Epipolar Fields on Surfaces

Peter J. Giblin[1]
Richard S. Weiss

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[1]Dept. of Pure Math, University of Liverpool L693BX, England

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Peter J. Giblin
Dept. of Pure Math
University of Liverpool L693BX, England

Richard S. Weiss *
Dept. of Computer Science
University of Massachusetts, Amherst, MA 01003

Abstract

The view lines associated with a family of profile curves of the projection of a surface onto the retina of a moving camera defines a multi-valued vector field on the surface. The integral curves of this field are called epipolar curves and together with a parametrization of the profiles provide a parametrization of regions of the surface. In addition, one has the epipolar constraints which define curves in the images. These image curves are related to the epipolar curves on the surface but not by a simple projection. We present an investigation of epipolar curves on the object surface, in the spatio-temporal surface and the traces in the images. We address the question of when there is an epipolar parametrization. We have obtained detailed results which depend on a classification [4] of vector fields on surfaces with boundary. These results give a systematic way of detecting the gaps left by reconstruction of a surface from profiles. They also suggest methods for filling in these gaps.

keywords: surface reconstruction, epipolar constraint, epipolar curve, spatio-temporal surface

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1 Introduction

This paper is concerned with some aspects of the reconstruction of surfaces from a sequence of profiles (also called apparent contours, outlines and occluding contours) where the motion of the observer is known. Such a reconstruction was introduced in [7] for a simple class of motions, and was generalised in [1, 3, 13] to arbitrary motion. For the general motion case, the *epipolar correspondence* played an important role in matching points from one profile to the next as well as in the parametrization of the surface. The first question one must ask is, "When is this parametrization possible?" The parametrization breaks down when profile is *singular* (has a cusp or worse) when the profile is occluded at a t-junction and, as we explain below, at 'frontier points'. In these circumstances a 'patching' operation is needed to fill in the missing pieces of the surface, which failed to be reconstructed, and to do this we need to know the nature of these missing pieces. The object of the present paper is to throw light on this by a thorough study of the epipolar curves close to the places where the prametrization breaks down. In addition one must consider the accuracy of determining the epipolar correspondence. This is problem is also studied here by examining the constraint in the image plane.

Recall that the profile of a surface M from a given viewpoint can be defined either for parallel or for perspective projection. For parallel projection and a view *direction* (unit vector) w we consider the *critical set* Σ_w on M to be the set of points where the normal to M is perpendicular to w. (If w is a function of t then we would write Σ_t rather than $\Sigma_{w(t)}$.) Projecting in the view direction w onto a view-plane perpendicular to w gives the profile. Thus profile points p and surface points r are related by an equation of the form

$$r = p + \lambda w \tag{1}$$

where λ is the distance from p to r.

For perspective projection, we follow [1, 3] in taking a centre of projection (camera centre) c not on or inside of M and defining the critical set Σ_c to be the set of points r of M where the normal is perpendicular to the line segment ('viewline') from c to r. (When c is a function of t we write Σ_t for $\Sigma_{c(t)}$.) The critical set is then projected along the visual rays onto a unit sphere centred at c to give the profile points $c + p$ in this sphere (the 'image sphere'). (See Figure 1.) Thus

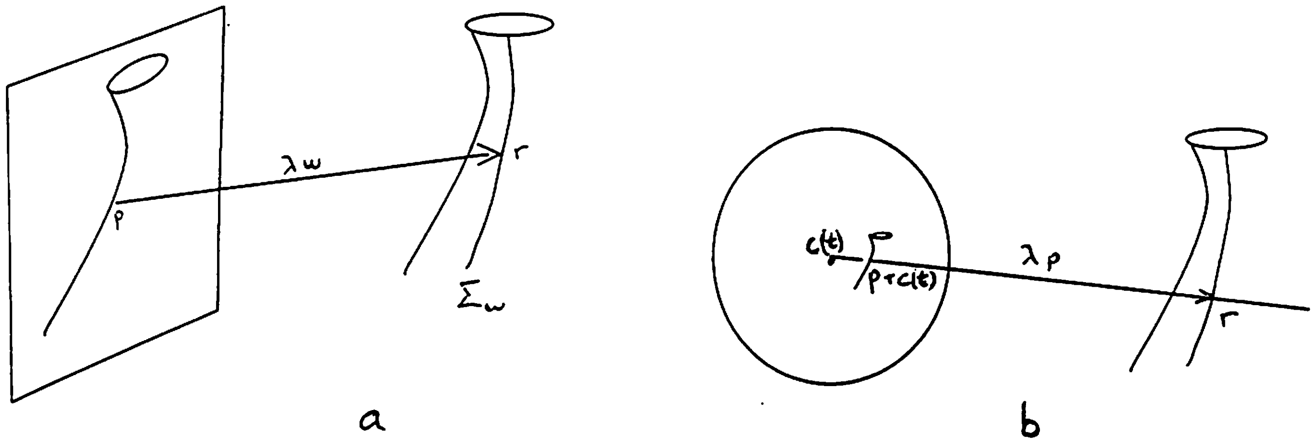


Figure 1: a) Parallel projection. b) Perspective projection.

p is regarded as a unit vector giving the direction of the viewline. We have

$$r = c + \lambda p \tag{2}$$

where here λ is the distance from c to r (the distance from the profile point $c + p$ to r being $\lambda - 1$). To allow for camera rotation we can take new coordinates q on the unit sphere, where $p = Rq, R$ being a rotation matrix and $R(0) =$ identity.

From (2) we have (suffices denoting derivatives)

$$r_t = c_t + \lambda_t p + \lambda p_t, \tag{3}$$

so that by taking the scalar product with the normal n to the surface at r, and using $r_t.n = 0, p.n = 0$, we obtain the distance formula (as in [3])

$$\lambda = \frac{-c_t.n}{p_t.n}. \tag{4}$$

Note that when we use rotated coordinates q instead of p, we have

$$p_t = Rq_t + \Omega \times Rq, \tag{5}$$

where Ω is parallel to the instantaneous rotation axis of R, with length equal to the instantaneous angular speed. It is usual to take $t = 0$ in (5), so that the equation reads $p_t = q_t + \Omega \times q$, since $R(0)$ is the identity.

We are interested in families of profiles, so that our view direction w or camera centre c will be functions of a variable t, and the rotation matrix R is also allowed to be a function of t. We assume this from now on. Given such a motion, a correspondence can be set up on the profiles as described below.

The **epipolar plane** at a profile point p is the plane spanned by the view direction w and tangent vector w_t, or the viewline $r - c$ and the tangent c_t to the motion. Note that the epipolar plane is *undefined* in the perspective case if $r - c, c_t$ are parallel. (Of course, for parallel projection w, w_t could only be parallel if $w_t = 0$.) The term 'epipolar plane' is borrowed from stereo, where the epipolar plane is the plane spanned by a viewline and the base line between the two camera positions. It is shown in [3, 13, 12] that there is considerable advantage to be gained from taking a parametrization of the surface M in which one set of parameter curves ($t =$ const.) consists of critical sets and the other set of parameter curves ($u =$ const. say) consists of **epipolar curves**, that is curves whose tangent vector is always along the viewline (or view direction). In the last paper, it is shown that the advantages of using the epipolar correspondence as opposed to other methods for defining a correspondence between points on two or more profiles is that the reconstruction can be transformed readily into an optimal estimation problem. Since the epipolar plane turns out to be the osculating plane for the epipolar curve through a given point, the viewlines are approximately coplanar and can be used to give a second order approximation of the curve. Some of the important geometric properties of reconstruction using the epipolar correspondences are presented in §2.

Note that when the epipolar parametrization is used, we have (for perspective projection)

$$r_t \| p, \tag{6}$$

since the curves $u =$ const. are tangent to the viewlines. For parallel projection we have r_t parallel to w.

There are circumstances where this desirable parametrization is impossible. One of these occurs when the critical set is *tangent* to the viewline: tangents to the two parameter curves at a point are not allowed to be equal in a regular parametrization. In that case it is well-known that the profile is *singular*, that is, has a cusp or worse singularity. (This is because the critical set, which is a space curve, is being projected in the direction of one of its tangents, resulting in a singular projected curve.) A similar problem will arise if the critical set itself becomes singular. This occurs when the point r on the surface is parabolic and in addition the viewline (or view direction) is along the unique asymptotic direction at r: see for example [9, p.458], 'lips and beaks' singularities, where the author refers to parabolic points as 'spinodal points'.

There is one other circumstance where the parametrization breaks down. This occurs when *the epipolar plane at p is the tangent plane to M at r*, i.e. when c_t is in the tangent plane at r, which is the same as saying that c_t is perpendicular to the surface normal n at r. See §3 below for a precise statement of when the epipolar parametrization fails.

There is another very geometrical way of describing this situation. Let $r(u,v)$ be a (regular, local) parametrization of the surface M, and let $n(u,v)$ be a nonzero normal vector at $r(u,v)$ (choosing some orientation for the normal such as that of $r_u \times r_v$). Then the family of critical sets is given by the equation

$$(r(u,v) - c(t)).n(u,v) = 0. \tag{7}$$

Regarding (7) as the equation of a family of curves in the (u,v) plane, parametrized by t, the *envelope* of the family is given by solving (7) simultaneously with the derivative of this equation with respect to t, namely $c_t.n(u,v) = 0$.

We call the envelope of critical sets on M the *frontier* of M, since, locally at least, the envelope separates those points of M which lie on some critical set (the points forming the **visible region**) from those which don't (forming the **invisible region**). See Figure 2. It is possible that, at some later time, critical sets will come to cover part of the invisible region; hence the emphasis here on localness. Note that Figure 2 shows the situation at as generic point of the frontier. In Figure 3 below we show what happens at a parabolic point of the frontier. To summarize:

Definition 1.1 *The frontier of M (relative to the given motion $c(t)$) is the envelope of critical sets on M, that is the points satisfying (7)*

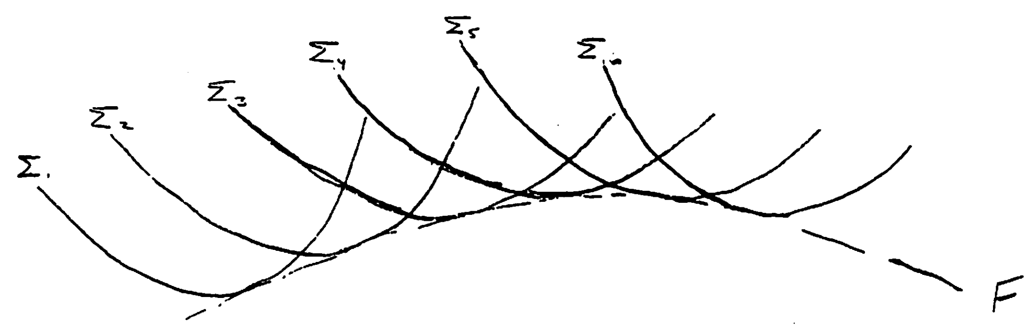


Figure 2: The frontier: envelope of critical sets on M, the generic case.

and $c_t.n(u,v) = 0$. *For parallel projection, the frontier points satisfy* $w_t.n(u,v) = 0$ *and* $w.n(u,v) = 0$. *The frontier contains the set of points* r *of* M *where the epipolar plane coincides with the tangent plane to* M.

The last statement follows because, at points where the epipolar plane coincides with the tangent plane to M, the vector c_t in the epipolar plane must be also a tangent vector to M, and hence perpendicular to n.

1.2 *Remark*

The only additional points of the frontier are those where $r - c$ and c_t (equivalently p and c_t) are parallel: here, the epipolar plane is undefined. These additional points are those where the camera motion is ***directly along the visual ray***. Although an epipolar parametrization of the surface is not possible here, one can consider $r(u,t)$ as a family of curves on the surface given by the critical sets. Similarly, $p(u,t)$ is a family of curves on the unit sphere given by the profiles. Even though there is no epipolar plane, the epipolar constraint (6) is still defined and can be used with (3) to show that p_t is parallel to p. But p_t is also perpendicular to p, since the latter is a unit vector, so, at such points when the epipolar constraint is applied, $p_t = 0$. Our analysis shows that the epipolar curves and critical sets at these points behave like those at a parabolic point on the frontier.

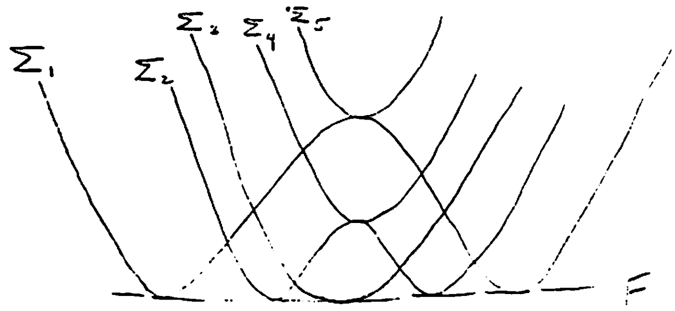


Figure 3: The frontier: envelope of critical sets on M, at a parabolic point.

It is geometrically clear that, along the envelope, the critical sets cannot be part of a coordinate grid on M: coordinate grid lines corresponding to constant values of one coordinate function are not allowed to intersect each other. We shall see later that the epipolar curves go badly wrong along the envelope too: they are in fact singular there. However, the spatio-temporal surface described in the next section does admit an epipolar parametrization. Before giving this construction, we will look at a surface which will be used as a running example for the ideas in this paper.

Example 1.3 ***The paraboloid***

Consider the surface $M: \; z = x^2 + y^2$, parametrized by $r(u,v) = (u, v, u^2 + v^2)$, and let $c(t) = (1, t, t^2)$ be the path traced out by the camera centres. (*Remark* It is no use having a straight line for the path of camera centres. For a straight line gives $c_{tt} \equiv 0$ which makes every point of the envelope a 'point of regression' [2, §5.26]. Indeed the envelope can reduce to isolated points instead of a curve: try $c(t) = (1, t, 0)$.)

For $c(t) = (1, t, t^2)$ the equation (7) becomes $f(u,v,t) = 0$ where

$$f(u,v,t) = (u-1)^2 + (v-t)^2 - 1. \qquad (8)$$

The critical sets in the plane of the parameters u, v are circles between the lines $u = 0, u = 2$ in the (u,v)-plane and these two lines form the envelope. That is, the frontier on the surface itself consists of the set of points of the form $(0, v, v^2), (2, v, 4+v^2)$ for arbitrary v. See Figure 4. The visible region in the parameter plane lies between the two lines $u = 0, u = 2$. ◇

2 Geometry of the viewlines

Suppose one has a regular parametrization of the surface M by $r(u,t)$ and a camera trajectory $c(t)$ such that for some fixed u_0 the view lines $c(t) + \lambda(u_0,t)p(u_0,t)$ are tangent to the surface at $r(u_0,t)$. Note that Proposition 3.4 gives the precise conditions under which this is possible. Thus, for the parameter curve $r(u_0,t)$, parametrized by t, there is a one parameter family of viewlines $l(t)$ such that $l(t)$ is tangent to the surface at $r(u_0,t)$. Intuitively, reconstruction algorithms are based on intersections of viewlines. In practice, these viewlines may not intersect and the points where they are closest can be considered as an approximation. In general, such an approximation may not even approach a point on the surface. However, there are cases where the closest points do converge to a point on the surface. In particular, if $l(t)$ is a tangent line in a family, then the point on $l(t)$ closest to the line $l(t+\epsilon)$ approaches $r(u_0,t)$ as $\epsilon \to 0$. There are two parametrizations which have this geometric property. One of these is the epipolar parametrization and the other is the normal parametrization. More formally, this can be stated as follows: the distance from the camera center to the surface at the point of tangency is to first order given by the distance from the camera center to the point where this viewline is closest to the viewline for a nearby camera position if and only if the parametrization satisfies either the epipolar or normal constraint. What is interesting about this result is that it gives a *new formula for the depth.*

Suppose the two viewlines are determined by $(c(t), r(u_0,t))$ and $(c(t+\epsilon), r(u_0, t+\epsilon))$ respectively. These lines in three-space will not in general intersect, and is possible to solve for the points where they are closest to each other. Let a be a point on the first line and b a point on the second line.

$$a = c(t) + \alpha p(u_0,t) \tag{9}$$
$$b = c(t+\epsilon) + \beta p(u_0, t+\epsilon) \tag{10}$$

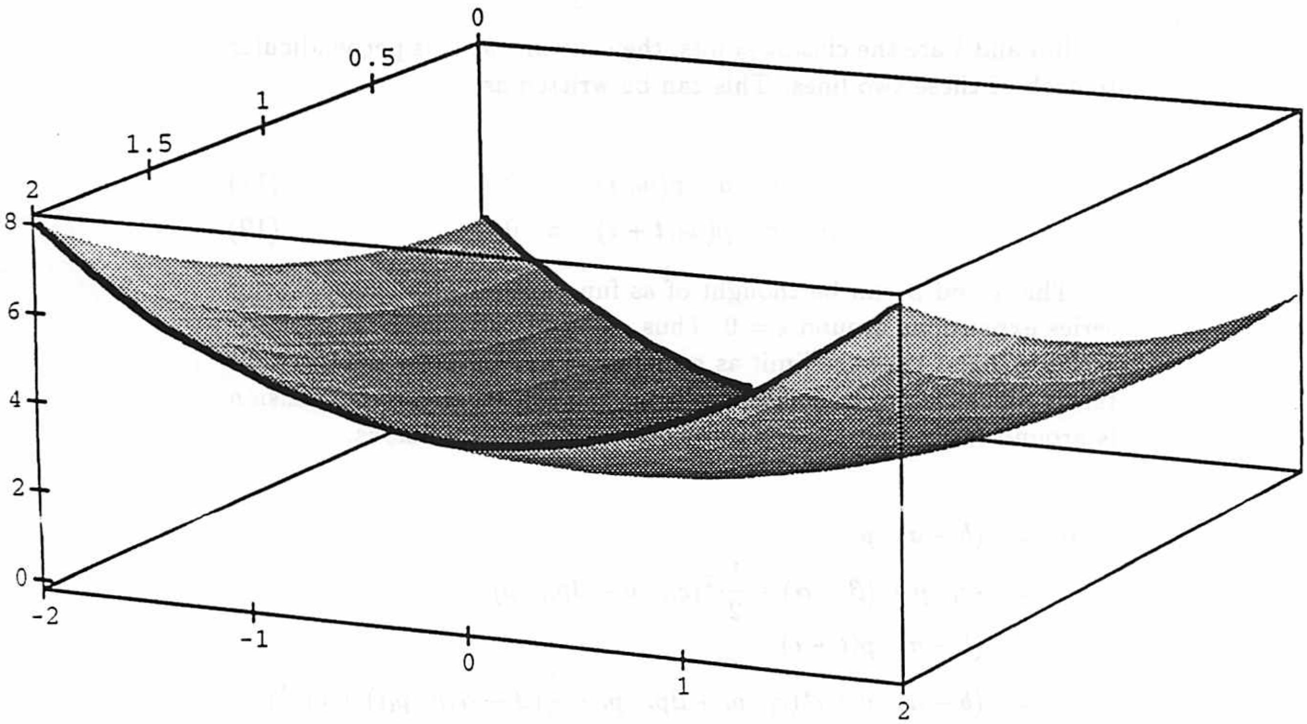


Figure 4: Visible region and frontier on the surface and in the parameter space of the surface of Example 1.3

$$b - a = \epsilon c_t + (\beta - \alpha)p + \beta\epsilon p_t + \frac{1}{2}\epsilon^2(c_{tt} + \beta p_{tt}) + o(\epsilon^3)$$

If a and b are the closest points, then the line $b-a$ is perpendicular to each of these two lines. This can be written as

$$(b - a) \cdot p(u_0, t) = 0 \quad (11)$$
$$(b - a) \cdot p(u_0, t + \epsilon) = 0 \quad (12)$$

The α and β can be thought of as functions of ϵ and have Taylor series expansions around $\epsilon = 0$. Thus, one can solve for α as a Taylor series in ϵ and take the limit as ϵ goes to 0, i.e. solve for the constant term. Since we are considering a fixed value of u_0 and the expansion is around t, we omit those parameters from the expressions.

$$\begin{aligned} 0 &= (b - a) \cdot p \\ &= \epsilon c_t \cdot p + (\beta - \alpha) + \frac{1}{2}\epsilon^2(c_{tt} \cdot p + \beta p_{tt} \cdot p) \\ 0 &= (b - a) \cdot p(t + \epsilon) \\ &= (b - a) \cdot p + \epsilon^2(c_t \cdot p_t + \beta p_t \cdot p_t + \frac{1}{2}(\beta - \alpha)p \cdot p_t t) + o(\epsilon^3) \end{aligned}$$

Manipulating these equations to solve for α gives, for $\epsilon \to 0$,

$$\alpha = \frac{-c_t \cdot p_t}{p_t \cdot p_t} \quad (13)$$

This is therefore the distance from the camera centre to the point a of closest approach of the two viewlines, in the limit. Note that this is defined whenever $p_t \neq 0$, including most points of the frontier, whereas formula (4) is not defined at the frontier. We now show that this is equal to the distance to the surface if and only if $r(u, t)$ is an epipolar or normal parametrization, i.e. $r_t \| p$ or $p_t \| n$.

Proposition 2.1 *If $r(u, t)$ is a parametrization of M with t being the parameter of a moving camera center, then for a point not on the frontier*

$$\frac{-c_t \cdot p_t}{p_t \cdot p_t} = \frac{-c_t \cdot n}{p_t \cdot n} \quad (14)$$

iff $r_t \| p$ or $p_t \| n$.

Proof: $\Leftarrow$ Case 1: If $r_t \| p$ then differentiating equation (2) one can see that c_t is in the plane of p and p_t, one writes

$$c_t = \xi p + \eta p_t$$

One can then take dot products with the surface normal and the tangent to the trace of the epipolar curve:

$$c_t \cdot n = \eta p_t \cdot n \quad (15)$$

$$c_t \cdot p_t = \eta p_t \cdot p_t \quad (16)$$

Eliminating η gives the desired equation.

Case 2: If $p_t \| n$ then substitution gives the desired equation.

$\Rightarrow$ Equation (14) can be rewritten as

$$(-c_t \cdot p_t)(p_t \cdot n) - (-c_t \cdot n)(p_t \cdot p_t) = 0 \quad (17)$$

This can be written in terms of cross products as

$$(p_t \times (c_t \times p_t)) \cdot n = 0 \quad (18)$$

This implies that $v = (p_t \times (c_t \times p_t))$ is in the tangent plane to the surface. Since this is a tangent vector and perpendicular to p_t, either $p_t \| n$ and every tangent vector is perpendicular to p_t or v is a multiple of p.

- Case 1: $(p_t \times (c_t \times p_t)) \| p$. By the fact that the cross product of two vectors is always perpendicular to each of the factors, $(p_t \times (c_t \times p_t))$ is always in the plane spanned by c_t and p_t. Then it follows that the triple product $[p, p_t, c_t] = 0$. This says that since generically $p_t \neq 0$, c_t is in the plane spanned by p and p_t. Using the equation

$$r_t = c_t + \lambda_t p + \lambda p_t,$$

  it follows that $[p, p_t, r_t] = 0$. Thus, r_t is in the epipolar plane as well, and by definition it is a tangent vector. Except at the frontier, the intersection of the epipolar plane and the tangent plane is just a line in the direction of p. This shows that $r_t \| p$.

- Case 2: $p_t \| n$. For the spherical image, the normal to the profile is the normal to the surface, so p_t is normal to the profile. This is the normal correspondence used in [3] for stationary curves on the surface and results in a parametization when tracking the profiles of critical sets, except when the profiles are singular.

3 The spatio-temporal surface

We introduce an auxiliary surface which will prove to be very useful in subsequent discussions. We work in the slightly more complex situation of perspective projection and make the following definition, which is given in two forms since M may be specified by a parameterization or implicitly as the zero locus of a function.

Definition 3.1 *Let M be a smooth surface, defined locally by a parametrization $(u, v) \to r(u, v)$. The **spatio-temporal surface** $\tilde{M}$ is defined to be the surface in R^3 (coordinates t, u, v) given by the equation*

$$(r(u, v) - c(t)).n(u, v) = 0, \tag{19}$$

where $n(u, v)$ is a nonzero normal vector at the point $r(u, v)$ of M. Compare the identical equation (7): the only difference here is that we make t an extra coordinate direction.

If M is defined by an equation $g(x, y, z) = 0$ (where we assume the partial derivatives g_x, g_y, g_z do not all vanish simultaneously on M, to ensure it is smooth), then the surface $\tilde{M}$ is defined in R^4 (coordinates t, x, y, z) by the equations $g = 0$ and

$$((x, y, z) - c(t)).(g_x, g_y, g_z) = 0. \tag{20}$$

The surface $\tilde{M}$ is closely allied to the 'spatio-temporal surface' of Faugeras [8]. Unless otherwise stated we use the 'parametrized' version (19) in what follows. Intuitively, we lift the critical set Σ_t on M to a 'height' t in a new coordinate direction, spreading out the critical sets in the t-direction, to make $\tilde{M}$.

A short calculation with the implicit function theorem [2] shows the following.

Proposition 3.2 *The spatio-temporal surface $\tilde{M}$ is smooth except at points (u, v, t) where all of the following happen: $r(u,v)$ is parabolic, the viewline $r - c$ is in the unique asymptotic direction at r, and r is a frontier point of M (i.e. $c_t \cdot n = 0$).*

The situation where $\tilde{M}$ is *not* smooth therefore corresponds to the case of 'lips/beaks singularities occurring on the frontier', which is not generic for a 1-parameter family of camera centres $c(t)$ and a generic smooth surface M. (As already cited, see [9, p.458] for more information on lips/beaks singularities.) For the paraboloid we have the following construction for $\tilde{M}$.

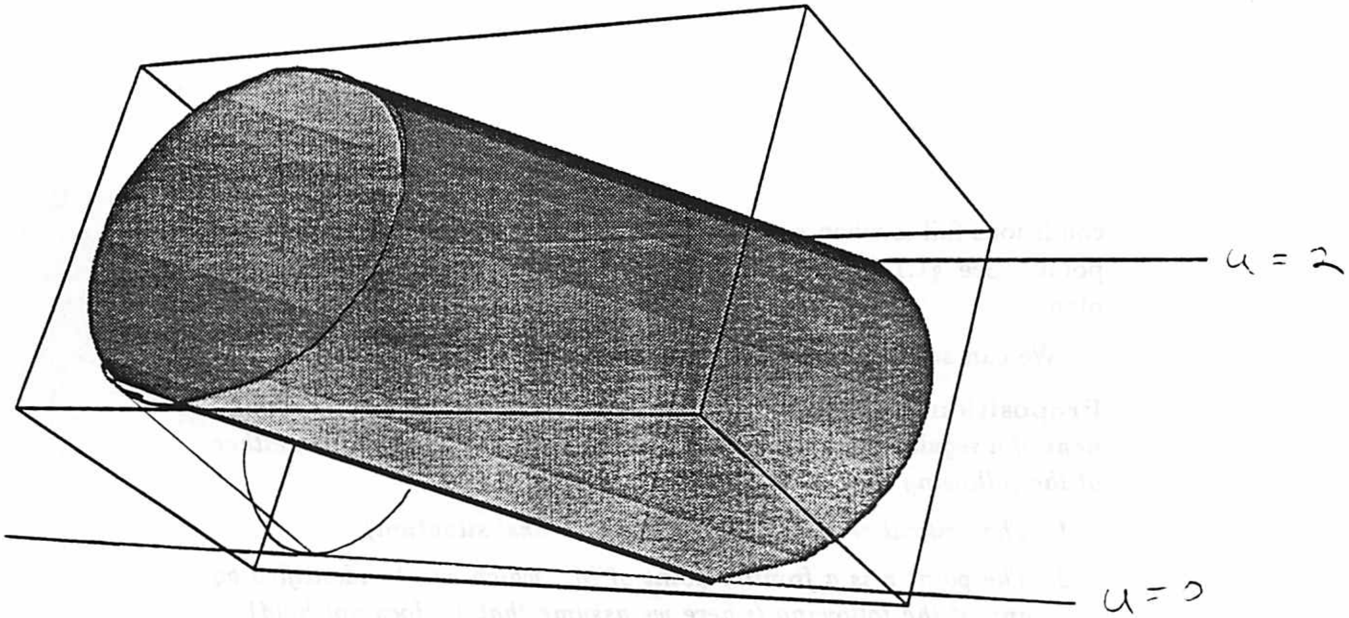


Figure 5: The spatio-temporal surface for the paraboloid example

Example 3.3 ***The Paraboloid, continued***

The equation (8) in addition to giving the critical set for each value of t is also the equation of $\tilde{M}$. Thus $\tilde{M}$ is a slanted cylinder all of whose horizontal sections are circular. See Figure 5. The projection from $\tilde{M}$ to M is not a diffeomorphism at points where $u = 0$ or $u = 2$; in fact it is a fold mapping, which is geometrically clear from the figure. The critical sets on M 'lift' to sets $\tilde{\Sigma}_t$ on $\tilde{M}$, namely the horizontal circular sections of $\tilde{M}$. ◇

The surface $\tilde{M}$ will be useful in several ways. For a first application, let us ask when we can use t and some other parameter u as a (local) regular parametrization of M. That is, when can the critical sets form one family of coordinate grid curves on M? This amounts to deciding the following:

1. When is $\tilde{M}$ parametrized locally by t, u or t, v?
2. When is the projection from $\tilde{M}$ to M given by $(u, v, t) \rightarrow r(u, v)$ a local diffeomorphism?

A routine calculation using the implicit function theorem shows that, assuming the critical sets to be smooth (i.e., avoiding the 'lips and beaks' situation), the only circumstance where both the above conditions fail is when $c_t.n(u,v) = 0$, that is when $r(u,v)$ is a frontier point. See §1.1 for the interpretation of this in terms of epipolar planes.

We can sum up the above conclusions as follows.

Proposition 3.4 *It is possible to use the parameter t as one component of a regular parametrization of M, locally at r, so long as neither of the following happens:*

1. *The critical set is singular ('lips or beaks' situation);*
2. *The point r is a frontier point of M, which can be identified by any of the following (where we assume that 1. does not hold):*
   - *The epipolar plane, which if it exists is spanned by the camera motion vector c_t and the viewline $r - c$, is the tangent plane to M at r,*
   - *The vector c_t is in the tangent plane to M at r,*
   - *The vector c_t is perpendicular to the profile normal at p (which is parallel to the surface normal at r),*
   - *The point r lies on the envelope of critical sets on M.*

If c_t is in the direction $r - c$, then the epipolar plane does not exist, but the point is still on the frontier. This can happen at isolated points on the frontier for isolated values of t. The critical sets of a surface r(u,v) are given by (r(u,v)-c(t)).n(u,v)=0, giving a 2-parameter family of solutions. The condition $c_t \| (r - c)$ is two constraints, so there can be isolated solutions. Note that, so far as determining the profile normal is concerned, it does not matter whether we use the p coordinates on the image sphere or the 'rotated coordinates' q. The converse of Proposition 3.4 is not true in the sense that the critical set can be singular at a point where the projection from $\tilde{M}$ to M is non-singular. However, t still cannot be used as a local parameter. This can be seen in the following example of a parabolic point which is not at the frontier.

Example 3.5 ***A non-frontier but parabolic point***

Consider the surface $M: z = x^2 + y^3$ which can be parametrized by $r(u,v) = (u, v, u^2 + v^3)$. Take the camera centre curve to be $c(t) = (0, 1, t)$. The equation (19) of $\tilde{M}$ is $f(u,v,t) = 0$, where

$$f(u,v,t) = ((u,v,u^2+v^3)-(0,1,t)).(-2u,-3v^2,1) = u^2-3v^2+2v^3+t.$$

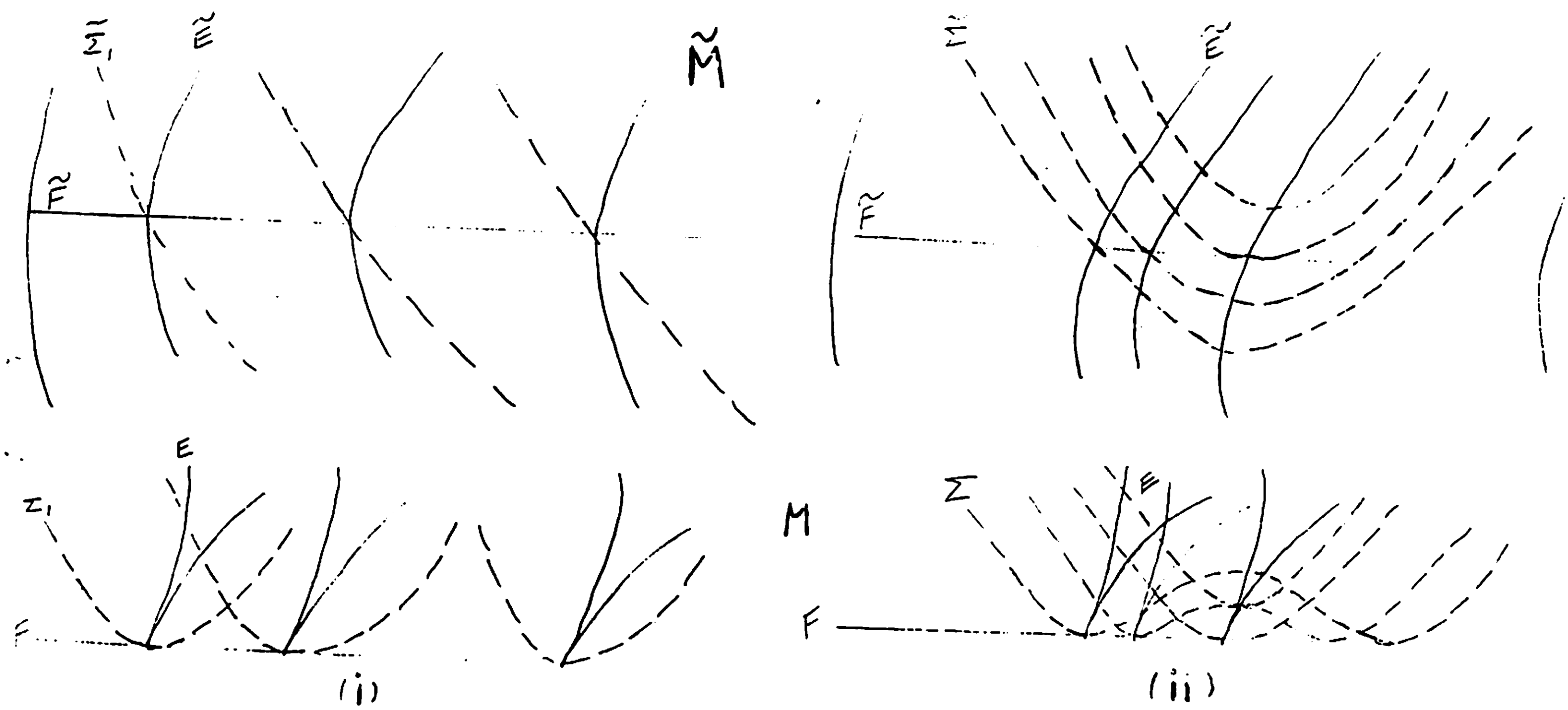


Figure 6: Projection from $\tilde{M}$ to M showing the critical sets and the frontier, (i) at a generic point of the frontier; (ii) at a parabolic point of the frontier

For $t = 0$ the critical set $f(u, v, 0) = 0$ is singular and the viewline is along the unique asymptotic direction to the surface M at the origin, namely the y-axis. To see this note that the normal plane containing the y-axis intersects the suface in the curve $(0, t, t^3)$, whose first and second derivatives vanish at $t = 0$. However, the projection from $\tilde{M}$ to M is actually nonsingular here: this is because $c_t(0).n(0,0) \neq 0$ is just the condition that the jacobian has full rank, which follows from the fact that the origin is *not* a frontier point. $\diamond$

Here is a second application of the surface $\tilde{M}$. In Figure 2 we have shown the pattern of critical sets along the frontier of M. Using $\tilde{M}$ we can find whether there are any exceptions to this simple picture. On $\tilde{M}$ the 'lifted critical sets' $\tilde{\Sigma}_t$ are simply the curves given by $t =$ constant. The 'lifted frontier' $\tilde{F}$ is the curve on $\tilde{M}$ given by the condition $c_t.n = 0$ as in §1.1. The interpretation of Figure 2 is that the $\tilde{\Sigma}_t$ are *transverse* (non-tangent) to $\tilde{F}$, so that the projection $\tilde{M}$ to M looks like Figure 6, left.

We can easily find the condition for the lifted critical set $\tilde{\Sigma}_t$ to be *tangent* to the lifted frontier $\tilde{F}$. For $\tilde{\Sigma}_t$ is given by the two equations

$(r-c).n = 0, t = \text{const.}$ and $\tilde{F}$ is given by the two equations $(r-c).n = 0, c_t.n = 0$. We simply require that the 3×3 matrix obtained by differentiating the three different equations here with respect to the three variables u, v, t should have determinant zero.

Proposition 3.6 *The lifted critical set $\tilde{\Sigma}_t$ and lifted frontier $\tilde{F}$ are tangent on $\tilde{M}$ if and only if the triple scalar product $[c_t, r-c, n_u \times n_v]$ is zero. This is so if and only if either*
(i) c_t is parallel to $r-c$ (the camera is heading straight along the view line), or
(ii) r is a parabolic point of M.

Proof: Evaluating the 3×3 determinant gives $(c_t.n_v)((r-c).n_u) - (c_t.n_u)((r-c).n_v)$, which rearranges to the triple scalar product. For the second assertion, note that c_t and $r-c$ are both tangent vectors (since we are considering a frontier point) and $n_u \times n_v$ is either zero or parallel to the normal vector n. If it is zero we have a parabolic point, and if not then the only way the triple scalar product can be zero is for c_t and $r-c$ to be dependent. □

The pattern on lifted critical sets on $\tilde{M}$, when $\tilde{\Sigma}_t$ is tangent to $\tilde{F}$, is therefore as shown in Figure 6, right, and the situation for the critical sets and frontier on M is easily derived by projection; it is also shown in the figure. There follows a simple example where the condition c_t parallel to $r-c$ holds. The pattern of critical sets for this case is the same as for a parabolic point on the frontier.

Example 3.7 ***The paraboloid, continued***

We consider again the paraboloid $z = u^2 + v^2$ but this time use the path $c(t) = (1-t, t^2, t^2)$, so that $c_t(0) = (-1, 0, 0)$ is along the visual ray from $c(0)$ to the origin. In this case the equation of $\tilde{M}$ turns out to be

$$(u-(1-t))^2 + (v-t^2)^2 = 1 - 2t + t^4,$$

and the additional equation of $\tilde{F}$ is $u - 2tv + t = 0$. Note that the critical sets, projected into the (u, v)-plane, are all circles. The frontier is not in this case parametrized by t near to $t = 0$, since there are *two* points of the frontier for each nonzero t close to 0. In fact, these are given by

$$v^2(1+4t^2) - v(2t^2+4t) + 2t = 0, u = (2v-1)t.$$

This equation has real distinct roots for v provided $t < 0$ (also for $t > 2.83$ approximately), so that each critical set for $t < 0$ is tangent twice to the frontier, and the critical sets then move away from the frontier for $t > 0$ (returning to it for $t > 2.83$). ◇

4 Epipolar curves

The epipolar field on M is, roughly speaking, the multi-valued vector field which associates to each point r of the visible region a vector in the viewline direction $r - c(t)$ for each t with r on the critical set Σ_t. The epipolar field becomes single-valued on the spatio-temporal surface $\tilde{M}$, which is another reason for studying that surface. The epipolar curves on M or $\tilde{M}$ are integral curves of the epipolar field. Before being more precise, we return to the paraboloid example.

Example 4.1 ***The paraboloid, continued***

We continue to take the curve of centres as $c(t) = (1, t, t^2)$. Given a point $r(u, v) = (u, v, u^2 + v^2)$, lying on a critical set Σ_t, we want the tangent vector to M which is along the viewline at r, i.e., along the direction $(u, v, u^2 + v^2) - (1, t, t^2)$. The required vector in parameter space is therefore simply along $(u - 1, v - t)$. Of course, we can eliminate t, but at the expense of making the multi-valuedness explicit: using (8) we find that the vector at (u, v) in parameter space is $(u - 1, \pm\sqrt{u(2 - u)})$, which happens to be of unit length.

To find the epipolar field on $\tilde{M}$ we need to find a tangent vector to $\tilde{M}$ at (u, v, t) which projects to a vector parallel to $(u - 1, v - t)$ under the projection $(u, v, t) \to (u, v)$. Using the gradient of f from equation (8) as the normal to $\tilde{M}$, we want a vector parallel to $(u - 1, v - t, \xi)$ satisfying

$$(u - 1, v - t, \xi).(u - 1, v - t, -(v - t)) = 0.$$

The solution for ξ is $1/(v - t)$, and the vector solution can be written so that the curves are parameterized as $(u(t), v(t), t)$. Such a vector is $((u - 1)(v - t), (v - t)^2, 1)$: we can take the epipolar field on $\tilde{M}$ to be given by this formula. (Below, in Proposition 4.4, we give a general prescription for finding the epipolar field in $\tilde{M}$.)

To find the epipolar curves on $\tilde{M}$ we want the solutions of the differential equation

$$\frac{dv}{dt} = (v-t)^2.$$

Substituting $w = v - t$ turns this into $dw/dt = w^2 - 1$, which gives $w = -\tanh(t+k)$ for any constant k, i.e. $v = t - \tanh(t+k)$. There are two 'exceptional' solutions, namely $v = t \pm 1$, which correspond to '$k = \mp\infty$'. Using equation (8), the corresponding solutions for u are $u = 1 \pm sech(t+k)$. The exceptional solutions for v both give $u = 1$. So the epipolar curves on $\tilde{M}$ are (for any constant k)

$$\begin{aligned} (u,v,t) &= (1 \pm sech(t+k), t + \tanh(t+k), t); \\ (u,v,t) &= (1, t \pm 1, t). \end{aligned} \tag{21}$$

Note that these curves are always nonsingular and are necessarily transverse to the 'lifted critical sets' $\tilde{\Sigma}_t$, which are given by $t =$ constant. This says that we can always parametrize $\tilde{M}$ locally with a coordinate grid consisting of the $\tilde{\Sigma}_t$ and the epipolar curves: 'the epipolar parametrization always works (locally) on $\tilde{M}$.'

The frontier is given by $c_t \cdot n = 0$, where $c_t = (0,1,2t)$ and $n = (-2u, -2v, 1)$. The epipolar field on M is obtained by projection from $\tilde{M}$, (so of course it becomes zero on the frontier, since $v = t$ there). The epipolar curves on M are obtained by treating the first and second components in (21) as parametrizations with respect to t. For example, consider the curve which, at $t = 0$, passes through $u = v = 0$. This is the curve

$$u = 1 - secht, v = t - \tanh t,$$

which has initial terms in its MacLaurin expansion

$$u = \frac{1}{2}t^2 + \ldots, \quad v = -\frac{1}{3}t^3 + \ldots.$$

This curve, like all the epipolar curves on M apart from the 'exceptional' curve $u = 1$, has an ordinary cusp where it meets the frontier. (The exceptional curve does not meet the frontier.) The shape of the epipolar curves in M is shown in Figure 4. The visible region here is that between the lines $u = 0, u = 2$.

The example is 'non-generic' to the extent that two epipolar curves on $\tilde{M}$ project to *indentically the same* epipolar curve on M . These are the curves $u = 1, v = t \pm 1$, which project to $u = 1$. Note that the two critical sets on M through points with $u = 1$ are *tangential.* This implies that the corresponding epipolar curves through such a point will also be tangential (the epipolar direction is determined by, indeed conjugate to, the tangent to the critical set). In the present case the epipolar curves are not merely tangential: they coincide. Note also that although the line $u = 1$ is in some sense exceptional, this does not prevent the use of local coordinates on M at a point on this line, with coordinate grid given by the epipolar lines and the critical sets. This ability to use an 'epipolar parametrization of M ', which is so important in [1, 3], depends only on the two sets of lines being *transversal* (non-tangent) at any point. Of course it fails hopelessly along the frontier. $\diamond$

The following definition has already been used informally in the above discussion.

Definition 4.2 *In the setup used above, with a surface M and a family of viewpoints $c(t)$ (resp. a family of view directions $w(t)$), an epipolar field on the visible region of M (the region covered by the critical sets) is a smooth nonzero (multi-valued) vector field which at any point r has vectors along the viewlines at r (resp. the view directions at r) corresponding to the critical set(s) passing through r.*

Let $\tilde{r} = (u, v, t)$ be a point of the spatio-temporal surface $\tilde{M}$. Thus, under the map $\pi : \tilde{M} \to M$ given by

$$(u, v, t) \to r(u, v),$$

the point $\tilde{r}$ goes to a point of the critical set Σ_t . An epipolar field in defined on the *whole* of $\tilde{M}$.

Definition 4.3 *An epipolar field on $\tilde{M}$ is a smooth vector field which associates to the point $\tilde{r}$, the tangent vector to $\tilde{M}$ projecting under π to a nonzero multiple of the viewline vector $r - c$.*

The above definitions do not specify the *lengths* of the vectors giving the vector field. This will not matter for us since we are only interested in the integral curves of the vector field, which we call the **epipolar curves** on (the visible part of) M or on $\tilde{M}$. Changing the lengths of the vectors only changes the parametrization of these curves. To make the vector fields definite we could normalise the vectors to length 1, say. Below, we give an explicit formula for this tangent vector field on $\tilde{M}$.

It is not hard to find explicitly the epipolar field on $\tilde{M}$ for a parametrized surface $r(u,v)$, where no assumption is made about this parametrization beyond that of regularity (r_u, r_v independent). The result is as follows.

Proposition 4.4 *An epipolar field on $\tilde{M}$ has the form*

$$((c_t \cdot n)\left(\frac{[r-c, r_v, n]}{\| n \|^3}\right), (-c_t \cdot n)\left(\frac{[r-c, r_u, n]}{\| n \|^3}\right), -II(r-c, r-c)) \quad (22)$$

Here, II is the second fundamental form of M (see e.g. [9, 10]), and $n = r_u \times r_v$ is normal to M. Note that the last component of this vector is zero if and only if the viewline is in an *asymptotic* direction at the surface point. This implies that the profile is singular at the corresponding point. At a frontier point, the first two components are zero and the epipolar field is 'vertical'. This says that *the epipolar curves on M are singular along the frontier of M '.*

Proof The equation of $\tilde{M}$ is given in 19 so that a general tangent vector to $\tilde{M}$ at (u, v, t) is say α, β, τ where

$$\alpha(r-c).n_u + \beta(r-c).n_v - \tau c_t.n = 0. \quad (23)$$

The image of the vector α, β, τ under the projection $M \to M$ is $\alpha r_u + \beta r_v$. We want this to equal $r - c$, which determines α and β since r_u, r_v are independent. Thus (23) determines τ so long as $c_t.n \neq 0$; the contrary case of course occurs precisely at the frontier. The final formula is independent of the frontier restriction since we can clear denominators.

Now $r - c = \alpha r_u + \beta r_v$ gives

$$(r-c) \times r_u = -\beta r_u \times r_v; \; (r-c) \times r_v = \alpha r_u \times r_v. \quad (24)$$

Hence

$$[r-c, r_u, r_u \times r_v] = -\beta \| r_u \times r_v \|^2,$$
$$[r-c, r_v, r_u \times r_v] = \alpha \| r_u \times r_v \|^2 .$$

The lifted tangent vector is therefore

$$(\alpha, \beta, \tau) = (\alpha, \beta, (\alpha(r - c).n_u + \beta(r - c).n_v)/c_t.n),$$

which is proportional to

$$c_t.n\alpha, c_t.n\beta, \alpha(r - c).n_u + \beta(r - c).n_v).$$

We can now substitute for α, β from (24). Note that n can here be *any* nonzero normal vector, for example $r_u \times r_v$.

Now $n_u.(r - c) = -II(r_u, r - c)$ provided n is a *unit* normal (see [10, p.190]). Using the linearity of II it is a simple matter to reduce the tangent vector to $\tilde{M}$ to the form given in the statement of the proposition.

4.5 *Notes*

1. The formula for the epipolar field on $\tilde{M}$ looks impressive, but if we regard $\tilde{M}$ as contained in $M \times R$ then it really says that the epipolar tangent vector to $\tilde{M}$ is along $(c_t.n(r-c), -II(r-c, r-c))$
2. It is a standard fact of surface geometry (see e.g. [3, Eq.(9)]) that $II(v, v)$, for a tangent vector v, is just the sectional curvature of M in the direction v, scaled by $\| v \|^2$. Thus, in our case, the term $II(r - c, r - c)$ in (22) can be rewritten κ^t/λ^2 where κ^t is the 'transverse curvature', i.e. the sectional curvature of M in the direction of viewing. Both quantities here can be measured from the image; see [3, §4].
3. Of course there is a similar formula to (22) in the case of parallel projection with variable viewing direction $w(t)$. In fact it is identical to the above formula, replacing $r - c$ by w and c_t by w_t, except that, for reasons of orientation, the sign in front of II becomes $+$. In the reinterpretation as in Note 2 above, we have simply $II(w, w) = \kappa^t$.

5 Epipolar constraints in the image sphere

In summary of reconstruction via epipolar curves, each point on the profile in the initial image frame is tracked to a point on the profile in each subsequent image using the epipolar constraint. The epipolar constraint is a line in the image which is intersected with a profile to give the correspondence to a point on a profile in a previous image. The correspondence is used to reconstruct a 3-D epipolar curve on the surface, which is part of a family of curves that can be used to parametrize the surface except at the frontier or when the profile is singular. While the camera is moving in space, the epipolar curves are generated on the surface, but we also are interested in what happens in the viewplane since all our measurements as well as the tracking are done there. In this analysis it should be kept in mind that because the camera frame is moving, a point in the viewplane at one instant in time is a different point in space at another. As the epipolar curve is traced each point appears on the profile at different time instant. One could look at the 3-D curve that is traced out by connecting the points in the different viewplanes as the camera moves. In the case of planar motion, this gives the pedal curve of the epipolar curve. If one identifies the different viewplanes using the coordinate system that is transported by the camera, then this trace of epipolar points is a curve in the viewplane, but *it is not the projection of a curve on the surface for any single view.* In addition, it depends strongly on the rotational motion of the camera. Nevertheless, we want to analyze this curve and the epipolar constraints that are used for tracking. As before the equation for the surface can be expressed in a rotating cordinate system as

$$r(u,t) = c(t) + \lambda(u,t)R(t)q(u,t) \tag{25}$$

From equation (5) we have

$$q_t = p_t - \Omega \times q \tag{26}$$

Assume that $c_t \neq \alpha p$ so that the epipolar plane is well-defined. Equation (3) together with the constraint $r_t \| p$ implies that p_t is in the epipolar plane spanned by c_t and p. But since p is a unit vector, it is also true that $p \cdot p_t = 0$. This means that the direction $\hat{p_t}$ of p_t is determined but not the magnitude. From equation (26) we have

$$\begin{aligned} q_t &= \alpha \hat{p_t} - \Omega \times q & (27) \\ q(t+\delta t) &= q(t) + (\alpha \hat{p_t} - \Omega \times q)\delta t + O((\delta t)^2) & (28) \end{aligned}$$

Thus, it can be seen that although q_t is determined by p_t, the direction of q_t cannot be obtained directly. The correct direction for q_t only emerges once the correct value for α is found. The epipolar constraint does not pass through the point q in the later image. From $q(t)$ go along the vector $(-\Omega \times q)\delta t$ and, at the endpoint, take the line in the direction p_t (this all takes place, strictly, in the tangent plane to the image sphere). This line meets the profile for $t + \delta t$ in the point satisfying the epipolar constraint. The next section shows more clearly what this looks like for a restricted type of motion.

6 An analysis of epipolar curves for circular motion and parallel projection

In this section we show what the epipolar constraint looks like in the case of *circular motion*, which is where the camera moves in circle on a sphere with the view direction passing through the center of the sphere and the viewplane rotating with the motion. This can also be thought of as an object rotating on a turntable. We also assume parallel projection to simplify the equations. First we re-derive the essential formulae for the frontier, spatio-temporal surface and epipolar curves. It turns out that for circular motion these formulae are particularly simple.

Let the point $a(t)$ rotate on a circle, which we regard as a circle of latitude at latitude β, on a sphere of radius ρ. Thus

$$a(t) = -\rho w(t), \text{ where } w(t) = (-c\cos t, -c\sin t, -s),$$

where $c = \cos\beta, s = \sin\beta$. Take a moving image plane with origin at $a(t)$ and normal vector $w(t)$. (We can also take $\rho = 0$, meaning that all the image planes pass through the origin. This is often done when setting up the distance formula in this context.)

Let us use coordinates (U, V) in this image plane, where a point p in space satisfies

$$p = a + Ue_1 + Ve_2,$$

$$e_1 = (-\sin t, \cos t, 0), e_2 = (s\cos t, s\sin t, -c),$$

so that e_1, e_2, w form a right-handed triad in 3-space. Note that

$$e_{1t} = cw - se_2, e_{2t} = se_1, w_t = -ce_1.$$

Thus by differentiation

$$p_t = (\rho c + U_t + Vs)e_1 + (V_t - Us)e_2 + Ucw.$$

The epipolar constraint is that r_t is parallel to w, and

$$r = p + \lambda w, r_t = p_t + \lambda_t w + \lambda w_t,$$

so this implies that p_t is in the plane of w and w_t, that is in the w, e_1 plane; hence the coefficient of e_2 in the expression for p_t is zero, i.e.

$$V_t = Us.$$

Going back to the formula for r_t (parallel to w) we deduce also that

$$U_t = \lambda c - Vs.$$

So we have the direction of the epipolar curve in the image plane given by $(U_t, V_t) = (\lambda c - Vs, Us)$. How do we construct this curve, which gives us the epipolar correspondence in the image plane? The construction is very similar to that for rotated coordinates in the image sphere given in §5 above. We have

$$U_t e_1 + V_t e_2 = p_t - a_t - U e_{1t} - V e_{2t},$$

where all the terms on the right-hand side are known besides p_t. We also know that p_t is in the plane of w, e_1, so that projecting p_t to the image plane we obtain a vector in the direction of e_1. The *length* of this vector is unknown. Projecting the other terms on the right-hand side into the image plane gives a known vector, namely $(\rho c - Vs)e_1 + Use_2 = b$, say. So from the current profile point p we take a vector $b\delta t$ and at the end of this vector we take the line in the e_1 direction. Where this meets the profile for $t+\delta t$ is the point of this next profile satisfying the epipolar constraint. We can see from the equations that this constraint in the rotating coordinate system is not a horizontal line nor does it pass through p.

7 Special circumstances where the epipolar parametrization fails

We have seen in Proposition 3.4 when it is possible to use the parameter t (which governs the camera centre motion) as one the the local parameters defining the surface M. Then the curves t = cont. on the surface M are simply the critical sets. We can use the epipolar

parametrization, in which the epipolar curves form the other set of parameter curves $u = \text{const.}$, provided the epipolar curves are (i) nonsingular and (ii) transverse (non-tangential) to the critical sets. In this section we list the possibilities where these conditions break down, so that an epipolar parametrization is impossible locally. The point of this is that when the epipolar parametrization is used to reconstruct the surface from the profiles, we will then be able to predict the way in which 'gaps', occurring through the failure of this parametrization, should be optimally filled.

According to Proposition 3.4, the 'bad' situations occur when the epipolar curves on M are singular, or when they fail to be transverse to the critical sets. The last simply means that the view direction and critical set tangent coincide, which is the same as saying that the profile is singular. So we need to consider the following special situations:

1. **Special non-frontier points r of the visible part of M :**
   (a) An epipolar direction at r is asymptotic, making the profile singular. Special cases of this are:
   (b) The point r is parabolic and one of the epipolar directions at r is asymptotic (creating a 'lips/beaks' transition on the profiles).
   (c) One of the epipolar directions at r is asymptotic with four-point contact (creating a 'swallowtail' transition on the profiles).
2. **Frontier points r of M :**
   (a) A general frontier point r.
   (b) A parabolic point r on the frontier.
   (c) A point on the frontier where c_t is along the viewline. (See Remark 1.2.)
   (d) The profile is singular at the point p in the viewplane corresponding to a frontier point r, i.e. an epipolar direction at r is asymptotic. Generically the frontier point will *not* be parabolic.

Most of these have been considered above, or are easy to deal with. In the next subsection, we consider the special cases in which the profile is singular. For example, in the 'lips/beaks' situation the only special thing is that the critical sets become singular: the epipolar curves are nonsingular. The most difficult case is the last one above, and we give brief details of that in §7.2. The figures 10-12 show the patterns of critical sets, the frontier and the epipolar curves in M and $\tilde{M}$.

7.1 Special non-frontier points

As noted above, if the profile has a cusp, then the critical set is tangent to the viewing direction, so there can be no epipolar parametrization at such points. As the camera moves, the point on the surface which generates this singular profile traces out a curve called the *cusp trajectory* on M. Unless the cusp point is a parabolic point, the critical sets are non-singular and transverse to the cusp trajectory. Recall that we are not at a frontier point here. Thus, in a neighborhood of the cusp trajectory away from parabolic points or endpoints, there is a parametrization such that the cusp trajectory is a parameter curve and the other parameter curves can be either critical sets or epipolar curves. If the surface were transparent, then this would be a parametrization of a surface, otherwise it is a surface with boundary, and the boundary is one of the parameter curves, namely the cusp trajectory. This boundary is also called the *natural boundary*. Note that the natural boundary separates the self-occluded points from the rest of the surface. In contrast with the frontier, which separates points which *can potentially* appear in profile from those that cannot, the natural boundary separates those points which actually do appear in profile from those that are obscured by another part of the surface. Thus, this type of boundary can only occur for non-convex objects or configurations of objects. The natural boundary can appear or disappear at swallowtail, lips, and beaks transitions.

For the 'lips/beaks' case, the critical set itself is singular, so it cannot be part of a parametrization. However, the epipolar curves are non-singular. Thus, it is necessary to find another family of curves transverse to the epipolar curves. The cusp trajectory is transverse to the epipolar curves, so there is a parametrization such that one family of curves is the epipolar family and the other contains the cusp trajectory which is the natural boundary (see Figs. 7,8). These figures also show the sequence of profiles for circular motion. Both the profiles and the trace of the epipolar curves have a cusp at the natural boundary.

A swallowtail point occurs when the tangent ray has order of contact four at a hyperbolic point, i.e. there are nearby tangents intersecting the surface at four points. This occurs along a flecnodal curve on the surface, and the camera center must lie on an asymptotic ray [9]. In general, the camera trajectory will only intersect the asymptotic developable surface of this flecnodal curve at isolated points. For opaque surfaces, a cusp trajectory and a t-junction trajectory will end at a swallowtail point. These two curves form the natural boundary (see Fig. 9.) Note that for the transparent surface, the cusp trajectory and t-junction trajectory each have two parts, so there are four curves that meet at the swallowtail, but only two of these are visible. In the viewplane for circular motion, the epipolar curves have cusps where the profiles have cusps, and they undergo a swallowtail transition as well. On the surface M, both the critical sets and the epipolar curves are non-singular at the swallowtail point; however, they are both tangent to the natural boundary at this point. In this case, only one of the three can be used.

7.2 Special frontier points

We can examine the case of a frontier point giving a singular profile point by considering a general surface $M: z = f(x, y)$, tangent to the (x, y)-plane at the origin, and a path $c(t) = (d + c_1(t), c_2(t), c_3(t))$ near to $t = 0$. We assume that $f_{xx}(0) = 0$ and $c_3'(0) = 0$, to make the profile singular and the origin on the frontier. Let us parametrize $\tilde{M}$ locally by x and t, which is valid provided $f_{xy}(0) \neq 0$. This just says that the origin is not parabolic on M. A straightforward, if lengthy, calculation shows that the epipolar field on $\tilde{M}$ close to $(x, y, t) = (0, 0, 0)$ (using x and y as local coordinates on M) has the following form, up to linear terms (all derivatives are at 0):

$$dx/dt = -d(c_2')f_{xy}x + d(c_3'')t,$$

$$dy/dt = d^2 f_{xxx}x + 2d(c_2')f_{xy}t.$$

This vector field has a *singularity* (a zero) at $x = t = 0$. Hence there are generically three possibilities for the nature of the integral curves: a node, a focus and a saddle (see for example [11, Ch.4] or any book

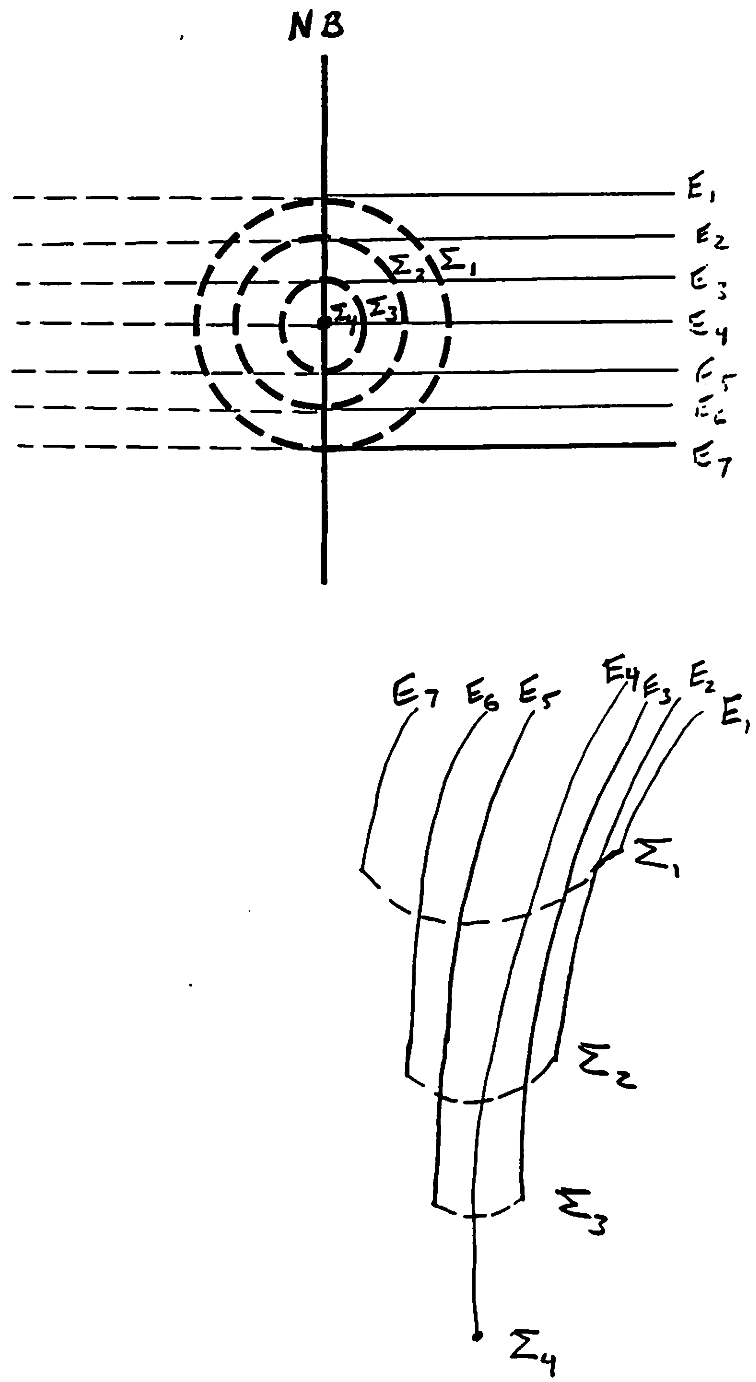
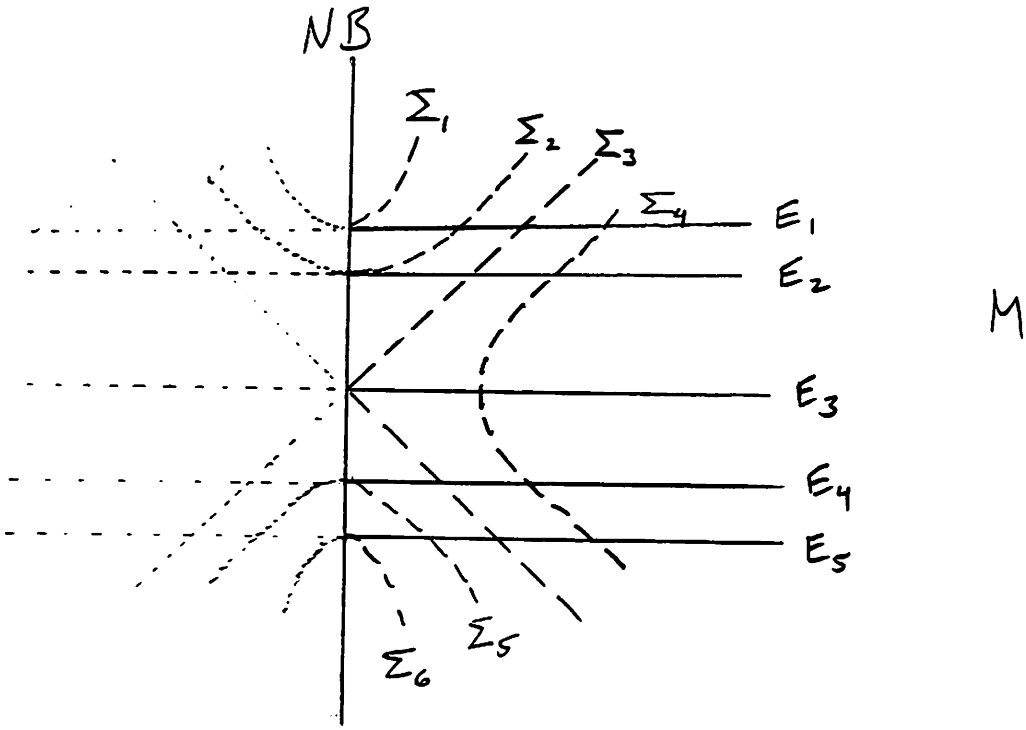


Figure 7: Lips: the top figure shows the critical sets, epipolar curves and natural boundary on M for a lips transition. At the transition the critical set is a point. Everything to the left of the natural boundary is occluded. The picture is the same for $\tilde{M}$ since the point is not on the frontier. The bottom picture shows the profiles and traces of epipolar curves in the image plane.



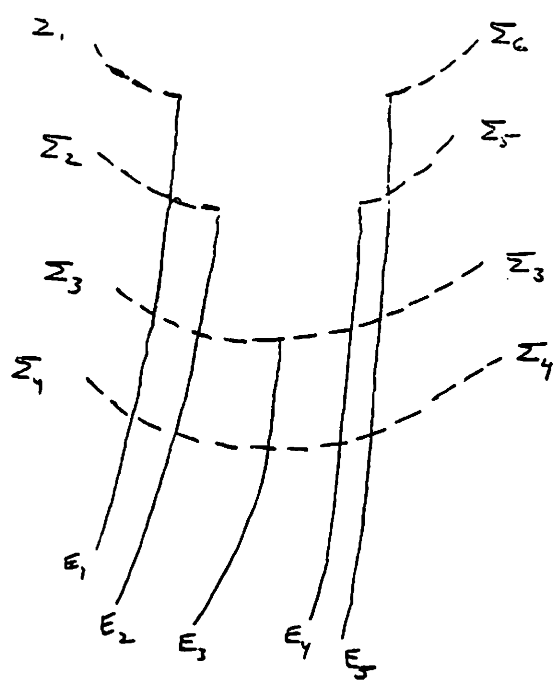


Figure 8: Beaks: the top figure shows the critical sets, epipolar curves and natural boundary on M for a beaks transition. At the transition the critical set consists of two straight lines. Everything to the left of the natural boundary is occluded. The picture is the same for $\tilde{M}$ since the point is not on the frontier. The bottom picture shows the profiles and traces of epipolar curves in the image plane.

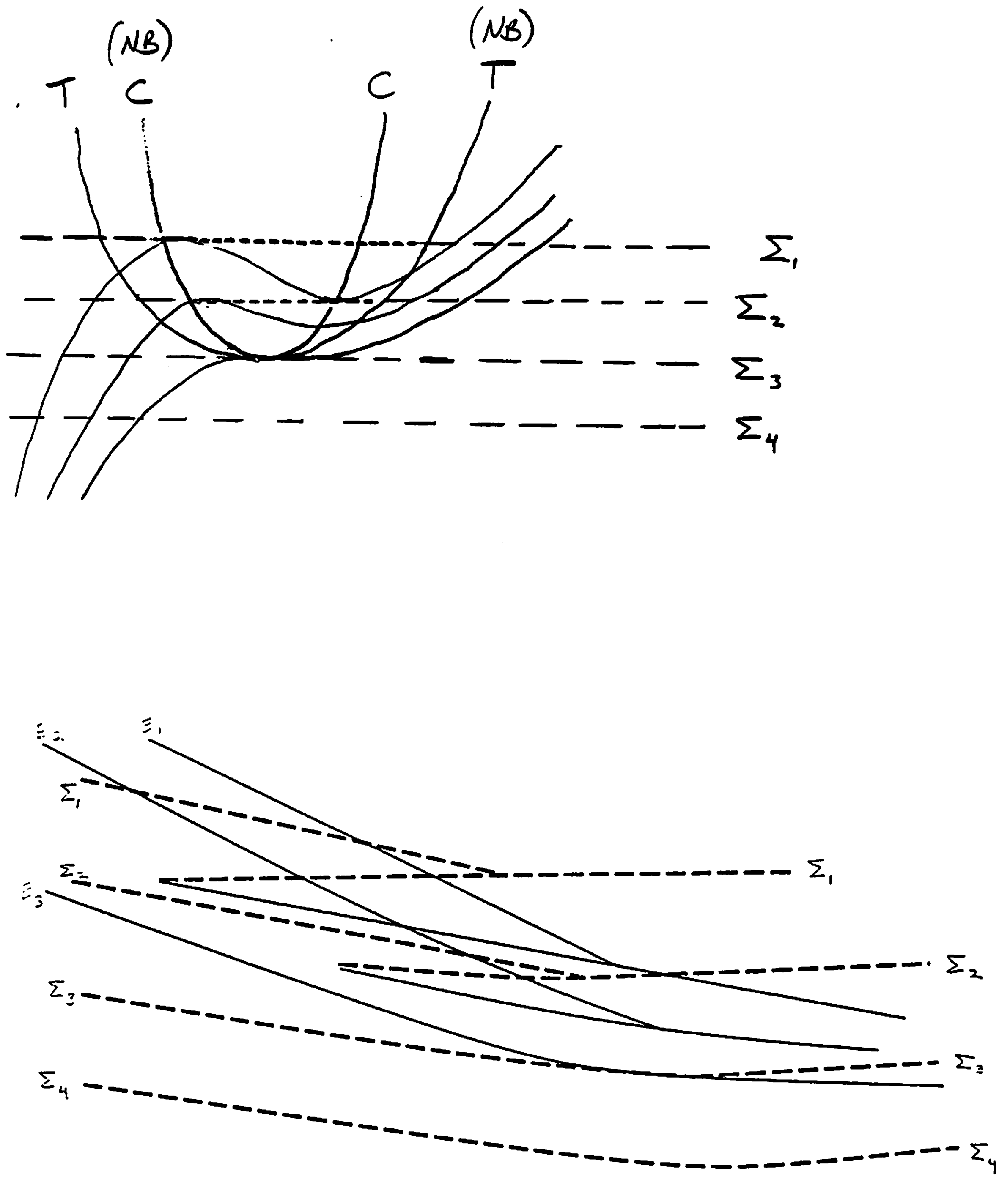


Figure 9: Swallowtail: the top figure shows the critical sets, epipolar curves and natural boundary on M for a swallowtail transition. Everything between the two branches of the natural boundary is occluded. The picture is the same for $\tilde{M}$ since the point is not on the frontier. The bottom picture shows the profiles and traces of epipolar curves in the image plane.

on elementary differential equations). Write A for the quantity

$$\frac{d(c_3'')f_{xxx}}{f_{xy}^2 c_2'^2}.$$

Then the distinction between the three cases is as follows:

1. **node**, i.e. the matrix of coefficients in the linearized vector field above has real distinct eigenvalues of the same sign, if and only if $-9/4 < A < -2$,
2. **saddle**, i.e. the matrix has real distinct eigenvalues of opposite signs, if and only if $A > -2$,
3. **focus**, i.e. the matrix has complex conjugate eigenvalues, if and only if $A < -9/4$.

The pattern of epipolar curves on the surface M (or rather on the surface with boundary which is the visible part of M) is shown also in Figures 10-12. Compare [4], where the classification of such vector fields on surfaces with boundary is given.

8 Conclusion

The epipolar constraint has been used for establishing correspondences between profiles of surfaces and results in an epipolar parametrization of regions of the surface. We have showed that such a parametrization breaks down at the frontier or when the profile is singular. A complete analysis of the frontier is given based upon the classification of vector fields on surfaces with boundary. In addition, even though the epipolar parametrization cannot be used at the frontier, the epipolar constraint still makes sense and the parametrization can be used for the spatio-temporal surface except when the profile is singular.

The analysis of the frontier presented in this paper is only one part of the problem of detecting gaps in the surface reconstruction. The analysis is also presented in this paper for the problem of occlusion. For example at T-junctions and cusps, part of the profile is occluded by the another part of the surface. This creates a *natural boundary* of visibility in addition to the frontier. The natural boundary can appear or disappear at swallowtail, lips, and beaks transitions.

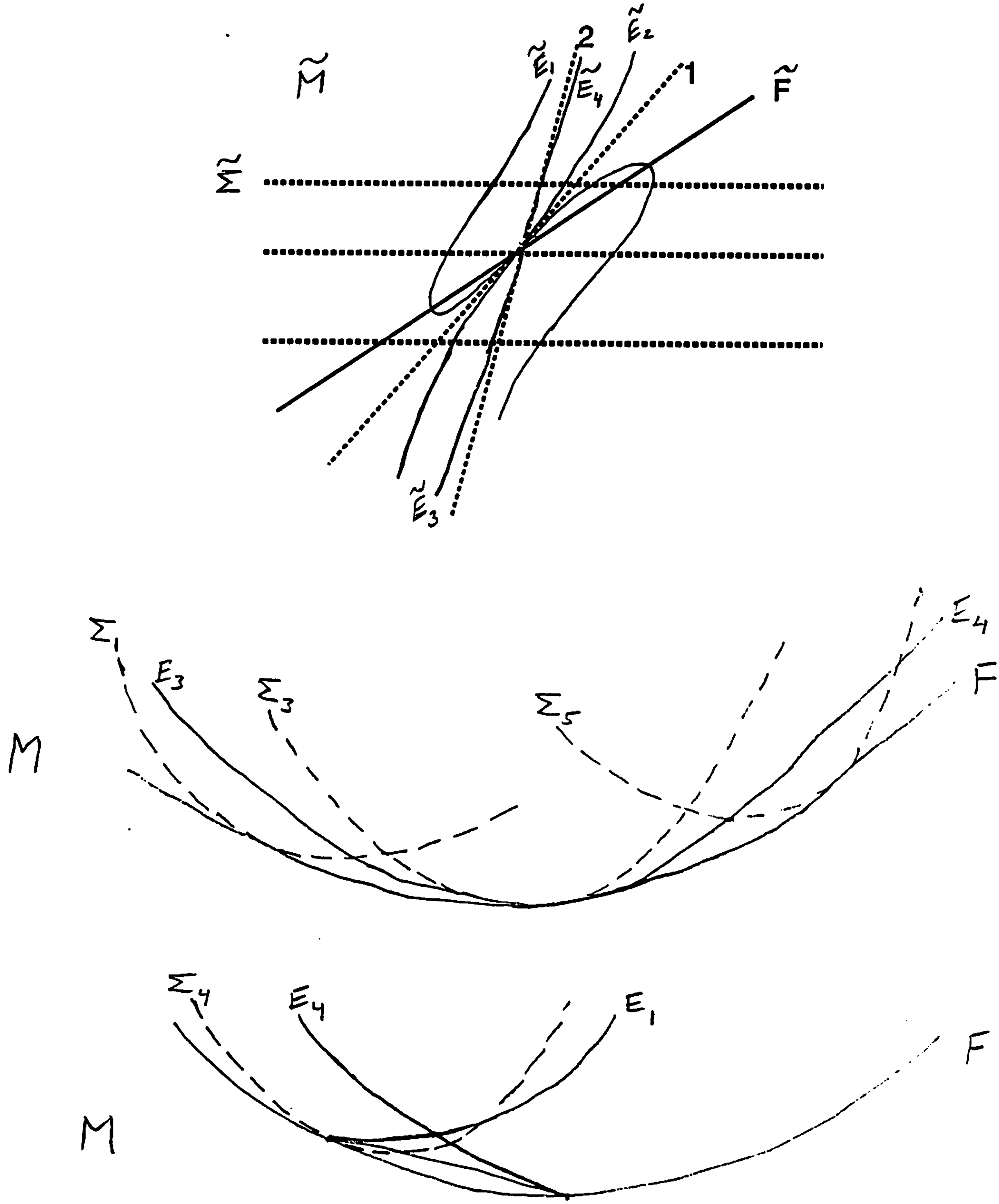


Figure 10: Case 1: Node of the epipolar field when the profile is singular at the frontier. There are two distinct eigenvalues with the same sign. The two eigendirections are labeled 1 and 2. Direction 2 is the exceptional direction, since there are only two epipolar curves tangent to it. All other epipolar curves are tangent to direction 1. On M, the top picture shows the exceptional case, and the bottom the general case.

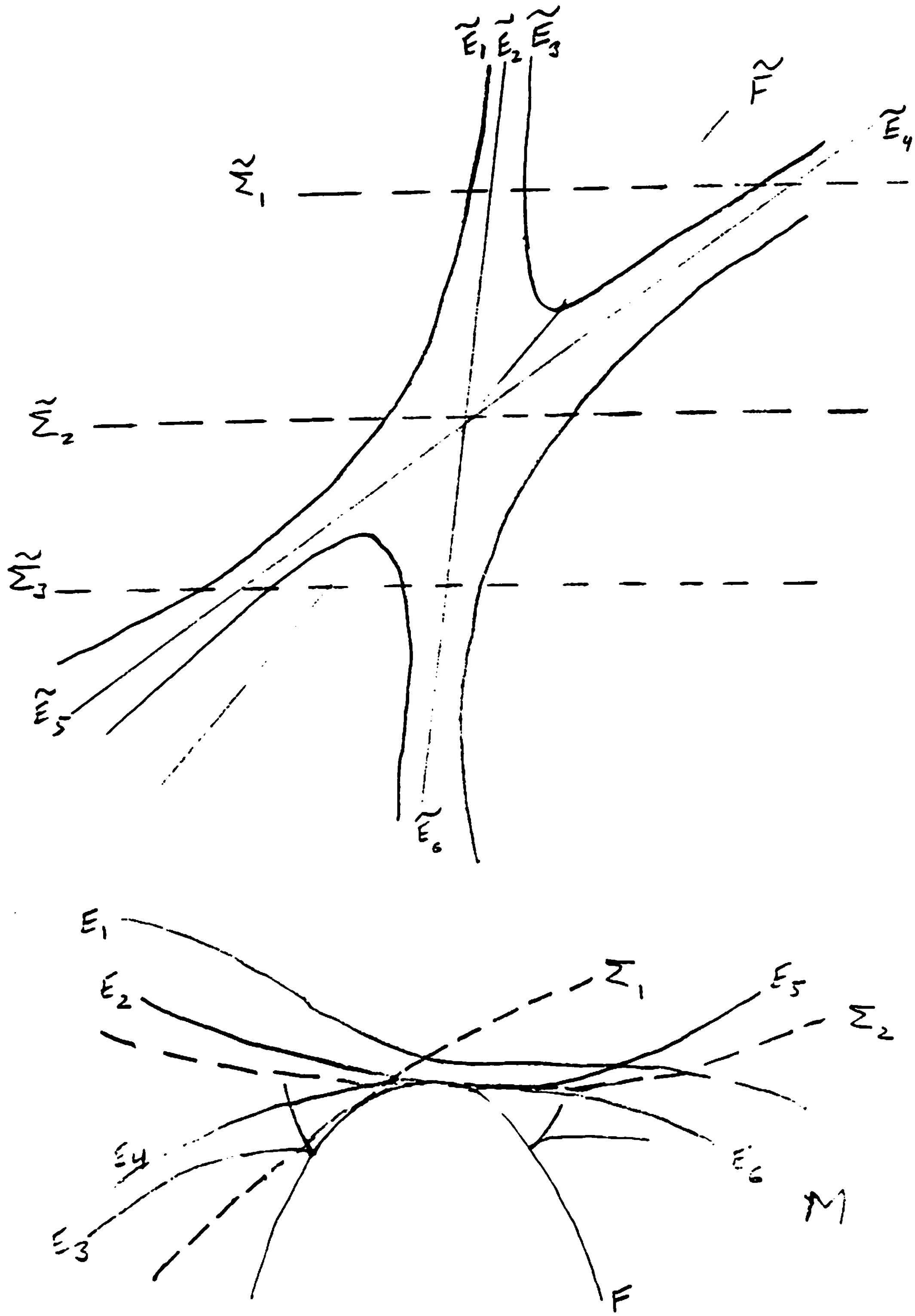


Figure 11: Case 2: Saddle of the epipolar field when the profile is singular at the frontier. Both of the epipolar lines through p_0 map to curves in M which are tangent to the frontier. In general neither eigenvector is in the kernel of the projection, so each epipolar curve maps to a smooth curve in M which is tangent to F.

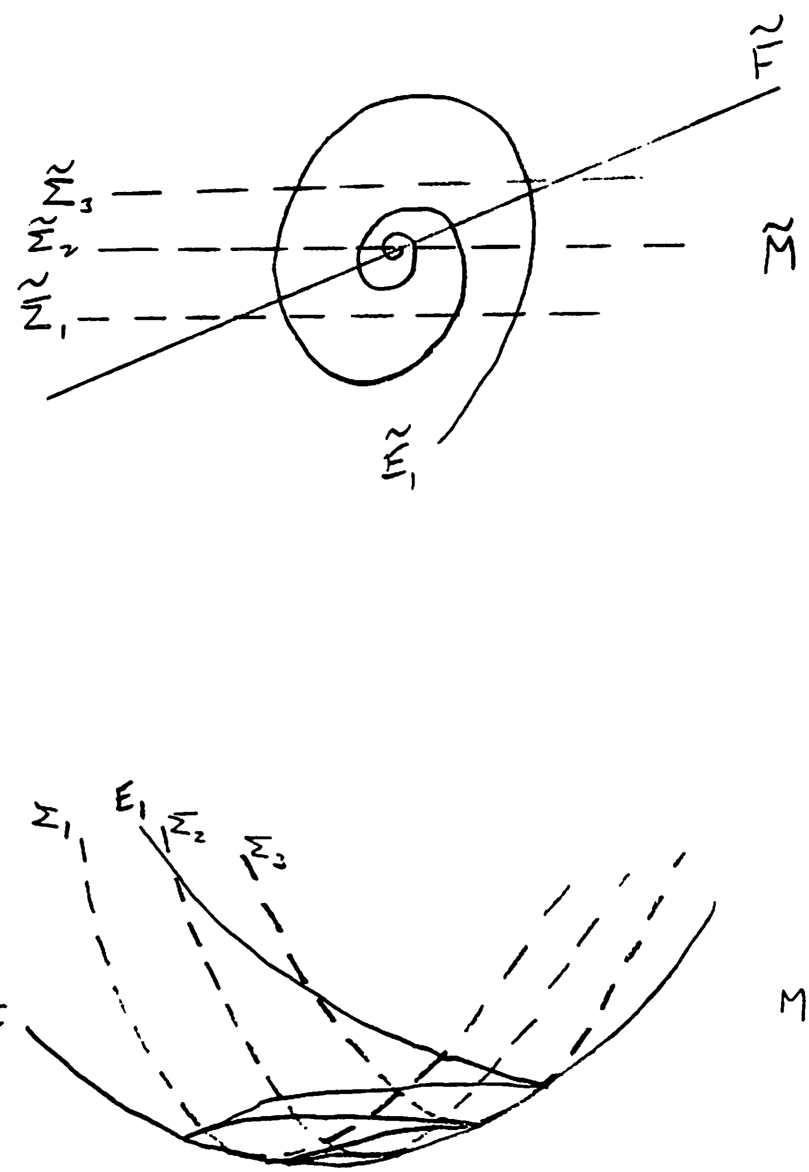


Figure 12: Case 3: Focus of the epipolar field when the profile is singular at the frontier. The eigenvalues are complex.

In addition to the natural boundary, which depends on the trajectory, there is a *global boundary of visiblity* which is trajectory independent, i.e. there may be some points which cannot be observed from any trajectory. This phenomenon is determined both by local and global geometry. The relevant local geometric information is the intersection of the tangent plane at a point with the inside of the surface. For an elliptic point, the tangent plane is on one side of the surface. Thus, it is outside for convex points and inside for concave points. Convex points will be visible if there are some rays in the tangent plane which do not intersect the surface elsewhere. Concave points will never be visible. Locally at hyperbolic points the tangent plane cuts the surface in two curves that cross, and whose tangent directions are the asymptotic directions. There will be a cone of directions in the tangent plane such that if some ray in this cone is not obscured by another part of the surface, then the point will be visible, i.e. this cone is the sector of the tangent plane which is outside of the surface. Parabolic and planar points are more difficult to characterize. For example, parabolic points on the 'outside' of a surface of revolution are visible, those on the 'inside' are not, e.g. the inside of a cup. Planar points are even more complicated, but for an ideal plane, the points are not visible in that the critical set moves infinitely fast over the surface. This topic is the subject of future work.

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