of the compound feature) will reverse the orientations at each point of the feature.

In the following we shall proceed to prove that the symmetry group of a compound feature is determined by the intersection of the symmetry groups of its primitive features. The first case we consider is when a compound feature F is composed of n pairwise *distinct* features.

Proposition 2.2.9 *Given a compound feature $F = (S, \rho)$ of primitive features $F_1 = (S_1, \rho_1), ..., F_n = (S_n, \rho_n)$ where $F_1, ..., F_n$ are pairwise distinct primitive features with symmetry groups $G_1, ...G_n$ respectively. Then the symmetry group G of F is $G = G_1 \cap ... \cap G_n$.*

Proof:

Let $g \in G$, then $g(S) = S$. Thus $g(S_1 \cup ... \cup S_n) = g(S_1) \cup ... \cup g(S_n) = S_1 \cup ... \cup S_n$. Then $g(S_i) \subseteq S_1 \cup ... \cup S_n$.

From Lemma 2.2.3 and the definition of distinct features (Definition 2.2.2) we know that $\forall g \in G, g(S_i) = S_i, i = 1...n$.

By Lemma 2.2.8 we have for all the non-planar primitive features $g * \rho_i = \rho_i$. Since $F_1 ... F_n$ are pairwise distinct there is at most one planar feature whose orientation has to be mapped to itself.

Therefore $g \in G_i$ for $i = 1, ..., n$. Thus $g \in G_1 \cap ... \cap G_n \Rightarrow G \subseteq G_1 \cap ... \cap G_n$.

For all $g \in G_1 \cap ... \cap G_n, g(S) = g(S_1 \cup ... \cup S_n) = g(S_1) \cup ... \cup g(S_n) = S_1 \cup ... \cup S_n = S$ and $g * \rho = g * (\rho_1 \cup ... \cup \rho_n) = g * \rho_1 \cup ... \cup g * \rho_n = \rho_1 \cup ... \cup \rho_n = \rho \Rightarrow g \in G \Rightarrow G_1 \cap ... \cap G_n \subseteq G$.

Therefore $G = G_1 \cap ... \cap G_n$. □

The following definition and three theorems are from [2]. We shall use these in our proofs.

[3] If R is a rotation and $\vec{u}, \vec{v}$ are vectors in Euclidean space, then the vector cross product obeys: $R(\vec{u}) X R(\vec{v}) = R(\vec{u} X \vec{v})$

Definition 2.2.10 *Two sets H, K are* **separated** *if*

$$\bar{H} \cap K = H \cap \bar{K} = \emptyset.$$

Theorem 2.2.11 *A set $M \subset X$ is* **connected** *if and only if M is not the union of two nonempty separated sets.*

Theorem 2.2.12 *For sets, connectivity is preserved by surjective mappings.*

Theorem 2.2.13 *If H and K are separated, then every connected subset M of $H \cup K$ lies either in H or in K.*

Next we examine what happens when a compound feature F is composed of a pair of *1-congruent* features. What is the symmetry group of F? Let us first prove one useful lemma:

Lemma 2.2.14 *For any pair of primitive features $F_1 = (S_1, \rho_1), F_2 = (S_2, \rho_2)$ where $S_1 \neq S_2$, if there exists a $g \in \mathcal{E}^+$ such that $g(S_1 \cup S_2) = S_1 \cup S_2$ then $g(S_1) = S_1, g(S_2) = S_2$ or $g(S_1) = S_2, g(S_2) = S_1$.*

Proof:

There are two possibilities for S_1 and S_2:

- $S_1 \cap S_2 = \emptyset$.
  Since $g(S_1 \cup S_2) = g(S_1) \cup g(S_2) = S_1 \cup S_2$, and $g(S_1)$ is a connected subset of $S_1 \cup S_2$ (Theorem 2.2.12), by Theorem 2.2.13 $g(S_1) \subseteq S_1$ or $g(S_1) \subseteq S_2$. If $g(S_1) \subseteq S_1$ then, due to connectivity, $g(S_2) \subseteq S_2$. Since g is a bijection $g(S_1) = S_1, g(S_2) = S_2$. Similarly, $g(S_1) = S_2, g(S_2) = S_1$.

- $S_1 \cap S_2 \neq \emptyset$.
  If there exist open sets $O_1 \subset g(S_1) \cap S_1$ and $O_2 \subset g(S_1) \cap S_2$. Then by Lemma 2.2.3 $g(S_1) = S_1$ and $g(S_1) = S_2$. Thus $S_1 = S_2$, a contradiction. Thus either $g(S_1)$ and S_1 share an open set such that $g(S_1) = S_1, g(S_2) = S_2$ or $g(S_1) = S_2, g(S_2) = S_1$.

Therefore $g(S_1) = S_1, g(S_2) = S_2$ or $g(S_1) = S_2, g(S_2) = S_1$. □

The proposition for finding the symmetry group of a pair of 1-congruent features follows:

Proposition 2.2.15 *Let a compound feature $F = (S, \rho)$ be composed of a pair of primitive features $F_1 = (S_1, \rho_1)$ and $F_2 = (S_2, \rho_2)$ which are 1-congruent of each other. If G_1, G_2 are the symmetry groups of F_1, F_2 respectively, and G is the symmetry group of F then $G = G_1 \cap G_2$.*

Proof:

For all $g \in G, g(S) = g(S_1 \cup S_2) = g(S_1) \cup g(S_2)$ and $g * \rho = g * (\rho_1 \cup \rho_2) = g * \rho_1 \cup g * \rho_2$. By Lemma 2.2.14,

- $g(S_1) = S_1, g(S_2) = S_2$:

  If F_1, F_2 are planar features, they have to be parallel planes with their normals pointing to the same direction, i.e. $\rho = \rho_1 = \rho_2$. Thus $g * \rho = \rho \Rightarrow g * \rho_1 = \rho_1$ and $g * \rho_2 = \rho_2$. For non-planar features $g * \rho_1 = \rho_1, g * \rho_2 = \rho_2$ (Lemma 2.2.8).

- $g(S_1) = S_2, g(S_2) = S_1$:

  If $g * \rho_1 = \rho_2$ then F_1, F_2 are 2-congruent; if $g * \rho_1 = -\rho_2$ then F_1, F_2 are complementary; both contradict the fact that F_1, F_2 are 1-congruent.

Then $g \in G_1 \cap G_2$. So we have $G \subseteq G_1 \cap G_2$.

On the other hand, for all $g \in G_1 \cap G_2$, $g(S) = g(S_1 \cup S_2) = g(S_1) \cup g(S_2) = S_1 \cup S_2 = S$; $g * \rho = g * (\rho_1 \cup \rho_2) = g * \rho_1 \cup g * \rho_2 = \rho_1 \cup \rho_2 = \rho$. Therefore $g \in G \Rightarrow G_1 \cap G_2 \subseteq G$. Thus we conclude $G = G_1 \cap G_2$. □

Lastly we consider the symmetry group of a compound feature F which is composed of a pair of *2-congruent* features.

Proposition 2.2.16 *Let a compound feature $F = (S, \rho)$ be composed of a pair of primitive features F_1 and F_2 which are 2-congruent of each other via g_c (Definition 2.2.2). If $F_1 = (S_1, \rho_1), F_2 = (S_2, \rho_2)$ have symmetry groups G_1, G_2 respectively, and G is the symmetry group of F then $G = \langle g_c \rangle (G_1 \cap G_2)$ where $\langle g_c \rangle$ denotes the subgroup of $\mathcal{E}^+$ generated by g_c.*

Proof:

If $g \in G$ then by Lemma 2.2.14 either

- $g(S_1) = S_1$ and $g(S_2) = S_2$:

  By Lemma 2.2.8, taking planar feature case into consideration also, $g * \rho_1 = \rho_1, g * \rho_2 = \rho_2$. Thus $g \in G_1$ and $g \in G_2 \Rightarrow g \in G_1 \cap G_2$; or

- or $g(S_1) = S_2$ and $g(S_2) = S_1 \Rightarrow g^2 = 1$:

  g can be written as $g = g_c g_c^{-1} g$. Let $g_0 = g_c^{-1} g$. $g_0(S_1) = g_c^{-1} g(S_1) = g_c^{-1}(S_2) = S_1, g_0 * \rho_1 = (g_c^{-1} g) * \rho_1 = g_c^{-1} * \rho_2 = g_c * \rho_2 = \rho_1$ (Lemma 2.2.4).

  Therefore $g_0 \in G_1$. Similarly we can prove $g_0 \in G_2$. Thus $g_0 \in G_1 \cap G_2 \Rightarrow g \in \langle g_c \rangle (G_1 \cap G_2) \Rightarrow G \subseteq \langle g_c \rangle (G_1 \cap G_2)$;

Therefore $G \subseteq \langle g_c \rangle (G_1 \cap G_2)$.

On the other hand, if $g \in \langle g_c \rangle (G_1 \cap G_2)$ then $g = g' g_{12}$ where $g' \in \langle g_c \rangle$ and $g_{12} \in G_1 \cap G_2$. Then $g(S) = g(S_1 \cup S_2) = g(S_1) \cup g(S_2) = g' g_{12}(S_1) \cup g' g_{12}(S_2) = g'(S_1) \cup g'(S_2)$. By lemma 2.2.14, either $g'(S_1) \cup g'(S_2) = S_1 \cup S_2 = S$ or $g'(S_1) \cup g'(S_2) = S_2 \cup S_1 = S$. For orientations $g * \rho = g * (\rho_1 \cup \rho_2) = g' g_{12} * \rho_1 \cup g' g_{12} * \rho_2 = g' * \rho_1 \cup g' * \rho_2$. Since $g' \in \langle g_c \rangle$, by definition of 2-congruent (Definition 2.2.2) either $g' * \rho_1 \cup g' * \rho_2 = \rho_1 \cup \rho_2 = \rho$ or $g' * \rho_1 \cup g' * \rho_2 = \rho_2 \cup \rho_1 = \rho$. Therefore $g \in G \Rightarrow \langle g_c \rangle (G_1 \cap G_2) \subseteq G$.

Thus we conclude $G = \langle g_c \rangle (G_1 \cap G_2)$. □

The fact that this proposition implies that the product of two groups, $< g_c >$ and $G_1 \cap G_2$, *is* a group is worthy of note — this is not in general to be expected of the product of groups. With this proposition we end this section where propositions are proved for the symmetry groups of all the possible pairs of the oriented primitive features.

3 Conclusion

In this paper we have carefully examined the representation and computation aspects of oriented surfaces. Special attention is given to the characterization of symmetry groups for contacting surfaces among solids.

The next step is to further study those compound features with more complicated inner structures. For example, one may define a concept of n-congruence on n features $F_1 \dots F_n$ as requiring that there exists $g \in \mathcal{E}^+$ such that $g(F_i) = F_{(i \bmod n)+1}$; this is a natural extension of 2-congruence. Such congruences will give rise to new symmetries of the compound feature. However Proposition 2.2.16 is not trivially generalized to cover this case.

Nevertheless, these results lay out a realistic and precise group theoretic framework for characterizing surfaces of solids and capture the very nature of surface contact — the state of being complementary. Under this formalization surface contact can be treated conceptually effectively and computationally efficiently [5].

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