ON OPTIMAL CALL ADMISSION CONTROL IN CELLULAR NETWORKS

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Abstract

Two important Quality-of-Service (QoS) measures for current cellular networks are the fraction of new and handoff "calls" that are blocked due to unavailability of "channels" (radio and/or computing resources). Based on these QoS measures, we consider optimal admission control policies for three problems: minimizing a linear objective function of the new and handoff call blocking probabilities (MINOBJ), minimizing the new call blocking probability with a hard constraint on the handoff call blocking probability (MINBLOCK) and minimizing the number of channels with hard constraints on both the blocking probabilities (MINC). We show that the well-known *Guard Channel policy is optimal for the MINOBJ problem*, while a new *Fractional Guard Channel policy is optimal for the MINBLOCK and MINC problems*. The Guard Channel policy reserves a set of channels for handoff calls while the Fractional Guard Channel policy effectively reserves a non-integral number of guard channels for handoff calls by rejecting new calls with some probability that depends on the current channel occupancy. It is also shown that the Fractional policy results in significant savings (20-50%) in the new call blocking probability for the MINBLOCK problem and provides some, though small, gains over the integral guard channel policy for the MINC problem. We see that the Fractional Guard Channel policy offers more flexibility than the Guard Channel policy in the sense of a richer set of parameters but the algorithms developed in the paper for determining the optimal parameter settings for the fractional policy are computationally inexpensive. Finally, we briefly explore the possibility of exploiting the combination of these features of the Fractional Guard Channel policy and its concomitant algorithms for real-time control of cellular networks.

Keywords: Wireless Networks, Call Admission, Personal Communication Services

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1 Introduction

During the last few years there has been tremendous interest and strides in the field of wireless communications. There is, hence, a great demand for Personal Communication Services (PCS) which will provide reliable voice and data communications, anytime and anywhere, via small lightweight, pocket-size terminals. PCS can be provided, in theory, by the current cellular technology and infrastructure and hence will be the focus of this paper. The service area in a PCS network is partitioned into cells[1]. In each cell, a Base Station (BS) manages the allocation of channels [2] to the Mobile Subscriber (MS) enabling the MS to communicate with other MS's or PSTN users. Note that the BS itself is assigned a set of channels and this assignment could be static or dynamic. We primarily assume a static assignment of channels for this paper but the ideas in the paper can be extended easily to the dynamic assignment scenario as well.

As the MS moves from one cell to another, any active call needs to be allocated a channel in the destination cell. This event, termed the handover or handoff, must be transparent to the MS. If the destination cell has no available channels, a call is terminated. The disconnection in the middle of a call is highly undesirable and one of the goals of the network designer is to keep the probability of such occurrences, also termed the handoff blocking probability, low. On the other hand, reserving channels for handoff traffic could increase blocking for new calls. As a result, there is a trade-off between the two Quality-of-Service (QoS) measures, the handoff and the new call blocking probabilities. In this paper, we consider the optimal admission control policies for three problems based on these two QoS measures:

MINOBJ: Minimizing a linear objective function of the two blocking probabilities

MINBLOCK: For a given number of channels, minimizing the new call blocking probability subject to a hard constraint on the handoff blocking probability.

MINC: Minimizing the number of channels subject to hard constraints on the new and handoff call blocking probabilities.

MINOBJ appeals to the network provider in terms of maximizing the revenue obtained. MINBLOCK

[1]Both micro and macro cells are likely.
[2]Channels could be frequencies, time slots or codes depending on the radio technology used.

guarantees a particular level of service to already admitted users while trying to maximize the revenue obtained. MINC is more of a network engineering problem where resources need to be allocated apriori based on, for example, traffic projections.

We show that the well-known Guard Channel policy is optimal for the MINOBJ problem. The notion of guard channels was introduced in the mid-80s [7, 8] as a call admission mechanism to give priority to handoff calls over new calls. In this policy, a set of channels called the guard channels are permanently reserved for handoff calls. We then introduce the Fractional guard channel policy which effectively reserves a non-integral number of guard channels for handoff calls by rejecting new calls with some probability that depends on the current channel occupancy. We show that a restricted version of the this Fractional policy is optimal for the MINBLOCK and MINC problems. We also develop computationally inexpensive algorithms for determination of the optimal parameters of the Fractional policy. Further, in order to ensure that the QOS requirements are continuously met, it may be necessary to adapt to variations in traffic load. For example, in the Guard Channel policy it may be necessary to dynamically change the number of guard channels with the traffic load. In this sense, the admission control policy can also be viewed as a control mechanism which may, for example, increase the blocking of new calls, as the traffic load increases, to ensure that existing calls can be served, limiting the handoff blocking probability to its prescribed level [1]. We briefly explore this possibility, highlighting the necessary mechanisms and advantages in using the Fractional policy as a control mechanim in a real-time environment.

The remainder of the paper is organized as follows. In Section 2, we describe the Guard Channel and the Fractional Guard Channel policies and compute blocking under these two policies. In Section 3, we consider the problem of minimizing a linear objective function of the new and handoff call blocking probabilities and show that the Guard Channel policy is optimal for this problem. In Section 4, we consider the MINBLOCK problem. We introduce a Limited Fractional Guard Channel policy, show it is optimal, and develop an algorithm for the solution of the MINBLOCK problem. We show that the algorithm gives us significant improvement over the Guard Channel policy. In Section 5, we consider the MINC problem and develop an algorithm for its solution. In Section 6, we explore, briefly, the suitability of the Fractional Guard Channel policy for cellular network control. We present our conclusions in the final section and mention some further ideas that we are currently exploring.

2 Admission Control Policies and Blocking Performance

We introduce the Guard Channel and Fractional Guard Channel admission control policies in this section. The computation of blocking under these policies is detailed and will be useful in subsequent sections for proving the optimality of these policies.

Consider a cellular network with C channels in a given cell. The Guard channel policy reserves a subset of these channels (say $C - T$) for handoff calls. Whenever the channel occupancy exceeds a certain threshold (T), the Guard channel policy rejects all new calls. In the Fractional Guard channel policy, new calls are accepted with a certain probability that depends on the current channel occupancy. Note that both these policies accept handoff calls as long as channels are available. These policies are illustrated algorithmically in Figure 1. We next focus on the new and handoff call blocking under these policies.

2.1 Blocking Performance

We compute performance of the admission policies based on the following assumptions:

- The arrival process of new and handoff calls is Poisson with rate λ_1 and λ_2 respectively. Let $\lambda = \lambda_1 + \lambda_2$ and $\lambda_2 = \alpha * \lambda$.
- The channel holding time for both type of calls is exponentially distributed with mean $1/\mu$ and let $\rho = \frac{\lambda}{\mu}$.
- The busy-line effect [13] is negligible, i.e., the interval between two calls from a MS is much greater than the mean call holding time.

This set of assumptions have been found to reasonable as long as the number of mobiles in a cell is much greater than the number of channels and have been used in the models in [7, 11, 15].

Define the state of a cell at time t by the total number of occupied channels[3]. Thus, the cell channel occupancy can be modeled by a continuous time Markov chain with C states. The state transition rate

[3]One could possibly enhance the state description by keeping track of new calls and handoff calls separately, rather than the total occupancy alone. However, this new state descriptor is not expected to change any of the conclusions of the paper given the memoryless nature of the arrival process.

```
/*********************/
/* Guard Channel Policy */
/*********************/
if (NEW CALL) then
   if (NumberOfOccupiedChannels < T)
      admit call;
   else
      reject call;
if (HANDOFF CALL) then
   if (NumberOfOccupiedChannels < C)
      admit call;
   else
      reject call;
/*****************************/
/* Fractional Guard Channel Policy */
/*****************************/
/* random(0,1) returns a uniformly generated
random number in the interval [0,1] */
if (NEW CALL) then
   if (random(0,1) ≤ β(NumberOfOccupiedChannels))
      admit call;
   else
      reject call;
if (HANDOFF CALL) then
   if (NumberOfOccupiedChannels < C)
      admit call;
   else
      reject call;
```

Figure 1: **Call Admission Policies**

diagram of a cell with C channels and $C - T$ guard channels is shown in Figure 2(a). Given this, it is straight forward to derive the steady-state probabilities that j channels are busy (P_j) [6, 7]:

$$P_j = \begin{cases} \frac{\rho^j}{j!} P_0, & 0 \leq j \leq T \\ \frac{\rho^j \alpha^{j-T}}{j!} P_0, & T \leq j \leq C \end{cases} \quad with \quad P_0 = \frac{1}{\Sigma_{j=0}^{T} \frac{\rho^j}{j!} + \Sigma_{j=T+1}^{C} \frac{\rho^j \alpha^{j-T}}{j!}}.$$

The state transition rate diagram for the Fractional Guard Channel policy is shown in Figure 2(b). At each state i, we now accept new calls with a probability β_i and handoff calls with probability 1. Thus the arrival rate at state i is $(\alpha + (1 - \alpha) * \beta_i) * \lambda$. The steady-state probabilities are calculated similarly as:

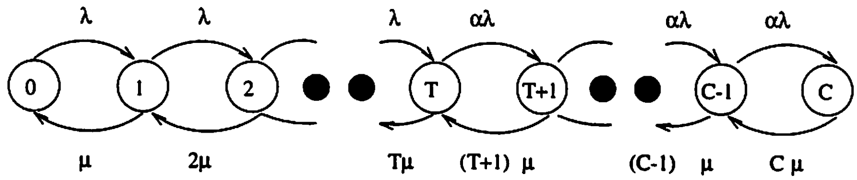

Fig 2(a). State transition diagram (Guard channel scheme).

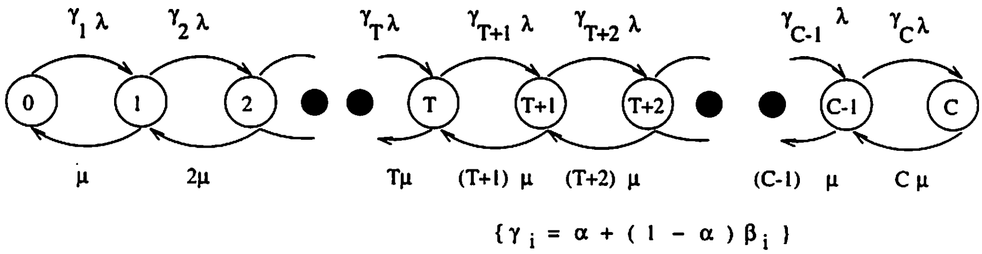

Fig 2(b). State transition diagram (Fractional guard channel scheme).

$$P_j = \left\{ \frac{\rho^j \Pi_{i=1}^j \gamma_i}{j!} P_0, \qquad 0 \le j \le C \qquad with \quad P_0 = \frac{1}{\Sigma_{j=0}^C \frac{\rho^j \Pi_{i=1}^j \gamma_i}{j!}}, \right.$$

$$and \quad \gamma_i = \alpha + (1-\alpha)\beta_i, \qquad 1 \le i \le C.$$

Define $\boldsymbol{\beta} = (\beta_1, \beta_2, ..., \beta_C)$ as the new call acceptance probability vector. Given the state probabilities, we can find the handoff blocking probability $B_h(C, \boldsymbol{\beta})$ and new call blocking probability $B_n(C, \boldsymbol{\beta})$ as

$$B_h(C, \boldsymbol{\beta}) = P_C, \tag{1}$$

$$B_n(C, \boldsymbol{\beta}) = \Sigma_{j=0}^C (1 - \beta_{j+1}) P_j \qquad where \quad \beta_{C+1} = 0. \tag{2}$$

Note that the blocking for the Guard Channel policy, with threshold T, can be calculated from the above by setting $\beta_i = 1, \ 1 \le i \le T$ and $\beta_i = 0, \ T+1 \le i \le C$.

3 Minimizing a Linear Objective Function

In this section, we consider the problem of finding an admission control policy that minimizes a linear objective function of the new and handoff call blocking probabilities.

Problem 1: MINOBJ: Minimize $F = A_1 * B_n(C,\beta) + A_2 * B_h(C,\beta)$ for a given C, and given constants A_1 and A_2 with $0 < A_1 < A_2$.

The above problem can be posed as an average cost problem[9]:

$$\text{Min} \lim_{N\to\infty} \frac{1}{N} E(\Sigma_{n=0}^{N-1} A_1 e^{-\eta T_{n_1}} + \Sigma_{n=0}^{N-1} A_2 e^{-\eta T_{n_2}}),$$

where η is the discount factor and T_{n_1} and T_{n_2} are the rejection times for the new and handoff calls respectively. Before we proceed to solve this problem, we consider the η-discounted finite horizon problem [9]:

$$\text{Min } E(\Sigma_{n=0}^{N-1} A_1 e^{-\eta T_{n_1}} + \Sigma_{n=0}^{N-1} A_2 e^{-\eta T_{n_2}}).$$

We denote by $V_k^\eta(i)$ the optimal cost for the k-stage problem starting in state i which is the number of allocated channels. We note that the control policy that achieves the minimum cost is a stationary policy, i.e., it depends only on the current state, which we denote as $u(i)$. Given the stationary control policy, the state of the system now evolves as a Markov decision process [9] and in particular our interest will be only in stationary policies that lead to irreducible Markov chains.

We discretize the discounted return problem above by using the standard uniformization technique [2] and by scaling time, one can assume that $C\mu + \lambda_1 + \lambda_2 + \eta = 1$[4]. The optimal cost function $V(\cdot)$ then satisfies:

$$\begin{aligned} V_k^\eta(i) &= \lambda_1 \min(V_{k-1}^\eta(i+1), A_1 + V_{k-1}^\eta(i)) + \lambda_2 \min(V_{k-1}^\eta(i+1), A_2 + V_{k-1}^\eta(i)) \\ &\quad + i\mu V_{k-1}^\eta(i-1) + (C-i)\mu V_{k-1}^\eta(i), \quad 1 \le i \le C-1. \end{aligned} \tag{3}$$

The first term is the contribution to the cost if the next transition is the arrival of a new call. Here, we have the option of rejecting the arrival in which case a cost A_1 is incurred. The second term is the contribution to the cost for a handoff call arrival. Here, we have the option of rejecting the handoff call in which case a cost A_2 is incurred. The third term is the contribution to the cost due to call completions and the fourth term is a consequence of the uniformization.

The boundary condition is handled by defining $V_k^\eta(C) = C \max(A_1, A_2)$ and $V_k^\eta(0) = 0$. Note that we assume that we are controlling both the new and handoff calls here. We will show later that controlling handoff calls is not beneficial as it leads to some channels being idle.

[4]Transitions associated with the discount cost η can be thought of as terminating the process of channel occupancy evolution

Lemma 1: $V_k^\eta(i)$ is monotonically non-decreasing and convex in i for all k.

Proof: In Appendix A.

Under minor technical conditions, it can be shown that the optimal cost function for the infinite-horizon problem, $V^\eta(i) = \lim_{k\to\infty} V_k(i)$, is also non-decreasing and convex [12, Assumption 8.28]. Also, since the state space is finite and the Markov chain induced by the control policy is irreducible, $V^\eta(i) - V^\eta(0)$ is bounded uniformly [9]. We now return to consider the average cost problem.

Since $V^\eta(i) - V^\eta(0)$ is bounded, the optimal control policy for the average cost problem is also stationary and the optimality equation is given as [9]:

$$\begin{aligned} g + h(i) &= \lambda_1 \min(h(i+1), A_1 + h(i)) + \lambda_2 \min(h(i+1), A_2 + h(i)) \\ &\quad + i\,\mu\, h(i-1) + (C-i)\,\mu\, h(i), \quad 1 \le i \le C-1, \end{aligned} \tag{4}$$

where $g = \lim_{\eta\to 1}(1-\eta)V^\eta(0)$ is a constant and $h(i) = \lim_{r\to\infty}[V^{\eta_r}(i) - V^{\eta_r}(0)]$ for some sequence $\eta_r \to 1$. Note that $h(i)$ inherits the structural properties of $V^\eta(\cdot)$ and hence is non-decreasing and convex. The optimal average cost policy is one which minimizes the right-hand side of the above equation. The following theorem shows that a threshold policy, the Guard Channel policy, is optimal for the average cost problem and hence for the MINOBJ problem.

Theorem 1: The Guard Channel policy is optimal for the MINOBJ problem.

Proof:

The optimal control policy $u(i)$ that incurs the minimum cost (see Equation (4)) is given as,

$$u(i) = \begin{cases} 1 & if\ h(i+1) - h(i) \le A_1, \\ 0 & Otherwise. \end{cases}$$

where $u(i) = 1(0)$ corresponds to accepting (rejecting) a new call when i channels in the cell are busy. Since $h(i)$ is non-decreasing and convex, $h(i) - h(i-1)$ is nondecreasing in i, and hence there exist integers i_0 and i_1 where $i_0 = Arg\inf\{i : h(i) - h(i-1)) > A_1\}$ and $i_1 = Arg\inf\{i : h(i) - h(i-1)) > A_2\}$. Thus, for new calls we have $u(i) = 1\ for\ i < i_0$ and $u(i) = 0\ for\ i \ge i_0$ and hence the optimal control policy for new calls is of the threshold type. Let us now consider the policy for handoff calls. Since $A_2 > A_1$, $i_1 > i_0$. If $i_1 < C$, then this will result in idling one or more channels. We can, hence, do better by not blocking the handoff traffic and letting them use these

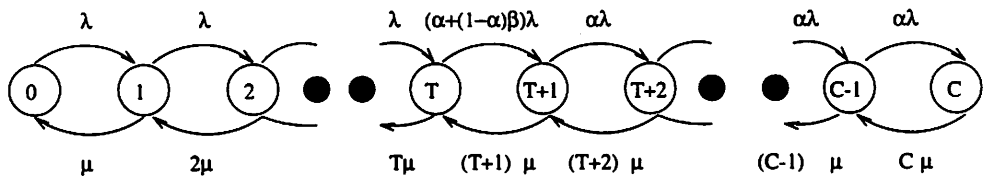

Figure 3: **State transition diagram (limited fractional guard channel scheme).**

channels, i.e., $i_1 = C$. Thus the overall optimal policy is to block all new calls when the channel occupancy reaches or exceeds i_0 and not block any handoff calls. Or in other words, the optimal policy is to reserve $C - i_0$ channels for handoff calls. Thus, the optimal policy for the average cost and MINOBJ problem is the Guard Channel policy. □

In the following sections, we consider the MINBLOCK and MINC problems. For these problems, we restrict our attention to a Limited Fractional Guard Channel policy and show that this policy is optimal.

4 Minimizing new call blocking with a hard constraint

Problem 2: MINBLOCK: Given C, Minimize $B_n(\cdot)$ such that

$$B_h(\cdot) \leq P_h. \tag{5}$$

For the above problem we consider a restricted version of the Fractional Guard Channel policy and show that it is the optimal policy for the MINBLOCK problem.

4.1 Limited Fractional Guard Channel Policy

Figure 3 shows the state transition rate diagram of a system with C channels for the Limited Fractional Guard Channel (LFG) policy and Figure 4 outlines this call admission algorithm. At state T, we now accept new calls with a probability β. From states $T + 1$ to C, we accept only handoff calls and at states 0 to $T - 1$, we accept both types of calls. The handoff and blocking probabilities for the

```
/* random(0,1) returns a uniformly generated
random number in the interval [0,1] */
if (NEW CALL) then
  if (NumberOfOccupiedChannels < T) then
    admit call;
  else if (NumberOfOccupiedChannels == T) AND (random(0,1) ≤ β)
    admit call;
  else
    reject call;
if (HANDOFF CALL) then
  if (NumberOfOccupiedChannels < C)
    admit call;
  else
    reject call;
```

Figure 4: **Call Admission with Limited Fractional Guard Channel Policy**

LFG policy can be easily calculated using Equations (1) and (2) by setting $\beta_{T+1} = \beta$ and the values of $\beta_i = 1, \;\; 1 \le i \le T$ and $\beta_i = 0, \;\; T+1 \le i \le C$. We now show that the Limited Fractional Guard channel policy is optimal for the MINBLOCK problem.

4.2 Optimality of the LFG policy

Let g_i represent the integral Guard Channel policy with threshold i. Thus, we have a set of policies $g_0, g_1, ...g_C$ for a C channel system. We know that, for the MINOBJ problem, $\exists i$ such that g_i is optimal. We show below that a policy that randomizes between two static policies g_{i-1} and g_i is optimal for the MINBLOCK problem.

Let us first define some notation:

$$U_i = \begin{cases} \Sigma_{j=0}^{i} \frac{\rho^j}{j!} & 0 \le i < C, \\ 0 & Otherwise. \end{cases} \quad V_i = \begin{cases} \Sigma_{j=i}^{C-1} \frac{\rho^j}{j!} \alpha^{j-i} & 0 \le i < C, \\ 0 & Otherwise. \end{cases} \quad W_i = \begin{cases} \frac{\rho^C}{C!} \alpha^{C-i} & 0 \le i \le C, \\ 0 & Otherwise. \end{cases}$$

and let $X_i = U_{i-1} + V_i + W_i$.

Consider the MINOBJ problem with $A_1 = 1$. The objective function E_T for a given threshold T, or policy g_T, is

$$E_T = \frac{V_T + W_T(A_2 + 1)}{X_T}, \qquad T = 0, 1, ..., C. \tag{6}$$

We first show that g_T is optimal for MINOBJ if $A_2^{T+1} \leq A_2 \leq A_2^T$ where

$$A_2^T = \begin{cases} \frac{V_{T-1}X_T - V_T X_{T-1}}{W_T X_{T-1} - W_{T-1} X_T} - 1 & T = 1, ..., C, \\ 0 & T = C+1, \\ \infty & T = 0. \end{cases}$$

We will first need the following property of the function A_2^T.

Property 1: $A_2^T > A_2^{T+1}$ for $T = 0, 1, ..., C$.

Proof: In Appendix A.

Lemma 2: If $A_2^T \leq A_2 \leq A_2^{T-1}$, then policy g_T is optimal for the MINOBJ problem.

Proof:

$$\begin{aligned} & A_2 \geq A_2^T \\ & \quad = \frac{V_T X_{T+1} - V_{T+1} X_T}{W_{T+1} X_T - W_T X_{T+1}} - 1 \\ \Rightarrow \quad & \frac{V_{T+1} + W_{T+1}(A_2 + 1)}{X_{T+1}} \geq \frac{V_T + W_T(A_2+1)}{X_T} \\ \Rightarrow \quad & E_{T+1} \geq E_T. \end{aligned}$$

and

$$\begin{aligned} & A_2 \leq A_2^{T-1} \\ & \quad = \frac{V_{T-1} X_T - V_T X_{T-1}}{W_T X_{T-1} - W_{T-1} X_T} - 1 \\ \Rightarrow \quad & \frac{V_{T-1} + W_{T-1}(A_2 + 1)}{X_{T-1}} \geq \frac{V_T + W_T(A_2+1)}{X_T} \\ \Rightarrow \quad & E_{T-1} \geq E_T. \end{aligned}$$

Also, if $E_T \leq E_{T+1}$, then $E_T \leq E_{T+i}$, $i = 1, ..., C-T$ since $E_{T+i-1} \leq E_{T+i}$ if $A_2 \geq A_2^{T+i}$ and this is true from Property 1.

Similarly, if $E_T \leq E_{T-1}$, then $E_T \leq E_{T-i}$, $i = 1, ..., T$ since $E_{T-i+1} \leq E_{T-i}$ if $A_2 \geq A_2^{T-i+1}$ and this follows from Property 1.

Thus, E_T is minimum among all E_i, $i = 0, 1, ..., C$ and hence policy g_T is optimal for MINOBJ problem.

□

Consider the Limited Fractional Guard Channel policy

$$f_{k,q} := [g_{k-1}, g_k, q],\ q \in [0, 1] \tag{7}$$

which admits new calls at state $k-1$ with probability q, and admits only handoff calls from states k through C. Let $B_h(u)$ and $B_n(u)$ be the handoff and new call blocking probabilities for a stationary (possibly randomized) policy u. Observe that, $B_h(f_{k,0}) = B_h(g_{k-1})$, $B_h(f_{k,1}) = B_h(g_k)$ and $B_h(f_{k,q})$ is a continuous function of q over interval [0,1] for $k = 0, 1, ..., C$. Now, we show that $f_{k,q}$ is optimal for the MINBLOCK problem. We need the following lemma for the proof.

Lemma 3: Consider $J^\omega(u) = B_n(u) + \omega B_h(u)$. Then if $A_2^T \leq \omega \leq A_2^{T-1}$, policy g_T is optimal by the previous lemma. In particular, if $\omega = A_2^{T-1}$, policies g_{T-1}, g_T and $f_{T,q}$ are all unconstrained optimal.

Theorem 2: If $B_h(g_0) \leq P_h \leq B_h(g_C)$, then for some $q \in [0, 1]$, $f_{j,q}$ is constrained optimal where $j = \min\{i : B_h(g_i) \leq P_h\}$.

Proof:

Suppose $B_h(g_{j-1}) \leq P_h \leq B_h(g_j)$ with j as given above. By continuity of $f_{j,q}$, $\exists q \in [0, 1]$ such that $B_h(f_{j,q}) = P_h$. Moreover, $f_{j,q}$ minimizes $J^\gamma(u)$ where $\gamma = A_2^{j-1}$ by Lemma 3. Thus, for any policy u,

$$\begin{aligned} & B_n(f_{j,q}) + \gamma P_h \\ = \ & J^\gamma(f_q) \\ \leq \ & J^\gamma(u) \\ \leq \ & B_n(u) + \gamma B_h(u). \end{aligned}$$

Thus $B_h(f_q) - B_n(u) \leq \gamma(B_h(u) - P_h)$ which implies $B_n(f_q) \leq B_n(u)$ for any policy which is feasible (i.e., $B_h(u) \leq P_h$).

Thus, $f_{j,q}$ is constrained optimal. □

```
/* Returns T+β */
/* We assume that this algorithm is called only when the constraint on
the handoff blocking probability can be met with C channels. Also, the traffic
parameters ρ and α are assumed to be built in the evaluation of function B_h */
1. Resolution = 0.0001; /* Change this for different resolutions, if desired */
2. U = C; L = 0; MaxIter = 15; Iter = 0; i = (U + L)/2;
3. if ((B_h(C, C, 0) <= P_h) return C;
4. while ((Iter < MaxIter) AND ((U - L) > Resolution)) {
   if (B_h(C, whole(i), frac(i)) > P_h) {
     U = i;
     i = (U + L)/2;
   } else {
     L = i;
     i = (U + L)/2;
   }
   Iter++;
}
5. while (B_h(C, whole(i), frac(i)) > P_h) {
   U = i;
   i = (U + L)/2;
}
6. return i;
```

Figure 5: **Algorithm MINB**

4.3 Algorithm for Optimal Parameters of LFG Policy

We now present an algorithm (Fig. 5), labeled MINB, that minimizes the new call blocking probability for LFG with the constraint that the handoff call blocking probability must be at most P_h, i.e., it determines the optimal values of parameters T and β for a given constraint P_h.

We first establish some properties of $B_n(\cdot)$ and $B_h(\cdot)$ which will be useful in proving that algorithm MINB minimizes the new call blocking probability while satisfying the constraint on the handoff call blocking probability.

Lemma 4: $B_n(\cdot)(B_h(\cdot))$ is a monotonically decreasing (increasing) function of both T and β and hence also of $T + \beta$.

Proof: Oh and Tcha ([6]) have shown that $B_h(C, T+1, \beta) > B_h(C, T, \beta)$ and $B_n(C, T+1, \beta) < B_n(C, T, \beta)$ for the integral guard channel policy. Thus, we only need to show that for a given T, B_n

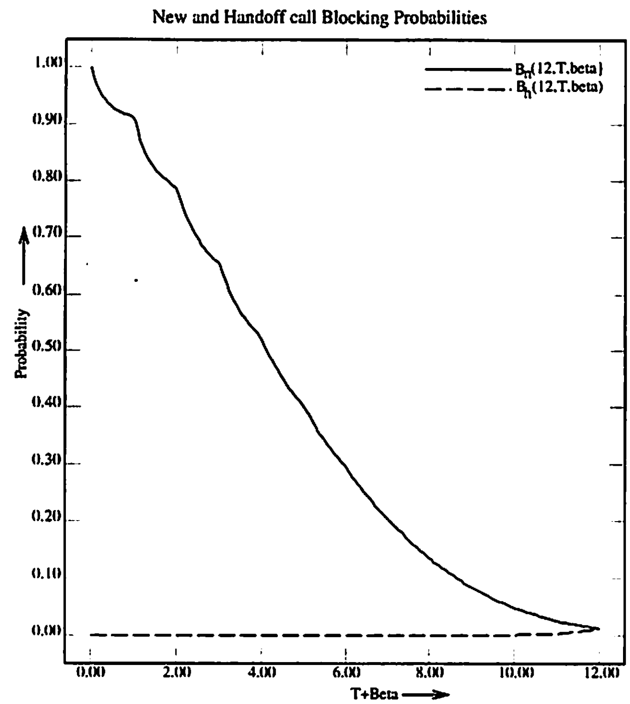

Figure 6: B_h **and** B_n **as functions of** $T + \beta$

(B_h) decreases (increases) monotonically with β. Differentiating B_h with respect to β,

$$\frac{\partial B_h(\cdot)}{\partial \beta} = \frac{(\Sigma_{j=0}^{T} \frac{\rho^j}{j!}) * (\frac{\rho^C}{C!}) * (1-\alpha)}{(\Sigma_{j=0}^{T} \frac{\rho^j}{j!} + \Sigma_{j=T+1}^{C} \frac{\gamma \rho^j \alpha^{j-T-1}}{j!})^2} > 0 \tag{8}$$

Since the derivative of B_h with respect to β is postive, B_h is a monotonically increasing function of β. Similarly, differentiating B_n with respect to β, we can show that the derivative is negative. Thus, we can prove that B_n decreases monotonically with β.

Theorem 3: Algorithm MINB minimizes the new call blocking probability ($B_n(\cdot)$) while satisfying the constraint on handoff call blocking probability ($B_h(\cdot) \leq P_h$).

Proof: Clearly, if the condition in Step 3 is true, we are done; we don't have any guard channels in this case and this would result in the least blocking probability for new calls.

The algorithm uses the fact that finding the value of $T+\beta$ which satisfies the equation $B_h(\cdot) = P_h$ is precisely the value for which $B_n(\cdot)$ is minimized. This is based on the fact that $B_h(\cdot)$ is monotonically increasing function of $T + \beta$ and hence the value of $T + \beta$ which satisfies the equality on the handoff constraint is the largest admissible value for $T + \beta$. This also minimizes the new call blocking probability as $B_n(\cdot)$ is monotonically decreasing function of $T + \beta$. The while loop in Step 4 performs a binary search to locate the value of $T + \beta$ which satisfies the equation $B_h(\cdot) = P_h$. A graph of the functions B_h and B_n versus $T + \beta$ is shown in Figure 6 for illustration. The traffic parameters correspond to those of Case I in Table 1 with 12 channels.

4.4 Numerical Examples

We show below that the introduction of Fractional Guard Channels gives us significant savings in terms of new call blocking probability as compared to the integral Guard channel approach. The load values and the number of channels in Table 1 are taken from [6]. Column 4 of Table 1 is obtained by computing the minimum new call blocking probability B_n for the integral guard channel policy subject to the constraint $B_h \leq 0.01$. Column 5 of Table 1 is obtained by via algorithm MINB and calculating the new call blocking probability. The last column in the table lists the percentage gain in new call blocking for LFG over the integral guard channel policy. We see significant savings in cases I through IV. The values in Case V has been particularly chosen to illustrate a case when the constraint on handoff probability can be met without the use of any guard channels. In this case, as expected, there is no gain for the fractional policy.

Item	Arrival Rate (λ)	Handoff Prob. (α)	C	Integral Policy (B_n)	Fractional Policy (B_n)	Percentage Gain for Fractional
Case I	6	1/6	12	0.024859	0.0013313	46.45
Case II	7	2/7	13	0.031118	0.021538	30.78
Case III	12	1/6	19	0.029255	0.022536	22.97
Case IV	14	2/7	22	0.023117	0.015211	34.20
Case V	14	2/7	23	0.0074454	0.007454	0.0

Table 1: MINBLOCK: Minimize B_n such that $B_h \leq 0.01$

5 Minimizing the number of Channels with hard constraints

Lastly, we consider the problem of finding an admission control policy that minimizes the number of channels while satisfying the blocking constraints for both the new and handoff calls. Formally,

Problem 3: MINC: Minimize C such that

$$B_n(C, T, \beta) \leq P_n \tag{9}$$

$$and\, B_h(C,T,\beta) \le P_h \quad (10)$$

with $0 \le T \le C$ (T,C $\in \mathcal{I}$) and $0 \le \beta < 1$.

Note that the LFG policy is optimal for this problem also. This follows easily from the fact the LFG policy is optimal for MINBLOCK problem and MINC problem can be reformulated as find the Minimum C such that the minimum B_n (from the MINBLOCK problem) is smaller than P_n, given $B_h \le P_h$.

5.1 Algorithm for Optimal Parameters of LFG Policy

We now present an algorithm (Fig. 4) that calculates the minimum number of channels which satisfy the QOS constraints that new call blocking probability must be at most P_n and handoff call blocking probability must be at most P_h. Note that we could use algorithm MINB by starting with the value of C that meets the constraint for the handoff call blocking probability and increase C till the new call blocking probability also meets its constraint. Algorithm MIN detailed in Figure 7 is more efficient.

The following claims will be useful in proving that algorithm MIN finds the minimum number of channels that satisfies the constraints expressed in Equations (8) and (9).

Claim 1: There exists a value of $T+\beta$ for $C \ge C_{lb}$,where $C_{lb} = Arg\ \ Min\{C : B_n(C,C,0) \le P_n\}$, such that $B_n(\cdot)$ and $B_h(\cdot)$ either both violate their constraints, P_n and P_h, or both meet them.

Let $T_n + \beta_n$ be such that $B_n(C, T_n, \beta_n) = P_n$. Note that a feasible solution $T_n + \beta_n$ always exists in $[0, C]$ for $C > C_{lb}$ since $B_n(C,0,0) = 1.0$ and $B_n(C,C,0) \le P_n$ and $B_n(\cdot)$ is monotonically decreasing in $T+\beta$. Now consider $B_h(\cdot)$. If $B_h(C,T,\beta) > P_h, \quad \forall T+\beta \in [0,C]$, then in the interval $[0, T_n+\beta_n)$ both $B_n(\cdot)$ and $B_h(\cdot)$ violate their constraints. On the other hand, if $B_h(C,T,\beta) \le P_h, \quad \forall T+\beta \in [0,C]$, then in the interval $[T_n+\beta_n, C]$ both $B_n(\cdot)$ and $B_h(\cdot)$ meet their constraints. Finally, if there is a feasible solution to $B_h(C,T_h,\beta_h) = P_h$ for $T_h+\beta_h \in [0,C]$, then if $T_h+\beta_h > T_n+\beta_n$, the interval $[T_n+\beta_n, T_h+\beta_h]$ includes all points $T+\beta$ where both $B_h(\cdot)$ and $B_n(\cdot)$ meet their constraints. If $T_h+\beta_h \le T_n+\beta_n$, then both $B_h(\cdot)$ and $B_n(\cdot)$ will violate their constraints for $T+\beta \in [T_h+\beta_h, T_n+\beta_n]$.

Claim 2: If there exists a value of $T+\beta$ for a given C where both the constraints given by

```
/* We assume that the traffic parameters ρ and α are assumed to be
built in the evaluation of functions B_h and B_n, which are evaluated based on
Equations (1) and (2) respectively. Also, whole(i) returns the integer part of i and
frac(i) returns the fractional part of i. */
1. C = 1;
2. while (B_n(C, C, 0) > P_n) C = C + 1;
3. if (B_h(C, C, 0) ≤ P_h) return C;
4. U = C; L = 0;
5. i = (U + L)/2;
6. while ((B_h(C, whole(i), frac(i)) > P_h) XOR (B_n(C, whole(i), frac(i)) > P_n)) {
   if (B_n(C, whole(i), frac(i)) > P_n) {
     L = i;
     i = (U + L)/2;
   }
else if (B_h(C, whole(i), frac(i)) > P_h) {
     U = i;
     i = (U + L)/2;
   }
}
7. if ((B_h(C, whole(i), frac(i)) ≤ P_h) AND (B_n(C, whole(i), frac(i)) ≤ P_n))
   return C;
else {
   C = C + 1
   goto step 3.
}
```

Figure 7: **Algorithm MIN**

Equations (8) and (9) are violated, then there is no value of $T + \beta$ for the given C where both the constraints can be met.

Claim 2 can be inferred from the arguments for different cases in proof of Claim 1. A simpler argument is that if a value of $T + \beta$ violates both the constraints, reducing (increasing) the value will lead to larger violations of $B_n(\cdot)(B_h(\cdot))$.

Theorem 4: Algorithm MIN finds the minimum number of channels that satisfies the constraints given by Equations (8) and (9).

Proof: The initial assignment in Step 2 of the algorithm is a lower bound, C_{lb}, on the minimum number of channels. If the condition at Step 3 succeeds, we are done. If not, the while loop in Step 6 tries to locate a value for T and β (i in algorithm MIN represents $T + \beta$) which satisfies the given constraints. Since we consider only values of $C > C_{lb}$, Claim 1 holds and hence the algorithm is guaranteed to terminate provided the algorithm can find the point that either violates or meets the constraints. In the following, we show that the algorithm finds such a point. To do that, we show that at each iteration of the while loop, the following invariant holds.

Invariant: The interval represented by $[L, U]$ is such that for $T + \beta = L$, the constraint given by Equation (9) is met and for $T + \beta = U$, the constraint given by Equation (8) is met.

This is due to the fact that the iteration condition of the while loop ensures that exactly one of the constraints is met at the midpoint of the interval and the body of the while loop resets the endpoint where that condition had been met previously to the midpoint. For example, if the constraint given by Equation (8) is violated at the midpoint while the constraint given by Equation (9) is satisfied. the lower bound of the interval is moved to the midpoint resulting in a new interval where the invariant still holds.

Thus, an interval $[L_1, U_1]$ is halved at each iteration, to form a new interval $[L_2, U_2]$, while the invariant is maintained. Since the invariant is maintained, it is easy to see that a point that either meets the constraints or violates them together cannot be in the part of the interval, $[L_1, U_1] - [L_2, U_2]$, that is eliminated from future consideration due to the monotonicity of $B_h(\cdot)$ and $B_n(\cdot)$. Hence, the algorithm is guaranteed to find the point (which is guaranteed to exist by Claim 1) that either meets the constraints or violates it. In the latter case, by Claim 2, there is no feasible solution for C and hence we increment C by 1 and go to Step 3. Thus, Algorithm MIN finds the minimum number of channels that satisfies the required QOS constraints.

A graph of the functions B_h and B_n versus $T + \beta$ is shown for illustrative purposes in Figure 6. The traffic parameters correspond to those of Case I in Table 2 with 12 channels. The figure also highlights the region in which the QoS constraints are met.

5.2 Numerical Examples

We show below that the introduction of Fractional guard channels gives us smaller values for the number of channels compared to the integral Guard channel. The load values in Table 2 are taken from [6]. Column 3 is obtained via an algorithm detailed in [6] which evaluates the minimum value of C for the integral Guard Channel policy while meeting the blocking constraints. Column 4 is obtained via algorithm MIN. From Table 2, we can see that the LFG policy gives us a smaller value of C in case I (12 instead of 13) and case IV (22 instead of 23). However, the LFG policy in general appeared to

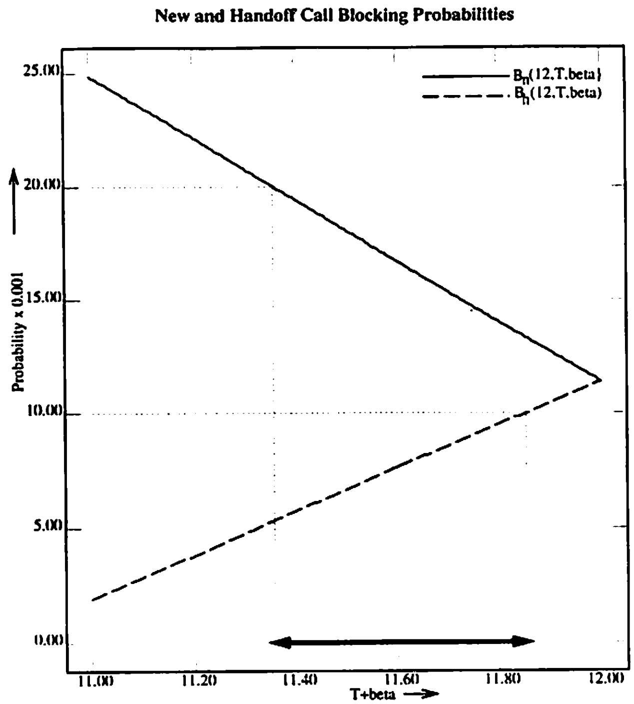

Figure 8: **The Region[11.37-11.84] where B_h <= 0.01 and B_n <= 0.02**

provide only small improvements (1-2 channels) over the integral guard channel policy.

Item	Arrival Rate (λ)	Handoff Prob. (α)	Integral Guard Channels (C, T, B_n, B_h)	Fractional Guard Channels $(C, T+\beta, B_n, B_h)$
Case I	6	1/6	(*13*,12,0.012,0.00087)	(*12*,11.625,0.016,0.0078)
Case II	7	2/7	(14,13,0.016,0.002)	(14,14.0,0.007,0.007)
Case III	12	1/6	(20,19,0.018,0.00165)	(20,20.0,0.0098,0.0098)
Case IV	14	2/7	(*23*,22,0.0145,0.002)	(*22*,21.313,0.0197,0.0063)

Table 2: Minimize C such that $B_n \leq 0.02$ and $B_h \leq 0.01$

6 Dynamic Fractional Guard Channel Policy

The importance of adaptability in wireless call admission algorithms and architecture have been stressed in [1, 5]. Also, call admission (whether new or handover) has to be done in a quick and timely fashion so that the user is not disconnected, excessively delayed or does not perceive glitches in the conversation [10]. Thus it is necessary that any proposed policy be able to adapt with possible changes in the traffic load while maintaining the desired QOS levels. We focus on the new and handoff call blocking in this section. The previous sections considered static optimal choices of parameters and policies for minimizing or meeting certain objective functions of these QOS measures. In this section, we consider, briefly, the scenario where the traffic loads are changing (as would be the case in reality).

We propose to adapt to changes in the traffic load by dynamically changing the parameters of the Fractional Guard Channel policy. This adaptability will require

- Estimation of traffic loads and blocking
- Fast computation of T and β

The estimation of the mean values of traffic arrival rate can be done using , for example, an exponential smoothing model [3]. Based on the traffic estimates and the current values of T and β, we can calculate the blocking probabilities $B_h(\cdot)$ and $B_n(\cdot)$ (or they can be estimated directly). Based on these probabilities, we can detect whether these values violate their respective QOS values P_h and P_n. If a violation is detected, we can use algorithm MIN to recompute values of T and β for the updated traffic estimates, thereby ensuring that the system remains within its prescribed QOS guarantees. Thus, the LFG policy can serve as a control mechanism which may, for example, automatically increase the blocking of new calls as overload conditions set in. This provides a way of bringing the cell back to stable condition, maintaining the low probability of dropping handoff calls. Note that an uncontrolled mechanism may lead to thrashing in overload conditions, where users are being connected and dropped off shortly afterwards, when the cells where the call originated and where the call was handed off are both overloaded. While the integral Guard Channel policy can also serve as a control mechanism by adjusting the value of T, the LFG policy has the advantage of two levels of control; parameter β can be used for fine-grained control while parameter T can serve as a coarse-grained control.

Any computation required for adaptive control of real-time systems must be fast. To illustrate the computational speed of algorithms MINB and MIN, consider Case IV from Table 2. Algorithm MINB takes about 0.8 milliseconds and algorithm MIN takes about 0.5 milliseconds on a Sparc-10 workstation. Thus, the LFG policy can be easily implemented at the Base Station and can be used to adaptively control the blocking probabilities experienced by the Mobile Subscriber.

7 Conclusions and Future Work

In this paper, we considered optimal admission control policies in cellular networks in the light of three problems: minimizing a linear objective function of the new and handoff call blocking probabilities (MINOBJ), minimizing the new call blocking probability with a hard constraint on handoff call blocking probability (MINBLOCK) and minimizing the number of channels with hard constraints on both the blocking probabilities (MINC). We showed that the well-known Guard Channel policy was optimal for MINOBJ. We defined a new Fractional Guard Channel policy and showed that a restricted version of it was optimal for the MINBLOCK and MINC problems. We also showed that the Fractional guard channel policy resulted in significant (20-50%) savings in the new call blocking probability over the integral Guard channel policy for the MINBLOCK problem and provided some, though small, improvement over the integral Guard channel policy for the MINC problem. Further, we showed that the algorithms developed in this paper for the solution of these problems were computationally inexpensive and have potential for use as a real-time control mechanism in cellular networks.

We are exploring the use of the Fractional Guard Channel policy to assist in Dynamic channel assignment and other hybrid policies [10]. One can also easily extend this policy to have finite buffers for handoff and/or new calls [14] in case delaying handoff and/or new calls is considered acceptable. We are also studying call admission policies when there are multiple classes of wireless traffic. Finally, we are also studying in greater detail the mechanisms for dynamically changing the parameters of the Fractional Guard Channel policy in order to adapt to changes in traffic load.

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Appendix A

Lemma 1: $V_k(i)$ is monotonically non-decreasing and convex in i for all k.

Proof: Part a)

We first show that V_k is monotonically non-decreasing by induction. The basis step is trivial since $V_0(i) = 0$. Assume that $g(i) = V_{k-1}(i)$ is monotonically non-decreasing. We need to show that $V_k(i)$ is monotonically non-decreasing. The first term in $\Delta(i) = V_k(i) - V_k(i-1)$ is

$$
\begin{aligned}
&min(g(i+1), A_1 + g(i)) - min(g(i), A_1 + g(i-1)) \\
&\quad = \; g(i) + min(g(i+1) - g(i), A_1) - g(i-1) - min(g(i) - g(i-1), A_1) \\
&\quad \geq \; g(i) - g(i-1) - min(g(i) - g(i-1), A_1) \quad \text{since g(i) is monotonically non-decreasing} \\
&\quad \geq \; 0 \quad \text{since g(i) is monotonically non-decreasing}
\end{aligned}
$$

Similarly, the second term of $\Delta(i)$ can also be shown to be monotonically non-decreasing. Let $f(i) = \mu V_{k-1}(i)$ and assume it is non-decreasing. Now, consider the third and fourth terms of $\Delta(i)$.

$$
\begin{aligned}
&(C-i)f(i) + i f(i-1) - (C-(i-1))f(i-1) - (i-1)f(i-2) \\
&\quad = \; (C-i)f(i) - (C-i)f(i-1) + (i+1)f(i-1) - (i-1)f(i-2) \\
&\quad \geq \; (i+1)f(i-1) - (i-1)f(i-2), \quad \text{since f(i) is non-decreasing} \\
&\quad \geq \; i f(i-1) - i f(i-2) \\
&\quad \geq \; 0.
\end{aligned}
$$

Hence, $\Delta(i) = V_k(i) - V_k(i-1) \geq 0$ and thus $V_k(i)$ is a monotonically non-decreasing function of i for all k.

Part b)

We show $V_k(i)$ is convex by induction. The basis step is again trivial since $V_0(i) = 0, 1 \leq i \leq C$. Assume $V_{k-1}(i)$ is convex. We need to show that $V_k(i)$ is convex. The first and second terms in Equation (3) can be shown to be convex by induction [8.35 in [12]]. The arguments for the 3rd and 4th terms follow. Let $f(i)$ be convex. Then consider $if(i-1) + (C-i)f(i)$. Let

$$
\begin{aligned}
\Delta(i) &= i f(i-1) + (C-i)f(i) - (i-1)f(i-2) - (C-(i-1))f(i-1) \\
&= (C-i)f(i) - (i-1)f(i-2) + (2i+1-C)f(i-1).
\end{aligned}
$$

Then,

$$\Delta(i) - \Delta(i-1)$$

$$= (C-i)f(i) - (i-1)f(i-2) + (2i+1-C)f(i-1)$$

$$- (C-(i-1))f(i-1) + (i-2)f(i-3) - (2i-1-C)f(i-2)$$

$$= (C-i)f(i) + (3i-2C)f(i-1) - (3i-2-C)f(i-2) + (i-2)f(i-3)$$

$$= (C-i)f(i) - 2(C-i)f(i-1) + (C-i)f(i-2)$$

$$+ i\,f(i-1) - 2(i-1)f(i-2) + (i-2)f(i-3)$$

$$\geq i\,f(i-1) - 2(i-1)f(i-2) + (i-2)f(i-3) \quad \text{by convexity of f(i),}$$

$$\geq (i-1)f(i-1) - 2(i-1)f(i-2) + (i-1)f(i-3) \quad \text{since f(i) is non-decreasing,}$$

$$\geq 0 \quad \text{by convexity of f(i).}$$

Thus, $V_k(i)$ is a convex function in i for all k.

□

Appendix B

Property 1: $A_2^T > A_2^{T+1}$ for T=0,1,...,C.

Proof: Case 1: $T = C$.

Assume $A_2^C > A_2^{C+1} = 0$

$$A_2^C > 0$$

$$\Rightarrow \frac{V_{C-1}X_C - 0X_{C-1}}{W_C X_{C-1} - W_{C-1}X_C} - 1 > 0$$

$$\Rightarrow X_C(V_{C-1} + W_{C-1}) > W_C X_{C-1}$$

$$\Rightarrow \Sigma_{j=0}^{C}\frac{\rho^j}{j!}\left(\frac{\rho^{C-1}}{(C-1)!} + \frac{\rho^C}{C!}\alpha\right) > \Sigma_{j=0}^{C-1}\frac{\rho^j}{j!}\frac{\rho^C}{C!} + \frac{\rho^C}{C!}\alpha\frac{\rho^C}{C!}$$

$$\Rightarrow \left(\Sigma_{j=0}^{C}\frac{\rho^j}{j!}\frac{\rho^{C-1}}{(C-1)!} - \left(\Sigma_{j=0}^{C-1}\frac{\rho^j}{j!}\frac{\rho^C}{C!}\right) + \left(\Sigma_{j=0}^{C}\frac{\rho^j}{j!}\frac{\rho^C}{C!}\alpha - \frac{\rho^C}{C!}\alpha\frac{\rho^C}{C!}\right) > 0\right.$$

The second term in the above equation is clearly greater than 0. The first term can further be expanded into

$$\{\frac{\rho^{C-1}}{(C-1)!}+(\frac{\rho^{C}}{1!(C-1)!}-\frac{\rho^{C}}{C!})+(\frac{\rho^{C+1}}{2!(C-1)!}-\frac{\rho^{C+1}}{1!C!})+...+(\frac{\rho^{2C-1}}{C!(C-1)!}-\frac{\rho^{2C-1}}{(C-1)!C!})\}$$

which is also greater than zero. Thus, the property holds for $T = C$.

Case 2: $T = 0$.

We need to show that $A_2^1 < A_2^0 = \infty$. It suffices to show that the denominator of A_2^1 is not zero since the numerator is a known finite quantity. The denominator is

$$\begin{aligned} & W_1X_0 - W_0X_1 \\ = \; & \frac{\rho^C}{C!}\alpha^{C-1}(\Sigma_{j=0}^{C}\frac{\rho^j}{j!}\alpha^j) - \frac{\rho^C}{C!}\alpha^C(1+\Sigma_{j=1}^{C}\frac{\rho^j}{j!}\alpha^{j-1}) \\ = \; & \frac{\rho^C}{C!}\alpha^{C-1}(1+\Sigma_{j=1}^{C}\frac{\rho^j}{j!}\alpha^j - \alpha - \frac{\rho^j}{j!}\alpha^j) \\ = \; & \frac{\rho^C}{C!}\alpha^{C-1}(1-\alpha) \\ > \; & 0 \end{aligned}$$

Thus, the property holds for $T = 0$.

Case 3: $T = 1, 2, ..., C-1$

Assume $A_2^{T-1} > A_2^T$.

$$\Rightarrow \frac{V_{T-1}X_T - V_TX_{T-1}}{W_TX_{T-1} - W_{T_1}X_T} - 1 > \frac{V_TX_{T+1} - V_{T+1}X_T}{W_{T+1}X_T - W_TX_{T+1}} - 1$$

$$\Rightarrow X_T^2V_{T-1}W_{T+1} - X_TX_{T+1}V_{T-1}W_T - X_{T-1}X_TV_TW_{T+1} + \underbrace{X_{T-1}X_{T+1}V_TW_T}$$

$$> X_T^2V_{T+1}W_{T-1} - X_TX_{T+1}V_TW_{T-1} - X_{T-1}X_TV_{T+1}W_T + \underbrace{X_{T-1}X_{T+1}V_TW_T}$$

$$\Rightarrow \quad X_T^2(V_{T-1}W_{T+1} - V_{T+1}W_{T-1}) > X_T X_{T+1}(V_{T-1}W_T - V_T W_{T-1}) + X_T X_{T+1}(V_T W_{T+1} - V_{T+1}W_T)$$

Now,

$$\begin{aligned}
& V_T W_{T+1} - V_{T+1} W_T \\
&= \Sigma_{j=T}^{C-1} \frac{\rho^j}{j!}\alpha^{j-T} \frac{\rho^C}{C!}\alpha^{C-T-1} - \Sigma_{j=T+1}^{C-1} \frac{\rho^j}{j!}\alpha^{j-T-1} \frac{\rho^C}{C!}\alpha^{C-T} \\
&= \frac{\rho^C}{C!}\alpha^{C-T-1}\frac{\rho^T}{T!}
\end{aligned}$$

and

$$\begin{aligned}
& V_{T-1} W_{T+1} - V_{T+1} W_{T-1} \\
&= \Sigma_{j=T-1}^{C-1} \frac{\rho^j}{j!}\alpha^{j-T+1} \frac{\rho^C}{C!}\alpha^{C-T-1} - \Sigma_{j=T+1}^{C-1} \frac{\rho^j}{j!}\alpha^{j-T-1} \frac{\rho^C}{C!}\alpha^{C-T+1} \\
&= \frac{\rho^C}{C!}\alpha^{C-T-1}\left(\frac{\rho^{T-1}}{(T-1)!} + \frac{\rho^T}{T!}\alpha\right)
\end{aligned}$$

Substituting,

$$\begin{aligned}
& \left(\Sigma_{j=0}^{T-1}\frac{\rho^j}{j!} + \Sigma_{j=T}^{C}\frac{\rho^j}{j!}\alpha^{j-T}\right)\frac{\rho^C}{C!}\alpha^{C-T-1}\left(\frac{\rho^T}{T!}\alpha + \frac{\rho^{T-1}}{(T-1)!}\right) \\
> & \left(\Sigma_{j=0}^{T}\frac{\rho^j}{j!} + \Sigma_{j=T+1}^{C}\frac{\rho^j}{j!}\alpha^{j-T-1}\right)\frac{\rho^C}{C!}\alpha^{C-T}\frac{\rho^{T-1}}{(T-1)!} + \left(\Sigma_{j=0}^{T-2}\frac{\rho^j}{j!} + \Sigma_{j=T-1}^{C}\frac{\rho^j}{j!}\alpha^{j-T+1}\right)\frac{\rho^C}{C!}\alpha^{C-T-1}\frac{\rho^T}{T!}
\end{aligned}$$

Rearranging and collecting $\frac{\rho^T}{T!}$ and $\frac{\rho^{T-1}}{(T-1)!}$ separately,

$$\begin{aligned}
& \frac{\rho^T}{T!}\frac{\rho^C}{C!}\alpha^{C-T-1}\left\{\left(\Sigma_{j=0}^{T-1}\frac{\rho^j}{j!} + \Sigma_{j=T}^{C}\frac{\rho^j}{j!}\alpha^{j-T}\right)\alpha - \left(\Sigma_{j=0}^{T-2}\frac{\rho^j}{j!} + \Sigma_{j=T-1}^{C}\frac{\rho^j}{j!}\alpha^{j-T+1}\right)\right\} \\
& + \frac{\rho^{T-1}}{(T-1)!}\frac{\rho^C}{C!}\alpha^{C-T-1}\left\{\left(\Sigma_{j=0}^{T-1}\frac{\rho^j}{j!} + \Sigma_{j=T}^{C}\frac{\rho^j}{j!}\alpha^{j-T}\right) - \alpha\left(\Sigma_{j=0}^{T}\frac{\rho^j}{j!} + \Sigma_{j=T+1}^{C}\frac{\rho^j}{j!}\alpha^{j-T-1}\right)\right\} > 0
\end{aligned}$$

$$\Rightarrow \quad \frac{\rho^T}{T!}\frac{\rho^C}{C!}\alpha^{C-T-1}\left\{\Sigma_{j=0}^{T-1}\frac{\rho^j}{j!}\alpha + \underbrace{\Sigma_{j=T}^{C}\frac{\rho^j}{j!}\alpha^{j-T+1}} - \Sigma_{j=0}^{T-1}\frac{\rho^j}{j!} - \underbrace{\Sigma_{j=T}^{C}\frac{\rho^j}{j!}\alpha^{j-T+1}}\right\}$$

$$+\frac{\rho^{T-1}}{(T-1)!}\frac{\rho^C}{C!}\alpha^{C-T-1}\{\Sigma_{j=0}^{T}\frac{\rho^j}{j!}+\underbrace{\Sigma_{j=T+1}^{C}\frac{\rho^j}{j!}\alpha^{j-T}}-\Sigma_{j=0}^{T}\frac{\rho^j}{j!}\alpha-\underbrace{\Sigma_{j=T+1}^{C}\frac{\rho^j}{j!}\alpha^{j-T}}\}>0$$

$$\Rightarrow \frac{\rho^{T-1}}{(T-1)!}\frac{\rho^C}{C!}\alpha^{C-T-1}(1-\alpha)\{\Sigma_{j=0}^{T}\frac{\rho^j}{j!}-\frac{\rho}{T}\Sigma_{j=0}^{T-1}\frac{\rho^j}{j!}\}>0$$

$$\Rightarrow \frac{\rho^{T-1}}{(T-1)!}\frac{\rho^C}{C!}\alpha^{C-T-1}(1-\alpha)\{1+(\rho-\frac{\rho}{T})+(\frac{\rho^2}{2!}-\frac{\rho^2}{T})+(\frac{\rho^3}{3!}-\frac{\rho^3}{T2!})+\cdots+(\frac{\rho^{T-1}}{(T-1)!}-\frac{\rho^{T-1}}{T(T-2)!})\}>0$$

The last equation is clearly true. Thus, property 1 holds for $T = 0, 1, \ldots, C$.

□