Large Deviations and the Generalized Processor Sharing Scheduling: Upper and Lower Bounds Part II: Multiple-Queue Systems

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Abstract

We prove asymptotic upper and lower bounds on the asymptotic decay rate of per-session queue length tail distributions for a single constant service rate server queue shared by multiple sessions with the *generalized processor sharing* (GPS) scheduling discipline. The special case of a two-queue GPS system has been dealt with separately in Part I of the paper [33], where exact bounds are obtained for each queue. A general multiple-queue GPS system is treated in this part (Part II) of the paper and tight upper and lower bound results are proved by examining the dynamics of bandwidth sharing nature of the GPS scheduling. We are not able to obtain exact bounds in this general case due to the complex nature of dynamic bandwidth sharing under the GPS scheduling. The proofs use sample-path large deviation principle and are based on some recent large deviation results for a single queue with a constant service rate server. These results have implications in call admission control for high-speed communication networks.

1 Introduction

In the future high speed digital networks, to support a variety of applications such as voice, video and datagram traffic with diverse traffic characteristics and *quality of service* (QoS) requirements, it has been suggested to provide different QoS service classes to accommodate this diversity of traffic characteristic and QoS requirement [1, 8, 28]. As a result, more sophisticated scheduling mechanism other than the simple First-In First-Out (FIFO) service discipline is needed to provide both protection and bandwidth sharing among service classes. For this purpose, the Generalized Process Sharing (GPS) service discipline [26, 25] (also known as *Weighted Fair Queueing*), has been proposed, one important feature of which is its ability to provide isolation among different classes, while, at the same time, allowing bandwidth sharing among classes.

More specifically, GPS is a work-conserving scheduling discipline, it assumes a fluid source model where source traffic is treated as infinitely divisible fluid (hence an ideal model). Consider n sessions sharing a GPS server with

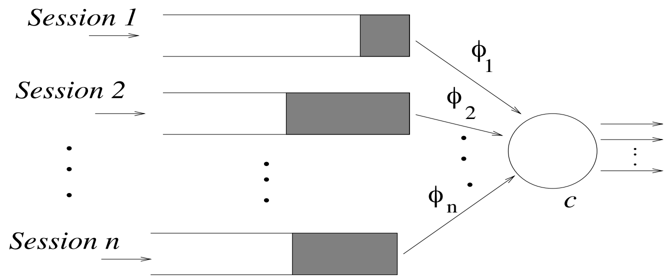

Figure 1: A two-queue GPS system.

rate c, each session with its own queue 1. Associated with the sessions are parameters $\{\phi_i\}_{1\leq i\leq n}$ (called *GPS assignment*) which determine the minimum sharing of bandwidth of each session. Each session is guaranteed a minimum service rate of $g_i = \frac{\phi_i}{\sum_{j=1}^{n}\phi_j}c$. More generally, if the set of sessions with queued packets at time t is $\mathcal{B}(t) \subseteq \{1,\ldots,n\}$, the session $i \in \mathcal{B}(t)$ receives service at rate $\frac{\phi_i}{\sum_{j\in\mathcal{B}(t)}\phi_j}$ at time t.

The performance of GPS has been studied in both deterministic [26, 27, 25] and stochastic setting [32, 35] and upper bounds on the interested metrics such as loss or delay are derived. In the deterministic case, Parekh and Gallager [26, 27, 25] show that the upper bounds are attainable in the worst-case. In the stochastic setting, how tight the upper bounds are is still an open question.

In this paper, we are interested in deriving upper and lower bounds on the asymptotic decay rate of the queue length tail distribution of each session. We consider a *discrete-time fluid model*, by which we mean that arrival and service happen at discrete-time slot indexed by integers, but arrival and service are in the form of *fluid*, *i.e.*, they are infinite divisible.

In Part I of the paper [33], we have looked at a two-queue GPS system. Due to the simpler bandwidth sharing mechanism of this special case, we are able to obtain exact bound on the asymptotic decay rate of the queue length tail distribution for each queue.

In this part (Part II) of the paper, we consider a general GPS system with multiple queues. Due to the complexity of the bandwidth sharing mechanism in the general GPS system, the upper and lower bounds we obtain do not match exactly. However, they have similar form, indicative of their tightness. In particular, if there are only two queues, the lower and upper bounds are the same. Our results are derived based on the sample-path large deviation principle and exploring the complicated bandwidth sharing structure in more details. A key concept introduced is *partial feasible sets* which captures the dynamics bandwidth sharing nature of the GPS scheduling. Our results are more general and includes those in [11] as a special case.

Study of asymptotic behavior of queueing systems has its implication in call admission control with QoS guarantees for the future high-speed networks. The theory of effective bandwidths (see [1], *e.g.*, [19, 18, 16, 20, 15, 23, 31, 4, 17, 13, 24]) developed in recent years exploits this asymptotics to provide a simple theoretical call admission control scheme for networks represented by a single server with a shared queue. This scheme is *asymptotically optimal*. For networks employing GPS service discipline, a theoretical admission control framework is laid out in [34] for various network service models based on the results in [35]. Optimal and sub-optimal call admission control schemes are designed using the stochastic envelope process model [4] and the theory of effective bandwidths. Although the upper bounds obtained in this paper are tighter than those in [35], they are generally impossible to compute effectively. In

[1] For an excellent survey on the theory of effective bandwidths, see [7].

[21], approximation methods are used to obtained tight bounds for the GPS system.

The rest of Part II of the paper is organized as follows. Section 2 lists some large deviation results regarding discrete-Time G/ D/ 1/ ∞ queueing systems which will be used later. Section 3 presents the assumptions, and some important sample-path relations regarding the GPS system, and states the main theorem. Section 4 proves the main theorem. The paper is concluded in Section 5.

2 Large Deviations for Discrete-Time G/ D/ 1/ ∞ Queueing Systems

In Section 2 of Part I of the paper, a brief overview of some key concepts and results from large deviation theory and its application to the study of discrete-time G/ D/ ∞ are presented. To save space, in this section we only list a few results that are to be used later in the paper [2] The following presentation follows closely the formulation in [4, 5].

We describe the arrival process to a discrete-time G/ D/ 1/ ∞ queueing system by a sequence of bounded, nonnegative random variables on $I\!R$, $\{a(t), t \in I\!N\}$, where $I\!N$ is the set of nonnegative integers. In other words, at time t [3], the amount of arrivals to the queue is $a(t)$. For any $\tau = 0, 1, 2, \ldots$ and any $t \in I\!N$, $t > \tau$, define $A(\tau, t) = \sum_{s=\tau}^{t-1} a(s)$, the number of arrivals during the time interval $[\tau, t)$. Also let $A(\tau, \tau) = 0$. We call A the cumulative arrival process. We make the following assumptions on the arrival process $\{a(t), t = 0, 1, 2, \ldots\}$ [4, 5].

(A1) The arrival process $\{a(t), t = 0, 1, 2, \ldots\}$ is ergodic and stationary.

(A2) For any $\theta \in I\!R$,

$$\Lambda_A(\theta) = \lim_{t \to \infty} \frac{1}{t} \log E e^{\theta A(0,t)} < \infty \tag{1}$$

and is differentiable.

(A3) $\{a(t), t = 0, 1, 2, \ldots\}$ is adapted to a filtration $\{\mathcal{F}_t^A, t \in I\!N\}$ with the following property: for any $\theta \in I\!R$, there exists a function $\Gamma_A(\theta)$, $0 \le \Gamma_A(\theta) < \infty$ such that for any $s = 0, 1, 2, \ldots$, $t \in I\!N$,

$$\Lambda_A(\theta)t - \Gamma_A(\theta) \le \log E(e^{\theta A(s,t+s)} | \mathcal{F}_s^A) \le \Lambda_A(\theta)t + \Gamma_A(\theta) \ \ a.s. \tag{2}$$

Note that (A3) implies (1) by taking $s = 0$ in (2). To emphasize (A2), we list it separately. Examples of random processes that satisfy (A1), (A2) and (A3) can be found in [6].

By Gärtner-Ellis Theorem, (A1) and (A2) imply that $\{A(0,t)/t, t \in I\!N\}$ satisfies the large deviation principle with the rate function [4]

$$\Lambda_A^*(x) = \sup_{\theta \in I\!R} \{\theta x - \Lambda_A(\theta)\}.$$

Moreover, if (A3) is also satisfied, then $\{a(t), t = 0, 1, 2, \ldots\}$ satisfies the sample path large deviation principle [5]. More precisely, for $t = 1, 2, \ldots$, define the scaled process

$$A^{(t)}(u) = \frac{1}{t} A(0, \lfloor tu \rfloor), \quad 0 \le u \le 1. \tag{3}$$

[2] Readers who are interested in general large deviation theory should consult [9, 14] or any other books on the subject. [30] gives an excellent survey of large deviation theory and its application to communication networks. [7] and reference therein is a good source on application of large deviation theory to effective bandwidths and queueing theory.

[3] Throughout the paper, whenever a discrete-time system is considered, all time indices are integers.

Let $\mu^{(t)}$ be the distribution of $A^{(t)}(u)$. Then $\{\mu^{(t)}, t \in I\!N\}$ satisfies the sample path large deviation principle with the rate function $I_A(\phi)$ defined as follows:

$$I_A(\phi) = \begin{cases} \int_0^1 \Lambda_A^*(\phi'(u))du, & \text{if } \phi \in AC_0([0,1], I\!R), \\ \infty, & \text{otherwise.} \end{cases}$$

Let c be the rate of the server in the G/D/1/∞ system and denote the backlog at time $t \in I\!N$ (or the queue length at time t) by $Q(t)$. A necessary and sufficient condition for the G/D/1/∞ queueing system to be stable is that the average arrival rate is less than the service rate, *i.e.*, $Ea(0) < c$. Under this stability condition, by Loynes' Theorem [22], assuming that the system starts with an empty queue at time 0, the distribution of $Q(t)$ increases monotonically to a stationary distribution $Q(\infty)$ as $t \to \infty$ and $Q(\infty) < \infty$ almost surely (*a.s.*).

Given that the assumptions (A1) and (A2) on the arrival process and the above stability condition are satisfied, it has been proven (see, *e.g.* [4]) that for any $x \geq 0$,

$$\lim_{x \to \infty} \frac{1}{x} \log Pr\{Q(\infty) > x\} = -\theta^* \tag{4}$$

where θ^* is the unique solution to the equation $\Lambda_A(\theta) = \theta c$ or $\theta^* = \sup\{\theta \in I\!R : \Lambda_A(\theta) < c\theta\}$.

Define $\alpha_A(\theta) = \Lambda_A(\theta)/\theta$. $\alpha_A(\theta)$ is called the *effective bandwidth* of the arrival process $\{a(t), t = 0, 1, 2, \ldots\}$ or the corresponding cumulative arrival process A.

For any $t \in I\!N$, let $S(0,t) = \sum_{\tau=0}^{t-1} b(\tau)$, where $b(\tau)$ is the number of departures at time τ. Thus $\{S(0,t), t \in I\!N\}$ is the (cumulative) departure process. Using the sample path large deviation principle, it is proved in [5] (see also [12]) that the $\{S(0,t)/t, t \in I\!N\}$ satisfies the large deviation principle with the rate function

$$\Lambda_D^*(\alpha) = \begin{cases} \Lambda_A^*(\alpha) & \text{if } \alpha \leq c \\ \infty & \text{otherwise.} \end{cases} \tag{5}$$

Thus

$$\Lambda_D(\theta) = \sup_{\alpha \in I\!R}\{\theta\alpha - \Lambda_D^*(\alpha)\} = \begin{cases} \Lambda_A(\theta) & \text{if } 0 \leq \theta \leq \tilde{\theta} \\ \theta c - \tilde{\theta} c + \Lambda_A(\tilde{\theta}) & \text{if } \theta > \tilde{\theta} \end{cases} \tag{6}$$

where $\tilde{\theta}$ is such that $\Lambda_A'(\tilde{\theta}) = c$, *i.e.*, $\Lambda_A^*(c) = c\tilde{\theta} - \Lambda_A(\tilde{\theta})$.

Therefore, $\alpha_D(\theta) = \Lambda_D(\theta)/\theta$ is the effective bandwidth of the departure process.

In this paper, we primarily interested in the *stationary* G/D/1/∞ queueing system. More specifically, we assume the backlog process of the system has reached its steady state, thus having the same distribution as $Q(\infty)$. We study the system at time 0 and look backward in time. Since the arrival process is stationary, this will have no effect on the assumptions (A1), (A2) and (A3). However, for easy reference, we re-state them from this point of view.

(A1′) The arrival process $\{a(-t), t = 0, 1, 2, \ldots\}$ is ergodic and stationary.

(A2′) For any $\theta \in I\!R$,

$$\Lambda_A(\theta) = \lim_{t \to \infty} \frac{1}{t} \log E e^{\theta A(-t,0)} < \infty$$

and is differentiable.

(A3′) $\{a(-t), t = 0, 1, 2, \ldots\}$ is adapted to a filtration $\{\mathcal{F}^A_{-t}, t = 0, 1, 2, \ldots\}$ with the following property: for any $\theta \in I\!R$, there exists a function $\Gamma_A(\theta)$, $0 \le \Gamma_A(\theta) < \infty$ such that for any $s = 0, 1, 2, \ldots$ and $t \in I\!N$,

$$\Lambda_A(\theta)t - \Gamma_A(\theta) \le \log E(e^{\theta A(-t-s,-s)} | \mathcal{F}^A_{-t-s}) \le \Lambda_A(\theta)t + \Gamma_A(\theta) \ \ a.s.$$

As $Q(0)$ has the same distribution as $Q(\infty)$, from (4), it can be proved that for any positive $\theta < \theta^*$,

$$Ee^{\theta Q(0)} < \infty. \tag{7}$$

The following lemmas are instrumental in proving the main theorem of the paper regarding the GPS system, the proofs of which can be found in the appendix of Part I of the paper [33].

Lemma 1 *Assume $Ea(0) < c$, then for any $\theta < \theta^*$,*

$$\lim_{t\to\infty} \frac{1}{t} \log E[e^{\theta(Q(-t)+A(-t,0))}] = \Lambda_A(\theta). \tag{8}$$

Let $x \wedge y = \min\{x, y\}$. For any $t \in I\!N$, define

$$D(-t) = [Q(-t) + A(-t, 0)] \wedge ct. \tag{9}$$

Lemma 2 *Assume $Ea(0) < c$. Then for any $\alpha \in I\!R$,*

$$\lim_{t\to\infty} \frac{1}{t} \log Pr\{D(-t)/t \ge \alpha\} = -\inf_{x\ge\alpha} \Lambda^*_D(x) \tag{10}$$

where

$$\Lambda^*_D(\alpha) = \begin{cases} \Lambda^*_A(\alpha) & \text{if } \alpha \le c \\ \infty & \text{otherwise.} \end{cases} \tag{11}$$

Moreover, let $\tilde\theta$ be such that $\Lambda'_A(\tilde\theta) = c$. Then for any $\theta \ge 0$,

$$\lim_{t\to\infty} \frac{1}{t} \log E[e^{\theta D(-t)}] = \Lambda_D(\theta) \tag{12}$$

where

$$\Lambda_D(\theta) = \sup_{\alpha \ge Ea(0)} \{\theta\alpha - \Lambda^*_D(\alpha)\} = \begin{cases} \Lambda_A(\theta) & \text{if } \ 0 \le \theta \le \tilde\theta \\ \theta c - \tilde\theta c + \Lambda_A(\tilde\theta) & \text{if } \ \theta > \tilde\theta. \end{cases} \tag{13}$$

For any $t \in I\!N$, define the scaled process $A^{(t)}(s) = \frac{1}{t} A(-\lfloor ts \rfloor, 0)$, $0 \le s \le 1$. Let

$$B(-t) = \min_{0\le\tau\le t} \{A(-\tau, 0) + c(t - \tau)\} = t \min_{0\le s\le 1} \{A^{(t)}(s) + c(1 - s)\}. \tag{14}$$

Lemma 3 *$\{B(-t)/t, t \in I\!N\}$ satisfies the large deviation principle with the rate function $\Lambda^*_B(x)$ defined as follows: If $Ea(0) < c$, then*

$$\Lambda^*_B(x) = \begin{cases} \Lambda^*_A(x) & \text{if } x \le c \\ \infty & \text{otherwise} \end{cases} \tag{15}$$

and if $Ea(0) \ge c$, then

$$\Lambda^*_B(x) = \begin{cases} 0 & \text{if } x = c \\ \infty & \text{otherwise.} \end{cases} \tag{16}$$

From Varadhan's Integral Lemma (see, *e.g.*, Theorem 4.3.1 in [9]), we have that for any $\theta \in I\!R$,

$$\lim_{t\to\infty} \frac{1}{t} \log E e^{\theta B(-t)} = \Lambda_B(\theta) = \sup_{x\in I\!R}\{x\theta - \Lambda_B^*(x)\}.$$

Therefore, if $Ea(0) < c$, then

$$\Lambda_B(\theta) = \begin{cases} \Lambda_A(\theta) & \text{if} \quad \theta \le \tilde{\theta} \\ \theta c - \tilde{\theta} c + \Lambda_A(\tilde{\theta}) & \text{if} \quad \theta > \tilde{\theta}. \end{cases}$$

If $Ea(0) \ge c$, then $\Lambda_B(\theta) = \theta c$ for all $\theta \in I\!R$.

3 Multiple-Queue GPS Systems: Assumptions, Sample Path Relations and Statement of the Main Theorem

Consider a GPS system with n queues, where $n \ge 2$ (Figure 1). Let c be the service rate of the GPS server and $\{\phi_i\}_{1\le i\le n}$ the GPS assignment for the n sessions sharing the GPS server where $\phi_i \ge 0, 1 \le i \le n$.

For any time t, let $a_i(t)$ denote the amount of arrival from session i to queue i at time t, and for any time interval $[\tau, t)$, let $A(\tau,t) = \sum_{s=\tau}^{t-1} a_i(s)$ denote the total amount of arrival during $[\tau, t)$. Similarly, let $b_i(t)$ denote the amount of service session i received at time t and $S_i(\tau,t) = \sum_{s=\tau}^{t-1} b_i(s)$ the total amount of service session i received during $[\tau, t)$. The backlog of queue i at time t is denoted by $Q_i(t)$.

An equivalent way to define GPS [26, 25] is that

$$\frac{S_i(\tau,t)}{S_j(\tau,t)} \ge \frac{\phi_i}{\phi_j}, \quad j = 1, 2, \ldots, n \tag{17}$$

for any session i that is *backlogged* throughout the interval $[\tau, t]$. A session is backlogged throughout an interval if there is always traffic from that session queued through the interval. From the definition of GPS scheduling, if session i is busy throughout $[\tau, t)$ (*i.e.*, $Q_i(s) \ne 0$ for $s \in [\tau, t)$), then $S_i(\tau,t) \ge \phi_i c(t-\tau)$. In other words, session i is guaranteed a service rate of $\phi_i c$ whenever it is busy.

Given that the arrival processes $\{a_i(t)\}, 1 \le i \le n$, are stationary, and that the stability condition, $\sum_{i=1}^n Ea_i(0) < c$, is satisfied, the GPS system is stable. In particular, the queue length process $Q_i(t)$ tends to a finite random variable Q_i *a.s.*, as $t \to \infty$. In the following exposition, we consider the *stationary* two-queue GPS system, *i.e.*, the system has reached its steady state. In particular, we assume the queue length distribution Q_i of each queue has reached its steady state at time 0 (hence it has the same distribution as Q_i). We examine the system at time 0 and look backward in time.

3.1 Assumptions

We make the following assumptions on the arrival processes [4]: for $i = 1, 2, \ldots, n$,

(A1′) The arrival process $\{a_i(-t), t = 0, 1, 2, \ldots\}$ is ergodic and stationary.

[4] The time index used reflects the point of view of looking backward in time. Recall that the set of assumptions (A1′), (A2′) and (A3′) is equivalent to (A1), (A2) and (A3).

(A2′) For any $\theta \in I\!R$,

$$\lim_{t\to\infty} \frac{1}{t} \log E e^{\theta A_i(-t,0)} = \Lambda_{A_i}(\theta)$$

exists and $\Lambda_{A_i}(\theta)$ is differential.

(A3′) $\{a_i(-t), t = 0, 1, 2, \ldots\}$ is adapted to a filtration $\{\mathcal{F}^{A_i}_{-t}, t = 0, 1, 2, \ldots\}$ with the following property: for any $\theta \in I\!R$, there exists a function $\Gamma_{A_i}(\theta), 0 \le \Gamma_{A_i}(\theta) < \infty$ such that for any $s = 0, 1, 2, \ldots$ and $t \in I\!N$,

$$\Lambda_{A_i}(\theta)t - \Gamma_{A_i}(\theta) \le \log E(e^{\theta A_i(-t-s,-s)} | \mathcal{F}^{A_i}_{-t-s}) \le \Lambda_{A_i}(\theta)t + \Gamma_{A_i}(\theta) \text{ a.s.}$$

We prove upper and lower bounds on the asymptotic decay rate of the stationary backlog process Q_i for each session i. Without loss of generality, we consider session 1. Before we state the results, some notation are necessary. We draw the reader's attention that in the definitions of the following sections, since session 1 is the queue under study, it is always treated specially, *e.g*, in the definition of partial feasible set, session 1 is excluded.

3.2 Partial Feasible Sets

Let $N_1 = N \setminus \{1\} = \{2, 3, \ldots, n\}$. For any fixed $\theta \ge 0$, we say a (possibly empty) set $F \subseteq N_1$ is a *partial feasible set* with respect to θ if F can be partitioned into $F_1, \ldots, F_l$ such that for any $i \in F_1$, $\alpha_i(\theta) < \frac{\phi_i}{\sum_{j\in N} \phi_j} c$, and for $2 \le l \le k$, if $i \in F_l$, then

$$\alpha_i(\theta) < \frac{\phi_i}{\sum_{j \in N \setminus F^{l-1}} \phi_j} (c - \sum_{j \in F^{l-1}} \alpha_j(\theta))$$

where $F^{l-1} = F_1 \cup \cdots F_{l-1}$.

$F_1, \ldots, F_k$ are called the *partial feasible partition* of F. For $1 \le l \le k$, let

$$\gamma^F_l(\theta) = \frac{\phi_i}{\sum_{j \in N \setminus F^{l-1}} \phi_j} (c - \sum_{j \in F^{l-1}} \alpha_j(\theta))$$

where $F^0 = \emptyset$ and for $1 \le l \le k$, F^l is defined above. Clearly if $i \in F_l$, then $\gamma^F_{l-1}(\theta) \le \alpha_i(\theta) < \gamma^F_l(\theta)$. $\{\gamma^F_l(\theta), 1 \le l \le k\}$ are called the *associated delimiting numbers* for F (or $F_1, \ldots, F_k$). In particular, we write $\gamma^F_k(\theta)$ as $\gamma_F(\theta)$.

Let $\mathcal{F}(\theta)$ be the collection of all partial feasible sets with respect to θ. Partial feasible sets have the following monotonicity properties.

Lemma 4 (Monotonicity Properties of Partial Feasible Sets)
(a) For any $F, F' \in \mathcal{F}(\theta)$ where $F \subseteq F'$, let $F_1, \ldots, F_k$ and $F'_1, \ldots, F'_{k'}$ be the partial feasible partitions of F and F' respectively. Then for $1 \le l \le k$ (note that $k \le k'$),

$$F^l = F_1 \cup \cdots F_l \subseteq F'^l = F'_1 \cup \cdots F'_l$$

and $\gamma^F_l(\theta) \le \gamma^{F'}_l(\theta)$. In particular, $\gamma_F(\theta) \le \gamma_{F'}(\theta)$.
(b) For any $0 < \theta < \theta'$, if $F \in \mathcal{F}(\theta')$, then $F \in \mathcal{F}(\theta)$. In other words, $\mathcal{F}(\theta') \subseteq \mathcal{F}(\theta)$. Moreover, $\gamma_F(\theta') \le \gamma_F(\theta)$.

From the above lemma, we see that $\gamma_F(\theta)$ is an increasing function in F and a decreasing function in θ.

For any $\theta > 0$, let $M(\theta)$ be the largest element in $\mathcal{F}(\theta)$. Then the partial feasible partition, $M_1(\theta), \ldots, M_k(\theta)$ of $M(\theta)$ and its associated delimiting numbers, $\gamma_1^M(\theta), \ldots, \gamma_k^M(\theta)$ are defined recursively as follows: for $l \geq 1$,

$$\gamma_l^M(\theta) = \frac{1}{\sum_{j \in N \setminus M^{l-1}(\theta)} \phi_j}(c - \sum_{j \in M^{l-1}(\theta)} \alpha_j(\theta))$$

and

$$M_l(\theta) = \{i \notin M^{l-1}(\theta) \cup \{1\} : \alpha_i(\theta) < \phi_i \gamma_l^M(\theta)\}$$

where $M^0(\theta) = \emptyset$ and $M^{l-1}(\theta) = M_1(\theta) \cup \cdots \cup M_{l-1}(\theta)$. Note that k is defined in such a way that if $M_1(\theta) = \emptyset$, then $k = 0$; otherwise k is the largest l such that $M_l \neq \emptyset$ but $M_{l+1}(\theta) = \emptyset$. Hence, for any $i \notin M(\theta)$ and $i \neq 1$, $\alpha_i(\theta) \geq \phi_i \gamma_M(\theta)$. In either case, $M(\theta) = M^k(\theta)$. It can be verified that $M(\theta)$ is the largest element in $\mathcal{F}(\theta)$. Therefore, $\gamma_M(\theta) \geq \gamma_F(\theta)$ for any $F \in \mathcal{F}(\theta)$.

3.3 Sample Path Relations

For any $\theta > 0$ and fix an $F \in \mathcal{F}(\theta)$. For $i \in F$, define $r_i^F(\theta) = \alpha_i(\theta)$ and for $i \in N_1 \setminus F$, $r_i^F(\theta) = \phi_i \gamma_F(\theta)$. We call $r_i^F(\theta)$ the *feasible rate* of session i with respect to F and θ.

Suppose $F_1, \ldots, F_k$ be the partial feasible partition of F, thus, $F = F_1 \cup \cdots F_k$. Let F_{k+1} and F_{k+2} be any partition of $N_1 \subseteq F$, *i.e.*, $F_{k+1} \cup F_{k+2} = N_1 \setminus F \cup F_{k+1}$, where either of them can be empty. Also let $F_0 = \emptyset$. Then from the definition of $R_i^F(\theta)$, we have that for $1 \leq l \leq k+2$ and for any $i \in F_l$,

$$r_i^F(\theta) \leq \frac{\phi_i}{\sum_{j \in N \setminus F^{l-1}} \phi_j}(c - \sum_{j \in F^{l-1}} r_j^F(\theta)). \tag{18}$$

In particular, for any $i \in F_{k+1} = G$ or $i \in F_{k+2} = E$, the equality holds, otherwise the strict inequality holds.

Following the notation in [35], the relation (18) says that $F_1, \ldots, F_k, F_{k+1}, F_{k+2}$ form a feasible partition of the sessions in N_1 with respect to $\{r_i^F(\theta), i \in N_1\}$. For $i \in N_1$, We consider a G/D/1/∞ queueing systems where the arrival process is the session i arrival process A_i and the service rate of the server is $r_i^F(\theta)$. Let $\delta_i^\theta(t)$ denotes the backlog of the queue at time t, *i.e.*,

$$\delta_i^\theta(t) = \max_{\tau \leq t}\{A_i(\tau, t) - r_i^F(\theta)(t - \tau)\}. \tag{19}$$

Clearly, $\delta_i^\theta(t) \geq 0$.

Now define

$$\eta_i^\theta(t) = Q_i(t) - \delta_i^\theta(t). \tag{20}$$

$\eta_i^\theta(t)$ is difference between the session i backlog of the GPS system and the backlog of the independent session i G/D/1/∞ queue.

From (19), it follows that for any integer τ and t such that $t_0 \leq \tau \leq t$,

$$A_i(\tau, t) \leq r_i^F(\theta)(t - \tau) + \delta_i^\theta(t) - \delta_i^\theta(\tau). \tag{21}$$

As $S(\tau, t) = Q_i(\tau) + A_i(\tau, t) - Q_i(t) \leq Q_i(\tau) + A_i(\tau, t)$, then from (21) and (20), we have

$$S_i(\tau, t) \leq r_i^F(\theta)(t - \tau) + \eta_i^\theta(\tau) - \eta_i^\theta(t). \tag{22}$$

We claim (see Lemma 1 in [35]) that

Lemma 5 *Let H be such that $F^{l-1} \subseteq H \subseteq F^l$, $1 \leq l \leq k+2$, then for any $t = 0, 1, 2, \ldots$,*

$$\sum_{i \in H} \eta_i^\theta(t) = \sum_{i \in H} Q_i(t) - \sum_{i \in H} \delta_i^\theta(t) \leq 0. \tag{23}$$

This lemma is very important, it suggests a way to compare the sample path behavior of GPS system with that of a decoupled system consisting of a set of (independent) single queue systems, where each session i is serviced by a server of rate $r_i^F(\theta)$. Lemma 5 says that if H is chosen as required, then the aggregate backlog over sessions in H of the GPS system is bounded above by the sum of the backlogs over the same set of sessions in the decoupled system.

Moreover, we have the following sample path lower bound on the output processes. The proof of the lemma is relegated to Appendix A.

Lemma 6 *Let H be such that $F^{l-1} \subseteq H \subseteq F^l$, $1 \leq l \leq k+2$, then for any $t = 0, 1, 2, \ldots$,*

$$\begin{aligned} \sum_{j \in H} S_j(\tau, t) &\geq \sum_{j \in H} (Q_j(\tau) + \min_{\tau \leq \tau_j \leq t} \{A_j(\tau, \tau_j) + r_j(t - \tau_j)\}) && (24) \\ &\geq \sum_{j \in H} \min_{\tau \leq \tau_j \leq t} \{A_j(\tau, \tau_j) + r_j(t - \tau_j)\} && (25) \end{aligned}$$

3.4 Statement of the Main Theorem

For any fixed $\theta \geq 0$, let $F \in \mathcal{F}(\theta)$ and $G \subseteq N_1 \setminus F$. For $i \in N_1$, $r_i^F(\theta)$ is the feasible rate of session i with respect to F and θ defined in § 3.3.

For $i \in N_1$ and for $\mu > 0$, define

$$\alpha_{D_i}(\mu, r_i^F(\theta)) = \begin{cases} \alpha_i(\mu) & \text{if } \mu \leq \tilde{\theta}_i \\ r_i^F(\theta) - \frac{1}{\mu}(\tilde{\theta}_i r_i^F(\theta) - \Lambda_{A_i}(\tilde{\theta}_i)) & \text{otherwise} \end{cases}$$

where $\tilde{\theta}_i$ is such that $\Lambda'_{A_i}(\tilde{\theta}_i) = r_i^F(\theta)$.

Note that $\alpha_i(\mu, r_i^F(\theta))$ is the effective bandwidth of the departure process when session i is serviced by a server of constant rate $r_i^F(\theta)$.

Let $r_G^F(\theta) = \sum_{i \in G} r_i^F(\theta)$, in particular, if $G = \emptyset$, $r_G^F(\theta) = 0$. For any $t \in I\!N$,

$$D_G^\theta(-t) = \left\{ \sum_{i \in G(\theta)} (A_i(-t, 0) + \delta_i^\theta(-t)) \right\} \wedge r_G^F(\theta) t. \tag{26}$$

For any $\mu > 0$, define

$$\Lambda_{D_G}^\theta(\mu) = \lim_{t \to \infty} \frac{1}{t} \log E e^{\theta D_G^\theta(-t)} \tag{27}$$

and

$$\alpha_{D_G}(\mu, r_G^F(\theta)) = \frac{\Lambda_{D_G}^\theta(\mu)}{\mu}. \tag{28}$$

From (26), clearly $D_G^\theta(-t) \le r_G^F(\theta)t$. Moreover, from the fact that $(a+b)\wedge(c+d) \ge a\wedge c + b\wedge d$ with $a, b, c, d \ge 0$, we have

$$D_G^\theta(-t) \ge \sum_{i\in G}\{(A_i(-t,0)+\delta_i^\theta(-t))\wedge r_i^F(\theta)t\}.$$

Therefore, for any $\mu > 0$,

$$\sum_{i\in G}\alpha_{D_i}(\mu, r_i^F(\theta)) \le \alpha_{D_G}(\mu, r_G^F(\theta)) \le r_G^F(\theta) = \sum_{i\in G} r_i^F(\theta). \tag{29}$$

Now we are in a position to state the main theorem of the paper.

Theorem 7 *Suppose that* $\{a_i(-t), t = 0, 1, 2, \ldots\}$, $i = 1, 2, \ldots, n$, *are independent and satisfy (A1), (A2) and (A3). Moreover, assume that the stability condition* $\sum_{i=1}^n Ea_i(0) < c$ *is satisfied. Then,*

$$\limsup_{x\to\infty}\frac{1}{x}\log Pr\{Q_1 > x\} \le -\theta^* \tag{30}$$

where

$$\theta^* = \arg\sup_{\theta\in I\!R}\left\{F \in \mathcal{F}(\theta) \text{ and } G \subseteq N_1 \setminus F : \alpha_1(\theta) < \frac{\phi_1}{\sum_{j\in N\setminus(F\cup G)}\phi_j}\left(c - \sum_{i\in F} r_i^F(\theta) - \alpha_{D_G}(\theta, r_G^F(\theta))\right)\right\} \tag{31}$$

and

$$\liminf_{x\to\infty}\frac{1}{x}\log Pr\{Q_1 > x\} \ge -\mu_* \tag{32}$$

where

$$\mu_* = \inf_{\theta\in I\!R}\min_{F\in\mathcal{F}(\theta)}\sup\{\mu \in R : \alpha_1(\mu) + \sum_{i\ne 1}\alpha_{D_i}(\mu, r_i^F(\theta)) < c\}. \tag{33}$$

4 Proof of the Main Theorem

The proof of the Main Theorem is divided in two parts. The upper bound part is proved in § 4.1 and the lower bound part is proved in § 4.2. In § 4.3, some ramifications of the Main Theorem are discussed.

4.1 Proof of the Upper Bound

To prove the upper bound (30), it suffices to prove the following lemma.

Lemma 8 *Given the assumption in Theorem 7, for any* $\theta > 0$, *let* $F \in \mathcal{F}(\theta)$ *and* $G \subseteq N_1 \setminus F$. *If*

$$\alpha_1(\theta) < \frac{\phi_1}{\sum_{j\in N\setminus(F\cup G)}\phi_j}(c - \sum_{i\in F} r_i^F(\theta) - \alpha_{D_G}(\theta, r_G^F(\theta))), \tag{34}$$

then

$$\limsup_{x\to\infty}\frac{1}{x}\log Pr\{Q_1 > x\} \le -\theta. \tag{35}$$

Proof: First observe that

$$Q_1(0) = \max_{t \in I\!N}\{A_1(-t,0) - S_1(-t,0)\}. \tag{36}$$

where the maximum is attained by any $t \in I\!N$ such that $Q_1(-t) = 0$.

Let t be the smallest $t \in I\!N$ such that $Q_1(-t+1) = 0$. In other words, session 1 is idle at time $-t$, but is busy throughout $(-t, 0)$: for any $\tau \in I\!N, 0 < \tau < t, Q_1(-\tau+1) > 0$.

Applying Lemma 5 with $H = F$ and using (22), we have,

$$\begin{aligned}\sum_{j \in F} S_j(-t,0) &\le \sum_{j \in F}(r_j^F(\theta)t + \eta_j^\theta(-t) - \eta_j^\theta(0)) \\ &\le \sum_{j \in F}(r_j^F(\theta)t + \delta_j^\theta(0)). \end{aligned} \tag{37}$$

as $\sum_{j \in F} \eta_j^\theta(-t) \le 0$ and $\eta_j^\theta(0) = Q_j(0) - \delta_j^\theta(0) \ge -\delta_j^\theta(0)$.

Then from the definition of GPS, we have

$$\begin{aligned} S_1(-t,0) &\ge \frac{\phi_1}{\sum_{j \in N \setminus F}\phi_j}(ct - \sum_{i \in F} S_i(-t,0)) \\ &\ge \frac{\phi_1}{\sum_{j \in N \setminus F}\phi_j}(ct - \sum_{i \in F} r_i^F(\theta)t - \sum_{i \in F}\delta_i^\theta(0)) \\ &= \frac{\phi_1}{\sum_{j \in N \setminus F}\phi_j}(ct - \sum_{i \in F} r_i^F(\theta)t) - \frac{\phi_1}{\sum_{j \in N \setminus F}\phi_j}\sum_{i \in F}\delta_i^\theta(0) \\ &\ge \frac{\phi_1}{\sum_{j \in N \setminus F}\phi_j}(ct - \sum_{i \in F} r_i^F(\theta)t) - \sum_{i \in F}\delta_i^\theta(0) \end{aligned}$$

where the second inequality follows from (37) and the last inequality holds as $\sum_{i \in F}\delta_i^\theta(0) \ge 0$.

Note that $\frac{\phi_1}{\sum_{j \in N \setminus F}\phi_j} = \frac{\phi_1}{\sum_{i \in N \setminus (F \cup G)}\phi_j}(1 - \frac{\sum_{i \in G}\phi_i}{\sum_{j \in N \setminus F}\phi_j})$, thus

$$\begin{aligned} S_1(-t,0) &\ge \frac{\phi_1}{\sum_{j \in N \setminus (F \cup G)}\phi_j}(ct - \sum_{i \in F} r_i^F(\theta)t - \frac{\sum_{i \in G}\phi_i}{\sum_{j \in N \setminus F}\phi_j}(ct - \sum_{i \in F} r_i^F(\theta)t)) - \sum_{i \in F}\delta_i^\theta(0) \\ &= \frac{\phi_1}{\sum_{j \in N \setminus (F \cup G)}\phi_j}(ct - \sum_{i \in F} r_i^F(\theta)t - r_G^F(\theta)t) - \sum_{i \in F}\delta_i^\theta(0) \end{aligned} \tag{38}$$

where we recall that $r_G^F(\theta) = \sum_{i \in G} r_i^F(\theta) = \frac{\sum_{i \in G}\phi_i}{\sum_{j \in N \setminus F}\phi_j}(c - \sum_{i \in F} r_i^F(\theta))$.

On the other hand, applying Lemma 5 for $H = F \cup G$, we have

$$\sum_{i \in F \cup G} S_i(-t,0) \le \sum_{i \in F \cup G}(Q_i(-t) + A_i(-t,0) \le \sum_{i \in F \cup G}(\delta_i^\theta(-t) + A_i(-t,0))$$

as $\sum_{i \in F \cup G} Q_i(-t) \le \sum_{i \in F \cup G}\delta_i^\theta(-t)$.

Using (21) for $i \in F$, we have

$$\sum_{i \in F \cup G} S_i(-t,0) \le \sum_{i \in F}(r_i^F(\theta)t + \delta_i^\theta(0)) + \sum_{i \in G}(\delta_i^\theta(-t) + A_i(-t,0)). \tag{39}$$

Therefore,

$$
\begin{aligned}
S_1(-t,0) &\geq \frac{\phi_1}{\sum_{j\in N\setminus(F\cup G)}\phi_j}\left(ct-\sum_{i\in F\cup G}S_i(-t,0)\right)\\
&\geq \frac{\phi_1}{\sum_{j\in N\setminus F}\phi_j}\left(ct-\sum_{i\in F}r_i^F(\theta)t-\sum_{i\in G}(\delta_i^\theta(-t)+A(-t,0))-\sum_{i\in F}\delta_i^\theta(0)\right)\\
&= \frac{\phi_1}{\sum_{j\in N\setminus(F\cup G)}\phi_j}\left(ct-\sum_{i\in F}r_i^F(\theta)t\sum_{i\in G}(\delta_i^\theta(-t)+A(-t,0))\right)-\frac{\phi_1}{\sum_{j\in N\setminus(F\cup G)}\phi_j}\sum_{i\in F}\delta_i^\theta(0)\\
&\geq \frac{\phi_1}{\sum_{j\in N\setminus(F\cup G)}\phi_j}\left(ct-\sum_{i\in F}r_i^F(\theta)t-\sum_{i\in G}(\delta_i^\theta(-t)+A(-t,0))\right)-\sum_{i\in F}\delta_i^\theta(0) \qquad (40)
\end{aligned}
$$

where the second inequality follows from (39) and the last inequality holds as $\sum_{i\in F}\delta_i^\theta(0)\geq 0$.
Combining (38) and (40), we have

$$
\begin{aligned}
S_1(-t,0) &\geq \left(\frac{\phi_1}{\sum_{j\in N\setminus(F\cup G)}\phi_j}\left\{ct-\sum_{i\in F}r_i^F(\theta)t-r_G^F(\theta)t\right\}-\sum_{i\in F}\delta_i^\theta(0)\right)\\
&\quad\vee\left(\frac{\phi_1}{\sum_{j\in N\setminus(F\cup G)}\phi_j}\left\{ct-\sum_{i\in F}r_i^F(\theta)t-\sum_{i\in G}(\delta_i^\theta(-t)+A_i(-t,0))\right\}-\sum_{i\in F}\delta_i^\theta(0)\right)\\
&= \frac{\phi_1}{\sum_{j\in N\setminus(F\cup G)}\phi_j}\left(ct-\sum_{i\in F}r_i^F(\theta)t-\left\{\sum_{i\in G}(\delta_i^\theta(-t)+A_i(-t,0))\right\}\wedge r_G^F(\theta)t\right)-\sum_{i\in F}\delta_i^\theta(0).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
Q_1(0) &= \max_{t\in I\!N}\{A_1(-t,0)-S_1(-t,0)\}\\
&\leq \max_{t\in I\!N}\Bigg\{A_1(-t,0)-\frac{\phi_1}{\sum_{j\in N\setminus(F\cup G)}\phi_j}\Bigg(ct-\sum_{i\in F}r_i^F(\theta)t\\
&\qquad -\left\{\sum_{i\in G}(\delta_i^\theta(-t)+A_i(-t,0))\right\}\wedge r_G^F(\theta)t\Bigg)+\sum_{i\in F}\delta_i^\theta(0)\Bigg\}\\
&= \max_{t\in I\!N}\left\{A_1(-t,0)-\frac{\phi_1}{\sum_{j\in N\setminus(F\cup G)}\phi_j}\left(ct-\sum_{i\in F}r_i^F(\theta)t-D_G^\theta(-t)\right)+\sum_{i\in F}\delta_i^\theta(0)\right\} \qquad (41)
\end{aligned}
$$

where the last equality follows from the definition of $D_G^\theta(-t)$.
For any $\mu>0$, recall that

$$\lim_{t\to\infty}\frac{1}{t}\log Ee^{\mu A_1(-t,0)}=\Lambda_{A_1}(\mu). \qquad (42)$$

and from the concavity of the function x^ψ where $0<\psi=\frac{\phi_1}{\sum_{j\in N\setminus(F\cup G)}\phi_j}<1$, we have

$$\lim_{t\to\infty}\frac{1}{t}\log Ee^{\mu\psi D_G^\theta(-t)}\leq\lim_{t\to\infty}\frac{1}{t}\log(Ee^{\mu D_G^\theta(-t)})^\psi=\psi\Lambda_{D_G^\theta}. \qquad (43)$$

(42) and (43) implies that for any $\epsilon > 0$, there exists a t_ϵ such that for any $t \geq t_\epsilon, t \in I\!N$,

$$Ee^{\mu A_1(-t,0)} \leq e^{(\Lambda_{A_1}(\mu)+\epsilon)t} \tag{44}$$

and

$$Ee^{\mu\psi D_G^\theta(-t)} \leq e^{(\psi\Lambda_{D_G^\theta}(\mu)+\epsilon)t}. \tag{45}$$

Moreover, for $i \in F$, if $\mu < \theta$, then $\alpha_i(\mu) < \alpha_i(\theta)$. Then, from (7), $Ee^{\mu\delta_i^\theta(0)} < \infty$. Therefore, by independence of $\delta_i^\theta(0)$, we have that for $0 < \mu < \theta$,

$$Ee^{\mu\sum_{i\in F}\delta_i^\theta(0)} = \prod_{i\in F} Ee^{\mu\delta_i^\theta(0)} = C_1 < \infty. \tag{46}$$

Now from (41), we have that for $0 < \mu < \theta$,

$$\begin{aligned} Ee^{\mu Q_1(0)} &\leq \sum_{t\in I\!N} E\exp\mu(A_1(-t,0) - \psi(c - \sum_{i\in F} r_i^F(\theta))t + \psi D_G^\theta(-t) + \sum_{i\in F}\delta_i^\theta(0)) \\ &\leq C_\epsilon + C_1\sum_{t\geq t_\epsilon}\exp t(\Lambda_{A_1}(\mu)+\epsilon)e^{t(\psi\Lambda_{D_G^\theta}(\mu)+\epsilon)}e^{-t\mu\psi(c-\sum_{i\in F} r_i^F(\theta))} \end{aligned} \tag{47}$$

where the last equality follows from (44), (45) and (46) with C_ϵ being a constant that depends on ϵ.
Note that if $\exp\{\Lambda_{A_1}(\mu) + \psi\Lambda_{D_G^\theta}(\mu) + 2\epsilon - \mu\psi(c - \sum_{i\in F} r_i^F(\theta))\} < 1$, then

$$\begin{aligned} &\sum_{t\geq t_\epsilon} e^{t(\Lambda_{A_1}(\mu)+\epsilon)}e^{t(\psi\Lambda_{D_G^\theta}(\mu)+\epsilon)}e^{-t\mu\psi(c-\sum_{i\in F} r_i^F(\theta))} \\ &\qquad = \frac{\exp\{(\Lambda_{A_1}(\mu) + \psi\Lambda_{D_G^\theta}(\mu) + 2\epsilon - \mu\psi(c - \sum_{i\in F} r_i^F(\theta)))t_\epsilon\}}{1 - \exp\{\Lambda_{A_1}(\mu) + \psi\Lambda_{D_G^\theta}(\mu) + 2\epsilon - \mu\psi(c - \sum_{i\in F} r_i^F(\theta))\}}. \end{aligned}$$

Therefore if $\exp\{\Lambda_{A_1}(\mu) + \psi\Lambda_{D_G^\theta}(\mu) + 2\epsilon - \mu\psi(c - \sum_{i\in F} r_i^F(\theta))\} < 1$ or $\Lambda_{A_1}(\mu) + \psi\Lambda_{D_G^\theta}(\mu) + 2\epsilon - \mu\psi(c - \sum_{i\in F} r_i^F(\theta)) < 0$, then $Ee^{\mu Q_1(0)} < \infty$.
Now by Chebyshev's Inequality, for any $x \geq 0$,

$$Pr\{Q_1(0) > x\} \leq e^{-\mu x}Ee^{\mu Q_1(0)}.$$

Thus if $\Lambda_{A_1}(\mu) + \psi\Lambda_{D_G^\theta}(\mu) + 2\epsilon - \mu\psi(c - \sum_{i\in F} r_i^F(\theta)) < 0$, then

$$\limsup_{x\to\infty}\frac{1}{x}\log Pr\{Q_1(0) > x\} \leq -\mu. \tag{48}$$

Taking $\epsilon \to 0$, and noting that $\alpha_{D_G}(\mu, r_G^F(\theta)) = \alpha_{D_G^\theta}(\mu) = \psi\Lambda_{D_G^\theta}(\mu)/\mu$, we have that if $\alpha_1(\mu) + \psi\alpha_{D_G}(\mu, r_G^F(\theta) - \psi(c - \sum_{i\in F} r_i^F(\theta)) < 0$, or $\alpha_1(\mu) < \psi(c - \sum_{i\in F} r_i^F(\theta) - \alpha_{D_G}(\mu, r_G^F(\theta))$, then, (48) holds for any positive μ such that $\mu < \theta$.
Lastly, taking $\mu \to \theta$, we have that if

$$\alpha_1(\theta) < \frac{\phi_1}{\sum_{j\in N\setminus(F\cup G)}\phi_j}(c - \sum_{i\in F} r_i^F(\theta) - \alpha_{D_G}(\theta, r_G^F(\theta)),$$

then

$$\limsup_{x\to\infty}\frac{1}{x}\log Pr\{Q_1(0) > x\} \leq -\theta.$$

■

4.2 Proof of the Lower Bound

To prove the lower bound (32), it suffices to prove the following lemma.

Lemma 9 *Given the assumption in Theorem 7, for any fixed $\theta > 0$ and $F \in \mathcal{F}(\theta)$, let $\mu_*^F(\theta) \in I\!R$ be such that*

$$\mu_*^F(\theta) = \sup\{\mu \in I\!R : \alpha_1(\mu) + \sum_{i \neq 1} \alpha_{D_i}(\mu, r_i^F(\theta)) < c\}, \tag{49}$$

i.e., *$\mu_*^F(\theta)$ is the solution to $\alpha_1(\mu) + \sum_{i \neq 1} \alpha_{D_i}(\mu, r_i^F(\theta)) = c$. Then*

$$\liminf_{x \to \infty} \frac{1}{x} \log Pr\{Q_1 > x\} \geq -\mu_*^F(\theta). \tag{50}$$

Proof: Recall that

$$Q_1(0) = \max_{t \in I\!N}\{A_1(-t, 0) - S_1(-t, 0)\}. \tag{51}$$

As $\sum_{i=1}^n S_i(-t, 0) \leq ct$, we have that $S_1(-t, 0) \leq ct - \sum_{i=2}^n S_i(-t, 0)$, hence

$$Q_1(0) \geq \max_{t \in I\!N}\{A_1(-t, 0) + \sum_{i=2}^n S_i(-t, 0) - ct\}.$$

Applying Lemma 6 to $H = F \cup G \cup E = \{2, \ldots, n\}$, we have

$$\sum_{i=2}^n S_i(-t, 0) \geq \sum_{i=2}^n \min_{0 \leq \tau_i \leq t}\{A_i(-t, -\tau_i) + r_i^F(\theta)\tau_i\}.$$

Hence

$$Q_1(0) \geq \max_{t \in I\!N}\{A_1(-t, 0) + \sum_{i=2}^n \min_{0 \leq \tau_i \leq t}\{A_i(-t, -\tau_i) + r_i^F(\theta)\tau_i\} - ct\}.$$

In other words, for any $t \in I\!N$, we have

$$Q_1(0) \geq A_1(-t, 0) + \sum_{i=2}^n \min_{0 \leq \tau_i \leq t}\{A_i(-t, -\tau_i) + r_i^F(\theta)\tau_i\} - ct. \tag{52}$$

For any $x \geq 0$, let $t = \lfloor \frac{x}{\beta} \rfloor$ where $\beta > 0$ is a constant fixed temporarily. From (52), we have

$$\begin{aligned}
&\liminf_{x \to \infty} \frac{1}{x} \log Pr\{Q_1(0) > x\} \\
&\quad = \frac{1}{\beta} \liminf_{t \in I\!N} \frac{1}{t} \log Pr\{Q_1(0) > \beta t\} \\
&\quad \geq \frac{1}{\beta} \liminf_{t \in I\!N} \frac{1}{t} \log Pr\left\{A_1(-t, 0) + \sum_{i=2}^n \min_{0 \leq \tau_i \leq t}\{A_i(-t, -\tau_i) + r_i^F(\theta)\tau_i\} - ct > \beta t\right\} \\
&\quad \geq \frac{1}{\beta} \liminf_{t \in I\!N} \frac{1}{t} \log Pr\left\{A_1(-t, 0)/t + \sum_{i=2}^n \bar{B}_i^\theta(-t)/t > c + \beta\right\} \qquad (53)
\end{aligned}$$

where $\bar{B}_i^\theta(-t) = \min_{0\le\tau_i\le t}\{A_i(-\tau_i,0) + r_i^F(\theta)(t-\tau_i)\} = \min_{0\le\tau_i\le t}\{A_i(-t,-\tau_i) + r_i^F(\theta)\tau_i\}$ by stationarity of the arrival processes.
From Lemma 3, $\{\bar{B}_i^\theta(-t)/t, t \in I\!N\}$ satisfies the LDP with the rate function $\Lambda^*_{\bar{B}_i^\theta}(x)$ as defined in Lemma 3 with c replaced by $r_i^F(\theta)$. Moreover, $\{A_1(-t,0)/t, t\in I\!N\}$ also satisfies the LDP with the rate function $\Lambda^*_{A_1}(x)$. Hence, by the Contraction Principle (see, [9]), we have

$$\begin{aligned}&\liminf_{x\to\infty}\frac{1}{x}\log Pr\{Q_1(0)>x\}\\ &\quad\ge -\frac{1}{\beta}\inf_{\{(\alpha_1,\alpha_2,\ldots,\alpha_n):\sum_{i=1}^n\alpha_i=c+\beta\}}\{\Lambda^*_{A_1}(\alpha_1)+\sum_{i=2}^n\Lambda^*_{\bar{B}_i^\theta}(\alpha_i)\}.\end{aligned}$$

As $\beta>0$ is arbitrary,

$$\begin{aligned}&\liminf_{x\to\infty}\frac{1}{x}\log Pr\{Q_1(0)>x\}\\ &\quad\ge -\inf_{\beta>0}\inf_{\{(\alpha_1,\alpha_2,\ldots,\alpha_n):\sum_{i=1}^n\alpha_i=c+\beta\}}\left\{\frac{\Lambda^*_{A_1}(\alpha_1)+\sum_{i=2}^n\Lambda^*_{\bar{B}_i^\theta}(\alpha_i)}{\beta}\right\}\\ &\quad= -\inf_{\{(\alpha_1,\alpha_2,\ldots,\alpha_n):\sum_{i=1}^n\alpha_i>c\}}\left\{\frac{\Lambda^*_{A_1}(\alpha_1)+\sum_{i=2}^n\Lambda^*_{\bar{B}_i^\theta}(\alpha_i)}{\sum_{i=1}^n\alpha_i-c}\right\}\end{aligned}$$

We claim that

$$\inf_{\{(\alpha_1,\alpha_2,\ldots,\alpha_n):\sum_{i=1}^n\alpha_i>c\}}\left\{\frac{\Lambda^*_{A_1}(\alpha_1)+\sum_{i=2}^n\Lambda^*_{\bar{B}_i^\theta}(\alpha_i)}{\sum_{i=1}^n\alpha_i-c}\right\}=\mu_*^F(\theta) \tag{54}$$

where, by definition, $\mu_*^F(\theta)=\sup_{\mu\in I\!R}\{\alpha_{A_1}(\mu)+\sum_{i=2}^n\alpha_{D_i}(\mu,r_i^F(\theta))\le c\}$.
For any $\alpha\in I\!R$, define

$$I(\alpha)=\inf_{\left\{\begin{array}{c}\alpha_1,\alpha_2,\ldots,\alpha_n\in I\!R\\ \sum_{i=1}^n\alpha_i=\alpha\end{array}\right\}}\left\{\Lambda^*_{A_1}(\alpha_1)+\sum_{i=2}^n\Lambda^*_{\bar{B}_i^\theta}(\alpha_i)\right\}$$

and let $I^*(\mu)$ be the Legendre-Fenchel transform of $I(\alpha)$, *i.e.*, $I^*(\mu)=\sup_{\alpha\in I\!R}\{\alpha\mu-I(\alpha)\}$. It is easy to see that $I^*(\mu)=\Lambda_{A_1}(\mu)+\sum_{i=2}^n\Lambda_{\bar{B}_i^\theta}(\mu)$ where $\Lambda_{\bar{B}_i^\theta}(\mu)=\sup_{\alpha\in I\!R}\{\alpha\mu-\Lambda^*_{\bar{B}_i^\theta}(\alpha)\}$. In particular, let $\tilde{\theta}_i$ be such that $\Lambda'_{A_i}(\tilde{\theta}_i)=r_i^F(\theta)$. Then for $i\in G$, as$Ea_i(0)<r_i^F(\theta)$,

$$\Lambda_{\bar{B}_i^\theta}(\mu)=\begin{cases}\Lambda_{A_i}(\mu) & \text{if } \mu\le\tilde{\theta}_i\\ r_i^F(\theta)\mu-r_i^F(\theta)\tilde{\theta}_i+\Lambda_{A_i}(\tilde{\theta}_i) & \text{otherwise.}\end{cases}$$

For $i\in E$, as $Ea_i(0)\ge r_i^F(\theta)$, $\Lambda_{\bar{B}_i^\theta}(\mu)=r_i^F(\theta)\mu$. $\alpha_{D_i}(\mu,r_i^F(\theta))=\Lambda_{\bar{B}_i^\theta}(\mu)/\mu$. From the definition of $\alpha_{D_i}(\mu,r_i^F(\theta))$, we see that in either case, $\alpha_{D_i}(\mu,r_i^F(\theta))=\Lambda_{\bar{B}_i^\theta}(\mu)/\mu$. Clearly $\mu_*^F(\theta)=\sup_{\mu\in I\!R}\{\alpha_{A_1}(\mu)+\sum_{i=2}^n\alpha_{D_i}(\mu,r_i^F(\theta))\le c\}=\sup_{\mu\in I\!R}\{I^*(\mu)\le c\mu\}$. To show (54), we note that

$$\inf_{\{(\alpha_1,\alpha_2,\ldots,\alpha_n):\sum_{i=1}^n\alpha_i>c\}}\left\{\frac{\Lambda^*_{A_1}(\alpha_1)+\sum_{i=2}^n\Lambda^*_{\bar{B}_i^\theta}(\alpha_i)}{\sum_{i=1}^n\alpha_i-c}\right\}=\inf_{\alpha>c}\left\{\frac{I(\alpha)}{\alpha-c}\right\}. \tag{55}$$

Then, for any μ such that $I^*(\mu) \leq c\mu$, $I(\alpha) \geq \mu\alpha - I^*(\mu) \geq \mu(\alpha - c)$. Hence,

$$\inf_{\alpha > c} \left\{ \frac{I(\alpha)}{\alpha - c} \right\} \geq \mu.$$

Since the above inequality is true for any μ such that $I^*(\mu) \leq c\mu$, we have

$$\inf_{\alpha > c} \left\{ \frac{I(\alpha)}{\alpha - c} \right\} \geq \mu_*^F(\theta).$$

Now let $\alpha^* = {I^*}'(\mu_*^F(\theta)) = \Lambda'_{A_1}(\mu_*^F(\theta)) + \sum_{i=2}^n \Lambda'_{\tilde{B}_i^\theta}(\mu_*^F(\theta))$, then $I(\alpha^*) = \alpha^*\mu_*^F(\theta) - I^*(\mu_*^F(\theta)) > 0$. But, from the definition of $\mu_*^F(\theta)$, we have $I^*(\mu_*^F(\theta)) = \mu_*^F(\theta)c$, therefore

$$\inf_{\alpha > c} \left\{ \frac{I(\alpha)}{\alpha - c} \right\} \leq \frac{I(\alpha^*)}{\alpha^* - c} = \mu_*^F(\theta).$$

Hence (54) holds. This completes the proof of the lemma. ■

4.3 Discussion

In this section, we discuss some ramifications of the main theorem (Theorem 7).

First we claim that for any θ, $0 < \theta \leq \theta^*$, there exists $F \in \mathcal{F}(\theta)$ and $G \in N_1 \setminus F$ such that the condition (34) in Lemma 8 holds. This is a consequence of the following lemma, the proof of which is relegated to Appendix B.

Lemma 10 *For any* $\theta > 0$*, if there exists* $F \in \mathcal{F}(\theta)$ *and* $G \in N_1 \setminus F$ *such that the condition (34) in Lemma 8 holds, then for any* $0 < \theta' < \theta$*, (34) also holds,* i.e.,

$$\alpha_1(\theta') \leq \frac{\phi_1}{\sum_{j \in N \setminus (F \cup G)} \phi_j}(c - \sum_{i \in F} r_i(\theta') - \alpha_{D_G}(\theta', r_G^F(\theta'))).$$

From the proof of Lemma 10, we see that as long as $F \in \mathcal{F}(\theta)$, the right hand side of (34) is an decreasing function of θ.

In particular, for any $\theta > 0$, as $\emptyset \in \mathcal{F}(\theta)$, we have

$$\theta^* \geq \theta_0^* = \sup\{\theta \in I\!R : \alpha_1(\theta) < c - \alpha_{D_{N_1}}(\theta, r_{N_1}^\emptyset(\theta))\} \tag{56}$$

where $r_{N_1}^\emptyset(\theta) = \frac{\sum_{i \in N_1} \phi_i}{\sum_{j=1}^n \phi_j} c$ is independent of θ. θ_0^* is the upper bound obtained in [11] for a general multiple-queue GPS system.

Now we fix a θ such that $0 < \theta \leq \theta^*$, and see how the choice of F and G affect the right hand side of (34).

Consider two arbitrary pairs of sets (F, G) and (F', G') where $F, F' \in \mathcal{F}(\theta)$ and $G \subseteq N_1 \setminus F$, $G' \subseteq N_1 \setminus F'$. Let $E = N_1 \setminus (F \cup G)$ and $E' = N_1 \setminus (F' \cup G')$. Define $\iota(F, F')$ and $\Delta(E, E')$ be such that if if $E \subseteq E'$, $\iota(E, E') = 1$ and $\Delta(E, E') = E' \setminus E$; whereas if $E \supseteq E'$, $\iota(E, E') = -1$ and $\Delta(E, E') = E \setminus E'$. Then

$$\frac{\phi_1}{\sum_{j \in N \setminus (F \cup G)} \phi_j} = \frac{\phi_1}{\sum_{j \in E' \cup \{1\}} \phi_j}(1 + \iota(E, E')\frac{\sum_{i \in \Delta(E,E')} \phi_i}{\sum_{j \in E \cup \{1\}} \phi_j}).$$

Using this relation, it can be easily checked that

$$\frac{\phi_1}{\sum_{j\in N\setminus(F\cup G)}\phi_j}(c-\sum_{j\in F}\alpha_j(\theta)-\alpha_{D_G}(\theta,r_i^F(\theta)))$$
$$<\frac{\phi_1}{\sum_{j\in N\setminus(F'\cup G')}\phi_j}(c-\sum_{j\in F'}\alpha_j(\theta)-\alpha_{D_{G'}}(\theta,r_i^{F'}(\theta))) \tag{57}$$

if and only if

$$\alpha_{D_{G'}}(\theta,r_i^{F'}(\theta))+\alpha_{D_G}(\theta,r_i^F(\theta))+\iota_{F,F'}\sum_{i\in\Delta(F,F')}\alpha_i(\theta)$$
$$+\iota_{E,E'}\frac{\sum_{i\in\Delta(E,E')}\phi_i}{\sum_{j\in E\cup\{1\}}\phi_j}(c-\sum_{j\in F}\alpha_j(\theta)-\alpha_{D_G}(\theta,r_i^F(\theta)))<0. \tag{58}$$

In the case where $F=F'$ and $G\subseteq G'$ (thus $E\supseteq E'$), then (57) holds if and only if

$$\alpha_{D_{G'}}(\theta,r_i^{F'}(\theta))<\alpha_{D_G}(\theta,r_i^F(\theta))+\frac{\sum_{i\in E\setminus E'}\phi_i}{\sum_{j\in E\cup\{1\}}\phi_j}(c-\sum_{j\in F}\alpha_j(\theta)-\alpha_{D_G}(\theta,r_i^F(\theta))).$$

On the other hand, in the case where $G=G'$ and $F'\subseteq F$ (thus $E\supseteq E'$), then (57) holds if and only if

$$\alpha_{D_{G'}}(\theta,r_i^{F'}(\theta))+\frac{\sum_{i\in E\setminus E'}\phi_i}{\sum_{j\in E\cup\{1\}}\phi_j}(c-\sum_{j\in F}\alpha_j(\theta)-\alpha_{D_G}(\theta,r_i^F(\theta)))<\sum_{i\in F\setminus F'}\alpha_i(\theta)+\alpha_{D_G}(\theta,r_i^F(\theta)).$$

In particular, let $G=\emptyset$ and $F=M:=M(\theta)$. Then for any $F'\in\mathcal{F}(\theta)$, we have $F'\subseteq M$, thus $\sum_{i\in M\setminus F'}\alpha_i(\theta)<\sum_{i\in M\setminus F'}\phi_i\gamma_M(\theta)$. Therefore,

$$\frac{\phi_1}{\sum_{j\in N\setminus F'}\phi_j}(c-\sum_{j\in F'}\alpha_j(\theta))<\frac{\phi_1}{\sum_{j\in N\setminus M}\phi_j}(c-\sum_{j\in M}\alpha_j(\theta)). \tag{59}$$

This fact can also be proved directly using Lemma 4(a).

Define

$$\hat{\theta}^*=\arg\sup_{\theta\in I\!R}\left\{F\in\mathcal{F}(\theta):\alpha_1(\theta)<\frac{1}{\sum_{j\in N\setminus F}\phi_j}(c-\sum_{i\in F}r_i^F(\theta))\right\}. \tag{60}$$

Then from (59), we have

$$\hat{\theta}^*=\sup\left\{\theta\in I\!R:\alpha_1(\theta)<\frac{1}{\sum_{j\in N\setminus H}\phi_j}(c-\sum_{i\in F}r_i^M(\theta))\right\}.$$

Hence, we have the following corollary.

Corollary 11 *Under the same assumptions in Theorem 7, we have that*

$$\limsup_{x\to\infty}\frac{1}{x}\log Pr\{Q_1>x\}\le-\hat{\theta}^*. \tag{61}$$

This corollary can be proved under much weaker assumptions than those of Theorem 7 (see Theorem 1 in [34]). In practice, since $\alpha_{D_G}(\theta, r_G^F(\theta))$ is usually impossible to compute, (60) may be more useful than (31). In [34], (60) is used as the basis for constructing feasibility tests and call admission control schemes under GPS scheduling. The interested reader is referred to the paper for more details.

We now turn our attention to the lower bound. For any $\theta > 0$ and $F \in \mathcal{F}(\theta)$, let $\mu_*^F(\theta)$ be defined in (49). Then by definition, $\mu_* = \inf_{\theta \in I\!R} \min_{F \in \mathcal{F}(\theta)} \mu_*^F(\theta)$.

Define

$$\theta_* = \arg\sup_{\theta \in I\!R} \left\{ \forall F \in \mathcal{F}(\theta) : \alpha_1(\theta) + \sum_{i \neq 1} \alpha_{D_i}(\theta, r_i^F(\theta)) < c \right\}.$$

For any $\theta \leq \theta_*$, by definition of θ_*, $\alpha_1(\theta) + \sum_{i \neq 1} \alpha_{D_i}(\theta, r_i^F(\theta)) < c$ for all $F \in \mathcal{F}(\theta)$. As for a fixed θ, $\alpha_{D_i}(\mu, r_i^F(\theta))$ is an increasing function of μ, we have $\mu_*^F(\theta) \geq \theta$ for all $F \in \mathcal{F}(\theta)$. Therefore $\min_{F \in \mathcal{F}(\theta)} \mu_*^F(\theta) \geq \theta$.

In particular, for any $\theta \leq \theta^*$, from the definition of θ^* and Lemma 10, there exists $F \in \mathcal{F}(\theta)$ such that the right hand side of (34) holds. By the following lemma (the proof is left to Appendix B), we have that $\theta^* \leq \theta_*$.

Lemma 12 *For any $\theta > 0$, if there exists $F \in \mathcal{F}(\theta)$ and $G \subseteq N_1 \setminus F$ such that*

$$\alpha_1(\theta) < \frac{\phi_1}{\sum_{j \in N \setminus (F \cup G)} \phi_j} (c - \sum_{i \in F} \alpha_i(\theta) - \alpha_{D_G}(\theta, r_G^F(\theta))),$$

then for any $F' \in \mathcal{F}(\theta)$,

$$\alpha_1(\theta) + \sum_{i \neq 1} \alpha_{D_i}(\theta, r_i^{F'}(\theta)) < c.$$

As a consequence, $\min_{F \in \mathcal{F}(\theta)} \mu_*^F(\theta) > \theta$.

On the other hand, for any $\theta > \theta_*$, there exists $F \in \mathcal{F}(\theta)$ such that $\alpha_1(\theta) + \sum_{i \neq 1} \alpha_{D_i}(\theta, r_i^F(\theta)) \geq c$. Again as for a fixed θ, $\alpha_{D_i}(\mu, r_i^F(\theta))$ is an increasing function of μ, we have $\mu_*^F(\theta) \leq \theta$. Therefore $\mu_* \leq \min_{F \in \mathcal{F}(\theta)} \mu_*^F(\theta) \leq \theta$. Therefore, $\mu_* \in [\theta^*, \theta_*]$.

In summary, we have the following relation.

$$-\theta_* \leq -\mu_* \leq \liminf_{x \to \infty} \frac{1}{x} \log Pr\{Q_1 > x\} \leq \limsup_{x \to \infty} \frac{1}{x} \log Pr\{Q_1 > x\} \leq -\theta^* \leq -\hat{\theta}^*.$$

Lastly, as a special case, we consider a two-queue GPS system. Without loss of generality, we assume $\phi_1 + \phi_2 = 1$.

First, if $\alpha_2(0) \geq \phi_2 c$, then for all $\theta > 0$, $\mathcal{F}(\theta) = \{\emptyset\}$. As $\gamma_2^{\emptyset}(\theta) = \phi_2 c$ and $\alpha_{D_2}(\theta, \gamma_2^{\emptyset}(\theta)) = \alpha_{D_2}(\theta, \phi_2 c) = \phi_2 c$, we see that θ^* and μ_* are equal and are the unique solution to $\alpha_1(\theta) = \phi_2 c$.

If $\alpha_2(0) < \phi_2 c$, for any $\theta > 0$, $\mathcal{F}(\theta)$ contains either $F = \emptyset$ or $F_1 = \emptyset$ and $F_2 = \{2\}$. As $\alpha_{D_2}(\theta, \phi_2 c) \leq \alpha_2(\theta)$, we see that θ^* and μ_* are again equal and are the solution to $\alpha_1(\theta) + \alpha_{D_2}(\theta, \phi_2 c) = c$.

Hence in the case of two-queue GPS system, the upper bound θ^* equals the lower bound μ_* and we arrive at the same conclusion as in Part I of the paper.

5 Conclusion

In this part of the paper, we present tight upper and lower bounds on the asymptotic decay rate of the queue length tail distribution for a general multiple-queue GPS system. When there are only two queues, the lower and upper bounds match, yielding exactly the same result proved separately in Part I of the paper [33]. The proofs are based on the sample-path large deviation principle and exploit the complicated bandwidth sharing structure of the GPS scheduling by introducing the notion of partial feasible sets. Our results are more general than the results of [11] on the multiple-queue GPS systems.

The GPS system we examined uses a discrete-time model. The results of the paper may be extended to the continuous-time model by imposing appropriate conditions (corresponding to (A1), (A2) and (A3)) on the continuous-time arrival processes. Then the arguments of this paper can be applied to pass from the discrete case to the continuous case (*cf.*, the proof of Theorem 5.1.19 in [9]). Methods, for instance, employed in [17, 2], may also be used to establish results for the continuous-time GPS system.

The paper deals only with the large buffer asymptotics under the GPS scheduling. Another future direction is to study the asymptotical behavior of the GPS scheduling with a large number of sources *à la* the methods of [29, 3].

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A Proof of Sample Path Lower Bound on Output Processes

Proof of Lemma 6: First note that (25) follows directly from (24) as $\sum_{i\in H} Q_i(\tau) \geq 0$.

Let $m = |H|$, we prove (24) by induction on m. The proof follows the same line of argument as in the proof of Lemma 1 in [35].

From the definition of $r_i^F(\theta)$, we see that there exists a partial feasible ordering among sessions in H. In particular, an ordering of the sessions in H such that any sessions in $F_l \cap H$ are ordered before sessions in $F_{l+1} \cap H$ for any l and sessions in the same $F_l \cup H$ are ordered arbitrarily is a partial feasible ordering. Fix such a partial feasible ordering. For simplicity in notation, we denote this ordering as $1, \dots, m$. We also drop F and θ in $r_i^F(\theta)$ in the following proof.

When $m = 1$, let $\tau_1, \tau \leq \tau_1 \leq t$, be such that for any $\tau_1', \tau_1 < \tau_1' \leq t, Q_1(\tau_1') > 0$ and $Q_1(\tau_1) \geq 0$ where $Q_1(\tau_1) > 0$ if and only if $\tau_1 = \tau$ and $Q_1(\tau) > 0$. In other words, session 1 is busy throughout the time interval $[\tau_1, t]$ which is contained in $[\tau, t]$. Note that by this choice of τ_1, we have

$$S_1(\tau, t) = S_1(\tau, \tau_1) + S_1(\tau_1, t) = Q_1(\tau) + A_1(\tau, \tau_1) + S_1(\tau_1, t).$$

By the definition of GPS and the fact that $r_1 < \frac{\phi_1}{\sum_{j=1}^N \phi_j}$, we have

$$S_1(\tau_1, t) \geq \frac{\phi_1}{\sum_{j=1}^N \phi_j}(t - \tau_1) > r_1(t - \tau_1).$$

Therefore,

$$S_1(\tau, t) \geq Q_1(\tau) + A_1(\tau, \tau_1) + r_1(t - \tau_1) \geq Q_1(\tau) + \min_{\tau \leq \tau_1 \leq t}\{A_1(\tau, \tau_1) + r_1(t - \tau_1)\}.$$

Now suppose the lemma is true for $m = 1, 2, \ldots, i-1$ over any time interval $[\tau, t]$, we show that it is also true for $m = i$.

Let $\tau_i \in [\tau, t]$ be such that for any $\tau_i' \in (\tau_i, t]$, $Q_i(\tau_i') > 0$ and $Q_i(\tau_i) \geq 0$ where $Q_i(\tau_i) > 0$ if and only if $\tau_i = \tau$ and $Q_i(\tau) > 0$. In other words, session i is busy throughout the time interval $[\tau_i, t]$ which is contained in $[\tau, t]$. Note that by this choice of τ_i, we have

$$S_i(\tau, t) = S_i(\tau, \tau_i) + S_i(\tau_i, t) = Q_i(\tau) + A_i(\tau, \tau_i) + S_i(\tau_i, t). \tag{62}$$

As

$$S_i(\tau_i, t) = Q_i(\tau_i) + A_i(\tau_i) - Q_i(t),$$

if $Q_i(t) \leq Q_i(\tau_i) + A_i(\tau_i, t) - r_i(t - \tau_i)$, then $S_i(\tau_i, t) \geq r_i(t - \tau_i)$.

Hence

$$S_i(\tau, t) \geq Q_i(\tau) + A_i(\tau, \tau_i) + r_i(t - \tau_i) \geq Q_i(\tau) + \min_{\tau \leq \tau_i \leq t} \{A_i(\tau, \tau_i) + r_i(t - \tau_i)\}.$$

Using the induction hypothesis, (24) then follows easily.

Now assume $Q_i(t) > Q_i(\tau) + A_i(\tau_i, t) - r_i(t - \tau_i)$. Let $x = Q_i(t) - Q_i(\tau) + A_i(\tau_i, t) - r_i(t - \tau_i)$, then $x > 0$ and

$$S_i(\tau_i, t) = r_i(t - \tau_i) - x. \tag{63}$$

As

$$r_i < \frac{\phi_i}{\sum_{j=i}^N \phi_j}(1 - \sum_{j=1}^{i-1} r_j),$$

from (63), we have

$$S_i(\tau_i, t) < \frac{\phi_i}{\sum_{j=i}^N \phi_j}(1 - \sum_{j=1}^{i-1} r_j)(t - \tau_i) - x. \tag{64}$$

Moreover, by the definition of GPS, for any j,

$$S_i(\tau_i, t) \geq \frac{\phi_i}{\phi_j} S_j(\tau_i, t).$$

Thus

$$\sum_{j=i}^N S_j(\tau_i, t) \leq (\sum_{j=i}^N \frac{\phi_j}{\phi_i}) S_i(\tau_i, t).$$

Using (64), we have

$$\begin{aligned}\sum_{j=i}^N S_j(\tau_i, t) &< (t - \tau_i)(1 - \sum_{j=1}^{i-1} r_j) - x \sum_{j=i}^N \frac{\phi_j}{\phi_i} \\ &\leq (t - \tau_i)(1 - \sum_{j=1}^{i-1} r_j) - x.\end{aligned} \tag{65}$$

On the other hand, since the system is busy throughout $[\tau_i, t]$,

$$\sum_{j=1}^{i-1} S_j(\tau_i, t) = t - \tau_i - \sum_{j=i}^N S_j(\tau_i, t).$$

From (65)

$$\sum_{j=1}^{i-1} S_j(\tau_i, t) > (t - \tau_i) \sum_{j=1}^{i-1} r_j + x. \tag{66}$$

Adding (63) to (66) yields

$$\sum_{j=1}^{i} S_j(\tau_i, t) > (t - \tau_i) \sum_{j=1}^{i} r_j. \tag{67}$$

Applying the induction hypothesis to sessions $1, 2, \ldots, i-1$ over $[\tau, \tau_i]$, we have

$$\sum_{j=1}^{i-1} S_j(\tau, \tau_i) \geq \sum_{j=1}^{i-1} (Q_j(\tau) + \min_{\tau \leq \tau_j \leq \tau_i} \{A_j(\tau, \tau_j) + r_j(\tau_i - \tau_j)\}).$$

This, combining with (62) and (67), yields

$$\begin{aligned}
\sum_{j=1}^{i} S_j(\tau, t) &= \sum_{j=1}^{i-1} S_j(\tau, \tau_i) + \sum_{j=1}^{i-1} S_j(\tau_i, t) + S_i(\tau, t) \\
&\geq \sum_{j=1}^{i} Q_j(\tau) + \sum_{j=1}^{i-1} \min_{\tau \leq \tau_j \leq \tau_i} \{A_j(\tau, \tau_j) + r_j(\tau_i - \tau_j)\} + A_i(\tau, \tau_i) + \sum_{j=1}^{i} r_j(t - \tau_i) \\
&\geq \sum_{j=1}^{i} Q_j(\tau) + \sum_{j=1}^{i} \min_{\tau \leq \tau_j \leq t} \{A_j(\tau, \tau_j) + r_j(t - \tau_j)\}.
\end{aligned}$$

This concludes the proof for the lemma. ■

B Proofs of the Two Lemmas in Section 4.3

Lemma 10 For any $\theta > 0$, if there exists $F \in \mathcal{F}(\theta)$ and $G \in N_1 \setminus F$ such that

$$\alpha_1(\theta) < \frac{\phi_1}{\sum_{j \in N \setminus (F \cup G)} \phi_j} (c - \sum_{j \in F} \alpha_j(\theta) - \alpha_{D_G}(\theta, r_i^F(\theta))), \tag{68}$$

then for any $0 < \theta' < \theta$,

$$\alpha_1(\theta') < \frac{\phi_1}{\sum_{j \in N \setminus (F \cup G)} \phi_j} (c - \sum_{j \in F} \alpha_j(\theta') - \alpha_{D_G}(\theta', r_i^F(\theta'))). \tag{69}$$

Proof: From Lemma 4(b), $F \in \mathcal{F}(\theta) \subseteq \mathcal{F}(\theta')$ and $\gamma_F(\theta) \leq \gamma_F(\theta')$. Therefore, for $i \in G$, $r_i^F(\theta) = \phi_i \gamma_F(\theta) = \phi_i \gamma_F(\theta') = r_i^F(\theta')$. Hence, for any $t \geq 0$,

$$\begin{aligned}
\delta_i^{\theta}(-t) &= \sup_{\tau \geq t} \{A_i(-\tau, -t) - r_i^F(\theta)(\tau - t)\} \\
&\geq \sup_{\tau \geq t} \{A_i(-\tau, -t) - r_i^F(\theta')(\tau - t)\}. \\
&= \delta_i^{\theta'}(-t)
\end{aligned} \tag{70}$$

Moreover,

$$
\begin{aligned}
\sum_{i\in G}\phi_i\gamma_F(\theta) &= \frac{\sum_{i\in G}\phi_i}{\sum_{j\in N\setminus F}\phi_j}(c-\sum_{j\in F}\alpha_j(\theta)) \\
&= \frac{\sum_{i\in G}\phi_i}{\sum_{j\in N\setminus F}\phi_j}(c-\sum_{j\in F}\alpha_j(\theta')) - \frac{\sum_{i\in G}\phi_i}{\sum_{j\in N\setminus F}\phi_j}\sum_{j\in F}(\alpha_j(\theta)-\alpha_j(\theta')) \\
&\geq \sum_{i\in G}\phi_i\gamma_F(\theta') - \sum_{j\in F}(\alpha_j(\theta)-\alpha_j(\theta')) \qquad (71)
\end{aligned}
$$

where the last inequality holds as $\alpha_i(\theta)-\alpha_i(\theta')>0$.
Now from the definition of $D_G^\theta(-t)$ and using (70) and (71), we have

$$
\begin{aligned}
D_G^\theta(-t) &= \left\{\sum_{i\in G}(A_i(-t,0)+\delta_i^\theta(-t))\right\}\wedge\left(\sum_{i\in G}\phi_i\gamma_F(\theta)t\right) \\
&\geq \left\{\sum_{i\in G}(A_i(-t,0)+\delta_i^{\theta'}(-t))\right\}\wedge\left(\sum_{i\in G}\phi_i\gamma_F(\theta)t - \sum_{j\in F}(\alpha_j(\theta)-\alpha_j(\theta'))t\right) \\
&\geq \left\{\sum_{i\in G}(A_i(-t,0)+\delta_i^{\theta'}(-t))\right\}\wedge\left(\sum_{i\in G}\phi_i\gamma_F(\theta)t\right) - \sum_{j\in F}(\alpha_j(\theta)-\alpha_j(\theta'))t
\end{aligned}
$$

as $a\wedge(b-c)\geq a\wedge b-c$ for $a,b,c\geq 0$.
Therefore,

$$
\alpha_{D_G}(\theta,r_G^F(\theta)) \geq \alpha_{D_G}(\theta,r_G^F(\theta')) - \sum_{j\in F}(\alpha_j(\theta)-\alpha_j(\theta')) \geq \alpha_{D_G}(\theta',r_G^F(\theta')) - \sum_{j\in F}(\alpha_j(\theta)-\alpha_j(\theta')). \qquad (72)
$$

substituting (72) in (68) and noting that $\alpha_1(\theta')<\alpha_1(\theta)$, we have (69). ■

Lemma 12 For any $\theta>0$, if there exists $F\in\mathcal{F}(\theta)$ and $G\subseteq N_1\setminus F$ such that

$$
\alpha_1(\theta) < \frac{\phi_1}{\sum_{j\in N\setminus(F\cup G)}\phi_j}(c-\sum_{i\in F}\alpha_i(\theta)-\alpha_{D_G}(\theta,r_G^F(\theta))), \qquad (73)
$$

then for any $F'\in\mathcal{F}(\theta)$,

$$
\alpha_1(\theta)+\sum_{i\neq 1}\alpha_{D_i}(\theta,r_i^{F'}(\theta)) < c. \qquad (74)
$$

As a consequence, $\min_{F\in\mathcal{F}(\theta)}\mu_*^F(\theta)>\theta$.

Proof: Let $E=N_1\setminus(F\cup G)$ and $E'=N_1\setminus(F'\cup G')$. We first observe that (73) implies (74) with $F'=F$, this is because

$$
\alpha_1(\theta) \;<\; c-\sum_{i\in F}\alpha_i(\theta)-\alpha_{D_G}(\theta,r_G^F(\theta)) - \frac{\sum_{i\in E}\phi_i}{\sum_{i\in E\cup\{1\}}\phi_i}(c-\sum_{i\in F}\alpha_i(\theta)-\alpha_{D_G}(\theta,r_G^F(\theta)))
$$

$$
\begin{aligned}
&\leq c - \sum_{i\in F}\alpha_i(\theta) - \sum_{i\in G}\alpha_{D_i}(\theta, r_i^F(\theta)) - \frac{\sum_{i\in E}\phi_i}{\sum_{i\in E\cup\{1\}}\phi_i}(c - \sum_{i\in F}\alpha_i(\theta) - \sum_{i\in G} r_i^F(\theta)) \\
&\leq c - \sum_{i\in F}\alpha_i(\theta) - \sum_{i\in G}\alpha_{D_i}(\theta, r_i^F(\theta)) \\
&\quad -\frac{\sum_{i\in E}\phi_i}{\sum_{i\in E\cup\{1\}}\phi_i}(c - \sum_{i\in F}\alpha_i(\theta) - \frac{\sum_{i\in G}\phi_i}{\sum_{i\in N\setminus F}\phi_i}(c - \sum_{i\in F}\alpha_i(\theta))) \\
&= c - \sum_{i\in F}\alpha_i(\theta) - \sum_{i\in G}\alpha_{D_i}(\theta, r_i^F(\theta)) - \sum_{i\in E} r_i^F(\theta))
\end{aligned}
$$

where the first inequality follows from (73) as $\frac{\phi_1}{\sum_{i\in N\setminus(F\cup G)}\phi_i} = 1 - \frac{\sum_{i\in E}\phi_i}{\sum_{i\in E\cup\{1\}}\phi_i}$, the second inequality follows from the fact that $\sum_{i\in G}\alpha_{D_i}(\theta, r_i^F(\theta)) \leq \alpha_{D_G}(\theta, r_G^F(\theta) \leq r_G^F(\theta) = \sum_{i\in G} r_i^F(\theta)$ and the last equality holds as for $i \in E$, $r_i^F(\theta) = \frac{\phi_i}{\sum_{i\in N\setminus F}\phi_i}(c - \sum_{i\in F}\alpha_i(\theta))$.

As for $i \in F$, $\alpha_i(\theta) \geq \alpha_i(\theta, r_i^F(\theta))$ and for $i \in E$, $r_i^F(\theta)) \geq \alpha_i(\theta, r_i^F(\theta))$, we have

$$
\alpha_1(\theta) + \sum_{i\neq 1}\alpha_{D_i}(\theta, r_i^F(\theta)) < c. \tag{75}
$$

If $F' \subseteq F$, from Lemma 4(a), $\gamma_{F'}(\theta) \leq \gamma_F(\theta)$, thus for any $i \in G\cup E$, $r_i^F(\theta) = \phi_i\gamma_F(\theta) \geq \phi_i\gamma_{F'}(\theta) = r_i^{F'}(\theta)$. Therefore, $\alpha_{D_i}(\theta, r_i^F(\theta)) \geq \alpha_{D_i}(\theta, r_i^{F'}(\theta))$. Moreover, for $i \in F$, $\alpha_i(\theta) \geq \alpha_{D_i}(\theta, r_i^{F'}(\theta))$. Therefore (74) holds for any $F' \subseteq F$.

Now we consider any $F' \in \mathcal{F}(\theta)$ such that $F' \supset F$. We first prove a somewhat different claim, then use this claim to show that (74) holds for any $F' \in \mathcal{F}(\theta)$ such that $F' \supset F$.

Let $\nabla F = F' \setminus F, \nabla G = G \cap \nabla F$ and $\nabla E = E \cap \nabla F$. Then $\nabla F = \nabla G \cup \nabla E$. Furthermore, define $G' = G \setminus \nabla G$ and $E' = E \setminus \nabla E$. Then $F' \cup G' \cup E' = N_1$.

Claim 13 *Let $F' \in \mathcal{F}(\theta)$ be such that $F' \supset F$ and for any $i \in \nabla F$, $\alpha_i(\theta) \leq r_i^F(\theta) = \phi_i\gamma_F(\theta)$. Then*

$$
\alpha_1(\theta) < \frac{\phi_1}{\sum_{j\in E\cup\{1\}}\phi_j}(c - \sum_{i\in F}\alpha_{D_i}(\theta, r_i^F(\theta)) - \alpha_{D_G}(\theta, r_G^F(\theta))), \tag{76}
$$

implies

$$
\alpha_1(\theta) < \frac{\phi_1}{\sum_{j\in E'\cup\{1\}}\phi_j}(c - \sum_{i\in F'}\alpha_{D_i}(\theta, r_i^{F'}(\theta)) - \alpha_{D_{G'}}(\theta, r_{G'}^{F'}(\theta))). \tag{77}
$$

To prove the claim, we first note that

$$
\begin{aligned}
\gamma_F(\theta) &= \frac{1}{\sum_{j\in N\setminus F}\phi_j}(c - \sum_{j\in F}\alpha_j(\theta)) = \frac{1}{\sum_{j\in N\setminus F'}\phi_j}(1 - \frac{\sum_{j\in\nabla F}\phi_j}{\sum_{j\in N\setminus F'}\phi_j})(c - \sum_{j\in F}\alpha_j(\theta)) \\
&= \frac{1}{\sum_{j\in N\setminus F'}\phi_j}(c - \sum_{j\in F'}\alpha_j(\theta) + \sum_{j\in\nabla F}\alpha_j(\theta) - \frac{\sum_{j\in\nabla F}\phi_j}{\sum_{j\in N\setminus F'}\phi_j}(c - \sum_{j\in F}\alpha_j(\theta))) \\
&= \frac{1}{\sum_{j\in N\setminus F'}\phi_j}(c - \sum_{j\in F'}\alpha_j(\theta) + \sum_{j\in\nabla F}\alpha_j(\theta) - \sum_{j\in\nabla F}\phi_j\gamma_F(\theta)) \\
&= \gamma_{F'}(\theta) - \frac{1}{\sum_{j\in N\setminus F'}\phi_j}\sum_{j\in\nabla F}[r_i^F(\theta) - \alpha_j(\theta)].
\end{aligned}
$$

Let $\zeta_F(\theta) = \frac{1}{\sum_{j\in N\setminus F'}\phi_j}\sum_{j\in\nabla F}(r_i^F(\theta)-\alpha_j(\theta))$, then $\gamma_F(\theta) = \gamma_{F'}(\theta) - \zeta_F(\theta)$. Now

$$\begin{aligned} r_G^F(\theta) &= \sum_{i\in G}\phi_i\gamma_F(\theta) = \sum_{i\in G'}\phi_i\gamma_F(\theta) + \sum_{i\in\nabla G}\phi_i\gamma_F(\theta) \\ &= \sum_{i\in G'}\phi_i\gamma_{F'}(\theta) - \sum_{i\in G'}\phi_i\zeta_F(\theta) + \sum_{i\in\nabla G}\phi_i\gamma_F(\theta), \end{aligned}$$

and

$$\begin{aligned} &\sum_{i\in\nabla G}\phi_i\gamma_F(\theta) - \sum_{i\in G'}\phi_i\zeta_F(\theta) = \sum_{i\in\nabla G}\phi_i\gamma_F(\theta) - \frac{\sum_{i\in G'}\phi_i}{\sum_{j\in N\setminus F'}\phi_j}\sum_{j\in\nabla F}(r_i^F(\theta)-\alpha_j(\theta)) \\ &= \sum_{i\in\nabla G} r_i^F(\theta) - \sum_{j\in\nabla F}(r_i^F(\theta)-\alpha_j(\theta)) + \frac{\sum_{i\in E'\cup\{1\}}\phi_i}{\sum_{j\in N\setminus F'}\phi_j}\sum_{j\in\nabla F}(r_i^F(\theta)-\alpha_j(\theta)) \\ &= \sum_{i\in\nabla G} r_i^{F'}(\theta) - \sum_{j\in\nabla E}(r_i^F(\theta)-\alpha_j(\theta)) \\ &\quad + \frac{\sum_{i\in E'\cup\{1\}}\phi_i}{\sum_{j\in N\setminus F'}\phi_j}\sum_{j\in\nabla G}(r_i^F(\theta)-\alpha_j(\theta)) + \frac{\sum_{i\in E'\cup\{1\}}\phi_i}{\sum_{j\in N\setminus F'}\phi_j}\sum_{j\in\nabla E}(r_i^F(\theta)-\alpha_j(\theta)) \\ &= \sum_{i\in\nabla G} r_i^{F'}(\theta) + \frac{\sum_{i\in G'}\phi_i}{\sum_{j\in N\setminus F'}\phi_j}\sum_{j\in\nabla G}(r_i^F(\theta)-\alpha_j(\theta)) + \frac{\sum_{i\in E'\cup\{1\}}\phi_i}{\sum_{j\in N\setminus F'}\phi_j}\sum_{j\in\nabla E}(r_i^F(\theta)-\alpha_j(\theta)) \\ &= \sum_{i\in\nabla G} r_i^{F'}(\theta) + \frac{\sum_{i\in G'}\phi_i}{\sum_{j\in N\setminus F'}\phi_j}\sum_{j\in\nabla G}(r_i^F(\theta)-\alpha_j(\theta)) \\ &\geq \sum_{i\in\nabla G} r_i^{F'}(\theta) \end{aligned}$$

where the last inequality holds as for $i\in\nabla F = \nabla G\cup\nabla E$, $r_i^{F'}(\theta) = \alpha_i(\theta) \leq r_i^F(\theta)$.
Hence,

$$r_G^F(\theta) \geq r_G^{F'}(\theta) + \sum_{i\in\nabla G} r_i^{F'}(\theta).$$

Now for any $t\geq 0$, recalling the definition of $D_G^\theta(-t)$, we have

$$\begin{aligned} D_G^\theta(-t) &= \left\{\sum_{i\in G}(A_i(-t,0)+\delta_i^\theta(-t))\right\} \wedge \left(\sum_{i\in G}\phi_i\gamma_F(\theta)t\right) \\ &= \left\{\left[\sum_{i\in G'}(A_i(-t,0)+\delta_i^\theta(-t))\right] + \left[\sum_{i\in\nabla G}(A_i(-t,0)+\delta_i^\theta(-t))\right]\right\} \\ &\quad \wedge \left(r_G^{F'}(\theta)t + \sum_{i\in\nabla G} r_i^{F'}(\theta)t\right) \\ &\geq \left\{\sum_{i\in G'}(A_i(-t,0)+\delta_i^\theta(-t))\right\} \wedge \left(r_G^{F'}(\theta)t\right) + \left\{\sum_{i\in\nabla G}(A_i(-t,0)+\delta_i^\theta(-t))\right\} \\ &\quad \wedge \left(\sum_{i\in\nabla G} r_i^{F'}(\theta)t\right) \end{aligned}$$

$$\geq \left\{\sum_{i\in G'}(A_i(-t,0)+\delta_i^\theta(-t))\right\}\wedge\left(r_G^{F'}(\theta)t\right)+\sum_{i\in\nabla G}\left([A_i(-t,0)+\delta_i^\theta(-t)]\wedge r_i^{F'}(\theta)t\right)$$

where the last two inequalities follow from the fact that $(a+b)\wedge(c+d)\geq a\wedge c+b\wedge d$ for $a\geq 0, b\geq 0, c\geq 0, d\geq 0$. Therefore,

$$\alpha_{D_G}(\theta, r_G^F(\theta)) \geq \alpha_{D_{G'}}(\theta, r_{G'}^{F'}(\theta)) + \sum_{i\in\nabla G}\alpha_{D_i}(\theta, r_{G'}^{F'}(\theta)). \tag{78}$$

Now from (76), we have

$$\begin{aligned}
\alpha_1(\theta) &< \frac{\phi_1}{\sum_{j\in E'\cup\{1\}}\phi_j}\Big(1-\frac{\sum_{i\in\nabla E}\phi_i}{\sum_{j\in E\cup\{1\}}\phi_j}\Big)\left(c-\sum_{i\in F}\alpha_{D_i}(\theta, r_i^F(\theta))-\alpha_{D_G}(\theta, r_G^F(\theta))\right)\\
&= \frac{\phi_1}{\sum_{j\in E'\cup\{1\}}\phi_j}\left\{c-\sum_{i\in F}\alpha_{D_i}(\theta, r_i^F(\theta))-\alpha_{D_G}(\theta, r_G^F(\theta))\right.\\
&\qquad \left. -\frac{\sum_{i\in\nabla E}\phi_i}{\sum_{j\in E\cup\{1\}}\phi_j}\left(c-\sum_{i\in F}\alpha_{D_i}(\theta, r_i^F(\theta))-\alpha_{D_G}(\theta, r_G^F(\theta))\right)\right\}.
\end{aligned}\tag{79}$$

Since for $i\in F$, $\alpha_{D_i}(\theta, r_i^F(\theta))\leq r_i^F(\theta)=\alpha_i(\theta)$, and $\alpha_{D_G}(\theta, r_G^F(\theta))\leq r_G^F(\theta)=\frac{\sum_{i\in G}}{\sum_{j\in N\setminus F}\phi_j}(c-\sum_{j\in F}\alpha_j(\theta))$, we have

$$\begin{aligned}
&\frac{\sum_{i\in\nabla E}\phi_i}{\sum_{j\in E\cup\{1\}}\phi_j}\left(c-\sum_{i\in F}\alpha_{D_i}(\theta, r_i^F(\theta))-\alpha_{D_G}(\theta, r_G^F(\theta))\right)\\
&\quad\geq \frac{\sum_{i\in\nabla E}\phi_i}{\sum_{j\in E\cup\{1\}}\phi_j}\frac{\sum_{i\in E\cup\{1\}}\phi_i}{\sum_{j\in N\setminus F}\phi_j}(c-\sum_{j\in F}\alpha_j(\theta))\\
&\quad= \sum_{i\in\nabla E}\phi_i\gamma_F(\theta)=\sum_{i\in\nabla E}r_i^F(\theta).
\end{aligned}\tag{80}$$

From the assumption in the claim, for $i\in\nabla E, r_i^F(\theta)\geq\alpha_i(\theta)\geq\alpha_{D_i}(\theta, r_i^{F'}(\theta))$. This, together with (79) and (80), yields that

$$\begin{aligned}
\alpha_1(\theta) &\leq \frac{\phi_1}{\sum_{j\in E'\cup\{1\}}\phi_j}\left\{c-\sum_{i\in F}\alpha_{D_i}(\theta, r_i^F(\theta))-\alpha_{D_G}(\theta, r_G^F(\theta))-\sum_{i\in\nabla E}\alpha_{D_i}(\theta, r_i^{F'}(\theta))\right\}\\
&\leq \frac{\phi_1}{\sum_{j\in E'\cup\{1\}}\phi_j}\left\{c-\sum_{i\in F}\alpha_{D_i}(\theta, r_i^{F'}(\theta))\right.\\
&\qquad \left.-\sum_{i\in\nabla G}\alpha_{D_i}(\theta, r_i^{F'}(\theta))-\alpha_{D_{G'}}(\theta, r_{G'}^{F'}(\theta))-\sum_{i\in\nabla E}\alpha_{D_i}(\theta, r_i^{F'}(\theta))\right\}\\
&= \frac{\phi_1}{\sum_{j\in E'\cup\{1\}}\phi_j}\Big(\sum_{i\in F'}\alpha_{D_i}(\theta, r_i^{F'}(\theta))+\alpha_{D_{G'}}(\theta, r_{G'}^{F'}(\theta))\Big)
\end{aligned}$$

where the second inequality follows from (78) and the fact that for $i\in F$, $\alpha_{D_i}(\theta, r_i^F(\theta))=\alpha_{D_i}(\theta, r_i^{F'}(\theta))$. This completes the proof of the claim.

We now use the claim to show that (74) holds for any $F' \in \mathcal{F}(\theta)$ such that $F' \supset F$.

Let $F'_1, \ldots, F'_k$ be the partial feasible partition of F' and $\gamma_1^{F'}(\theta), \ldots, \gamma_k^{F'}(\theta)$ be the associated delimiting numbers. Let $F'_0 = \emptyset$ and $\gamma_0^{F'}(\theta) = 0$. Then $F' = F'_1 \cup \ldots F'_k$ and for any $i \in F'_l, \gamma_{l-1}^{F'}(\theta) \leq \alpha_i(\theta) < \gamma_l^{F'}(\theta), l = 1, \ldots, k$. In particular, for all $i \in F', \alpha_i(\theta) < \gamma_k^{F'}(\theta) = \gamma_{F'}(\theta)$. Let $F'^l = F'_1 \cup \ldots F'_l$. Since $F \subset F'$, by Lemma 4(a), there exists $m, 1 \leq m \leq k$, such that $F \subseteq F'^m$ but $F \not\subseteq F'^{m-1}$. In other words, we must have $\gamma_{m-1}^{F'}(\theta) < \gamma_F(\theta) \leq \gamma_m^{F'}(\theta)$.

Let $G_0 = F, G_1 = F \cup F'^{l-1}$, and for $l = 2, \ldots, k-m+2, G_l = F'^{m+l-2}$ with $G_{k-m+2} = F'^k = F'$. Then we can check that for any $i \in \nabla G_l = G_{l+1} \setminus G_l, l = 0, 1, \ldots, k-1, \alpha_i(\theta) \leq \gamma_G(\theta)$. In particular, for $i \in \nabla G_0 = F'^{l-1} \setminus F$, $\alpha_i(\theta) < \gamma_{l-1}^{F'}(\theta) < \gamma_F(\theta) = \gamma_{G_0}(\theta)$ and $i \in \nabla G_1 = F'_l \setminus F$, $\alpha_i(\theta) \leq \gamma_l^{F'}(\theta) = \gamma_{F'^{l-1}}(\theta) \leq \gamma_{G_1}(\theta)$ where the last two steps follow from the definition of $\gamma_l^{F'}(\theta)$ and Lemma 4(a).

As $\sum_{i \in F} \alpha_i(\theta) \geq \sum_{i \in F} \alpha_{D_i}(\theta, r_i^F(\theta))$, clearly (73) implies that (76) holds with $G_0 = F$. Then by the claim, (76) is then true for G_1. Applying the claim recursively to $G_1, \ldots, G_{k-m+1}$, we have that (76) is true for $G_{k-m+2} = F'$. Therefore (with G' and E' appropriately defined)

$$\begin{aligned}
\alpha_1(\theta) &< \frac{\phi_1}{\sum_{j \in E' \cup \{1\}} \phi_j} (c - \sum_{i \in F'} \alpha_{D_i}(\theta, r_i^{F'}(\theta)) - \alpha_{D_{G'}}(\theta, r^{F'_{G'}}(\theta))) \\
&= c - \sum_{i \in F'} \alpha_{D_i}(\theta, r_i^{F'}(\theta)) - \alpha_{D_{G'}}(\theta, r^{F'_{G'}}(\theta)) \\
&\quad - \frac{\sum_{i \in E'} \phi_i}{\sum_{j \in E' \cup \{1\}} \phi_j} (c - \sum_{i \in F'} \alpha_{D_i}(\theta, r_i^{F'}(\theta)) - \alpha_{D_{G'}}(\theta, r^{F'_{G'}}(\theta))) \\
&\leq c - \sum_{i \in F'} \alpha_{D_i}(\theta, r_i^{F'}(\theta)) - \alpha_{D_{G'}}(\theta, r^{F'_{G'}}(\theta)) - \sum_{i \in E'} r_i^{F'}(\theta) \\
&\leq c - \sum_{i \neq 1} \alpha_{D_i}(\theta, r_i^{F'}(\theta)).
\end{aligned}$$

This completes the proof of the lemma. ∎

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