# Monochromatic Paths and Triangulated Graphs

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**Abstract**

This paper considers two properties of graphs, one geometrical and one topological, and shows that they are strongly related. Let $G$ be a graph with four distinguished and distinct vertices, $w_1, w_2, b_1, b_2$. Consider the two properties, $TRI^+(G)$ and $MONO(G)$, defined as follows.

$TRI^+(G)$: There is a planar drawing of $G$ such that:

- all 3-cycles of $G$ are faces;
- all faces of $G$ are triangles except for the single face which is the 4-cycle $(w_1 - b_1 - w_2 - b_2 - w_1)$.

$MONO(G)$: $G$ contains the 4-cycle $(w_1 - b_1 - w_2 - b_2 - w_1)$, and for any labeling of the vertices of $G$ by the colors {white, black}, such that $w_1$ and $w_2$ are white, while $b_1$ and $b_2$ are black, *precisely one* of the following holds.

- There is a path of white vertices connecting $w_1$ and $w_2$.
- There is a path of black vertices connecting $b_1$ and $b_2$.

Our main result is that a graph $G$ enjoys property $TRI^+(G)$ if, and only if, it is minimal with respect to property $MONO$. Building on this, we show that one can decide in polynomial time whether or not a given graph $G$ has property $MONO(G)$.

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# 1 Introduction

We consider drawings of simple graphs on the plane and on orientable surfaces. In a drawing $\mathcal{G}$ of a graph $G$ on an orientable surface, a vertex $v$ is represented by a point, and an edge between vertices $u$ and $v$ (denoted $u - v$) is represented by a curve joining its two endpoints. Two such curves do not intersect, except perhaps at their endpoints. When we delete from the surface all points and curves of $\mathcal{G}$, the surface is partitioned to (one or more) connected components called *faces*. If the topological boundary of a face is a cycle of $G$, we sometimes do not distinguish between the face and the cycle, referring to the cycle as a face.

Let $\Psi$ be a property of graphs. We say that a graph $G$ is *minimal with respect to* $\Psi$ if $G$ satisfies $\Psi$ and any proper subgraph of $G$ does not satisfy $\Psi$.

A *q-graph* is a simple graph with four distinguished and distinct vertices $w_1$, $w_2$, $b_1$, and $b_2$ which form a 4-cycle $(w_1 - b_1 - w_2 - b_2 - w_1)$. We refer to this 4-cycle as the *principal cycle* of $G$, and to the edges and vertices of the cycle as the *principal edges* and *vertices*. Other edges and vertices of $G$ are *nonprincipal*.

A *q-path* in a q-graph $G$ is a path[1] whose endpoints are either $(w_1, w_2)$ or $(b_1, b_2)$. A *valid coloring* of $G$ is a labeling of the vertices of $G$ by the colors {white, black} in such a way that vertices $w_1$ and $w_2$ are labeled white, and vertices $b_1$ and $b_2$ are labeled black. This paper is devoted to exposing strong interrelationships among the following properties of q-graphs.

**$TRI(G)$:** There is a planar drawing of $G$ in which all faces are triangles, except for one face, which is the principal cycle.

**$TRI'(G)$:** There is a drawing of $G$ on an orientable surface such that all faces are triangles, except for one face which is the principal cycle.

**$TRI^+(G)$:** There is a planar drawing of $G$ as per property $TRI$, and, in addition, every 3-cycle is a face. (Thomassen [2] used property $TRI^+$ to study 2-linked graphs. A graph satisfying $TRI^+$ is called there a *rib*.)

**$MONO'(G)$:** Any valid coloring of $G$ has a monochromatic q-path.

**$MONO(G)$:** $MONO'(G)$ holds, and, additionally, no valid coloring of $G$ has both a white q-path and a black q-path.

[1] A *path* in a graph $G$ is a sequence of vertices, wherein adjacent vertices are connected by an edge in $G$. A path is *simple* if no vertex occurs more than once. The *length* of the path is the number of edges, i.e., one less than the number of vertices.

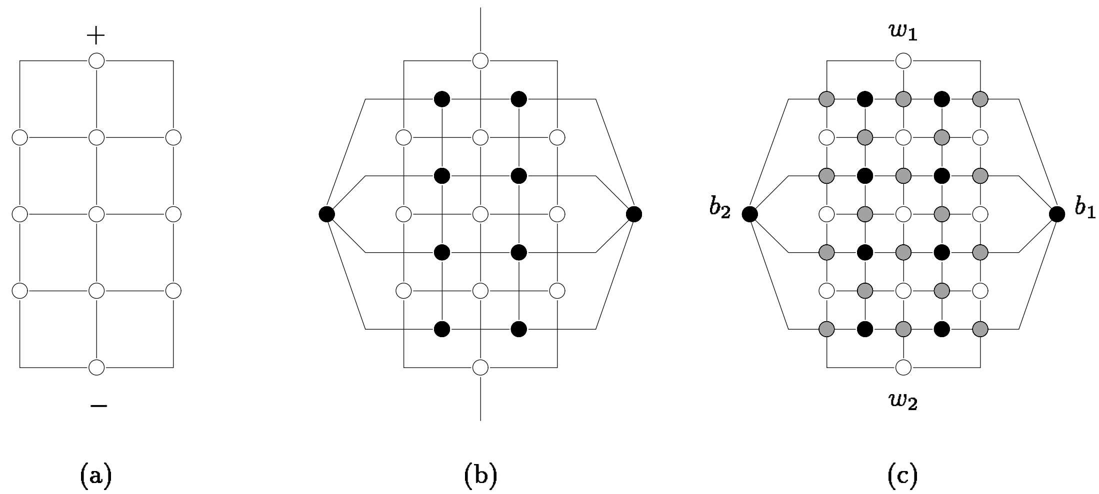


Figure 1: Transforming an instance of Shannon's game into a q-game. (a) Shannon's game. (b) Overlaying the dual graph. (c) Replacing crossings with vertices.

Our main results demonstrate the following relationships among these properties.

1. *If $TRI'(G)$ holds, then $MONO'(G)$ holds.*
2. *$TRI^+(G)$ holds if, and only if, $G$ is minimal with respect to property $MONO$.*

Building on these results, we show that one can decide in polynomial time whether or not a given graph $G$ enjoys property $MONO$.

The stimulus for this study comes from a two-player path-construction game that generalizes several other games, namely, Hex, Bridgit and the Shannon Switching Game [1]. This generalized game—let us call it a *q-game*—is played on a q-graph $G$. The game begins with $G$ in an *initial configuration:*

- Some vertices of $G$, in particular, $b_1$ and $b_2$, are colored black;
- some vertices of $G$, in particular, $w_1$ and $w_2$, are colored white;
- all other vertices of $G$ are uncolored.

The two players, called *Black* and *White*, alternately select an uncolored vertex and color it with their own color. The game concludes when all vertices are colored. Player Black (resp., Player White) wins if there is a black (resp., a white) q-path and no white (resp., no black) q-path in the fully colored $G$; otherwise, the game is a tie. In this context, property $MONO(G)$ means that there is no tie in a q-game based on the graph $G$.

Let us see how the q-game generalizes Shannon's Switching Game. Shannon's game is based on a graph $K$ having two distinguished vertices: $+$ and $-$. The two players, called *Short* and *Cut*, alternately select an unclaimed edge and claim it. The game concludes with Short winning if he owns a $+$ to $-$ path; otherwise, Cut wins.

If $K$ is a planar graph, drawn so that vertices $+$ and $-$ lie on the same face, then there is a q-game that generalizes the Shannon game. The q-graph and its initial coloring are constructed as follows. Add to the drawing of $K$ an edge from vertex $+$ to vertex $-$, drawn within the face that has both vertices. Call the augmented graph $K'$. On the drawing of $K'$, overlay the (geometric) dual of $K'$; see Figure 1. Now, replace each edge-crossing in the augmented drawing with a vertex, and add the principal edges (these edges are missing in Figure 1). In the initial configuration, color the vertices of $K'$ black and the vertices of the dual graph white, and leave the crossing vertices uncolored.

# 2 Property $TRI(G)$ Implies Property $MONO(G)$

**Theorem 2.1** *If $TRI'(G)$ holds, then $MONO'(G)$ holds.*

**Proof:** Let $\mathcal{G}$ be a drawing of $G$ as provided by property $TRI'$. Let us augment $\mathcal{G}$ by adding the edge $w_1 - w_2$ drawn within the face $(w_1 - b_1 - w_2 - b_2 - w_1)$. We call the augmented drawing $\mathcal{G}^+$, and, by extension, we call the new depicted graph $G^+$. (Note that graph $G^+$ may be not simple.) Clearly, all faces of $\mathcal{G}^+$ are triangles.

Consider now the multigraph $G^*$ that is the geometrical dual [3] of $\mathcal{G}^+$: the vertices of $G^*$ are the faces of $\mathcal{G}^+$, and the edges of $G^*$ are in one-to-one correspondence with edges shared by faces of $\mathcal{G}^+$. For each edge $e$ of $G^*$, let $e^\circ$ denote the corresponding edge of $G^+$.

Let a valid coloring of graph $G$ be given. In the resulting colored drawing, an edge or a face of $\mathcal{G}^+$ is called *bichromatic* if it has both black and white vertices. Define the *bichromatic subgraph* $K = \langle V, E \rangle$ of $G^*$ by:

$$
\begin{aligned}
V &= \{v \mid v \text{ is a bichromatic face of } \mathcal{G}^+\} \\
E &= \{e \mid e^\circ \text{ is a bichromatic edge of } G^+\}
\end{aligned}
$$

Since every face of $\mathcal{G}^+$ is a triangle, every vertex of $K$ has degree exactly two; hence, $K$ is a collection of disjoint simple cycles.

Let $C = (v_0 - v_1 - \cdots - v_{n-1} - v_0)$ be a cycle of $K$. Define the "circular list" $C_w \stackrel{\text{def}}{=} (x_0, x_1, \ldots, x_{n-1}, x_0)$ of white vertices of $G$ by

$$x_i = \text{ the white vertex on the edge } (v_i - v_{i+1})^\circ \text{ of } G^+$$

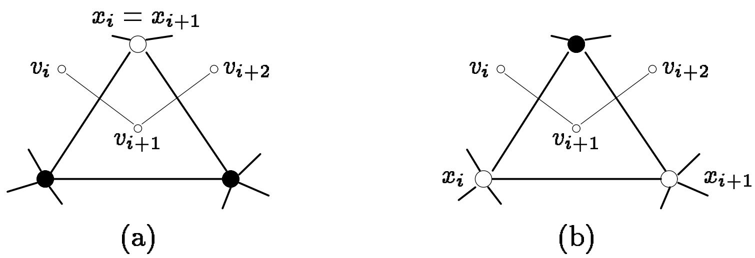


Figure 2: Construction of $C_w$.

(where addition on subscripts is modulo $n$). Let $x_i, x_{i+1}$ be any two consecutive vertices of $C_w$. If the face $v_{i+1}$ has exactly one white vertex, then $x_i = x_{i+1}$ (see Figure 2(a)); alternatively, if the face $v_{i+1}$ has two white vertices, then $x_i$ and $x_{i+1}$ are neighbors in $G^+$ (see Figure 2(b)). Let $\widehat{C}_w$ be the cycle in $G^+$ obtained by contracting each block of consecutive identical copies of a vertex $x$ in $C_w$ to a single copy of $x$. ($\widehat{C}_w$ is not necessarily a simple cycle.) Dually, we can define the "circular list" $C_b$ of black vertices of $C$, and the associated contracted cycle $\widehat{C}_b$ of black vertices in $G^+$.

Consider the triangles $t_1 = (w_1 - w_2 - b_1 - w_1)$ and $t_2 = (w_1 - w_2 - b_2 - w_1)$, which are faces of $\mathcal{G}^+$, hence vertices of $K$. Let $C$ be the cycle of $K$ which contains triangle $t_1$. On the one hand, if cycle $C$ also contains triangle $t_2$, then both $b_1$ and $b_2$ are in $\widehat{C}_b$, so that $\widehat{C}_b$ contains a path $P$ of black vertices connecting $b_1$ and $b_2$. Since path $P$ does not use the edge $w_1 - w_2$, it is a path in the original graph $G$. On the other hand, if cycle $C$ does not contain triangle $t_2$, then the edge $w_1 - w_2$ appears exactly once in $\widehat{C}_w$. Hence, $\widehat{C}_w$ contains a path of white vertices connecting $w_1$ and $w_2$ which does not use the edge $w_1 - w_2$.

We have thus shown that graph $G$ has property $MONO'(G)$. □

Note that the proof does not use the fact that graph $G$ is drawn on an orientable surface. Hence, Theorem 2.1 holds for graphs drawn on any two-dimensional manifold.

Since a planar q-graph clearly cannot have two disjoint q-paths, one with endpoints $(b_1, b_2)$ and one with endpoints $(w_1, w_2)$, Theorem 2.1 actually implies the following.

**Lemma 2.2** *If $TRI(G)$ holds, then $MONO(G)$ holds.*

# 3 $TRI^+(G)$ Holds iff $G$ Is $MONO$-Minimal

Let $G$ be a graph and $Q$ a set of vertices and edges of $G$. Let us denote by $G \backslash Q$ the subgraph of $G$ generated by removing: all edges of $Q$; all vertices of $Q$ as well as their incident edges.

Let $G$ be a q-graph. We say that $G'$ is a *q-subgraph* of $G$ if $G'$ is a subgraph of $G$ and a q-graph (having the same principal vertices as $G$).

A *trail* $T$ of a q-graph $G$ is a simple path in $G$ such that:

1. The endpoints of $T$ are $(w_1, w_2)$ or $(b_1, b_2)$; i.e., $T$ is a q-path.
2. $T$ does not contain the other two principal vertices.
3. For any vertices $u$ and $v$ of $T$: if $u$ and $v$ are adjacent in $G$, then $u$ and $v$ are adjacent in $T$.

**Lemma 3.1** *Let $G$ be a q-graph and $P$ a q-path in $G$ whose only principal vertices are its endpoints. Then there a trail $T$ in $G$ whose vertex-set is a subset of the vertex-set of $P$.*

**Proof:** We lose no generality by assuming that $P$ is a $b_1$-to-$b_2$ q-path. Let $P'$ be the subgraph of $G$ induced by the vertex-set of $P$. One verifies easily that any minimal-length $b_1$-to-$b_2$ path in $P'$ is a trail in $G$. □

A consequence of Lemma 3.1 is that any q-graph having property $MONO$ has a trail.

**Lemma 3.2** *Let $MONO(G)$ hold, and let $T$ be a $b_1$-to-$b_2$ trail in $G$. Then:*

***(a)*** *$w_1$ and $w_2$ are not connected in $G \backslash T$.*

***(b)*** *Any simple $b_1$-to-$b_2$ path in $G$ whose vertex-set is a subset of $T$'s coincides with $T$.*

***(c)*** *For any nonprincipal vertex $v$ of $T$, there is a $w_1$-to-$w_2$ trail $T'$ such that $v$ is the only vertex common to $T$ and $T'$.*

***(d)*** *Every q-subgraph of $G$ that has property $MONO(G)$ has trail $T$ as a subgraph.*

Clearly, by symmetry, we may interchange the roles of $(w_1, w_2)$ and $(b_1, b_2)$ in the lemma.

**Proof:**

**(a)** If $w_1$ and $w_2$ were connected in $G \backslash T$, then one would be able to color $G$ in a way that simultaneously produces a white and a black q-path, contradicting property $MONO(G)$.

**(b)** Let $P = (u_1 - u_2 - \cdots - u_m)$ be a simple $b_1$-to-$b_2$ path whose vertices all appear in trail $T = (v_1 - v_2 - \cdots - v_n)$. Assume that $P \neq T$, and let $i$ be the smallest index such that $u_i \neq v_i$. Clearly, then, vertices $u_{i-1}$ and $u_i$ are adjacent in $G$ but are not adjacent in $T$, contradicting the definition of "trail."

**(c)** Let $C$ be the valid coloring of $G$ whose only black vertices are the vertices of $T \backslash \{v\}$. By (b), $G$ cannot have a black q-path under coloring $C$. Because $MONO(G)$ holds, then, $G$ must have a white q-path under coloring $C$; in fact, by Lemma 3.1, $G$ must have a white trail $T'$. By (a), trail $T'$ must intersect trail $T$. Since $v$ is the only white vertex of $T$ under coloring $C$, it must be the only vertex common to trails $T$ and $T'$.

**(d)** Assume, for contradiction, that the q-subgraph $G'$ of $G$ has property $MONO(G')$ but does not contain trail $T$ as a subgraph. Let $C$ be the coloring of $G'$ that colors all vertices of $T \cap G'$ black and colors all other vertices white. By (b), $G'$ cannot have a black q-path under coloring $C$. Since $G'$ has property $MONO(G')$, it must contain a white q-path $P$ under coloring $C$. However, such a path $P$ would be a path in $G$ that is disjoint from trail $T$. By (a), such a path cannot exist. □

In what follows, we concentrate on $b_1$-to-$b_2$ trails for definiteness. The entire development dualizes to $w_1$-to-$w_2$ trails by interchanging the roles of the principal sets $\{b_1, b_2\}$ and $\{w_1, w_2\}$.

Let $G$ be a q-graph, and let $T$ be a $b_1$-to-$b_2$ trail in $G$. Define the following subgraphs of $G$.

$A_1^{(T)} \stackrel{\text{def}}{=}$ the connected component of $G \backslash T$ that contains principal vertex $w_1$.

$A_2^{(T)} \stackrel{\text{def}}{=}$ the connected component of $G \backslash T$ that contains principal vertex $w_2$.

$X^{(T)} \stackrel{\text{def}}{=} G \backslash (T \cup A_1^{(T)} \cup A_2^{(T)})$.

Note that the four graphs $T$, $A_1^{(T)}$, $A_2^{(T)}$, and $X^{(T)}$ form a partition of $G$: the graphs collectively contain all vertices of $G$, while property $MONO(G)$ implies that the graphs are disjoint.

Given $G$ and $T$ as above, define the q-graphs $G_1^{(T)}$ and $G_2^{(T)}$ as follows. For $i = 1, 2$, let $G_i' \stackrel{\text{def}}{=} G \backslash (A_i^{(T)} \cup X^{(T)})$. Construct the graph $G_i^{(T)}$ by adding to $G_i'$ the vertex $w_i$, as well as the edges $w_i - t$ for every vertex $t$ of $T$. See Figure 3.

**Lemma 3.3** *If $MONO(G)$ holds, and if $T$ is a trail in $G$, then both $MONO(G_1^{(T)})$ and $MONO(G_2^{(T)})$ hold.*

**Proof:** By symmetry, it suffices to establish that $MONO(G_1^{(T)})$ holds. To this end, let $C$ be a valid coloring of $G_1^{(T)}$. Extend $C$ to a coloring of $G$ by labeling all vertices of $A_1^{(T)} \cup X^{(T)}$ white.

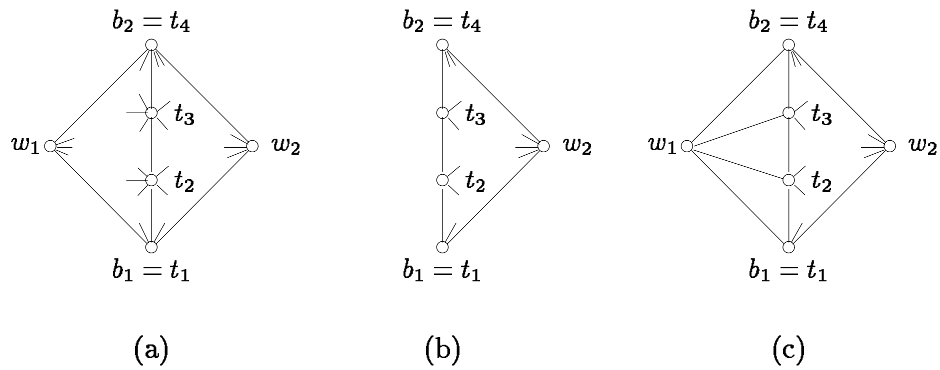


Figure 3: Construction of $G_1^{(T)}$: (a) $G$ and $T$, (b) $G_1'$, (c) $G_1^{(T)}$.

Now, if $MONO(G)$ holds, then $G$ has a monochromatic q-path $P$. If $P$ is a black q-path, then $P$ is a subgraph of $G_1^{(T)}$. Alternatively, if $P$ is a white q-path, then by Lemma 3.2.a, $P$ intersects $T$. From the way we have constructed $G_1^{(T)}$, it should be clear that, in this case, we can construct a white q-path in $G_1^{(T)}$ from $P$. (In short, we replace an initial segment of $P$ by the edge from $w_1$ to the last vertex of $T$ that appears in $P$.) We have thus shown that $MONO'(G_1^{(T)})$ holds.

To finish the proof, we must show that $G_1^{(T)}$ never simultaneously has both a black q-path and a white one. Assume, for contradiction, that, under coloring $C$, $G_1^{(T)}$ does simultaneously contain the black q-path $P^{(b)}$ and the white q-path $P^{(w)}$. Then $P^{(b)}$ is a black q-path in $G$; moreover, Lemma 3.2.c assures us that we can use $P^{(w)}$ to construct a white q-path in $G$ that coexists with $P^{(b)}$. This, however, contradicts property $MONO(G)$. □

We are finally ready for our weak converse to Lemma 2.2.

**Lemma 3.4** *If $MONO(G)$ holds, then $G$ has a q-subgraph $G'$ for which $TRI(G')$ holds.*

**Proof:** We prove the lemma by induction on the number of vertices of $G$. If $G$ has no more than four vertices, then direct inspection verifies that $TRI(G)$ holds. Henceforth, therefore, we assume that $G$ has more than four vertices, and we consider five exhaustive, but not necessarily disjoint, cases.

**Case 1**. *$G$ has a trail $T$ such that both $A_1^{(T)}$ and $A_2^{(T)}$ have at least two vertices.*

With no loss of generality, say that $T$ is a $b_1$-to-$b_2$ trail. Now, since $MONO(G)$ holds, Lemma 3.3 assures us that both $MONO(G_1^{(T)})$ and $MONO(G_2^{(T)})$ hold also. Let us

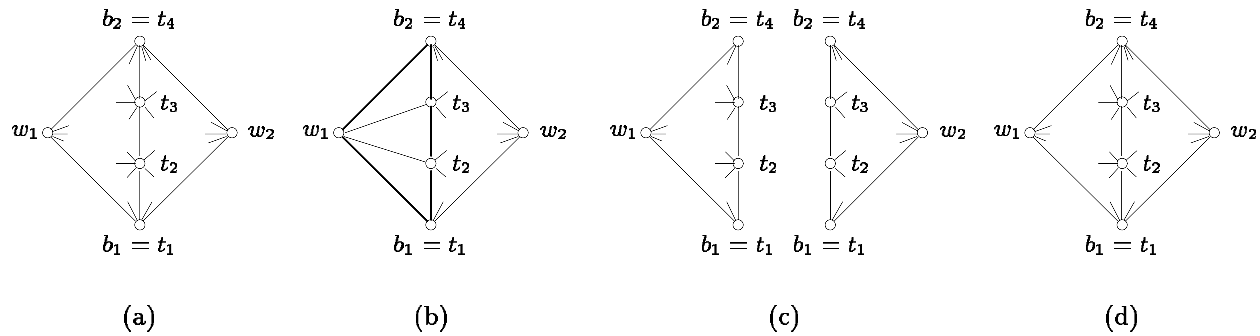


Figure 4: Construction of $\mathcal{G}$: (a) $G$ and $T$, (b) $\overline{\mathcal{G}}_1$, (c) $\mathcal{G}_2$ and $\mathcal{G}_1$, (d) $\mathcal{G}$.

focus on $G_1^{(T)}$. (A symmetric analysis can be done for for $G_2^{(T)}$.) Since $G_1^{(T)}$ has fewer vertices than $G$, our induction hypothesis guarantees that it has a q-subgraph $K_1$ that has property $TRI(K_1)$. By Lemma 2.2, then, $MONO(K_1)$ holds. Hence, by Lemma 3.2.d, trail $T$ is a subgraph of $K_1$, and $w_1 - T - w_1$ is a simple cycle in $K_1$.

Let $\overline{\mathcal{G}}_1$ be a planar drawing of $K_1$ whose external face is the principal cycle (of $G$), which goes in the clockwise direction. Generate $\mathcal{G}_1$ from $\overline{\mathcal{G}}_1$ by removing vertex $w_1$, all edges incident to $w_1$, and all other vertices and edges that reside in the internal domain of the plane bounded by the simple cycle[2] $(w_1 - T - w_1)$. See Figure 4. All faces of $\mathcal{G}_1$ are triangles, except for the external face $(w_2 - T - w_2)$. In a similar way, construct the planar drawing $\mathcal{G}_2$ whose external face is $(w_1 - T - w_1)$. Merge drawings $\mathcal{G}_1$ and $\mathcal{G}_2$ into a single drawing $\mathcal{G}$ by identifying each vertex of $T$ in $\mathcal{G}_1$ with the corresponding vertex in $\mathcal{G}_2$. Note that since $T$ is a trail, this merging does not duplicate edges of the original graph[3]. Hence, $\mathcal{G}$ is a planar drawing of a q-subgraph $G'$ of $G$ that witnesses property $TRI(G')$.

**Case 2**. *$G$ has a trail of length one.*

This case is immediate.

**Case 3**. *$G$ has a trail $T$ of length two.*

Say that $T = (b_1 - t - b_2)$ for some vertex $t$ of $G$. If each of $A_1^{(T)}$ and $A_2^{(T)}$ has at least two vertices, then we can just invoke Case 1. Assume, therefore, that one of these graphs, say $A_1^{(T)}$, has only one vertex—which must be principal vertex $w_1$. In this case, vertex $w_1$ has precisely three neighbors: vertices $b_1$, $b_2$, and $t$.

Consider the q-graph $G'' \stackrel{\text{def}}{=} G \backslash \{w_1\}$ where $t$ replaces $w_1$ as a principal vertex. We claim that $MONO(G'')$ holds. To verify this claim, let $C$ be any valid coloring of $G''$.

[2] Actually, there are no other vertices and edges, but we do not need this fact.
[3] Otherwise, edges short-circuiting vertices of $T$ may be duplicated.

Extend $C$ to a valid coloring of $G$ by labeling vertex $w_1$ white. Let $P$ be any monochromatic q-path in $G$ (which must exist by property $MONO(G)$). On the one hand, if $P$ is a black q-path in $G$, then it is a subgraph of $G''$; on the other hand, if $P$ is a white q-path in $G$, then $P\backslash\{w_1\}$ is a white q-path in $G''$. Now, $G''$ cannot simultaneously have both a white q-path $P^{(w)}$ and a black q-path $P^{(b)}$, or else $G$ would also have paths of both colors, contradicting property $MONO(G)$. To wit, path $P^{(b)}$ would be a black q-path in $G$, while path $(w_1 - P^{(w)})$ would be a white q-path in $G$.

Since $G''$ has one fewer vertex than $G$, there is a planar drawing $\mathcal{G}^{(3)}$ of a q-subgraph $G^{(3)}$ of $G''$, that witnesses property $TRI(G^{(3)})$. Let us alter this drawing as follows. In the face $(t - b_1 - w_2 - b_2 - t)$, add vertex $w_1$ and the edges $w_1 - b_1$, $w_1 - b_2$, and $w_1 - t$. Easily, this altered drawing depicts a q-subgraph $G^{(4)}$ of $G$, that witnesses property $TRI(G^{(4)})$.

**Case 4.** *$G$ has a trail $T$ of length greater than three.*

Denote $T$ by $T = (b_1 = t_1 - t_2 - t_3 - \cdots - t_n = b_2)$, where $n > 4$. Lemma 3.2.c assures us that there is a $w_1$-to-$w_2$ trail $T'$ that shares precisely vertex $t_3$ with $T$. It follows that vertex $t_2 \in A_1^{(T')}$, and vertex $t_4 \in A_2^{(T')}$; therefore, Case 1 applies.

**Case 5.** *All trails of $G$ are of length three.*

For any vertex $v$ of $G$, denote by $\Gamma(v)$ the set of neighbors of $v$, and define:

$$s(v) \stackrel{\text{def}}{=} \sum_{w_i \in \Gamma(v)} i + \sum_{b_i \in \Gamma(v)} i.$$

Let $C$ be the valid coloring of $G$ wherein a nonprincipal vertex $v$ is labeled white if, and only if, $s(v)$ is even. Since $G$ has a monochromatic q-path, it has a monochromatic trail $T$ as well. Let us discuss only the case where $T$ is a $b_1$-to-$b_2$ trail; the proof of the other case is similar. Let $T = (b_1 = t_1 - t_2 - t_3 - t_4 = b_2)$. We now infer several important facts about the adjacencies of the vertices of $T$.

**Fact 1.** *Vertex $t_2$ is adjacent to $b_1$ but not to $b_2$, and is adjacent to precisely one of $w_1, w_2$.*
By Lemma 3.2.c, there is a $w_1$-to-$w_2$ trail $T'$ (perforce, of length three) that shares precisely vertex $t_2$ with trail $T$. Therefore, vertex $t_2$ is adjacent to one of $w_1$ and $w_2$. Easily, $t_2$ cannot be adjacent to both $w_1$ and $w_2$, nor to both $b_1$ and $b_2$, since either such double-adjacency would lead to a trail of length two in $G$.

**Fact 2.** *Vertex $t_3$ is adjacent to $b_2$ but not to $b_1$, and is adjacent to precisely one of $w_1, w_2$.*
This follows by reasoning symmetric to Fact 1.

**Fact 3.** *$t_2$ and $t_3$ cannot be adjacent to the same vertex of $\{w_1, w_2\}$.*
Such common adjacency would imply that $s(t_2) \neq s(t_3)$ mod 2.

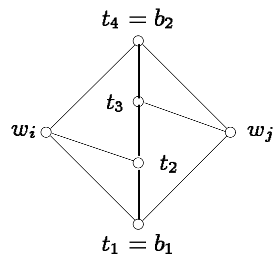


Figure 5: The subgraph induced by $T \cup \{b_1, b_2, w_1, w_2\}$.

Facts 1-3 imply that the subgraph of $G$ induced by the vertices of $T \cup \{b_1, b_2, w_1, w_2\}$ has the form depicted in Figure 5. One can see from this figure that both $A_1^{(T)}$ and $A_2^{(T)}$ must each have at least two vertices. Therefore, the scenario of Case 1 holds.

Since the list of cases above is exhaustive, the lemma is proved. □

**Lemma 3.5** *If $TRI^+(G)$ holds, then $G$ is minimal with respect to property $TRI$.*

**Proof:** Let $K$ and $K'$ be simple graphs (not q-graphs) such that $K'$ is a subgraph of $K$, $K'$ has at least four vertices, and both $K$ and $K'$ have planar drawings such that every face is a 3-cycle and every 3-cycle is a face. We claim that $K = K'$. To the end of proving this, focus on a vertex $v$ of $K'$, and let $N(v, K)$ (resp., $N(v, K')$) be the subgraph of $K$ (resp., of $K'$) induced by the neighbors of $v$. Clearly, $N(v, K)$ and $N(v, K')$ are simple cycles, and $N(v, K')$ is a subgraph of $N(v, K)$. Since no proper subgraph of a simple cycle is a cycle, we must have $N(v, K) = N(v, K')$. (We have subtly used the fact the $K'$ has at least four vertices; since if $K' = K_3$ then $N(v, K')$ is a single edge!) This establishes that every vertex $v$ of $K'$ has the same set of neighbors in $K'$ as it does in $K$. Since $K$ is connected and $K'$ not empty, we conclude that $K = K'$.

Let us return now to q-graphs. Assume that $TRI^+(G)$ holds, and let $G'$ be any q-subgraph of $G$ that enjoys property $TRI(G')$. We need to show that $G = G'$. Clearly, $G'$ has a q-subgraph $G''$ that enjoys property $TRI^+(G'')$. Let $\mathcal{G}$ be a planar drawing of $G$ that witnesses property $TRI^+(G)$. By drawing new vertices and edges in the principal face of $\mathcal{G}$, we can construct a planar drawing such that every face is a 3-cycle and every 3-cycle is a face. Let us call the graph depicted by this augmented drawing $K$. Now, add the same vertices and edges to graph $G''$, and call the resulting graph $K'$. $K$ and $K'$ satisfy the requirements of our claim in the preceding paragraph; hence, $K = K'$. The equality of $K$ and $K'$, however, implies that $G = G''$. □

The preceding series of lemmas allows us to prove our main theorem.

**Theorem 3.6** *$TRI^+(G)$ holds if, and only if, $G$ is minimal with respect to property $MONO$.*

**Proof:** Say first that $TRI^+(G)$ holds. By Lemma 2.2, $MONO(G)$ holds also. Assume then, for contradiction, that $G$ is not minimal with respect to property $MONO$; in particular, let $G'$ be a proper q-subgraph of $G$ that enjoys property $MONO(G')$. By Lemma 3.4, there exists a q-subgraph $G''$ of $G'$ that enjoys property $TRI(G'')$. But this contradicts Lemma 3.5.

Say next that $G$ is minimal with respect to property $MONO$. Then, by Lemma 3.4, $TRI(G')$ holds for some q-subgraph $G'$ of $G$. It follows that $TRI^+(G'')$ holds for some q-subgraph $G''$ of $G'$. We have just shown, though, that property $TRI^+(G'')$ implies that $G''$ is minimal with respect to property $MONO$. Since $G$ is also minimal with respect to the property, we must have $G = G''$. □

# 4 Property *MONO* is Decidable in Polynomial Time

Since property $TRI^+(G)$ can be verified in polynomial time, theorem 3.6 provides a polynomial-time algorithm for deciding whether or not a given q-graph is minimal with respect to property $MONO$. We need some more technical lemmas to establish that property $MONO$ itself is polynomial-time decidable.

Let $G$ be a q-graph. Define the q-subgraph $\widehat{G} \stackrel{\text{def}}{=} \langle \widehat{V}, \widehat{E} \rangle$ of $G$, by:

$$\begin{aligned} \widehat{V} &\stackrel{\text{def}}{=} \{v \mid v \text{ is a principal vertex or } v \text{ is on some trail}\} \\ \widehat{E} &\stackrel{\text{def}}{=} \{e \mid e \text{ is a principal edge or } v \text{ is on some trail}\} \end{aligned}$$

The next lemma establishes that any $G$ that enjoys property $MONO$ has exactly one $MONO$-minimal q-subgraph, which is the graph $\widehat{G}$.

**Lemma 4.1** *Assume that $MONO(G)$ holds. Then $\widehat{G}$ is the only q-subgraph of $G$ that is minimal with respect to property $MONO$.*

**Proof:** By Lemma 3.2.d, any q-subgraph of $G$ that enjoys property $MONO$ includes $\widehat{G}$ as a q-subgraph. Hence, we need only establish that $MONO(\widehat{G})$ holds.

To this end, let $C$ be a valid coloring of $\widehat{G}$. Extend $C$ in any way to a valid coloring of $G$. Since $MONO(G)$ holds, $G$ has a monochromatic q-path; hence, by Lemma 3.1, $G$ has a monochromatic trail. By definition, this trail is a subgraph—hence, a monochromatic q-path—of $\widehat{G}$. Of course, $\widehat{G}$ could not have two conflicting monochromatic q-paths, or else $G$ would also. We conclude that $\widehat{G}$ enjoys property $MONO$. □

Lemma 4.1 implies that, if $G$ is minimal with respect to property $MONO$, then any nonprincipal edge of $G$ is on a trail; moreover, the lemma combines with Lemma 3.2.c to imply that any nonprincipal vertex of $G$ is on both a $w_1$-to-$w_2$ trail and a $b_1$-to-$b_2$ trail.

**Lemma 4.2** *If $G$ is minimal with respect to property $MONO$, then any two distinct nonadjacent vertices of $G$ are separated by a trail.*[4]

**Proof:** Let $u$ and $v$ be any two distinct nonadjacent vertices of $G$. We consider the following two cases:

**Case 1:** *There is a trail $T$ that contains both $u$ and $v$.*

Let $x$ be a vertex of $T$ that lies between $u$ and $v$. By Lemma 3.2.c, there is a trail $T'$ that shares precisely vertex $x$ with $T$. By Lemma 3.2.a, $T'$ separates $u$ and $v$.

**Case 2:** *No trail of $G$ contains both $u$ and $v$.*

In this case, $u$ and $v$ must be nonprincipal vertices. Let $T$ be a $b_1$-to-$b_2$ trail that contains vertex $v$. Since $G$ is $MONO$-minimal, the arguments of Case 1 in the proof of Lemma 3.4 show that the graph $X^{(T)}$ is empty. Assume, with no loss of generality, that vertex $u$ belongs to the graph $A_2^{(T)}$. Let $P$ be a $v$-to-$w_1$ path such that $v$ is the only vertex common to $P$ and $T$. Consider now the q-graph $G_1^{(T)}$. It is easy to verify by geometrical arguments that $G_1^{(T)}$ enjoys property $TRI^+$. (Start with a drawing of $G$ that witnesses property $TRI^+$, and modify it as shown in Figure 3.) Let $T'$ be a $b_1$-to-$b_2$ trail in $G_1^{(T)}$ that passes through vertex $u$, and let $P'$ be a $u$-to-$w_2$ path in $G_1^{(T)}$ such that $u$ is the only vertex common to $T'$ and $P'$.

Let us return now to graph $G$. $T$ and $T'$ are trails in $G$ which, respectively, avoid vertices $u$ and $v$; $P$ and $P'$ are paths which both avoid $(T \cup T')\backslash\{u,v\}$. Consider the valid coloring of $G$ wherein vertices of $(T \cup T')\backslash\{u,v\}$ are black and all other vertices are white. Assume that $G$ has a white q-path and, therefore, a white trail $T''$. Now, trail $T''$ must intersect both $T$ and $T'$; hence, it must include both $u$ and $v$. But this contradicts the assumption that delineates this Case. We conclude, therefore, that $G$ has a black q-path and, therefore, a black trail $\overline{T}$. Since $\overline{T}$ separates vertices $w_1$ and $w_2$, and since vertices $u$ and $v$ are connected to $w_1$ and $w_2$, respectively, by paths that are disjoint to $\overline{T}$, we see that vertices $u$ and $v$ are separated by $\overline{T}$. □

**Lemma 4.3 (Thomassen [2])** *If $TRI^+(G)$ holds, then, for any edge $e$ that is* not *in $G$, the graph $(G \cup \{e\})$ contains two disjoint q-paths.*

[4]Vertices $u$ and $v$ of $G$ are *separated* by trail $T$ if any path connecting $u$ and $v$ contains a vertex of $T$.

**Proof:** We present here an alternative proof. Let $e$ be the edge $u - v$. By Lemma 4.2, there is a trail $T$ in $G$ that separates vertices $u$ and $v$. Say, with no loss of generality, that $u \in A_1^{(T)}$ and $v \in A_2^{(T)}$, and that $T$ is a $b_1$-to-$b_2$ trail. It follows that $G$ contains a $w_1$-to-$u$ path $P_1$ and a $v$-to-$w_2$ path $P_2$, such that both paths avoid trail $T$. One sees easily that, in the graph $(G \cup \{e\})$, trail $T$ and path $(P_1 - u - v - P_2)$ are disjoint q-paths. $\Box$

For any q-graph $G$, we defined $G^{\oplus}$ to be the graph generated from $G$ by adding a vertex $z$ and four edges connecting $z$ to the principal vertices. For any 3-cycle $t$ in $G$, let $G^{(t)}$ be the q-subgraph of $G$ generated by removing (from $G$) all vertices not connected to $z$ in $G^{\oplus} \backslash t$. A q-graph $G$ is *lean* if for any 3-cycle $t$: $G = G^{(t)}$. (In other words, for any such $t$, every vertex of $G \backslash t$ is connected (in $G \backslash t$) to some principal vertex.)

**Lemma 4.4** *For any q-graph $G$ and any 3-cycle $t$ in $G$: property $MONO(G)$ holds if, and only if, $MONO(G^{(t)})$ holds.*

**Proof:** Let $Y$ be the set of vertices removed in the construction of $G^{(t)}$. Let $P$ be a q-path in $G$ that uses vertices of $Y$. Then there is a q-path $P'$ in $G^{(t)}$ whose vertex-set is a subset of $P$'s, that has the same endpoints as $P$. This means that the vertices of $Y$ are superfluous, as far as monochromatic q-paths are concerned. $\Box$

**Lemma 4.5** *Any lean q-graph $G$ enjoys property $MONO(G)$ if, and only if, it enjoys property $TRI^{+}(G)$.*

**Proof:** By Lemma 2.2, property $TRI^{+}(G)$ implies property $MONO(G)$. To establish the converse, we assume that $MONO(G)$ holds and show that $G$ is $MONO$-minimal.

Assume, for contradiction, that $G \neq \widehat{G}$. We claim that there is a path $P$ in $G$ whose endpoints are nonadjacent vertices in $\widehat{G}$ and all of whose other ("internal") vertices are not in $\widehat{G}$. If $G$ has an edge that is not in $\widehat{G}$, then this edge is the required path; otherwise, $G$ has a vertex $v$ that is not in $\widehat{G}$. Let $X$ be the connected component of $G \backslash \widehat{G}$ that contains $v$, and let $\Gamma(X)$ denote the set of neighbors of $X$ in $\widehat{G}$. If $\Gamma(X)$ contains two nonadjacent vertices, then we are done. Alternatively, $\Gamma(X)$ is a clique in $\widehat{G}$. Since $\widehat{G}$ does not include the 4-clique $K_4$, $\Gamma(X)$ is either empty or is one of the smaller cliques $K_1$, $K_2$, or $K_3$. Since every edge and vertex of $\widehat{G}$ is on a 3-cycle, this contradicts the fact that $G$ is lean. This establishes the existence of the desired path $P$.

Now, if path $P$ is a single edge, then $G$ contains two disjoint q-paths by Lemma 4.3. However, the same also holds when $P$ is a path of several edges. This contradicts property $MONO(G)$. $\Box$

**Theorem 4.6** *Property $MONO$ is polynomial-time decidable.*

**Proof:** Let a q-graph $G$ be given. By Lemma 4.4, we can reduce $G$ in polynomial time to a lean q-graph $G'$ that has property $MONO$ if, and only if $G$ does. Having $G'$, we can decide $MONO(G)$ in polynomial time via Lemma 4.5. □

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