

Resolution for FO:

To show $\Gamma \vdash \alpha$, we show that $\Gamma \cup \{\neg\alpha\} \vdash \square$.

Let $\varphi \stackrel{\text{def}}{=} \Gamma \cup \{\neg\alpha\}$. (Assume that Γ is a finite set of formulas. Otherwise we would do this repeatedly for increasing subsets of Γ until we prove \square .)

Put φ into RPF-CNF form and then Skolemize, getting $\varphi_S \equiv \forall x_1 \cdots x_k (C_1 \wedge C_2 \wedge \cdots \wedge C_t)$.

We then try to use resolution to show that $\{C_1, \dots, C_t\} \vdash \square$, where we remember that all the variables are universally quantified, so at any step, for any clause, C_i , we may substitute any term, t for any variable, x_j , getting the new clause $C_i[t/x_j]$.

Goal: we will show that FO resolution is complete. (Gödel proved in his Ph.D. thesis that the proof system given in *Principia Mathematica* by Russell and Whitehead is complete. We will prove that Resolution is complete.)

Theorem 10.1 [Gödel's Completeness Theorem for First Order Logic, 1929] *for all $\varphi \in \mathcal{L}(\Sigma), \Gamma \subseteq \mathcal{L}(\Sigma)$,*

$$(\Gamma \models \varphi) \quad \Leftrightarrow \quad (\Gamma \vdash \varphi) .$$

[Recall that $\Gamma \vdash \varphi$ iff $\Gamma \wedge \neg\varphi \vdash \square$, i.e., when we put $(\Gamma \wedge \neg\varphi)$ into RPF-CNF and then Skolemize, we can derive the empty clause using resolution and substitution of terms for variables. All of the variables are understood to be universally quantified because the Skolemized formula is universal.]

Proof:

Soundness: $(\Gamma \vdash \varphi) \Rightarrow (\Gamma \models \varphi)$, because we saw in the propositional case that resolution preserves truth and substitution into universal formulas also preserves truth, i.e., for any variable v , formula, φ , and term, t ,

$$\forall v(\varphi) \models \varphi[t/v] .$$

We now show the opposite direction. Assume that $\Gamma \models \varphi$.

Let $\gamma = \Gamma \wedge \neg\varphi$. Thus, $\gamma \in \text{FO-UNSAT}$ ¹

We put γ into RPF-CNF form, and then Skolemize to obtain, $\gamma_S \in \text{FO-UNSAT}$,

$$\gamma_S = \forall x_1 x_2 \dots x_k (C_1 \wedge \dots \wedge C_r)$$

Since $\gamma_S \in \text{FO-UNSAT}$ it has no model, so certainly no Herbrand model.

Let $\mathcal{T} = \{C_i[t_1/x_1, \dots, t_k/x_k] \mid 1 \leq i \leq r, t_1, \dots, t_k \in \text{closedTerm}(\Sigma')\}$, where Σ' is Σ plus all the added Skolem constant and function symbols.

Suppose that $\mathcal{H} \models \mathcal{T}$. It would follow that $\mathcal{H} \models \gamma_S$. But $\gamma_S \in \text{FO-UNSAT}$. Therefore, \mathcal{T} is unsatisfiable.

¹Here we are assuming that Γ is a finite set of formulas, so its conjunction is a single formula. If Γ is infinite, then let $\Gamma = \{\alpha_1, \alpha_2, \dots\}$. We would put each α_i into RPF-CNF form and Skolemize it, and do the same for $\neg\varphi$. Then we would have an infinite set of clauses, which are all together unsatisfiable.

Observe that every atomic formula in \mathcal{T} is of the form $R(t_1, \dots, t_a)$ where $t_1, \dots, t_a \in \text{closedTerm}(\Sigma')$. Each such formula acts exactly like a propositional variable, $p_{R(t_1, \dots, t_a)}$. Thus, \mathcal{T} is an unsatisfiable set of propositional formulas. Therefore, by the Completeness Theorem for PropLogic, $\mathcal{T} \vdash \square$. \square

Definition 10.2 [Unification, Most General Unifier] A substitution, s , *unifies* a set of atoms, $\{A_1, \dots, A_k\}$, iff $A_1s = A_2s = \dots = A_k s$. If g is a unifier of a set of atoms, such that for any other unifier, s of these atoms, there is a substitution, s' , such that $s = gs'$, then g is called the *most general unifier* (mgu). \square

Example 10.3 Let $\varphi_0 = \{C_1, C_2\}$ where $C_1 = \{\neg P(x), \neg P(f(w)), Q(y)\}$, $C_2 = \{P(z)\}$ be a set of two (universally quantified) clauses.

The substitution, $s = [z/x]$, unifies $\{P(x), P(z)\}$. Thus, we can derive the clause

$$C_3 = \{\neg P(f(w)), Q(y)\} = \text{Res}(\{\neg P(x), \neg P(f(w)), Q(y)\}, \{P(x)\}) = \text{Res}(C_1[s], C_2[s])$$

Furthermore, the substitution, $t = [x/z, f(w)/x]$, unifies the set $\{P(x), P(f(w)), P(z)\}$. Thus, in one step, we can derive

$$C_4 = \{Q(y)\} = \text{Res}(\{\neg P(f(w)), Q(y)\}, \{P(f(w))\}) = \text{Res}(C_1t, C_2t)$$

In fact s and t are mgu's. \square

Algorithm 10.4 [Unification Algorithm]

Input: Set of atoms $S = \{A_1, A_2, \dots, A_r\}$ to be unified.

Output: Most General Unifier (mgu) for S if it exists, else "not unifiable"

1. $s := \emptyset$ # empty substitution
2. **while** ($|S| > 1$) :
3. *scan each atom from left to right until first difference*
4. **if** none of these symbols is a variable: **return**("not unifiable")
5. *Let x be the variable and $t \neq x$ be a term starting at same point*
6. **if** x occurs in t : **return**("not unifiable")
7. $s := s[t/x]$
8. **return**("mgu is ", s)

Example 10.5 [Running Unification Algorithm on Example 10.3]

First Iteration: $S = \{P(x), P(f(w)), P(z)\}$ Start scanning each character, the first difference is at the 3rd character: x , f , and z .

Choose variable x and term $f(w)$: $s := [f(w)/x]$

Second Iteration: $S = \{P(f(w)), P(z)\}$ Still at 3rd character, choose variable z and term $f(w)$: $s := [f(w)/x, f(w)/z]$

return(mgu is $[f(w)/x, f(w)/z]$) \square

Proposition 10.6 *If the set of atoms, S , is unifiable, then the Unification Algorithm returns the mgu, otherwise it returns "not unifiable"*

Running time: In the worst case can be exponential because we can have weird nested repeated substitutions (example in text).

Good news: With the right data structure, where atoms are represented with a DAG (circuit), the algorithm is linear because we progress by one character at each step and each substitution is constant time.

We now have a useful, complete proof system for FO Logic. In the following algorithm, **rename variables** if necessary so that **no variable occurs in more than one clause**. This will allow all possible unifications to be found.

Algorithm 10.7 *[Complete FO Resolution Algorithm]*

Input: $\psi \in \mathcal{L}(\Sigma)$

Output: *if $\psi \in \text{FO-VALID}$ then a proof of this fact; otherwise, algorithm might never halt*

1. $\psi' :=$ universal closure of ψ
2. $\varphi := \neg\psi'$
3. $\varphi' :=$ formula equivalent to φ but in RPF, with quantifier-free part in CNF
4. $\varphi_S := \forall x_1 \dots x_k (C_1 \wedge \dots \wedge C_m)$, the Skolemization of φ'
5. **# Note:** $\varphi_S \in \text{FO-UNSAT} \iff \psi \in \text{FO-VALID}$
6. $\text{ClauseSet} := \{C_1, \dots, C_m\}$
7. **while** ($\square \notin \text{ClauseSet}$):
8. *use Algorithm 10.4 to apply resolution, adding resulting clauses to ClauseSet*
9. **return**(" $\psi \in \text{FO-VALID}$ ")