# TCP Connection Game: A Study on the Selfish Behavior of TCP Users

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***Abstract*—**
**We present a game-theoretic study of the selfish behavior of TCP users when they are allowed to use multiple concurrent TCP connections so as to maximize their goodputs or other utility functions. We refer to this as the *TCP connection game*. A central question we ask is whether there is a Nash Equilibrium in such a game, and if it exists, whether the network operates efficiently at such a Nash Equilibrium. Combined with the well known PFTK TCP Model [13], we study this question for three utility functions that differ in how they capture user behavior. The bad news is that the loss of efficiency or price of anarchy can be arbitrarily large if users have no resource limitations and are not socially responsible. The good news is that, if either of these two factors are considered, efficiency loss is bounded. This may partly explain why there will be no congestion collapse if many users use multiple connections.**

## I. Introduction

The conventional wisdom is that the stability of the Internet is due to TCP's congestion control mechanisms. For example, [1] used a game-theoretic study to show that TCP-Reno and Drop-Tail buffer can make the network operate efficiently when end points are allowed to adjust the increase and decrease parameters of TCP. On the other hand, [11] argues that both network and user behavior should be considered regarding the stability issue of the Internet. This paper is an attempt to understand the impact of user's behavior on the efficiency of the Internet.

Using a game-theoretic framework, we evaluate the impact of greedy behavior of users when allowed to open multiple TCP connections. One of our motivations comes from the observation of the trend that more and more users use some software agents (e.g.,FlashGet [4]) for concurrent downloading in order to accelerate file transfers [8]. Specifically, we are interested in a scenario in which a number of users compete for the capacity of a single bottleneck link. Users have infinite amounts of data to send, and they are allowed to open a number of concurrent connections. This scenario can be modeled as a non-cooperative game in which players are individual users. The strategy of each player is the number of TCP connections. Each player tries to maximize its own utility. We call this game the *TCP Connection Game*.

For this type of general game, we study three specific games that differ from each other in their utility functions. In *game 1*, a user's utility function corresponds to the long-term average goodput (packets per second transferred by the bottleneck link without being discarded). In *game 2*, users take into account the potential cost incurred on the system and on themselves. This cost is assumed proportional to the aggregate sending rate of all connections opened by all users. The cost incurred by each user is then the aggregate sending rate of all connections opened by this user. This cost not only can be interpreted as the packet sending cost for a user, but also the system-wide network resource consumed by the offered traffic. Thus in this sense, a user concerned with this cost can also be thought of being socially responsible. In *game 3*, in addition to the packet sending cost, we introduce another term specific only to users and that accounts for the cost of maintaining open connections. We also allow users to have different computation powers. A more powerful user will have more computation resource to support more TCP connections. For all three games, the Nash Equilibrium (NE) can be thought of as a combination of the number of connections of all users, at which no users can benefit from increasing or decreasing its number of connections. In this paper, we only study pure strategy NE.

We are interested in the following questions. Do there exist Nash Equilibria (NE) for these TCP connection games? If so, what is the loss of efficiency and price of anarchy of the network operating at a NE? How can the behaviors of users potentially affect the efficiency of the NE? For example, how can the socially responsible behavior of users affect the efficiency of the NE? Are users treated fairly at NE? Are NEs stable in the sense that any deviation from NE will converge back to NE?

We find that, in game 1, when users do not have any resource constraints and are not socially responsible, the loss of efficiency or the price of anarchy can be arbitrarily large. This is in contrast to the conclusion in [1] that the network operates efficiently with TCP-Reno loss recovery mechanism and DropTail queue even when users are capable of freely choosing additive increase and multiplicative decrease parameters of TCP. However, we find that in games 2 and 3, the efficiency loss is bounded if users are resource constrained and socially responsible. We also show that there exists a unique NE for game 2 and that it is locally stable, namely, any small deviation from NE will eventually converge back to NE. And, we have observed that it is very likely that the NE is globally stable as well. We also observe that integer NEs very likely exist for the real case that users are only allowed to open an integer number of connections. Last, in game 3, we show that a user with greater computation power is able to obtain greater goodput at the NE than a user with smaller computation power.

The rest of the paper is organized as follows. Section II presents the related work. System optimization problem is addressed in Section III. In Section IV, we study game 1 in which users are only interested in maximizing their goodput. In Section V, we introduce the packet sending cost and social responsibility into user's utility function. Section VI studies game 3 in which both cost and computation power are considered. We present the simulation study with NS in Section VII, and conclude the paper in Section VIII.

## II. Related Work

There are numerous works on the stability of the Internet and TCP congestion control. The most relevant one is a recent game-theoretic study of TCP by Akella *et al* [1]. They studied a class of TCP games in which each user controls a single TCP connection, and the strategy of each user is a pair of values corresponding to the additive increase parameter $\alpha$ and multiplicative decrease parameter $\beta$. Our work differs from their work in that we let users use standard TCP but allow them to choose the number of concurrent connections. Morris and Tay [11] studied congestion collapse through a model of the interaction between network and user behavior. Liu *et al* [8] studied the impact of concurrent downloading on the fairness and system's transient behavior.

In our game-theoretic study, we investigate the effect of the number of connections on TCP performance. Regarding this subject, there are also several related works, such as [14] and [10]. Qiu *et al* [14] gave some simulation results on TCP performance when there are a large number of TCP connections. Morris [10] also did simulations on TCP performance when the number of flows is large, and he gave some recommendations on the buffer size for improving the bottleneck link throughput.

There are many TCP modeling works. Among them, we choose the known and the most accurate PFTK TCP model [13] as the basis of our analysis. To the best of our knowledge, our work is the first attempt to systematically study the competition among selfish users being capable of strategically adjusting the number of TCP connections in a game-theoretic framework.

## III. System Optimization

In this section, we first define the TCP connection game, with specific utility function definitions left for the following sections. Then we address the system optimization problem, which is applicable to all utility functions studied in this paper.

Formally, in the TCP connection game, there are $m$ ($m \geq 2$) TCP users with different Round Trip Time (RTT) $R_i$ competing for the capacity $C$ of a bottleneck link. Individual users are treated as players. A strategy $n_i$ available to a user $i$ is a feasible number of connections he/she can open concurrently. In practice, $n_i$ takes positive integer values. However, we will consider the case where $n_i$ is a real-valued number. Then we discuss the case where $n_i$ is a positive integer number. Let $S_i$ denote the feasible strategy set of player $i$, then $n_i \in S_i$. And the feasible strategy space of this game is $S = S_1 \times S_2 \cdots \times S_m$. Then, a feasible strategy tuple is a $m$-dimension vector $\mathbf{n} = (n_1, n_2, ..., n_m) \in S$. The objective of each player is to maximize its utility $U_i$ by adjusting $n_i$. Various utility functions $U_i$ are defined in the following sections to capture different user behaviors. Nash Equilibrium $\mathbf{n}_{ne}$ or $\mathbf{n}^* = (n_1^*, ..., n_m^*)$ [1] is defined as

$$n_i^* = \text{argmax}_{n_i \in S_i} U_i(n_1^*, n_2^*, ..., n_i, ..., n_m^*), \forall i$$

In the TCP connection game, from a system point of view, the aggregate goodput of all connections of all players is always the bottleneck link capacity. Thus, there is nothing to be optimized regarding the aggregate goodput. However, we note that there is a cost associated with this aggregate goodput. For each user $i$, each connection will have a sending rate or offered rate $B_i$. Then, $\sum_{i=1}^m n_i B_i$, the aggregate of all these offered rates, drives the bottleneck link to full utilization. The larger $\sum_{i=1}^m n_i B_i$ is, the more network resource utilized. Thus, we can think of $B_i$ as a "cost" from the system's point of view and $\sum_{i=1}^m n_i B_i$ as an aggregate cost $\Phi$. Thus, to get the highest efficiency, the system's optimization objective is to minimize this total cost while maintaining the bottleneck link fully utilized.

In the following, we first introduce the known TCP sending rate model and its related per-connection goodput model in [13]. Throughout this paper, we will base all our analysis on these two models.

The best known TCP sending rate model, full PFTK TCP model relating TCP sending rate $B_i$ to loss rate $p$ and RTT $R_i$, is given in [13], but is too complicated for analysis. Therefore, we use a simplified version (recommended in the TFRC standard proposal [5]) given as

$$B_i = 1/(\mu R_i \sqrt{p} + T_{0,i} \nu (p^{3/2} + 32 p^{7/2})) \qquad (1)$$

where $\mu = \sqrt{2b/3}$, $\nu = 3/2\sqrt{3b/2}$, $b = 1$ or 2, and $T_{0,i} = 4R_i$.

For per-connection goodput, we assume that the expected window size $W$ of each connection is the same for all flows going through a congested bottleneck link [13], because we expect all connections to incur the same loss rate $p$ at the bottleneck link queue. Let $\bar{W}$ denote this common window size. Then, the per-connection goodput of player $i$ will be $G_i = \bar{W}/R_i$. Suppose that player $i$ has $n_i$ connections. Then, based on the bottleneck principle, the sum of goodputs of all connections of all players equals the link capacity $C$, i.e., $\sum_{i=1}^m n_i G_i = C$. Then, we have $\bar{W} = C/(\sum_{i=1}^m n_i/R_i)$ and $G_i = (C/R_i)/(\sum_{i=1}^m n_i/R_i)$.

We have the following formulation of the system optimization problem:

$$\min_{\mathbf{n}} \Phi = \sum_{i=1}^m n_i B_i \qquad (2)$$

subject to

$$B_i = 1/(\mu R_i \sqrt{p} + T_{0,i} \nu (p^{3/2} + 32 p^{7/2})) \qquad (3)$$

$$B_i(1-p) = (C/R_i)/(\textstyle\sum_{j=1}^m n_j/R_j), \forall i \qquad (4)$$

$$n_i \in [1, ..., n_i^{max}]; m \geq 2; \mathbf{n} = \{n_1, ..., n_m\} \qquad (5)$$

Note, (4) indicates that per-connection goodput of a user is the product of its per-connection sending rate and the probability of successful transfer. Regarding the optimal operating point of the whole system, we have the following result.

[1] For notational convenience, in this paper, we use both $\mathbf{n}^*$ and $\mathbf{n}_{ne}$ interchangeably to denote the number of connections at NE. Similarly we use both $p^*$ and $p_{ne}$ to denote the loss rate at NE.

*Theorem 1:* In the TCP connection game, the system optimal cost is uniquely achieved at $\mathbf{n}_{opt} = (1, 1, ..., 1)$.

*Proof:* First, we transform the objective function into a simpler form: $\Phi = C/(1-p)$. It is easy to see that we need to find the minimal feasible $p$ to minimize $\Phi$.

Note that $p$ is a function of $\mathbf{n}$ (see (4)). If we take $T_{0,i} = 4R_i$, as recommended in [5], and let

$$\bar{\phi}(p) = \mu\sqrt{p} + 4\nu(p^{3/2} + 32p^{7/2})$$

(4) can be rewitten as

$$F(p, \mathbf{n}) = (1-p)(\sum_{j=1}^{m} n_j/R_j) - C\bar{\phi} = 0$$

Solving for $p$ given a specific $\mathbf{n}$ is actually equivalent to solving the above equation. First note that,

$$\lim_{p\to 0} F = \sum_{j=1}^{m} n_j/R_j > 0; \lim_{p\to 1} F = -C\cdot\bar{\phi}(1) < 0$$

Furthermore, $F(p, \mathbf{n})$ is a strictly monotonic decreasing function of $p$ since

$$\partial F/\partial p = -\sum_{j=1}^{m} n_j/R_j - C\bar{\phi} < 0$$

Thus, there must be a unique solution $p$ in $F(p, \mathbf{n}) = 0$ for any given feasible $\mathbf{n}$. That is, $p$ as a function of $\mathbf{n}$ is implicitly defined in $F(p, \mathbf{n}) = 0$. Note that $p$ is a increasing function of $\mathbf{n}$, thus, minimal $p_{opt}$ (satisfying $F(p, \mathbf{n}) = 0$) is uniquely achieved at $\mathbf{n}_{opt} = (1, 1, ..., 1)$. Then, we see that the system cost expressed in (2) uniquely achieves minimum value at $\mathbf{n}_{opt}$ with $\Phi_{opt} = C/(1-p_{opt})$. ■

## IV. Game 1: Aggressive Users

In this section, we study the TCP Connection Game with goodput as the utility function. Users with this utility function are *aggressive* in the sense that they only care about goodput and have no resource limitations and are not socially responsible. We first identify the Nash Equilibria of this game, and then study how badly the system's performance is influenced by this selfish user behavior.

### A. Nash Equilibrium

Recall the basic definition of the TCP connection game in Section III. The strategy set of player $i$ is $S_i = \{1, 2, 3, 4, ..n_i^{max}\}$, where $n_i^{max}$ is the maximum allowable number of connections for user $i$. We allow each player to maximize its aggregate goodput by adjusting its feasible number of connections $n_i$. Specifically, the utility of player $i$ is represented as:

$$U_i = n_i G_i = (Cn_i/R_i)/(\sum_{j=1}^{m} n_j/R_j) \quad (6)$$

We call this *utility function 1* and the game with this utility function *Game 1*. The following lemma shows that the game with this utility function has a unique Nash Equilibrium (NE) at a boundary point in the strategy space.

*Lemma 1:* There exists a unique Nash Equilibrium (NE) of the TCP connection game with utility function 1. At this NE, all players use their maximum number of allowable connections, that is, NE is $(n_1^{max}, n_2^{max}, ..., n_m^{max})$.

*Proof:* Note that the strategy set of each player is a discrete set. To make the analysis easier, we first relax the strategy set of any player to be a real interval $[1, n_i^{max}]$. This relaxed version is a continuous kernel game [2].

For player $i$, consider the partial derivative

$$\partial G_i/\partial n_i = ((C/R_i)\sum_{j\neq i} n_j/R_j)/(\sum_{j=1}^{m} n_i/R_j)^2 \quad (7)$$

Obviously, $\partial G_i/\partial n_i > 0$, thus, player $i$ always has an incentive to increase its number of connections regardless of the number of connections used by other players. Since this is true for all players, the only NE is $\mathbf{n}^* = (n_1^{max}, n_2^{max}, ..., n_m^{max})$. Since the strategy set of the original discrete game is a subset of this continuous kernel game, and this NE is a feasible strategy in the original discrete game, we conclude that the original discrete TCP connection game has a unique NE at $\mathbf{n}^*$. ■

**Remarks.** There is no fairness at the NE. Since the utility function is an increasing function of the number of connections opened by a user, user $i$ with larger $n_i^{max}/R_i$ will have a larger goodput than user $j$ with smaller $n_j^{max}/R_j$.

### B. Price of Anarchy and Loss of Efficiency

Price of Anarchy [6], is defined as the ratio of system performance at the worst NE and the system performance at the system optimal point. This value quantifies the loss of efficiency of the worst NE. In a TCP connection game with utility function 1, there is a unique NE. Then, price of anarchy is just the efficiency loss of this unique NE.

Let $p_{ne}$ denote the loss rate when the system is at NE. The system cost at NE is:

$$\Phi_{ne} = \sum_{i=1}^{m} n_i^{max} B_{i,ne} = C/(1-p_{ne})$$

Then, the price of anarchy is given by:

$$L_{eff} = \Phi_{ne}/\Phi_{opt} = (1-p_{opt})/(1-p_{ne}) \quad (8)$$

If the number of users $m$ is fixed, then $\Phi_{opt}$ is a constant regardless of the values of $n_i^{max}$. But $\Phi_{ne}$ is an increasing function of $n_i^{max}$. The reason is as follows. Based on the proof of Theorem 1, we know that $p$ is an increasing function of the number of connections and $p$ asymptotically approaches 1 as $n_i, \forall i$ goes to $\infty$. Thus, when $n_i^{max} \to \infty$, $p_{ne} \to 1$. Then (8) indicates that the price of anarchy becomes unbounded and arbitrarily large.

It is interesting to note that the price of anarchy asymptotically approaches a constant when the population of users increases. We assume that all users have the same RTT $\bar{R}$, and they have the same maximal allowable number of connections

$\bar{n}$, then $p_{ne}$ is the solution of $(1-p_{ne})B_{ne} = C/(m\bar{n})$, and $p_{opt}$ is the solution of $(1-p_{opt})B_{opt} = C/m$. The loss of efficiency is $L_{eff} = (1-p_{opt})/(1-p_{ne}) = \bar{n}B_{ne}/B_{opt}$.

Since $\lim_{m\to\infty} p_{opt} = 1$ and $\lim_{m\to\infty} p_{ne} = 1$, then

$$\lim_{m\to\infty} B_{opt} = 1/(\mu\bar{R} + 33T_0\nu) = \lim_{m\to\infty} B_{ne}$$

Thus, $\lim_{m\to\infty} L_{eff} = \bar{n}$. However we need to be cautious when interpreting this result. In this case, $m$ is so large that the network cannot even support the case where each user opens only one connection ($p_{opt} \to 1$). Thus, the network cannot operate efficiently even at the system optimal point.

## V. Game 2: Resource Constrained and Socially Responsible Users

The previous section shows that the price of anarchy can be arbitrarily large if users are only interested in maximizing their goodputs. In this section, we will show that if users have some resource constraints and take some social responsibility by considering the cost to the system in their utility functions, then the price of anarchy is bounded.

Recall that we treat the aggregate sending rate from all connections opened by a player as the efforts or cost incurred by that user. Let $n_iB_i$ denote this cost. Note, this cost not only represents a cost to the system but also can be interpreted as the cost to the user for sending data. Then a user $i$ may want to examine the tradeoff between the cost $n_iB_i$ and the achieved goodputs when making a decision on how many connections to open, thus we can derive a utility function as follows

$$U_i = C(n_i/R_i)/(n_i/R_i + \sum_{k=1,k\neq i}^{m} n_k/R_k) - \beta n_i B_i \quad (9)$$

We call this *utility function 2* and the game with this utility function *Game 2*. Here, coefficient $\beta \in (0,1)$ represents a user's weight on the efforts or cost. A smaller $\beta$ means a user is less resource constrained and less socially responsible. If $\beta = 0$, this utility function becomes just the goodput, the utility function 1.

In this section, first, we study a continuous kernel *symmetric* multiple player TCP connection game in which all users have the same Round Trip Time (RTT). We then consider two extensions. One is a discrete version of the *symmetric* multiple player TCP connection game. The other one is a continuous kernel *asymmetric* multiple player TCP connection game in which users have different RTTs.

### A. Continuous Kernel Symmetric TCP Connection Game

In this game, since all users have the same RTT, the per-connection sending rate from all users are all the same. Thus, an arbitrary player $i$ has the utility function given in (9) with $B$ replacing $B_i$ and all $R$ terms being canceled out. Note that $B$ is given in (3) and (4). $B$ is a function of $p$ which is in turn a function of $n_i, \forall i$. The strategy set for player $i$ is a real interval $S_i = [1, \infty)$. Since all players take a real-valued number as a feasible strategy and the identity of a player is not important, we call this game a continuous kernel symmetric [2] TCP connection game with utility function 2.

*Theorem 2:* There is a unique Nash Equilibrium (NE) $\mathbf{n}^*$ in the continuous kernel symmetric TCP connection game with utility function 2. At this NE, all players have the same number of connections. This NE is an interior point of the strategy space for $m < m_0$ and $\mathbf{n}^* = (1,1,...,1)$ for $m \geq m_0$. Threshold $m_0$ is the largest $m$ satisfying $m(1-p^*)B^* \leq C$ where $p^*$ and $B^*$ are respectively loss rate and per-connection sending rate at the NE.

*Proof:* The proof consists of two parts. In the first part, we prove that the unique Nash Equilibrium achieved at an interior point in the strategy space. In the second part, we present the results when the number of players is very large.

**Part 1:**

Each player $i$ tries to solve for its optimal strategy $n_i^*$, as a response to the strategies of all other players. Thus, if there is an interior point NE $\mathbf{n}^* = (n_1^*, ..., n_m^*)$, then it must be true that $\forall i, \partial U_i/\partial n_i^* = 0$, and $n_i^* = \text{argmax}_{n_i \in S_i} U_i(n_1^*, ..., n_i, ..., n_m^*)$.

In the following, we first introduce a fact indicating that the stationary point satisfying $\partial U_i/\partial n_i = 0$ is actually the maximum point if it is in $[1,\infty)$. Then we show that there is a unique $\mathbf{n}^*$ satisfying $\partial U_i/\partial n_i^* = 0, \forall i$.

First, we need to seek all vectors $\mathbf{n}^*$ satisfying a set of $m$ equations

$$\partial U_i/\partial n_i = 0, \forall i \in [1,2,3,...,m] \quad (10)$$

In the following, we first prove that if $\mathbf{n}^*$ exists, $n_i^* = n_j^*, \forall i, j$. Then, we show that such $\mathbf{n}^*$ is actually unique by proving that there is only one $p^*$ for which $n_i^* = n_j^*, \forall i, j$.

For an arbitrary player $i$, we have

$$\frac{\partial U_i}{\partial n_i} = \frac{Cn_{-i}}{(n_i+n_{-i})^2} - \frac{\beta}{\phi} - \frac{\beta n_i C\varphi}{(n_i+n_{-i})^2[(p-1)\varphi - \phi]} \quad (11)$$

where

$$\phi = \mu R\sqrt{p} + T_0\nu(p^{3/2} + 32p^{7/2}) = 1/B \quad (12)$$

$$\varphi = \frac{\mu R}{2\sqrt{p}} + T_0\nu(\frac{3}{2}\sqrt{p} + 112p^{5/2}) \quad (13)$$

and $n_{-i} = \sum_{k=1,k\neq i}^{m} n_k$ and $\varphi = d\phi/dp$.

**Fact 1: Best response of a player is unique and it is the stationary point if the stationary point is in $[1,\infty)$.** First we need to show that for any given $n_{-i}$, there is only one unique maximal point for $U_i$. In fact, player $i$ needs to solve the following equations to get a candidate for a maximal point $n_i^m$:

$$0 = \beta n_i - n_{-i}(1-p-\beta)[\varphi(1-p)/\phi + 1] \quad (14)$$

$$0 = C\phi - (n_i + n_{-i})(1-p) \quad (15)$$

where (14) is a simplification of $\partial U_i/\partial n_i = 0$. We can think of $n_i^m$ and $p$ are implicit functions of $n_{-i}$. We note that for any given $n_{-i}$, there is a unique pair of $(n_i^m, p)$ as the solution to (14) and (15). We can check that the unique stationary point $n_i^m$ obtained from this implicit function is indeed a maximal point. We can enlarge the domain of $U_i$ to be $(0,\infty)$, and notice that $n_i^m$ is also a unique stationary point for this enlarged domain. Since $U_i(0, n_{-i}) = 0$ and $\lim_{n_i\to\infty} U_i = -\infty$, they are not larger than $U_i(n_i^m, n_{-i})$ given that $n_i^m$ is indeed an interior point. Then we can conclude $n_i^m$ is indeed a maximal point in domain $(0,\infty)$. If it is still a stationary interior point in $[1,\infty)$, then it also must be a maximal point. Otherwise if it is smaller than 1, then maximal point is taken at the boundary 1, which is discussed in Part 2 of this proof. We can show that $n_i^m = f_i(n_{-i})$ and $p = f_p(n_{-i})$ are continuous functions[2] on domain $n_{-i} \in (0,\infty)$. In addition, from implicit function theorem, we know that they are continuously differentiable.

Now, we go on to prove the existence and uniqueness of NE. Consider two arbitrary players $i$ and $j$, and let

$$\delta_i n_i = \sum_{k=1,k\neq i}^{m} n_k; \quad \delta_j n_j = \sum_{k=1,k\neq j}^{m} n_k$$

When $\partial U_i/\partial n_i = \partial U_j/\partial n_j = 0$, we get

$$(1-p)[\delta_i + \beta\varphi/((1-p)\varphi+\phi)]/(1+\delta_i) - \beta{=}0 \quad (16)$$
$$(1-p)[\delta_j + \beta\varphi/((1-p)\varphi+\phi)]/(1+\delta_j) - \beta{=}0 \quad (17)$$

Let $\Delta = \beta\varphi/((1-p)\varphi+\phi)$, then combining (16) and (17) leads to

$$(\delta_i/(1+\delta_i) - \delta_j/(1+\delta_j)) + \triangle(1/(1+\delta_i) - 1/(1+\delta_j)) = 0 \quad (18)$$

For (18) to be true, we need either $\triangle = 1$ or $\delta_i = \delta_j$. We can show that $\Delta = 1$ cannot be true. We prove this by contradiction. Assume that it is true, then we can substitute it into (16), and get $\beta = 1-p$. Substituting $\beta = 1-p$ into $\Delta = 1$, we get $\phi = 0$. We know that $\phi = 0$ is impossible given that $p \in (0,1)$, thus $\Delta \neq 1$. Thus, the only possible solution is $\delta_i = \delta_j, \forall i,j$. This implies that $n_i^* = n_j^*$ at NE $\mathbf{n}^*$ if it exists.

In the following, we will prove that, when $n_i^* = n_j^*$, there exists a unique solution $p^*$ for (10). Then we can conclude that there is one unique $\mathbf{n}^*$.

Since at NE all players have the same number of connections, from (11), we obtain

$$(m-1)/\beta - m/(1-p) + \varphi/((1-p)\varphi+\phi) = 0 \quad (19)$$

Let $F(p)$ denote the LHS of (19). Ideally, solving equation (19) with $p$ as unknown, we can get loss rate at NE $p^*$. Then, substituting $p^*$ back into (4), we can get $\mathbf{n}^*$ as the number of connections of all users at NE. However, (19) contains several powers of $p$ such as $7/2$ and $5/2$, which makes it impossible to get an algebraic solution of $p$. Thus, in the following, we examine several properties of $F(p)$, and based on these properties make an inference about the behavior of NE. For a exact value of $p^*$ and $\mathbf{n}^*$ when given a network setting, we can use Matlab to numerically solve for them.

First, we will prove that (19) has only one solution for $p$ in $(0,1)$. We note that $F(p)$ is a continuous function, and the domain of $F(p)$ is $p \in (0,1)$, and $\lim_{p\to 0} F(p) > 0$ and $\lim_{p\to 1} F(p) < 0$. We claim that $F(p)$ is a strictly monotonic decreasing function. If this claim is true, then there must be a single solution $p^*$ for $F(p) = 0$. In the following, we prove this claim.

[2] $f_i(n_{-i})$ is referred to as the best response or reaction function in this paper.

Consider the derivative

$$\begin{aligned}\frac{dF}{dp} &= \frac{-m}{(1-p)^2} + \frac{\varphi'\phi}{[(1-p)\varphi+\phi]^2} \\ &< \frac{-1}{(1-p)^2} + \frac{\varphi'\phi}{[(1-p)\varphi+\phi]^2} \\ &= \frac{-\phi^2 - 2(1-p)\varphi\phi - (1-p)^2(\varphi^2 - \varphi'\phi)}{(1-p)^2[(1-p)\varphi+\phi]^2} \quad (20)\end{aligned}$$

Thus, to prove that $\frac{dF}{dp} < 0$, we only need to prove that $\varphi^2 > \varphi'\phi$. This can be easily proved. See Claim 1 in appendix for details.

If we substitute $p^*$ into (4), together with the result that all users have the same number of connection at $\mathbf{n}^*$, we conclude that there is only one NE for this game and it is symmetric. That is, $n_i^* = \text{argmax}_{n_i \in S_i} U_i(n_1^*, ..., n_i, ..., n_m^*)$ and $n_i^* = n^*, \forall i$.

**Part 2:**

Now, we will show that if $m \geq m_0$ where $m_0$ is the largest $m$ such that $m(1-p^*)B^* \leq C$, the NE is no longer an interior point of the strategy space. Instead, it is $\mathbf{n} = (1,1,...,1)$.

Recall (19), and let

$$F(p,m) = m(1/\beta - 1/(1-p)) - 1/\beta + \varphi/((1-p)\varphi+\phi) \quad (21)$$

Given a value of $m$, we can plot a curve for $F(p,m)$ with $p$ as x-axis and $F(p,m)$ as y-axis. Note that all these curves (with different $m$ values) all meet at a single common point $(p_0, F(p_0,m))$ with $p_0 = 1-\beta$. Take any $m_i$ and $m_j$ and check that $F(p,m_i) = F(p,m_j)$ implies $p = 1-\beta$.

Recall that $F(p,m)$ is a monotonic decreasing function of $p$, and $F(p^*,m) = 0$. Since $F(p_0,m) < 0$, so $p^*$ must be smaller than $p_0$. When $p < p_0$, we get

$$dF/dm = 1/\beta - 1/(1-p) = 1/(1-p_0) - 1/(1-p) > 0$$

Thus, as $m$ increases, $F(p,m)$ is strictly monotonic increasing, and since $F(p,m)$ is a monotonic decreasing function of $p$, thus, we see that as $m$ increases, for any given $F(p)$, $p$ will be strictly increasing towards $p_0$. Then, it must be also true that for $F(p^*) = 0$, as $m$ increases, $p$ approaches $p_0$.

Recall that at NE, we must have $(1-p^*)/\phi^* = C/(mn^*)$ Since all users must have at least one connection, i.e., $n^* \geq 1$, we have to make sure that $m(1-p^*)/\phi^* \leq C$. We know that as $m$ increases, $p^* \to p_0 = 1-\beta$, then $\phi^*$ as a function of $p^*$ also increases to $\phi_0$ (function of $p_0$). Thus, $(1-p^*)/\phi^*$ is bounded below by $(1-p_0)/\phi_0$. So, as $m$ becomes larger and larger, eventually, $m(1-p^*)/\phi^*$ will be larger than $C$, then NE is no longer an interior point. Let $m_0$ denote this threshold, then it is the largest $m$ satisfying

$$m(1-p^*)/\phi^* \leq C \quad (22)$$

Since it is difficult to obtain an explicit expression of $p$ as a function of $m$, we rely on numerical method to identify $m_0$. ∎

**Remarks:** Note that the utility function of this TCP connection game is not *concave* in general. But we can still get an alternative but non-constructive proof of the existence of NE of this TCP game by modifying the proof of a general result (Theorem

4.3, pp. 173 in [2]). We can replace the strict convexity of cost function in that proof with the uniqueness of best response in TCP game, then the existence of NE is immediately obtained.

**An illustrative example: NE as an interior point**. We present an example to illustrate an interior-point NE in a continuous kernel TCP connection game with utility function 2. There are two players competing for a bottleneck link with capacity $C = $ 10Mbps or 1250pkts/sec. They have the same RTT (240ms), and choose $\beta = 0.7$. To identify NE, we can plot the best response curves of these two players. For example, suppose we want to know the best response curve of player 2. Given a specific number $n_1$ of connections of player 1, player 2 uses the simplified PFTK TCP model to maximize its utility defined in (9). We use optimization toolbox in Matlab to solve this optimization problem to get $f_2(n_1)$ as the best response to $n_1$ and plot the best response curve $f_2(n_1)$. Similarly we can plot the best response curve $f_1(n_2)$ for player 1. The intersecting point of these two curves is the NE. Figure 1 shows the simulation result, and we see that there is indeed one unique NE. And, it can be easily verified that this NE is the same as that predicted in Theorem 2.

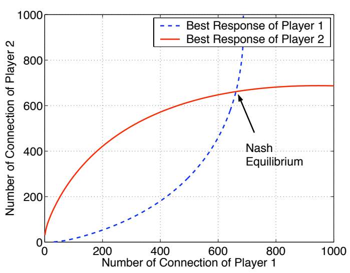


Fig. 1. Best response curves intersect at Nash Equilibrium.

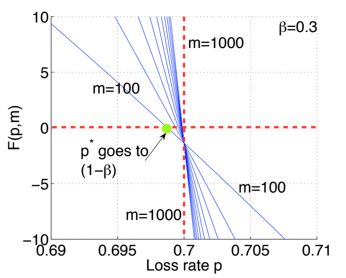


Fig. 2. $p^*$ as a function of $m$ when $\beta = 0.3$.

**An illustrative example: effects of population size**. In this example, we use the same network settings as before. We take $\beta = 0.3$, and vary the number of players $m$ from 100 to 1000. We plot all curves of $F(p, m)$ as a function of $p$ in Figure 2. All these curves intersect at $p = 0.7$ as predicted by Theorem 2. Note that $p^*$ are points where these curves intersecting with $F(p, m) = 0$. This figure shows that $n^*$ approaches $p_0 = 1 - \beta$ as $m$ increases. In Figure 3, for several different $\beta$ values, we plot the loss rate $p^*$ at NE when the number of users $m$ increases. As shown in this figure, for any given $\beta$, $p^*$ approaches to $1 - \beta$ when $m$ is not very large. However, when $m$ is very large, Figure 4 shows that $p$ grows more quickly, but still less than $\log m$. In summary, this example has verified NE's behavior predicted in part 2 of the proof of Theorem 2.

### *B. Loss of Efficiency*

As in Section III, we define the system optimization problem as minimizing the cost to maintain a busy bottleneck link. The loss of efficiency is defined as the ratio between the cost of the system at Nash Equilibrium and the system optimal cost. As before, the optimal system cost is $\Phi_{opt} = C/(1 - p_{opt})$. Then, we can naturally get the loss of efficiency of NE for this game.

*Corollary 1:* In the continuous kernel symmetric TCP connection game, the loss of efficiency is $L_{eff} = (1 - p_{opt})/(1 - p_{ne})$, and it is always larger than or equal to 1, but it is bounded.

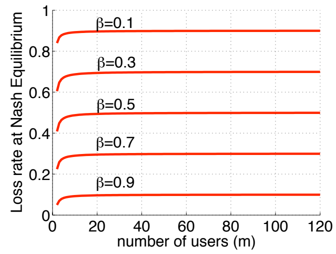


Fig. 3. $p^*$ as a function of $m$ for different $\beta$ values. $m$ varies from 2 to 120.

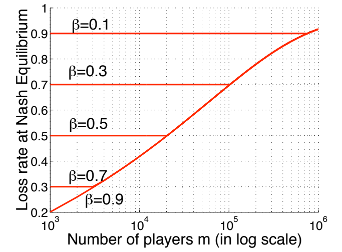


Fig. 4. $p^*$ as a function of $m$ for different $\beta$ values. $m$ varies from 1000 to 1000,000.

*Proof:* Consider that the system cost when the system is at NE:

$$\Phi_{ne} = B_{ne}(\sum_{i=1}^{m} n_i^*) = C/(1 - p*)$$

Then the loss of efficiency is given as

$$L_{eff} = \Phi_{ne}/\Phi_{opt} = (1 - p_{opt})/(1 - p_{ne}) \tag{23}$$

We note that the loss of efficiency is always larger than or equal to 1. Recall that $p_{opt}$ must satisfy $(1 - p)m/\phi = C$, and $p^*$ or $p_{ne}$ must satisfy $(1 - p^*)mn^*/\phi* = C$. Then we have

$$(1 - p)/\phi = n^*(1 - p^*)/\phi^*$$

Since $n^* \geq 1$, then $p^* \geq p_{opt}$. As $m$ increases, $n^*$ decreases. Before $n^*$ reaches 1, $p^*$ is strictly larger than $p_{opt}$, and after that, $p^* = p_{opt}$. Then, it must be true that the maximal efficiency loss occurs when $m$ is small.

Recall that $1 - p^* > \beta$ and $p_{opt}$ is an increasing function of $m$, thus, we have

$$L_{eff} = (1 - p_{opt})/(1 - p^*) < (1 - p_{opt,m=2})/\beta \tag{24}$$

This upper bound is a simple function of network parameters and user's aggressiveness coefficient $\beta$.

∎

**An illustrative example.** We take the network settings in the previous examples, and choose $\beta = 0.7$. In Figure 5, we plot the loss rate of NE and system optimal point. As predicted, the loss rate of the NE is always greater than and equal to the loss rate of system optimal point. When $m$ is sufficiently large, all users will just use one connection, then the trajectory of loss rate increase will be the same as that of system optimal point.

In Figure 6, for several different values of $\beta$, we plot the loss of efficiency of NE as a function of $m$. The solid lines are the actual loss of efficiency, and the dashed lines are the upper bound computed from (24). As expected, the loss of efficiency is always bounded by (24).

**Effects of user's aggressiveness** . $\beta$ represents a user's preference of how much effort he/she is willing to expend to get the desired goodput share of the bottleneck capacity. Intuitively, as $\beta$ gets larger, a user is likely to use less effort, then the number of connections at NE will be smaller, and loss rate of NE will be smaller.

This can be verified by looking at the relationship between loss rate of NE and $\beta$. Recall that $p^*$ is the solution to the following equation

$$F = (m - 1)/\beta - m/(1 - p) + \varphi/((1 - p)\varphi + \phi) = 0$$

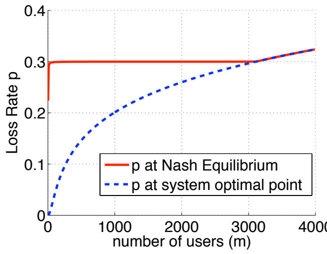


Fig. 5. Loss rate as a function of number of players $m$ when $\beta = 0.7$.

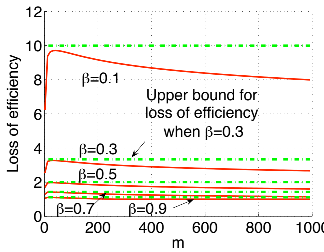


Fig. 6. Loss of efficiency of NE as a function of $m$ when $\beta = 0.7$.

Then

$$\partial p/\partial \beta = -F'_\beta / F'_p = \frac{(m-1)/\beta^2}{-\frac{m}{(1-p)^2} + \frac{\varphi' \phi}{((1-p)\varphi + \phi)^2}} < 0$$

This indicates that $p$ is a decreasing function of $\beta$. Since $n^*$ is a increasing function of $p$, we know that if $\beta$ increases, at the NE, users users open fewer and fewer connections.

We can expect the system cost (the aggregate costs of all users) to decrease as $\beta$ increases. Recall that the system cost at the NE is $f = \sum_{i=1}^{m} n_i^* B^* = C/(1-p^*)$. As $p^*$ is a decreasing function of $\beta$, we see that as $\beta$ becomes larger and larger (users becomes less and less aggressive), the required system effort will be smaller and smaller. In addition, since $p_{opt}$ is independent of $\beta$, from the above discussion on $p^*$, we see that $L_{eff}$ is a decreasing function of $\beta$. This is understandable, as users become less and less aggressive (larger $\beta$), NE will be more and more efficient.

As an example, we use the same network settings as before, and fix the number of users to be 100, but vary $\beta$ from 0.05 to 0.99. We expect that the loss rate at the NE to decrease as $\beta$ increases, and finally reach the loss rate of the system optimal point. This means that $\beta$ is so large that NE is no longer an interior point of the stategy space and all users are so conservative that everyone just opens one connection, as shown in Figure 7. On the other hand, as users becomes more and more aggressive ($\beta$ decreases), at the NE, users open more and more connections. This means that the whole system will need more and more effort to keep the same aggregate goodputs. Figure 8 shows the loss of efficiency decreases as users become less and less aggressive, as expected.

It would also be interesting to understand the situation where different users have different $\beta$. This will be a topic of our future research.

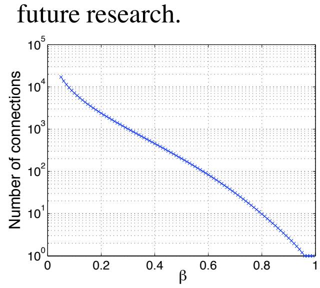


Fig. 7. Number of connections at NE as a function of $\beta$ when $m = 100$.

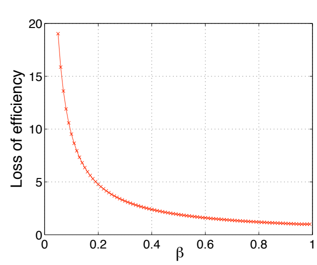


Fig. 8. Loss of efficiency at NE as a function of $\beta$ when $m = 100$.

## C. Stability of Nash Equilibrium

A natural question is whether the unique NE of this game is stable. As defined in [2], [7], if some player deviates from NE by an arbitrary amount in the feasible strategy set, and other players observe this and they adjust their responses optimally based on some fixed ordering of moves, and if this adjustment process converges to the original NE, then we say that this NE is *globally stable* with respect to this adjustment scheme. Correspondingly we can define local stability by restricting the stability domain to be some $\varepsilon$-neighborhood of NE. As pointed out in [2], stability condition is the same regardless of the adjustment schemes when there are only two players. Here, we only study the stability of a two-player TCP game, and this adjustment scheme is also called *best reply dynamics*. The response or reaction function of each player is determined by solving its optimization problem. Recall (14) and (15), from which we obtain $n_1(t+1) = f_1(n_2(t))$ and $n_2(t+1) = f_2(n_1(t))$ where $t$ indicates discrete time step.

Some sufficient conditions (needed for contraction mapping) for the global stability of NE is given in [7], but they are not satisfied in this game. Checking the best response curves in Figure 1 shows that contraction mapping is not true when the game is in state $(1, 1)$, namely both players using only one connection. Nevertheless, we are able to show that the unique NE is locally stable.

*Theorem 3:* In the two-player symmetric continuous kernel TCP connection game with utility function 2, the unique NE is locally stable.

The basic idea of the proof of this theorm is as follows. Since we can derive the exact form of the first derivative of the best reponse function and this derivative is continuous, we can check that the absolute value of this derivative is strictly smaller than 1 at the NE, and then we are able to show that locally at NE, there exists a contraction mapping driving the system to the NE if the deviation from the NE is sufficiently small, based on the Banach contraction mappling theorem [9] and the mean value theorem. The detailed proof is in appendix.

As for the global stability, we simulated a large range of network parameters, and found that $f_1(n_2)$ and $f_2(n_1)$ were always concave functions. Since the concavity of the best response function and the uniqueness of NE implies global stability (proved in appendix), we conjecture that the NE of TCP connection game is very likely to be globally stable.

## D. Extension 1: Integer TCP Connection Game

In this section, we consider a more practical TCP connection game in which each player can only choose a positive integer number of connections. That is, each player's strategy set is $\mathbb{N}$. We call this the *Integer TCP Connection Game*.

To study this integer TCP connection game, we use the results for the corresponding continuous kernel game. Note that if the pure strategy NE in the continuous game is an integer vector, then it must be a NE of the corresponding integer game. The more interesting case is where the pure strategy NE of the continuous game is a non-integer vector $\mathbf{n}^* = (n^*, n^*, ..., n^*)$. When this happens, we can approximate $n^*$ by taking floor $n_f^*$ and ceiling $n_c^*$ of $n^*$ to get $2^m$ integer-valued vectors.

In the following, first, we will show that, at these integer-valued vectors, the utility deviation of each player from the non-integer NE is bounded. As the number of users increases, this bound approaches zero. For convenience, we call such a vector an *approximate Nash Equilibrium*[3]. Next, we demonstrate that this integer game must have pure strategy NE(s) at some of these integer vectors given that some pathological cases never occur.

We start with a simple example of a two-player game. Figure 9 shows that in the continuous version of the game, the intersecting point of the best response curves of two players is a fraction number $(661.5, 661.5)$. If we restrict the strategy space of each player to be $\mathbb{N}$, we can approximate the continuous game NE with the floors and ceilings of the NE vector to get four vectors: $(661, 661)$, $(661, 662)$, $(662, 661)$ and $(662, 662)$.

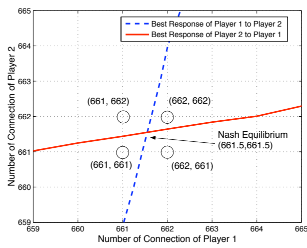


Fig. 9. An observed case where integer NE must exist.

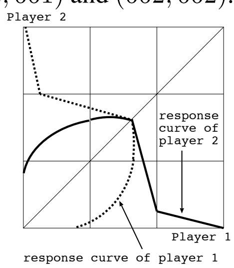


Fig. 10. A pathological case where no integer NE exists.

How much performance loss will be incurred due to this integer constraint on strategy space? Recall that for an arbitrary user $i$, its utility at the NE is given as

$$U_i^* = Cn^*/(mn^*) - \beta n^* B^* = C/m - \beta n^*/\phi^*$$

Among all such integer approximate NEs, the worst case *goodput loss* could happen when user $i$ opens $(n^* - 1)$ connections while all others open $(n^* + 1)$ connections. Let $G_l$ denote this goodput lower bound, then $G_l = C(n^* - 1)/(mn^* + m - 2)$. On the other hand, the worst case *cost increase* could happen when user $i$ opens $(n^* + 1)$ connections while all others open $(n^* - 1)$ connections. Then, the cost increase upper bound is $J_u = \beta(n^* + 1)/\phi_u$, where $\phi_u$ is the solution of

$$(1 - p_u)/\phi_u = C/(mn^* - m + 2)$$

Then, the upper bound of utility loss is given as

$$\begin{aligned}\Delta U =& G^* - G_l - \beta(J^* - J_u) \\ =& C\left(\frac{1}{m} - \frac{1 - 2/(n^* + 1)}{m - 2/(n^* + 1)}\right) + \beta\left[n^*\left(\frac{1}{\phi_u} - \frac{1}{\phi^*}\right) + \frac{1}{\phi_u}\right]\end{aligned}$$

Thus the utility loss of any user at any approximate integer NE is bounded by $\Delta U$, and this bound approaches zero as $m$ increases. The system performance loss at any approximate NE from that of the NE in continuous game also approaches zero.

Now the next question is whether these approximate NEs are possibly NEs in the integer game? For this question, we have the following result.

*Theorem 4:* In the integer symmetric TCP connection game with utility function 2, if the Nash Equilibrium of the corresponding continuous game is a non-integer vector $\mathbf{n}^* = (n^*, ..., n^*)$ with $n_f^*$ and $n_c^*$ denoting the floor and ceiling of $n^*$, then there must exist pure-strategy integer Nash Equilibrium provided that the following condition is satisfied: the best response for player $i, \forall i$ is always chosen from $n_f^*$ and $n_c^*$ given that all other players choose either $n_f^*$ and $n_c^*$.

*Proof:* We can form a new game $G$ by restricting each player have only two strategies $n_f^*$ and $n_c^*$. That is, if the original integer game has a positive integer strategy set $\mathbb{N}$ for each player, then in game $G$ we have $S_G = \{n_f^*, n_c^*\}$ for each player. We claim that any NE of $G$ must be a NE of the original integer game. If we pick any NE $\mathbf{n}_G^* = (n_{1,G}^*, ..., n_{m,G}^*)$ of game $G$ if it exists, we know that

$$U_i(n_{i,G}^*, \mathbf{n}_{-i,G}^*) \geq U_i(n_{i,G}, \mathbf{n}_{-i,G}^*); \quad n_{i,G} \in S_G$$

where $\mathbf{n}_{-i,G}^* = (n_1^*, ..., n_{i-1}^*, n_{i+1}^*, ..., n_m^*)$ and $n_{k,G}^* \in S_G, \forall k$. Based on the assumption, we immediately know that $U_i(n_{i,G}^*, \mathbf{n}_{-i,G}^*) \geq U_i(n_i, \mathbf{n}_{-i,G}^*)$ if $n_i \in \mathbb{N}, \forall i$. This implies $\mathbf{n}_G^*$ must be a NE in the original integer game.

In fact, there must be a NE in any multiple player symmetric game with each player having only two strategies [3]. Thus, we conclude that, in the original *integer* game, there must be a pure strategy NE achieved at some vectors formed by $n_f^*$ or $n_c^*$. ■

To illustrate this theorem, we recall Figure 9 and observe that when player 2 chooses 662 as its strategy, the best *integer* response of player 1 must be either 661 or 662, since for each strategy of player 2, player 1 has a unique best response and every other strategy monotonically decreases in utility as they get away from the best response (recall that there is only one unique interior-point maximum for utility function 2). Similarly we can have the following arguments: when player 2 chooses 661, the best *integer* response of player 1 must be either 661 or 662; when player 1 chooses 661 or 662, the best *integer* response of player 2 must be chosen from 661 and 662. This observation is exactly the condition required for Theorem 4.

There is some pathological case in which there could be no pure strategy NE. Figure 10 shows such an example. If we assume that the integer closest to the response curve is the integer with the highest uility, then, we can see that the game shown in this figure has no pure strategy integer NE even though there is a fractional NE in the continuous game. Actually, we never saw this case in our simulations.

Since the condition required for Theorem 4 is satisfied in all of our simulations, we conjecture that pure strategy integer NE always exists for the integer TCP connection game. This will be a topic of our future research.

## E. Extension 2: Asymmetric TCP Connection Game

The *asymmetric* game differs from the previous *symmetric* game in that users have different Round Trip Time (RTT). For player $i$, we have the following utility function

$$U_i = (Cn_i/R_i)(n_i/R_i + \sum_{k=1, k\neq i}^{m} n_k/R_k) - \beta n_i B_i \quad (25)$$

where $B_i$ is given in (3) and (4).

*Theorem 5:* There is a unique Nash Equilibrium (NE) in the continuous kernel *asymmetric* TCP connection game. This NE is an interior point of the strategy space given that the number of users is not larger than $m_0$ given in (26). At this interior-point NE, for any two players $i$ and $j$, we have $n_i^*/n_j^* = R_i/R_j$.

[3]Note, these approximate integer Nash Equilibria are different from $\varepsilon$-Nash Equilibrium defined in [2].

Since this proof is very similar to that of Theorem 2, we only sketch the basic idea as follows. First we need to set $\bar{\phi} = \mu\sqrt{p} + 4\nu(p^{3/2} + 32p^{7/2})$, $\phi_i = R_i\bar{\phi}$ and $\varphi_i = R_i\bar{\varphi}$. Then following a similar procedure as in the proof of Theorem 2, we can derive $\partial U_i/\partial n_i$ and $\partial U_j/\partial n_j$ for any two players, and let

$$\delta_i n_i/R_i = \sum_{k\neq i} n_k/R_k; \qquad \delta_j n_j/R_j = \sum_{k\neq j} n_k/R_k$$

Then we can show that $\delta_i = \delta_j$, thus, $n_i^*/n_j^* = R_i/R_j$.

If we sort the number of connections in an ascending order as $n_1^*, n_2^*, ..., n_m^*$, then as we increase the number of users, all $n_i^*$s will simultaneously decrease but maintain their relative proportional relationship. As $m$ reaches a large enough number where $n_1^*$ must be less than 1, then player 1 will just maintain one connection. From then on, as $m$ continues increasing, NE will no longer be an interior point. To maintain an interior-point NE, $m$ must be smaller than $m_0$, where $m_0$ is the largest $m$ satisfying

$$m(1-p^*)/(R_1\bar{\phi}^*) \leq C \qquad (26)$$

It is easy to see that at this interior-point NE, users have the same utility. And the efficiency loss of the NE is bounded.

## VI. Game 3

Recall that cost $\beta n_i B_i$ considered in Section V includes the cost to the whole system and the cost to a user at packet level. In this section, we introduce another term specific only to users and that accounts for the cost of maintaining open connections. Specifically, we use $\alpha n_i$ to represent the computation cost, and call $\alpha$ the computation power coefficient. Intuitively, the more connections a user opens, the more computation power he/she needs. $\alpha n_i$ can be thought of the resource requirement on CPU power, memory, etc. Thus, we can consider a more comprehensive utility function including both packet sending cost and computation resource limitation. We refer to this as *utility function 3*, given as:

$$U_i = (Cn_i)/(n_i + \sum_{k=1,k\neq i}^{m} n_k)(1-\beta/(1-p)) - \alpha n_i \quad (27)$$

We refer to the TCP game with this utility function as *Game 3*.

*Theorem 6:* There is a unique Nash Equilibrium (NE) $n_\alpha^*$ in the continuous kernel symmetric TCP connection game with utility function 3. At this NE, all players have the same number of connections. This NE is an interior point of the strategy space for $m < m_{0,\alpha}$ and $\mathbf{n}_\alpha^* = (1, 1, ..., 1)$ for $m \geq m_{0,\alpha}$, where $m_{0,\alpha}$ is the largest $m$ such that $m(1-p_\alpha^*)/\phi_\alpha^* \leq C$, and $p_\alpha^*$ is the loss rate at the NE.

Following a similar procedure in the proof of Theorem 2, we can prove that there is a unique NE $\mathbf{n}_\alpha^* = (n_\alpha^*, n_\alpha^*, ..., n_\alpha^*)$. See Appendix X-D for details. Since at the NE, we must have $(1-p_\alpha^*)/\phi_\alpha^* = C/(mn_\alpha^*)$, and since all users must have at least one connection, i.e., $n_\alpha^* \geq 1$, we have to make sure that $m \leq m_{0,\alpha}$ where $m_{0,\alpha}$ is the largest $m$ such that $m(1-p_\alpha^*)/\phi_\alpha^* \leq C$. We need to rely on numerical method to identify $m_{0,\alpha}$. Similar to Theorem 2, we have $p_\alpha^* < p_{0,\alpha}$ where $p_{0,\alpha}$ is the solution of $1 - p_{0,\alpha} = \alpha\phi_{0,\alpha} + \beta$. Thus, $1 - p_\alpha^* > \beta$. And we know that as $m$ increases, $p^* \to p_{0,\alpha}$, then $\phi_\alpha^*$ as a function of $p_\alpha^*$ also increases to $\phi_{0,\alpha}$ (function of $p_{0,\alpha}$). Thus, $(1-p_\alpha^*)/\phi_\alpha^*$ is bounded. So, when $m$ becomes larger and larger, eventually, $m(1-p_\alpha^*)/\phi_\alpha^*$ will be larger than $C$, which means that all users only use one connection at the NE.

**Comparison between Game 2 and Game 3**.

Since $\alpha$ represents user's computation power limitation, introducing $\alpha$ will make users more conservative. Thus, we might expect that at the NE of Game 3, users will open fewer number of connections than at the NE of Game 2. And, as the number of users increases, users will be more quickly to tend to open just one connection in Game 3. This intuition is formalized in the following lemma.

*Lemma 2:* The interior-point Nash Equilibrium (NE) of Game 3 will give a lower loss rate and smaller number of connections than the NE of Game 2. And, as the number of users increases, an interior-point NE of Game 3 will more quickly become the boundary NE $(1, 1, .., 1)$.

*Proof:* Recall that at NE, $p^*$ of Game 2 must satisfy $F(p,m) = 0$ given in (21). Similarly, for Game 3, $p_\alpha^*$ must satisfy

$$F_\alpha(p,m) = m(\frac{1}{\beta} - \frac{1}{1-p} - \frac{\alpha}{\beta}\frac{\phi}{1-p}) - \frac{1}{\beta} + \frac{\varphi}{(1-p)\varphi + \phi} = 0$$

We can compare $p_\alpha^*$ and $p^*$. First, note that, $\lim_{p\to 0} F_\alpha(p,m) = \lim_{p\to 0} F(p,m)$, and $F_\alpha(p_{0,\alpha}, m) = F(p_0, m) < 0$ where $p_{0,\alpha}$ and $p_0$ are the limits for $p^*$ (as $m$ increases) in Game 3 and Game 2 respectively. Second, we have $p_{0,\alpha} < p_0$, $p_\alpha^* < p_{0,\alpha}$ and $p^* < p_0$. Third, both $F_\alpha$ and $F$ are strictly monotonic decreasing function of $p$, and $F_\alpha(p)$ decreasing faster than $F(p)$. Thus, it must be true that $p_\alpha^* < p^*$ and $\mathbf{n}_\alpha^* < \mathbf{n}^*$. ■

**Loss of Efficiency.**

As before, the loss of efficiency of Game 3 is $L_{eff} = (1 - p_{opt})/(1-p_{ne,\alpha})$. Similar to Game 2, the loss of efficiency is always larger than or equal to 1, but it is upper-bounded. Recall that $1 - p_\alpha^* > \beta$. And since $p_{opt}$ is an increasing function of $m$, thus, we have

$$L_{eff} = (1-p_{opt})/(1-p_\alpha^*) < (1-p_{opt,m=2})/\beta$$

Even though this upper bound is the same as that of Game 2, we see that the actual efficiency loss of this game is smaller than that of Game 2 since $p_\alpha^* < p^*$.

The findings in this section again indicate that we might not expect large efficiency loss or congestion collapse in reality.

**Users with different computation power.**

We might as well be interested in the case where users have different computation power. Then, we can represent a user's utility function as: $U_i = n_i C/(n_i + \sum_{k=1,k\neq i}^{m} n_k) - \alpha_i n_i$ where $\alpha_i \neq \alpha_j, \forall i \neq j$. For this game, we have the following result. Detailed proof is in Appendix X-E.

*Theorem 7:* In the continuous kernel multiple player TCP connection game with users having different computation power, when $m < m_0$, there exists an interior-point NE, where

$m_0$ is the largest $m$ such that $\frac{C\phi^*(\sum \alpha_k-(m-1)\alpha_1)}{(1-p^*)\sum \alpha_k} \geq 1$. At this NE, the more powerful user will have more connections and higher goodput and utility.

## VII. NS Simulations

We use NS simulations in this section to verify the analytical results derived from previous sections. We consider a single bottleneck link with capacity 10Mbps or 1250pkt/sec, competed by users who are allowed to open multiple concurrent connections. Due to space limitation, we only present here an example simulation result on utility function 2.

**Checking PFTK Model with NS Simulations.** First, we show to what extent the simplified PFTK model captures the TCP behavior observed in NS simulations. Figure 11 and Figure 12 show respectively the comparison of loss rate and goodput among those measured in NS simulation, estimated by simplified PFTK model [13], and estimated by Square-Root-P model [12]. In Figure 11, to compute the estimated loss rate $p$ of the simplified PFTK TCP model, we numerically solve for $p$ by using the measured TCP sending rate $B$ in NS. Similarly, we use $B$ to solve for $p$ for Square-Root P model. Figure 11 shows that Square-Root P model is completely useless when the number of connections gets large. The simplified PFTK TCP model gives a good estimate of loss rate. In addition, Figure 12 shows that the simplified PFTK TCP gives a very good estimate of measured per-connection goodput.

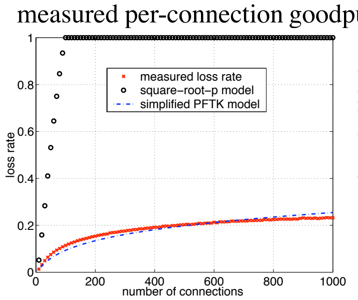

Fig. 11. Loss rate comparison.

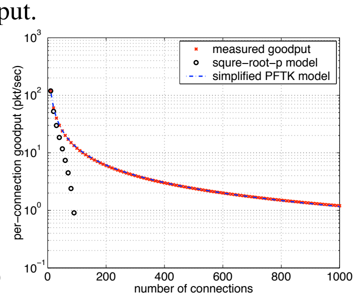

Fig. 12. Goodput (pkt/sec) comparison.

**Measuring Nash Equilibrium from NS Simulations.** Figure 13 illustrates the existence of a unique Nash Equilibrium observed in NS simulation of a two-player symmetric TCP connection game. Both users have the same two-way propagation delay 40ms. The bottleneck link queue is a RED queue with a target queuing delay 10ms. Each user uses utility function 2 with aggressive coefficient $\beta = 0.8$. Figure 14 shows that predicted NE by our analysis is very close to the one observed by NS simultion in Figure 13.

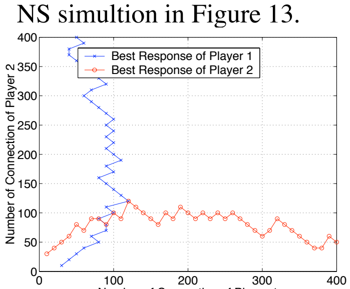

Fig. 13. Nash Equilibrium observed in NS simulation. Utility function 2 with $\beta = 0.8$.

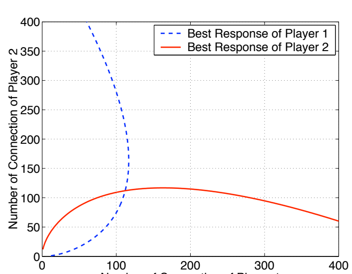

Fig. 14. Nash Equilibrium computed by PFTK model. Utility function 2 with $\beta = 0.8$.

## VIII. Conclusions

In this paper, we studied a particular selfish behavior of TCP users in which users are allowed to open multiple concurrent connections to maximize their individual goodputs or other utilities. Since such a strategic usage of TCP is easy to realize through some software agents (e.g.,FlashGet [4]) and its potential impact could be harmful [8], it is important to understand its implication on the stability of the Internet. To this end, we modeled users as players in a non-cooperative non-zero-sum game competing for the capacity of a single bottleneck link, referred to as the *TCP Connection Game*. We use different utility functions to model different user behaviors, and use the well known PFTK TCP model [13] as the basis of our analysis.

We demonstrated analytically that there was always a unique Nash Equilibrium (NE) in all variants of TCP connection games we studied. Our results indicate that, at the NE, the loss of efficiency or price of anarchy can be arbitrarily large if users have no resource limitations and are not socially responsible. However, if either of these two factors are considered, the efficiency loss is bounded. And in game 2, the game capturing the user's cost and social responsibility, we have also shown that the unique NE is always locally stable and is globally stable if the game satisfying certain conditions which are actually observed in all our simulations. And the integer NEs always exist when users are restricted to use only an integer number of connections if some pathological case never occurs. In summary, the general message is that this selfish usage of TCP might not lead to the congestion collapse.

## IX. Acknowledgments

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## X. Appendix

### A. Appendix 1

*Claim 1:* In the continuous kernel symmetric TCP connection game with utility function 2, we have $\varphi^2 > \varphi'\phi$, where

$$\phi = \mu R\sqrt{p} + T_0\nu(p^{3/2} + 32p^{7/2}) \tag{28}$$

$$\varphi = \frac{\mu R}{2\sqrt{p}} + T_0\nu(\frac{3}{2}\sqrt{p} + 112p^{5/2}) \tag{29}$$

and $T_0 = 4R$; $\mu = \sqrt{2b/3}$; $\nu = 3/2\sqrt{3b/2}$, and $\varphi' = d\varphi/dp$.

*Proof:*

First, we have

$$\begin{aligned}\varphi^2 =& \mu^2R^2/4p^{-1} + 6\mu\nu R^2 + 36R^2\nu^2 p \\ &+ 448\mu\nu R^2p^2 + 5376R^2\nu^2p^3 + 16\cdot 112^2R^2\nu^2p^5 \\ \varphi'\phi =& (-1/4)\mu^2R^2p^{-1} + 2R^2\mu\nu + 12R^2\nu^2p \\ &+ 1119\mu\nu R^2p^2 + 4864R^2\nu^2p^3 + 143360R^2\nu^2p^5\end{aligned}$$

Then,

$$\begin{aligned}\varphi^2 - \varphi'\phi =& \frac{1}{2}\mu^2R^2p^{-1} + 4\mu\nu R^2 + 24R^2\nu^2p - 671\mu\nu R^2p^2 \\ &+ 512R^2\nu^2p^3 + 57344R^2\nu^2p^5 \\ >& \frac{1}{2}\mu^2R^2p^{-1} + 4\mu\nu R^2 - 671\mu\nu R^2p^2 + 57344R^2\nu^2p^5\end{aligned}$$

Let $f(p)$ denote RHS of this inequality. We need to show that $f(p) > 0$. First, note that $\lim_{p\to 0} f(p) = \infty$, and $\lim_{p\to 1} f(p) > 0$, so, if we can show that all interior maximal and minimal values of $f(p)$ are larger than zero, we can conclude that $f(p) > 0$.

Now consider

$$\frac{df}{dp} = bR^2((-1/3)p^{-2} - 2013p + 967680p^4)$$

All interior maximal and minimal points of $f(p)$ must satisfy $\frac{df}{dp} = 0$, which implies

$$-1/3 - 2013p^3 + 967680p^6 = 0$$

Let $x = p^3$, then we can solve

$$-1/3 - 2013x + 967680x^2 = 0$$

There are only two solutions, $x_1 = 0.0022$, $x_2 = -1.5417e-04$, since $p \in (0,1)$, then we know that only $x_1$ is a valid solution, so, $p = x_1^{1/3} = 0.1301$. At this point, we have $f(p) = 7.4605 > 0$. So far, we have shown that values of $f(p)$ at boundary points and the interior maximal(or minimal) points all larger than zero, thus we conclude that since $\forall p \in (0,1)$, we have $f(p) > 0$ and $\varphi^2 - \varphi'\phi > 0$. ∎

### B. Appendix 2

*Theorem 8:* In the continuous kernel multiple player symmetric TCP connection game with utility function 2, the best response (or reaction) function of any player is well defined and continuously differentiable.

*Proof:* Recall the notations introduced in the proof of Theorem 2, we need to show that for any given $n_{-i}$, there is only one unique optimal point for $U_i$. In fact, player $i$ needs to solve the following equations to get an optimal point $n_i^m$:

$$0 = \beta n_i - n_{-i}(1-p-\beta)[\varphi(1-p)/\phi + 1] \tag{30}$$

$$0 = C\phi - (n_i + n_{-i})(1-p) \tag{31}$$

where (30) is a simplification of $\partial U_i/\partial n_i = 0$.

In the following three steps, we will show that there exist well defined functions $n_i = f(n_{-i})$ and $p = g(n_{-i})$ for all $n_i \in (0,\infty)$. Then we will prove that these two functions are continuously differentiable by using implicit function theorem. Finally we will prove that the stationary point obtained from solving equations (30) and (31) is actually the maximal point.

Note that here we work in the enlarged domain $n_i \in (0,\infty), \forall i$, and the result here can be directly applied to the original domain $n_i \in [1,\infty), \forall i$.

To simplify notation, lets consider a two-player game and check for the existence and continuous differentiability of the reaction function of player 1.

Let

$$\begin{aligned}F(n_2,n_1,p) =& \beta n_1 - n_2(1-p-\beta)[\varphi(1-p)/\phi + 1] \\ G(n_2,n_1,p) =& C\phi - (n_1+n_2)(1-p)\end{aligned}$$

The domain $\mathbb{D}$ of $F$ and $G$ is

$$n_2 \in (0,\infty), n_1 \in (0,\infty), p \in (0,1)$$

**Step 1: Uniqueness of $(n_1, p)$ given $n_2$.**

We want to show that $\forall n_2 \in (0,\infty)$, there always exists a unique pair $(n_1,p)$ such that $F(n_2,n_1,p) = 0$ and $G(n_2,n_1,p) = 0$ with $(n_2,n_1,p) \in \mathbb{D}$.

We can transform (30) and (31) into

$$n_1 = \frac{1}{\beta}n_2(1-p-\beta)[\varphi(1-p)/\phi + 1] \tag{32}$$

$$n_1 = \frac{C\phi}{1-p} - n_2 \tag{33}$$

Given any feasible $n_2$, if we can find a unique intersecting point of the curves defined by function (32) and (33), then this implies there is a unique pair $(n_1,p)$ such that $F(n_2,n_1,p) = 0$ and $G(n_2,n_1,p) = 0$ are satisfied simultaneously.

To prove that there exists a unique intersecting point is equivalent to proving $H(p) = 0$ has a unique solution $p$, with $H(p)$ is given as (by combining (32) and (33))

$$H(p) = \frac{C\phi}{1-p} - n_2 - \frac{1}{\beta}n_2(1-p-\beta)[\frac{\varphi}{\phi}(1-p) + 1]$$

Note that $H(p)$ is continuous in $p \in (0,1)$ and

$$\lim_{p\to 0} H = -\infty; \quad \lim_{p\to 1} H = +\infty$$

And,

$$\frac{\partial H}{\partial p} = \frac{C\varphi(1-p)+C\phi}{(1-p)^2} + \frac{1}{\beta}n_2[\frac{\varphi}{\phi}(1-p)+1] - \frac{1}{\beta}n_2(1-p-\beta)[\frac{\varphi'\phi-\varphi^2}{\phi^2}(1-p)-\frac{\varphi}{\phi}] \quad (34)$$

Since $\varphi'\phi - \varphi^2 < 0$ (proved in Claim 1), we have $\frac{\partial H}{\partial p} > 0$. This implies that $H(p)$ is monotonically increasing from $-\infty$ to $\infty$. Thus, we know there must exist a unique $p$ for $H(p) = 0$. Thus, we have proved that, for any $n_2 \in (0, \infty)$, there exists a unique pair of $(n_2, p)$ satisfying (30) and (31).

**Step 2: Continuous differentiability of reaction function.**

In domain $\mathbb{D}$, take any point $n_{2,0}$, then based on the above result, we can find a unique pair $n_{1,0}, p_0$ such that $F(n_{2,0}, n_{1,0}, p_0) = 0$ and $G(n_{2,0}, n_{1,0}, p_0) = 0$. Then we can take some open set $N$ containing this point and $N \subset \mathbb{D}$, based on implicit function theorem, if all the following conditions are satisfied in $N$:

1) $F$ and $G$ and all their partial derivatives w.r.t. $n_2, n_1, p$ are continuous;
2) $F(n_{2,0}, n_{1,0}, p_0) = 0$ and $G(n_{2,0}, n_{1,0}, p_0) = 0$;
3) $J(n_{2,0}, n_{1,0}, p_0) \neq 0$ (defined below).

then we know that there exists an open interval $I$ containing $n_{2,0}$ and two well defined functions

$$n_1 = f(n_2) : I \rightarrow (0, \infty)$$

$$p = g(n_2) : I \rightarrow (0, 1)$$

such that:

$$F(n_2, f(n_2), g(n_2)) = 0, G(n_2, f(n_2), g(n_2)) = 0$$

and $n_{1,0} = f(n_{2,0})$ and $p_0 = g(n_{2,0})$. And in this interval $I$, we have

1) $f$ and $g$ are continuous;
2) derivatives $df/dn_2$ and $dg/dn_2$ are continuously differentiable.

Since the above result is true for all $n_2 \in (0, \infty)$, we know that the reaction function $n_1 = f(n_2)$ is continuously differentiable in its domain $n_2 \in (0, \infty)$.

We now show that all above three conditions are satisfied.

The first and second condition are obviously true. Now, we will prove the third condition. That is, we need to show the following is true.

$$J(n_2, p, n_1) = \begin{vmatrix} F'_p \, F'_{n_1} \\ G'_p G'_{n_1} \end{vmatrix} \neq 0$$

We can get

$$J = \begin{vmatrix} \delta - n_2(1-p-\beta)\delta'_p & \beta \\ C\varphi + (n_1+n_2) & -(1-p) \end{vmatrix} = -(\varphi(1-p)/\phi + 1 - n_2(1-p-\beta)\delta'_p)(1-p) - \beta(C\varphi + n_1 + n_2)$$

where

$$\delta = \varphi(1-p)/\phi + 1$$

and

$$\delta'_p = (1/\phi^2)(\varphi'\phi(1-p) - \varphi\phi - \varphi^2(1-p))$$

Note that $\delta'_p < 0$ due to Claim 1 and $\phi > 0, \varphi > 0$ in domain $\mathbb{D}$. Then, it follows that $J < 0$ everywhere in its domain $\mathbb{D}$.

Note, we can prove the reaction function is continuous by modifying the proof for the continuity of reaction function in [2] (p. 173, Theorem 4.3).

In summary, we have proved all three conditions, thus we can conclude that there exist two well defined and continuously differentiable functions $n_1 = f(n_2)$ and $p = g(n_2)$ for all $n_2 \in (0, \infty)$.

**Step 3: Stationary point is the maximal point when it is an interior point of strategy set.**

We now show that the unique stationary point $n_1$, obtained from above by solving equations (30) and (31) for any given $n_2$, is indeed a maximal point.

This is because that if that stationary point is a candidate optimal point, we must have $1 - p > \beta$.

To see why this is true, recall that at the stationary point, we have

$$n_1 = 1/\beta n_2(1-p-\beta)[(1-p)\varphi/\phi + 1]$$

plus one constraint

$$(1-p)/\phi = C/(n_1+n_2)$$

We let $F = n_1 - 1/\beta n_2(1-p-\beta)[(1-p)\psi/\phi + 1]$ and $G = (1-p)/\phi - C/(n_1+n_2)$. Then we solve these two functions $F = 0$ and $G = 0$ to get solutions for $p$ and $n_1$ given $n_2$.

$F = 0$ is actually

$$n_1 = 1/\beta n_2(1-p-\beta)[(1-p)\psi/\phi + 1]$$

In this formula, the only possible term that can be smaller than 0 is $(1-p-\beta)$. Then whenever we get $p$ such that $(1-p-\beta) < 0$, it must be necessarily true that $n_1 < 0$. However, because of the definition of our game, $n_1$ can only take a value from $[1, \infty)$, thus, $n_1$ is then forced to be 1, namely we no longer take that stationary point as a candidate for our optimal point. Thus, if this occurs, there is no point to look at that stationary point.

On the other hand, if this does not happen, namely we still take that stationary point as our candidate for optimal point, then it must be true that $1 - p > \beta$. Thus, if we examine whether it is a maximum or minimum or inflection point, we need to keep in mind the fact that $1 - p > \beta$. Then, we can check at this point, $U_1 > 0$, that is,

$$U_1(n_1) = (1 - \beta/(1-p))Cn_1/(n_1+n_2) > 0$$

If we compare it with two boundary values (given $n_2$):

$$\lim_{n_1 \rightarrow 0} U_1 = 0; \lim_{n_1 \rightarrow \infty} U_1 = -\infty$$

So, $U_1(n_1)$ is larger than these two boundary values in the enlarged domain $(0, \infty)$, we then see that this stationary point is indeed a maximal point (note, we already proved the uniqueness of this stationary point and the function is continuous.) If we restrict the domain to be $[1, \infty)$ and if this stationary is still a candidate for optimal point, then we see it is also a maximal point.

■

### C. Stability Proof

First, we prove the local stability of game 2. Theorem 3 is restated as: In the two-player symmetric continuous kernel TCP connection game with utility function 2, the unique NE is locally stable.

*Proof:* We prove this theorem by checking the derivative of reaction (or best response) function and using Banach contraction mapping theorem. This proof follows a similar argument in [7].

Recall that in Theorem 8, we have shown that the reaction function is continuously differentiable. Lets look at the derivative of the reaction function of player 1. It is given as the solution to the following equation arrays:

$$0=[n_2\delta - n_2(1-p-\beta)\delta'_p]\frac{dp}{dn_2} + \beta\frac{dn_1}{dn_2} - (1-p-\beta)\delta \tag{35}$$

$$0=[C\varphi + n_1 + n_2]\frac{dp}{dn_2} - (1-p)\frac{dn_1}{dn_2} - (1-p) \tag{36}$$

where $\delta = \varphi(1-p)/\phi + 1$ and $\delta'_p = (1/\phi^2)[\varphi'_p\phi(1-p) - \varphi\phi - \varphi^2(1-p)]$

Solving this equation array we can get $dp/dn_2$ and $dn_1/dn_2$. We are interested in $dn_1/dn_2$, which is

$$\frac{dn_1}{dn_2} = \frac{(1-p-\beta)\delta B - (1-p)A}{\beta B + (1-p)A}$$

where $A = n_2\delta - n_2(1-p-\beta)\delta'_p$ and $B = C\varphi + n_1 + n_2$.

Recall the proof of Theorem 2, and we know that at NE loss rate $p$ must satisfy

$$(1-p) - 2\beta + \frac{\varphi\beta(1-p)}{(1-p)\varphi + \phi} = 0$$

which can be transformed into

$$[(1-p) - \beta](1-p)\frac{\varphi}{\phi} = 2\beta - (1-p)$$

We can use this equation to evaluate the derivative of reaction function $n_1 = f(n_2)$ at NE to get

$$\frac{dn_1^*}{dn_2^*} = \frac{\beta B^* - (1-p^*)A^*}{\beta B^* + (1-p^*)A^*}$$

Notice that $0 \le |\frac{dn_1^*}{dn_2^*}| < 1$.

Because of symmetry of the game and the continuity of the derivative of reaction function, we know that there must exist an open set $N = (n^* - \epsilon, n^* + \epsilon) \times (n^* - \epsilon, n^* + \epsilon)$ containing $(n_1^*, n_2^*) \in N$ such that

$$0 \le |\frac{dn_1}{dn_2}| < 1; 0 \le |\frac{dn_2}{dn_1}| < 1; \forall (n_1, n_2) \in N$$

Now we can prove that: 1) if any player has any deviation from NE, the system will stay in $N$; 2) as long as the system state is in $N$, the best reply dynamics always converges to the NE. Then, local stability of NE is established.

We will check two cases, and in each case we can show that the above two results are actually true.

Since the stability condition is independent of adjustment schemes for a two-player game [2], it is sufficient to examine a serial adjustment scheme in which player 1 moves first, then followed by player 2.

First we assume $0 < |\frac{dn_1}{dn_2}| < 1$. Note here we assume that $|\frac{dn_1}{dn_2}| \neq 0$. Suppose player 2 deviates from NE by $0 < \eta \le \epsilon$, then because $0 \le |\frac{dn_1}{dn_2}| < 1$, best response move by player 1 leads to a new $n_1$ with $n_1 \in (n^* - \epsilon, n^* + \epsilon)$, namely, the system state is still in $N$. Because of the symmetry, we can say that as long as any deviation $\eta$ of any player from the NE is smaller than $\epsilon$, the system remains in $N$ in all the subsequent moves.

Now, we can show that the infinite sequence of moves is a contraction mapping. Take any $\bar{n}_1 \in N$, and define an operator

$$T : (n^* - \epsilon, n^* + \epsilon) \to (n^* - \epsilon, n^* + \epsilon),$$

$$T = (f_1 \circ f_2)(n_1) = f_1(f_2(n_1))$$

where $f_1$ and $f_2$ are the reaction functions of player 1 and player 2. To simplify notation, let

$$f'_1 = dn_1/dn_2; \qquad f'_2 = dn_2/dn_1$$

Because $f_1$ and $f_2$ are continuously differentiable, we know that $T'(\bar{n}_1) = f'_1(n_2)f'_2(\bar{n}_1)$ where $n_2 = f_2(\bar{n}_1)$. And since $0 < |f'_1(n_2)| < 1$ and $0 < |f'_2(n_1)| < 1$ when $(n_1, n_2) \in N$, by the mean value theorem, for any $n_{1,k}$ and $n_{1,m} \in N$, we have

$$|T(n_{1,k}) - T(n_{1,m})| \le |n_{1,k} - n_{1,m}| \sup_{0<\theta<1} |T'(n_{1,k} + \theta n_{1,m})| < |n_{1,k} - n_{1,m}| \tag{37}$$

Thus, $T$ defines a contraction mapping from a complete space $N$ into itself.

Next, we check the case where $|\frac{dn_1}{dn_2}|$ could be zero when $(n_1, n_2) \in N$. If $|\frac{dn_1}{dn_2}| = 0$ only occurs when $(n_1, n_2) = (n^*, n^*)$, then the above contraction mapping argument can also be applied. If $|\frac{dn_1}{dn_2}| = 0$ when $(n_1, n_2) \in M \subseteq N$ (occurs on more than one point), we can use simple argument to show the best-reply dynamics converges to NE. Without loss of generality, suppose player 1 deviates from $n^*$ to $n_{1,new}$ in $M$, since the best response of player 2 is still $n^*$ because of $|\frac{dn_2}{dn_1}| = 0, \forall (n_1, n_2) \in M$, then when player 1 moves again in the next round, it will choose $n^*$ again because this is the best response. Thus, the process converges back to NE in two steps.

■

*Theorem 9:* In the two-player symmetric continuous kernel TCP connection game with utility function 2, if the response function is concave, then the unique NE is globally stable.

*Proof:*

To study the stability of Nash Equilibrium or the convergence of best-reply dynamics, we need to think of the game playing process as a discrete time dynamic system. At each time step

$t$, each player will make a move. Since stability condition for a two-player game is independent of the ordering of movement of players [2], we can assume at each step $t$, player 1 makes the move first, then followed by player 2.

Whenever a player makes a move, it will follow its best response function. To simplify notations, let response functions be $n_1 = f_1(n_2)$ and $n_2 = f_2(n_1)$.

Since it is almost impossible to get the closed form of the $f_1$ and $f_2$. We will rely on the properties of these functions to make inference on the system.

Since this game has a unique NE (shown in Theorem 2), and $f_1$ and $f_2$ are concave (assumed in this proposition and demonstrated through simulations), we can show that the NE is globally stable or the best-reply dynamics always converges to the NE.

Let $n_0$ denote the number of connections where either $f_1$ or $f_2$ reaches maximum. Since either $f_1$ or $f_2$ is concave, then there are only three situations that could happen for NE: 1) NE is achieved before $f_1$ and $f_2$ reach maximum; 2) NE is achieved after $f_1$ and $f_2$ reach maximum; 3) NE is achieved when $f_1$ and $f_2$ reach maximum. We study these three cases in the following.

**Case 1**.

As shown in Figure 15, we can restrict ourself to two subregion $A_1$ and $A_2$ of the original space, since all other states outside $A_1$ and $A_2$ will fall into them after at most two moves. $A_1$ is formed by $\{(1, 1), (1, n^*), (n^*, n^*), (n^*, 1)\}$, and $A_2$ is formed by $\{(n^*, n^*), (n^*, n_0), (n_0, n^*), (n_0, n_0)\}$.

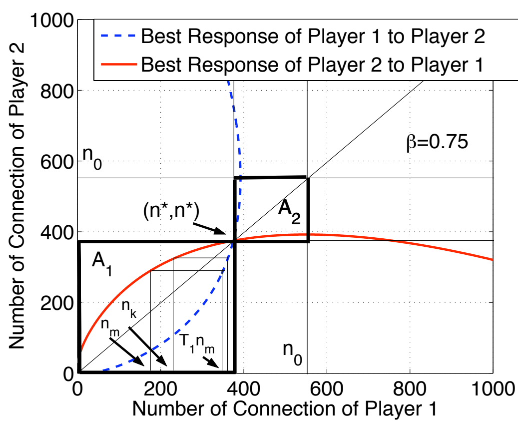


Fig. 15. Case 1. Maximum point $n_0$ of response function is larger than $n^*$.

We can show that if the system starts within $A_1$ then it will converges to NE, and the same is true for $A_2$.

In region $A_1$, by assumption, we know that $f_1$ and $f_2$ are both increasing and bijective functions, and $f_2(n_1) > f_1^{-1}(n_1)$ and $f_1(n_2) > f_2^{-1}(n_2)$.

Now suppose that we look at player 1 at step $t + 1$, we have

$$n_1(t + 1) = (f_1 \circ f_2)(n_1(t)) \quad (38)$$
$$= f_1(f_2(n_1(t))) \quad (39)$$
$$> f_2^{-1}(f_2(n_1(t))) = n_1(t) \quad (40)$$

Thus, $n_1(t)$ is strictly increasing. Eventually, $n_1(t)$ will enter the region where the derivative of $f_2(n_1)$ will be smaller than 1 because $f_2$ has to go across $n_2 = n_1$ line in order to have the unique symmetric NE. Let $n_c$ be the point where this region starts. Let $T_1 = f_1 \circ f_2$.

Now we will show that once $n_1(t) > n_c$, $T_1$ will be a contraction mapping, then based on the Banach contraction mapping theorem [9], $n_1(t)$ will converges to a fixed point where $T_1(n_1) = n_1$, namely, the Nash Equilibrium.

When $n_1(t) > n_c$, the derivative of $f_2(n_1)$ is smaller than 1, correspondingly, the derivative of $f_1(n_1)$ is smaller than 1 due to symmetry, thus, the derivative of $f_1^{-1}(n_1)$ is larger than 1. And since $T_1(n_1)$ is an increasing function and $f_1^{-1}(n_1)$ is a convex and increasing function, then at $T_1(n_1)$, $f_1(n_1)$ also has a derivative larger than 1. Thus, take any two points $n_k \in (n_c, n^*)$ and $n_m \in (n_c, n^*)$, and without loss of generality, assume $n_k > n_m$, then we have

$$\frac{f_2(n_k) - f_2(n_m)}{n_k - n_m} < \frac{f_1^{-1}(T_1(n_k)) - f_1^{-1}(T_1(n_m))}{T_1(n_k) - T_1(n_m)}$$

Since

$$f_2(n_k) - f_2(n_m) = f_1^{-1}(T_1(n_k)) - f_1^{-1}(T_1(n_m))$$

thus, $T_1(n_k) - T_1(n_m) < n_k - n_m$, namely, operator $T_1$ defines a contraction mapping on the close subset $[n_c, n^*]$ of Banach space $\mathbb{R}$. Thus, Banach contraction mapping theorem can be applied here. It naturally follows that the the best-reply dynamics (game playing process) converges to the NE.

Similarly, we can show that if the system starts in region $A_2$, then $n_1$ and $n_2$ strictly decreases as time goes by, and they converges to $n^*$.

In summary, this NE is globally stable.

**Case 2**.

As shown in Figure 16, we can restrict ourself to two subregion $A_1$ and $A_2$ of the original space, since all other states outside $A_1$ and $A_2$ will fall into them after at most three moves. $A_1$ is formed by $\{(n_0, n^*), (n^*, n^*), (n^*, \infty), (n_0, \infty)\}$, and $A_2$ is formed by $\{(n^*, n^*), (n^*, \infty), (\infty, n_0), (n^*, n_0)\}$.

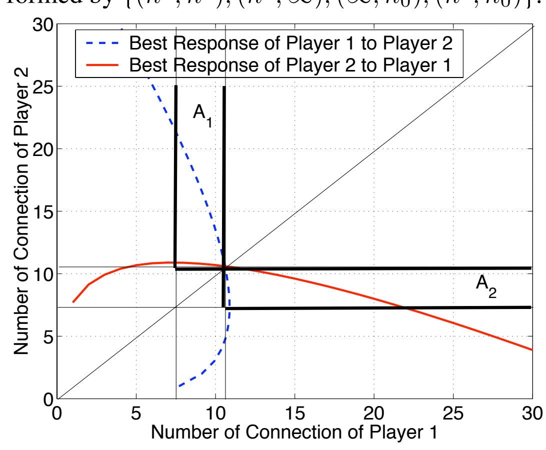


Fig. 16. Case 2. Maximum point $n_0$ of response function is smaller than $n^*$.

Following a similar line of reasoning to **Case 1**, we can show that if the system starts within $A_1$ then it will converge to NE, and the same is true for $A_2$.

By assumption and the uniqueness of NE, $f_1$ and $f_2$ are both decreasing functions in $A_1$, and $f_1(n_2) > f_2^{-1}(n_2)$; $f_1^{-1}(n_1) > f_2(n_2)$ and $df_1^{-1}(n_2)/dn_2 < df_2(n_1)/dn_2 < 0$.

Again, it is sufficient to examine only player 1. At step $t+1$, we have

$$n_1(t+1) = (f_1 \circ f_2)(n_1(t)) \tag{41}$$
$$= f_1(f_2(n_1(t))) \tag{42}$$
$$> f_2^{-1}(f_2(n_1(t))) = n_1(t) \tag{43}$$

Thus, $n_1(t)$ is strictly increasing.

Again, let $T_1 = f_1 \circ f_2$, we can show that $T_1$ is a contraction mapping. Take any two points $n_k \in (n_0, n^*)$ and $n_m \in (n_0, n^*)$, and without loss of generality, assume $n_k > n_m$, then we have

$$0 < \frac{f_2(n_m) - f_2(n_k)}{n_k - n_m} < \frac{f_1^{-1}(T_1(n_m)) - f_1^{-1}(T_1(n_k))}{T_1(n_k) - T_1(n_m)}$$

Since

$$f_2(n_m) - f_2(n_k) = f_1^{-1}(T_1(n_m)) - f_1^{-1}(T_1(n_k))$$

thus, $T_1(n_k) - T_1(n_m) < n_k - n_m$, namely, operator $T_1$ defines a contraction mapping on the close subset $[n_0, n^*]$ of Banach space $\mathbb{R}$. Thus, by applying Banach contraction mapping theorem, we know that the best-reply dynamics (game playing process) converges to the NE. Similarly, we can show that if the system starts in region $A_2$, best-reply dynamics also converges to NE. Thus, this NE is globally stable.

**Case 3**.

As shown in Figure 17, we can restrict ourself to two subregion $A$ of the original space, since all other states outside $A$ will fall into them after at most two moves. $A$ is formed by $\{(1, n^*), (n^*, n^*), (n^*, 1), (1, 1)\}$. Note that $A$ is the same as $A_1$ in **Case 1**, thus, convergence and stability is true.

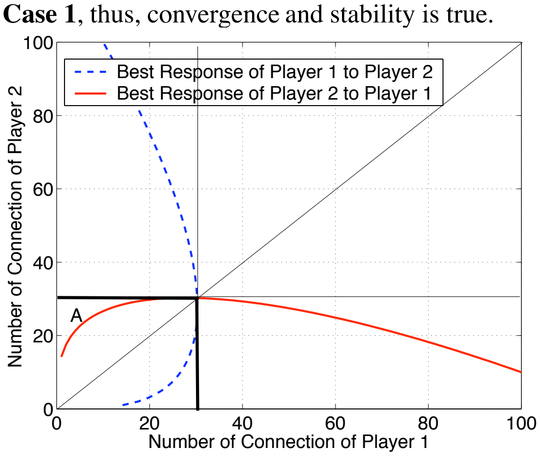


Fig. 17. Case 3. Maximum point $n_0$ of response function equals $n^*$.

■

Note, in our proof of Proposition 9, we do not use anything specific to TCP connection game. Thus, we have the following result for a more general game.

*Proposition 1:* In the two-player symmetric continuous kernel game with concave response function and NE being unique and symmetric, then the NE is globally stable.

**Remarks.** We use a similar argument to the proof of uniqueness and stability of NE for a two-player game in Theorem 1 in [7]. Li and Basar in [7] dealt with general cost function and the sufficient conditions given there are very difficult to check for our TCP game. They proved the existence, uniqueness and stability of NE all together using Banach contraction mapping by giving stringent sufficient conditions. In our TCP game, we avoid the difficulty to check convexity of the utility function and prove the uniqueness of the NE (Theorem 2). Relying on numerical simulations, we observe that the best reply function is always concave for a large range of network parameters, and since this property together with the uniqueness of NE indicates the global stability of NE (Proposition 9), we conjecture that it very likely that TCP game has a global stable NE and the best-reply dynamics always converges to the NE.

### D. *Appendix 3: Proof for Theorem 6*

Theorem 6 is restated as follows: There is a unique Nash Equilibrium (NE) $n_\alpha^*$ in the continuous kernel symmetric TCP connection game with utility function 3. At this NE, all players have the same number of connections. This NE is an interior point of the strategy space for $m < m_{0,\alpha}$ and $\mathbf{n}_\alpha^* = (1, 1, ..., 1)$ for $m \geq m_{0,\alpha}$, where $m_{0,\alpha}$ is given in (50).

*Proof:* .

Following a similar procedure in the proof of Theorem 2, we take derivative of utility function of player $i$ with respect to its number of connections $n_i$ to get

$$\frac{\partial U_i}{\partial n_i} = \frac{C\sum_{k=1,k\neq i}^m n_k}{(n_j + \sum_{k=1,k\neq i}^m n_k)^2} - \frac{\beta}{\phi} - \frac{\beta n_i C\varphi}{(n_j + \sum_{k=1,k\neq i}^m n_k)^2[(p-1)\varphi - \phi]} - \alpha\phi \tag{44}$$

Let $\delta_i n_i = \sum_{k=1,k\neq i}^m n_k$, then we get

$$\frac{1-p}{1+\delta_i}[\delta_i + \frac{\beta\varphi}{(1-p)\varphi + \phi}] - \beta - \alpha\phi = 0 \tag{45}$$

Similarly, we can prove that $\delta_i = \delta_j, \forall i, j$, which means that at NE, all users have the same number of connections. Thus, (44) can be rewritten as

$$\frac{m-1}{\beta} - \frac{m}{1-p} + \frac{\varphi}{(1-p)\varphi + \phi} - \frac{\alpha}{\beta}\frac{m\phi}{1-p} = 0 \tag{46}$$

Let $F(p)$ denote the LHS of this equation. We can get

1) as $p \to 0$, $F(p) \to \frac{m-1}{\beta} - (m-1) > 0$.
2) as $p \to 1$, $F(p) \to \frac{m-1}{\beta} - \infty + \frac{\varphi_1}{\phi_1} - \frac{alpha}{\beta}\infty < 0$.

As in Theorem 2, we have

$$\frac{dF}{dp} = \frac{-m}{(1-p)^2} + \frac{\varphi'\phi}{[(1-p)\varphi + \phi]^2} - \frac{\alpha}{\beta}\frac{\varphi + \phi}{(1-p)^2} < 0 \tag{47}$$

Thus, as $F(p)$ is a decreasing function, there must be unique $p_\alpha^*$ such that $F(p^*) = 0$. Solve (46) for $p_\alpha^*$ and substitute it into $\frac{\partial U_i}{\partial n_i}$, we can get a unique $\mathbf{n}_\alpha^* = (n_\alpha^*, n_\alpha^*, ..., n_\alpha^*)$, namely, the unique NE.

We also can show that if $m$ gets very large, this unique NE will no longer be an interior point of the strategy space. Instead, it will be $\mathbf{n} = \{1, 1, ..., 1\}$. Recall (46), and let

$$F(p, m) = m(\frac{1}{\beta} - \frac{1}{1-p} - \frac{\alpha}{\beta}\frac{\phi}{1-p}) - \frac{1}{\beta} + \frac{\varphi}{(1-p)\varphi + \phi} \tag{48}$$

Similarly as in the proof of Theorem 2, given a value of $m$, we can plot a curve for $F(p, m)$ with $p$ as x-axis and $F(p, m)$ as y-axis. It can be easily verified that these curves (with different $m$ values) all meet at a single common point $(p_0, F(p_0, m))$ with $p_0$ as a solution to $1 - p = \alpha\phi + \beta$.

Recall that $F(p, m)$ is a monotonic decreasing function of $p$, and $F(p^*_\alpha, m) = 0$. Since $F(p_0, m) < 0$, so $p^*_\alpha$ must be smaller than $p_0$.

When $p < p_0$, we get

$$\frac{dF}{dm} = \frac{1}{\beta} - \frac{1}{1-p} - \frac{\alpha}{\beta}\frac{\phi}{1-p} > 0 \tag{49}$$

Combined with the result that $F(p, m)$ is a monotonic decreasing function of $p$, then it must be $p$ is an increasing function of $m$, and for $F(p^*_\alpha) = 0$, as $m$ increases, $p$ approaches $p_0$.

Since in NE, we must have $B^*_\alpha \cdot (1 - p^*_\alpha) = C/(m \cdot n^*_\alpha)$, and since all users must have at least one connection, i.e., $n^*_\alpha \geq 1$, we have to make sure that $m \leq m_0$ where $m_0$ is the solution of

$$m(1 - p^*_\alpha)/\phi^*_\alpha = C \tag{50}$$

Similarly as in Theorem 2, we know that as $m$ increases, $p^* \to p_{0,\alpha}$, then $\phi^*_\alpha$ as a function of $p^*_\alpha$ also increases to $\phi_{0,\alpha}$ (function of $p_{0,\alpha}$). Thus, $(1 - p^*_\alpha)/\phi^*_\alpha$ is bounded. So, as $m$ becomes larger and larger, eventually, $m(1 - p^*_\alpha)/\phi^*_\alpha$ will be larger than $C$, which means that all users only use one connection at NE. When this happens, the NE is no longer an interior point. After that, as $m$ keeps increasing, $p^*$ is larger than $p_0$ and eventually approaches 1. Let $m_{0,\alpha}$ denote this threshold value, then we also need to rely on numerical method to identify $m_{0,\alpha}$. ■

### E. Appendix 4

Proof for Theorem 7. This theorem is restated as follows. In the continuous kernel multiple player TCP connection game with computation power consideration, when the number of users is not larger than $m_0$ given in (52), there exists an interior-point NE in which the more powerful user will have more connections and higher goodput and utility.

*Proof:* For any two arbitrary players $i$ and $j$. Let $\sum_{k=1,k\neq i}^{m} n_k = \delta_i n_i$ and $\sum_{k=1,k\neq j}^{m} n_k = \delta_j n_j$, then consider

$$\partial U_i/\partial n_i = ((1-p)/\phi)(\delta_i/(1+\delta_i)) - \alpha_i = 0$$
$$\partial U_j/\partial n_j = ((1-p)/\phi)(\delta_j/(1+\delta_j)) - \alpha_j = 0$$

We get $\alpha_i/\alpha_j = (N - n_i)/(N - n_j)$ where $N = \sum_{k=1}^{m} n_k$. Then we have $m - 1$ independent such equations, and use the fact that $N = \sum_{k=1}^{m} n_k$, we can get

$$n^*_i = (\sum_{k=1}^{m} \alpha_k - (m-1)\alpha_i)N/(\sum_{k=1}^{m} \alpha_k) \tag{51}$$

Substituting (51) into $\partial U_i/\partial n_i = 0$, we get

$$(1 - p^*)/\phi = \sum \alpha_k/(m-1)$$

and solve this equation we can get $p^*$. Let $F(p^*, m) = (1 - p^*)(m - 1) - \phi^* \sum \alpha_k$, then note that $\lim_{p^*\to 0} F > 0$ and $\lim_{p^*\to 1} F < 0$ and $F(p^*, m)$ is a decreasing function of $p^*$, then, there must be a unique solution $p^*$ for this equation. Since any $p^*$ must satisfy $(1 - p^*)/\phi^* = C/N^*$, then

$$N^* = C(m-1)/\sum \alpha_k$$

together with (51), we can find $\mathbf{n}^*$.

Without loss of generality, we can let $n^*_1$ to be the smallest number of connections (corresponding to the largest $\alpha_i$). Then, in order for $p^*$ to be an interior point (make sure $n^*_1 \geq 1$), $m$ should be smaller than $m_0$, where $m_0$ is the largest $m$ such that

$$\frac{C\phi^*(\sum \alpha_k - (m-1)\alpha_1)}{(1-p^*)\sum \alpha_k} \geq 1 \tag{52}$$

Note that at this interior-point NE, more powerful users (with smaller $\alpha$) will have more connections. Since goodput and utility are both increasing functions of $n_i$, it follows that more powerful users have higher goodput and higher utility at NE. ■